Fracton topological order, generalized lattice gauge theory, and duality

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We introduce a generalization of conventional lattice gauge theory to describe fracton topological phases, which are characterized by immobile, pointlike topological excitations, and subextensive topological degeneracy. We demonstrate a duality between fracton topological order and interacting spin systems with symmetries along extensive, lower-dimensional subsystems, which may be used to systematically search for and characterize fracton topological phases. Commutative algebra and elementary algebraic geometry provide an effective mathematical tool set for our results. Our work paves the way for identifying possible material realizations of fracton topological phases.

I. INTRODUCTION

Topological phases of matter are currently attracting tremendous interest from diverse disciplines such as theoretical physics, quantum information, and quantum materials. Gauge theory provides a unified framework for understanding several important topological phases, from quantum Hall states to quantum spin liquids [1–11]. The fractional statistics of topological excitations [12,13], or anyons, can be understood from the Aharanov-Bohm phase of a charge moving around a flux. The topological degeneracy of the ground state is characterized by the holonomy of a locally flat gauge connection around noncontractible loops.

Recently, a new kind of topological phase which does not fit into the framework of gauge theory has been discovered in exactly solvable lattice models in three dimensions [14–18]. A remarkable property of this new phase is the existence of pointlike fractional excitations termed "fractons" [18], which can be created only at the corners of membrane- or fractal-like operators, unlike anyons that are created at the ends of Wilson lines. The creation of anyons at the two ends of a Wilson line immediately implies that anyons can move by repeated application of a local, linelike operator. In contrast, the absence of any operator that can create a pair of fractons implies that a single fracton cannot move without creating additional excitations; that is, fractons are fundamentally *immobile*. Thus far, two broad classes of fracton topological orders have been found:

Type I fracton phases, such as the Chamon-Bravyi-Leemhuis-Terhal (CBLT) model [14,15] and the Majorana cubic model [18], have fracton excitations appearing at the corners of *membrane* operators; composites of fractons form topological excitations that are mobile only within lower-dimensional subsystems.

Type II fracton phases, such as Haah's code [16] and related models [17], have fracton excitations that appear at the corners of *fractal* operators [19]. All topological excitations are strictly localized, and there are no mobile topological quasiparticles.

Fracton topological order provides an exciting development in the search for new quantum phases of matter, for new schemes for quantum information processing [20], and in the investigation of glassy dynamics in interacting quantum systems [21]. Fractons enable new forms of electron fractionalization [18] and provide an alternative to Fermi or Bose

statistics in three dimensions. Fractons may be used to build a robust, finite-temperature quantum memory, as theoretically demonstrated for Haah's code [16,22]. The innately slow dynamics of fractons provides an intriguing connection with quantum glasses, many-body localization, and a new testing ground for the postulates of quantum statistical mechanics.

Research on fracton topological phases is in its early stages and has been based on studies of specific lattice models. It is thus highly desirable to find a more unified theoretical framework for fracton topological order. In this work, we demonstrate that fracton topological phases can be obtained as the quantum dual of d-dimensional systems that possess "subsystem symmetries", namely, a set of symmetries associated with subsystems of dimension $2 \leqslant d_s < d$. Specifically, we establish an exact duality relating both type-I and type-II fracton topological orders to symmetry-breaking order in quantum systems with subsystem symmetries along planes and fractals, respectively. This duality between fracton topological order and subsystem symmetry breaking, hereafter referred to as the *F-S* duality, is naturally obtained from a generalized lattice gauge theory which we introduce. Instead of placing a gauge field on links between neighboring sites as in standard lattice gauge theory, we introduce a field to mediate *multibody* interactions between matter fields on a cluster of neighboring sites. This yields an interacting quantum system with a generalized Gauss's law that characterizes the fracton topological phase.

Before describing our construction in generality, we present a concrete example that yields a different class of type-I fracton topological phases. Consider a model of Ising spins at the sites of a three-dimensional cubic lattice, whose Hamiltonian H_{plaq} is defined to be a sum of four-spin interactions at each plaquette, as shown in Table I. This classical "plaquette Ising model" is invariant under a spin flip $\tau \to -\tau$ along any xy, yz, or xz plane of the cubic lattice. The plaquette Ising model has a rich history of study, attracting interest as a model for the statistical mechanics of smooth surfaces and as a lattice regularization of string theory [23–28].

We introduce a generalized lattice gauge theory to construct the quantum dual of the plaquette Ising model in a transverse field. This generalizes Wegner's duality [1] between the d-dimensional transverse-field Ising model and Ising lattice gauge theory [2]. Wegner's duality is motivated by the observation that a configuration of Ising spins may be specified by

the locations of the *domain walls* between symmetry-breaking states of the Ising model. As a result, a dual representation of the Ising matter is given by Ising "domain wall" fields on the links of the lattice. Furthermore, since domain walls form closed, $(d-1)$-dimensional surfaces, physical states in the domain wall Hilbert space must satisfy a local "zero-flux" condition that the lattice curl of the domain wall spins vanishes around each plaquette. In this way, the d-dimensional transverse-field Ising model is dual to $\mathbb{Z}_2$ lattice gauge theory.

Our duality between fracton topological order and subsystem symmetry breaking is obtained by a similar observation. A configuration of Ising spins may, equivalently, be specified by the eigenvalue of each *interaction term* in the Hamiltonian [29]. For example, to obtain the dual of the plaquette Ising model, we are naturally led to introduce the Ising fields $\{\sigma\}$ at the center of each plaquette. Physically, the σ field labels the presence or absence of a domain wall between the subsystem symmetry-breaking ground states of the plaquette Ising Hamiltonian. While domain walls in the ordinary Ising model form closed surfaces, the σ fields in our model must satisfy more exotic local constraints due to the geometry of the plaquette interactions to ensure a one-to-one correspondence with the physical space of domain walls in H_{plaq}.

As we demonstrate below, the quantum dual of the plaquette Ising Hamiltonian, in terms of the σ fields, exhibits fracton topological order. The resulting fracton Hamiltonian has a solvable limit, analogous to the deconfined phase of a conventional gauge theory, which is given by a Hamiltonian for the Ising fields σ, now placed at the links of the dual cubic lattice. As shown in Table I, this fracton Hamiltonian consists of two types of terms: (1) a 12-spin σ^x interaction for the spins surrounding a dual cube and (2) four-spin σ^z interactions at each vertex of the dual cubic lattice that are aligned along the xy, yz, and xz planes. The cubic and crosslike geometries of the interactions motivate the name "X-cube" model. The ground state is topologically ordered, as the ground states are locally indistinguishable, and one of the fundamental excitations, obtained by flipping the eigenvalue of the cubic interaction term, is a fracton. This can be seen by observing that there is no local operator that can create a single pair of cube excitations. For example, the operator σ_n^z creates four cube excitations when acting on the ground state. Repeated application of σ_n^z over a membrane separates the four cube excitations to each *corner*, as shown in Fig. 1(a). Therefore, a single cube excitation is fundamentally immobile and cannot move without creating additional cube excitations. Pairs of cube excitations, however, can be moved by sequentially applying a local, membranelike operator.

The quasiparticle content of the X-cube model is summarized in Table I, along with other fracton phases such as Haah's code and a spin model which we introduce in this work and call the "checkerboard model"; these and other fracton phases such as the CBLT model are discussed in the Supplemental Material [30]. All of these phases are obtained by applying our generalized lattice gauge theory prescription to spin models with subsystem symmetries. As we will demonstrate, a simple property of the classical spin model, that no product of interaction terms acts exclusively on a pair of isolated spins, guarantees that its quantum dual exhibits fracton topological order.
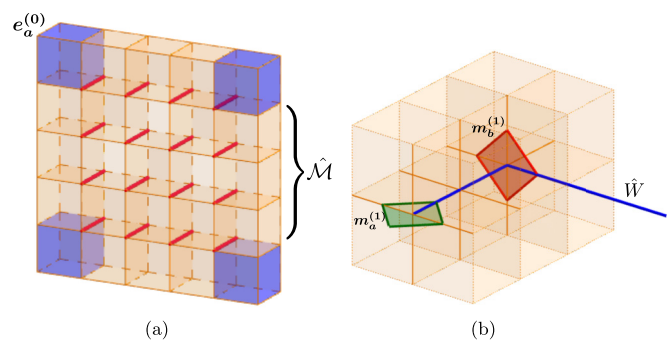

FIG. 1. The fundamental excitations of the X-cube model are shown in (a) and (b). Acting on the ground state of the X-cube model with a product of σ^z operators along the colored red links that lie within a flat, rectangular region $\mathcal{M}$ generates four fracton cube excitations $e_a^{(0)}$ at the corners of the region. A straight Wilson line of σ^x operators acting on the blue links in (b) isolates a pair of quasiparticles ($m_a^{(1)}$ or $m_b^{(1)}$) at the ends that are only free to move along the line. Attempting to move these quasiparticles in any other direction by introducing a corner in the Wilson line creates a topological excitation at the corner, as shown in (b).
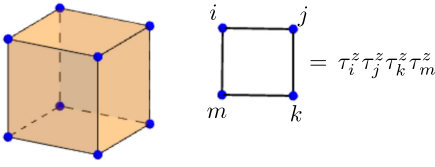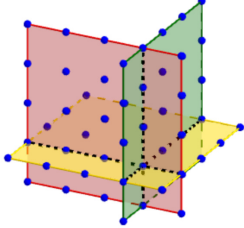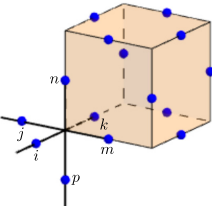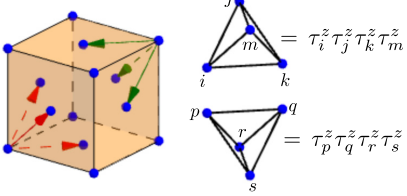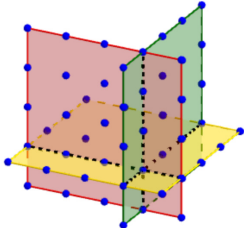
More generally, consider a classical Hamiltonian for Ising spins τ_i at the sites of a three-dimensional Bravais lattice. We assume, for simplicity of presentation, that there is a single spin at each lattice site; the case where the unit cell is larger is explained in the Supplemental Material [30]. The Hamiltonian consists of ℓ types of interactions at each lattice site i and may be written in the form
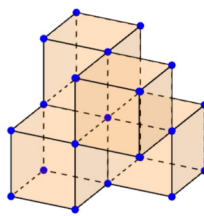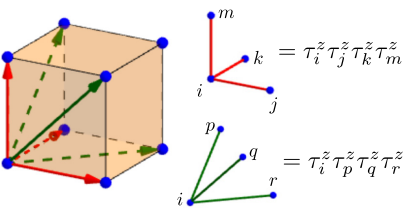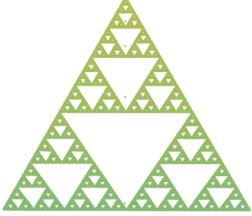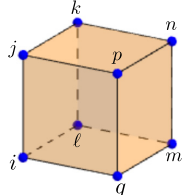
$$H_0 = -t \sum_i \left(\mathcal{O}_i^{(1)}[\tau] + \cdots + \mathcal{O}_i^{(\ell)}[\tau] \right), \tag{1}$$

with the constant $t > 0$. We demonstrate that a classical spin Hamiltonian (1) satisfying certain simple properties may be used to build a topologically ordered, quantum system with fracton excitations. First, we require that the spin Hamiltonian (1) has a *subsystem symmetry* under which the spin-flip transformation $\tau \to -\tau$ along nonlocal subsystems of the lattice, i.e., subsystems that scale with the system size, leaves H_0 invariant. We further require that H_0 has no *local* symmetries. In this sense, a subsystem symmetry is "intermediate" between local and global symmetries [31–33]. For the remainder of this work, we will refer to the plaquette Ising model H_{plaq} and the tetrahedral Ising Hamiltonian H_{tetr} as concrete examples. As shown in Table I, the Hamiltonian H_{tetr} is defined on the fcc lattice and consists of nearest-neighbor four-spin interactions that form elementary tetrahedra. The tetrahedral Ising model has two interaction terms per site on the fcc lattice. Both the tetrahedral and plaquette Ising models have a subsystem symmetry, as they are invariant under spin flips along orthogonal planes (xy, yz, or xz).

An important consequence of the subsystem symmetry of the spin Hamiltonian H_0 is that the resulting ground state has subextensive *classical* degeneracy D, taking the form $\log_2 D \sim O(L)$ on the length-L three-torus. Since the degeneracy is classical in nature, each ground state may be distinguished by a local order parameter. Transitioning between ground states, however, requires performing a spin flip along a subsystem. While a local perturbation can lift the classical degeneracy, no local operator can connect distinct ground states.

TABLE I. Representative examples of fracton topological orders built from classical spin systems with subsystem symmetry. The classical models shown above are the plaquette, tetrahedral, and fractal Ising models (H_{plaq}, H_{tetr}, and H_{frac}, respectively). In the plaquette Ising model, spins are placed on the sites of a simple cubic lattice as shown, and the Hamiltonian is a sum of four-spin interactions at the face of each cube. In the tetrahedral Ising model, spins are arranged on an fcc lattice, and each spin participates in four-spin interactions coupling neighboring spins that form a tetrahedron, as indicated. Finally, the fractal Ising model consists of two types of four-spin interactions at each cube. The model has a fractal symmetry and is invariant under a spin flip along a three-dimensional Sierpinski triangle, as elaborated in the Supplemental Material [30]. The X-cube, checkerboard, and Haah's code fracton models ($H_{\text{X-Cube}}$, H_{check}, and H_{Haah}, respectively) are solvable limits of fracton phases that are obtained by promoting the subsystem symmetry of the indicated spin models to a local symmetry. The classical spin Hamiltonian with subsystem symmetries that corresponds to the CBLT model is presented in the Supplemental Material [30]. The fracton model $H_{\text{X-Cube}}$ is naturally represented by placing spins σ on links of the cubic lattice and is a sum of a 12-spin σ^x operator at each cube and the indicated planar, four-spin σ^z operators. The checkerboard model is a sum of eight-spin σ^x and σ^z interactions over cubes arranged on an fcc lattice and is self-dual under the exchange $\sigma^x \longleftrightarrow \sigma^z$. Only the fundamental excitation types are indicated above. Here, we have adopted the notation $e_a^{(j)}$ ($m_a^{(j)}$) to refer to a dimension-j excitation, i.e., an excitation that is only free to move within a dimension-j subsystem without creating additional excitations, that is obtained by flipping a σ^x-type (σ^z-type) interaction. For type-I fracton phases, bound states of fracton excitations can form mobile quasiparticles; these mobile composite excitations are not indicated here.

Classical Spin System	**Subsystem Symmetry**	**Fracton Topological Phase**
$= \tau_i^z \tau_j^z \tau_k^z \tau_m^z$		$A_c = \prod_{n \in \partial c} \sigma_n^x$
		$B_s^{(xz)} = \sigma_i^z \sigma_p^z \sigma_n^z \sigma_k^z$
		$B_s^{(xy)} = \sigma_i^z \sigma_j^z \sigma_k^z \sigma_m^z$
		$B_s^{(yz)} = \sigma_j^z \sigma_n^z \sigma_m^z \sigma_p^z$
Plaquette Ising Model	Planar	**X-Cube Model** $\left[\text{Type I}:\ e_a^{(0)},\ m_a^{(1)},\ m_b^{(1)}\right]$
$= \tau_i^z \tau_j^z \tau_k^z \tau_m^z$ $= \tau_p^z \tau_q^z \tau_r^z \tau_s^z$		$A_c = \prod_{n \in \partial c} \sigma_n^x$ $B_c = \prod_{n \in \partial c} \sigma_n^z$
Tetrahedral Ising Model	Planar	**Checkerboard Model** $\left[\text{Type I}:\ e_a^{(0)},\ m_a^{(0)}\right]$
$= \tau_i^z \tau_j^z \tau_k^z \tau_m^z$ $= \tau_i^z \tau_p^z \tau_q^z \tau_r^z$		$A_c = \mu_i^z \sigma_j^z \mu_k^z \mu_m^z \sigma_n^z \mu_p^z \sigma_p^z \sigma_q^z$ $B_c = \mu_i^x \sigma_j^x \mu_k^x \sigma_\ell^x \mu_\ell^x \mu_m^x \sigma_n^x \mu_p^x \sigma_q^x$
Fractal Ising Model	Fractal	**Haah's Code** $\left[\text{Type II}:\ e_a^{(0)},\ m_a^{(0)}\right]$

II. GENERALIZED LATTICE GAUGE THEORY AND THE *F-S* DUALITY

We now build a quantum Hamiltonian with fracton topological order by promoting the subsystem symmetry of the spin system (1) to a local symmetry. We begin by adding a transverse field at each lattice site to allow the classical spins to exhibit quantum fluctuations. Next, we introduce additional Ising spins $\sigma_{i,a}$ at the center of each multispin interaction appearing in H_0; these spins appear at the sites of a lattice with an ℓ-site basis. We introduce a minimal coupling H_c between the σ spins and the Ising matter fields by coupling each σ to its corresponding multispin interaction:

$$H_c \equiv -t \sum_i \left(\sigma_{i,1}^z \mathcal{O}_i^{(1)}[\tau^z] + \cdots + \sigma_{i,\ell}^z \mathcal{O}_i^{(\ell)}[\tau^z] \right). \quad (2)$$

After this minimal coupling, the Hamiltonian describing both σ and Ising matter fields is given by

$$H = H_c - h \sum_i \tau_i^x. \quad (3)$$

We refer to σ as the *nexus field*, as each σ is placed at the center of an elementary multispin interaction of the classical spin Hamiltonian. We will soon observe that the nexus field provides a natural generalization of a gauge field in a conventional lattice gauge theory. In contrast to our construction, applying the standard gauging procedure to any of the spin models shown in Table I by introducing a gauge field on the links of the cubic lattice would result in a Hamiltonian with conventional $\mathbb{Z}_2$ topological order. Our procedure is also distinct from discretizations of "higher-form" gauge theories, in which interactions between $(n-1)$-form matter fields are mediated by an n-form gauge field [34,35].

The subsystem symmetry of the classical spin system (1) has now been promoted to a local spin-flip symmetry of the Hamiltonian (3). While τ_n^x, the generator of a single spin flip, anticommutes with several multispin interactions in (1), this can be compensated by acting with the nexus field $\sigma_{j,a}^x$ on the lattice sites associated with these interactions. As a result, the operator

$$G_i = \tau_i^x A_i, \quad (4)$$

where

$$A_i \equiv \prod_{(j,a) \in P(i)} \sigma_{j,a}^x \quad (5)$$

generates a local symmetry of the Hamiltonian ($[G_i, H] = 0$). The set $P(i)$ specifies the locations of multispin interactions that anticommute with τ_i^x.

We proceed to add all other interaction terms involving the nexus field and the Ising spins that are consistent with this local spin-flip symmetry. To lowest order, we include a transverse field for the matter and nexus fields:

$$H = -t \sum_{i,a} \sigma_{i,a}^z \mathcal{O}_i^{(a)}[\tau^z] - h \sum_i \tau_i^x - J \sum_{i,a} \sigma_{i,a}^x. \quad (6)$$

Since the operator G_i generates a local symmetry of the Hamiltonian, we may impose the condition $G_i |\Psi\rangle = |\Psi\rangle$ on the Hilbert space, which amounts to a generalized Gauss's law. In analogy with conventional gauge theory, we will refer to A_i

as the *nexus charge* operator. As an example, the nexus charge operator in the X-code model is given by the product of σ^z on 12 spins sitting at the links surrounding a cube, as shown in Table I.

Since the generalized Gauss's law condition commutes with the spin-nexus Hamiltonian (6), it is possible to choose a "gauge" that completely eliminates the Ising matter fields and yields a Hamiltonian exclusively for the nexus spins. First, we impose the generalized Gauss's law $\tau_i^x |\Psi\rangle = A_i |\Psi\rangle$ to obtain a Hamiltonian acting within the constrained Hilbert space,

$$H = -t \sum_{i,a} \sigma_{i,a}^z \mathcal{O}_i^{(a)}[\tau^z] - h \sum_i A_i - J \sum_{i,a} \sigma_{i,a}^x. \quad (7)$$

Since each τ_i^z operator commutes with the Hamiltonian, we may restrict our attention to states in the constrained Hilbert space that satisfy $\tau_i^z = +1$. This yields the gauge-fixed Hamiltonian

$$H = -t \sum_{i,a} \sigma_{i,a}^z - h \sum_i A_i - J \sum_{i,a} \sigma_{i,a}^x. \quad (8)$$

When $t/h \ll 1$, it is convenient to identify an effective Hamiltonian that takes the form

$$H_{\text{eff}} = -K \sum_{i,k} B_i^{(k)} - h \sum_i A_i - J \sum_{i,a} \sigma_{i,a}^x, \quad (9)$$

where we have introduced operators $B_i^{(k)}$ at each lattice site i. These operators are determined in perturbation theory by computing the simplest product of σ^z terms near a given lattice site that commute with the nexus charge $[B_i^{(k)}, A_j] = 0$. As an example the $B_i^{(k)}$ operators obtained by applying this construction to the plaquette Ising model are shown in Table I.

Our proposal bears a resemblance to the construction of a conventional lattice gauge theory. First, the Hamiltonian for the $\mathbb{Z}_2$ gauge theory is recovered from the general form of the Hamiltonian for the Ising matter and nexus fields (6) if the matter fields couple through nearest-neighbor two-body interactions, so that $\mathcal{O}_i^{(a)} = \tau_i^z \tau_{i_a}^z$, where i_a is the nearest neighbor to site i. This is in contrast to the multibody interactions that are present in our models with subsystem symmetry. In the gauge-fixed Hamiltonian (8), A_i then becomes the familiar $\mathbb{Z}_2$ charge operator, while the operator B_n, appearing in the effective Hamiltonian H_{eff}, precisely measures the $\mathbb{Z}_2$ flux through an elementary plaquette.

Within our construction, the $B_i^{(k)}$ operators provide the natural generalization of the flux in a lattice gauge theory. As the excitation obtained by flipping the eigenvalue of a $B_i^{(k)}$ operator is often pointlike, we will refer to the excitation as a "generalized monopole". While the flux is always a linelike excitation in a three-dimensional Abelian lattice gauge theory, the behavior of the generalized monopole can be quite varied. As an example, the generalized monopole is a fracton in both the checkerboard spin model and Haah's code but is free to move along a line without creating additional excitations in the X-cube model. We refer to such an excitation as a dimension-1 quasiparticle [18], as the excitation is only mobile along a line. As shown in Fig. 1(b), a straight Wilson line can create an isolated pair of the generalized monopole excitations in the X-cube model, which can move only along the line without creating additional excitations. Within type-I fracton

topological order, composites of the fracton charge excitations that are mobile in two dimensions can have nontrivial mutual statistics with the generalized monopole. This is true in both the checkerboard and X-cube models, where an anyon formed from a composite of two fracton charges has π statistics with a generalized monopole in its plane of motion.

We now consider the generalized lattice gauge theory (8) when $J = 0$, so that the nexus field (defined in the σ^z basis) has no dynamics. The resulting Hamiltonian

$$H_{\text{nexus}} = -t \sum_{i,a} \sigma^z_{i,a} - h \sum_i A_i \qquad (10)$$

has a local symmetry, as the $B_i^{(k)}$ operators commute with each term in (10). We refer to the emergent local constraints on the Hilbert space

$$B_i^{(k)} |\Psi\rangle = |\Psi\rangle \qquad (11)$$

as the generalized "flatness" condition, analogous to a flat connection in a continuum gauge theory, as these constraints are obtained in the limit that there is zero "flux" of the nexus field.

Our construction of a generalized lattice gauge theory implies that the quantum dual of the Ising matter in the presence of a transverse field,

$$H_{\text{spin}} = -t \sum_{i,a} \mathcal{O}_i^{(a)}[\tau^z] - h \sum_i \tau^x_i, \qquad (12)$$

is precisely given by the nexus Hamiltonian (10), combined with the generalized flatness condition (11).

Without appealing to the generalized lattice gauge theory, the duality can be obtained directly from the Hamiltonian H_{spin}. A dual representation is constructed by placing the nexus spins at the centers of the interactions $\mathcal{O}_i^{(a)}$. The nexus spins are now interpreted as domain wall variables for the ordered phase ($t/h \gg 1$) of the spin model H_{spin}. The nexus spins must satisfy local constraints due to the geometry of the multispin interactions $\mathcal{O}_i^{(a)}[\tau]$ in order to correspond to the physical space of domain walls between ground states of H_0. These local constraints are precisely given by the generalized flatness condition (11). As an example, the generalized monopole operators $B_i^{(k)}$ for the plaquette Ising model are obtained by noting that the product of four plaquette interactions that wrap a cube is equal to the identity. Our *F-S* duality implies a map between local operators in the Hilbert spaces of the Ising matter and nexus fields, as summarized in Table II. As an example, a domain wall in the ground state of the plaquette Ising model is shown in Fig. 2.

III. FRACTON TOPOLOGICAL ORDER

We now invoke the *F-S* duality to demonstrate that the commuting Hamiltonian

$$H_{\text{fracton}} = -\sum_{i,k} B_i^{(k)} - \sum_i A_i \qquad (13)$$

exhibits fracton topological order. We argue that (i) the spectrum of H_{fracton} has subextensive topological degeneracy and (ii) the nexus charge is fundamentally immobile. We
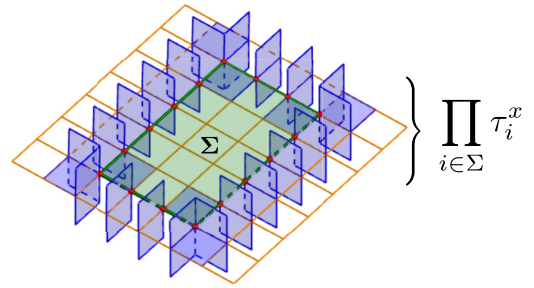

FIG. 2. A "domain wall" in the ground state of the plaquette Ising model H_{plaq} is depicted by coloring the plaquette interactions that have been flipped by the action of a spin-flip transformation along a planar region Σ. The *F-S* duality implies that the ground state for the X-code fracton phase is given by an equal superposition of a dual representation of these domain walls.

provide rigorous proofs of these statements in the Supplemental Material [30] using techniques in commutative algebra and elementary algebraic geometry, which provide effective mathematical tools to study the subsystem symmetries of classical spin models, as well as the ground-state degeneracy and excitation spectrum of fracton topological phases. An algebraic representation of a classical Ising system defines an algebraic variety over the field of characteristic 2 ($\mathbb{F}_2$), defined by $\mathbb{Z}_2$ addition and multiplication [36]. Two conditions on this variety, as derived in the Supplemental Material [30] from the Buchsbaum-Eisenbud criterion [37,38] for the exactness of a complex of free modules, guarantee that the quantum dual exhibits fracton topological order.

We begin by using the *F-S* duality to demonstrate that the subextensive degeneracy of the classical, $h = 0$ ground state of H_{spin} implies that the Hamiltonian H_{fracton} has subextensive *topological* ground-state degeneracy on the torus. Recall that a product of τ^x_i operators along an appropriate subsystem Σ generates a symmetry of the Hamiltonian H_{spin}. When $t/h \gg 1$, the ground state exhibits classical, subextensive degeneracy since there are $O(L)$ independent subsystems along which a spin flip commutes with all of the interaction terms $\mathcal{O}_i^{(a)}$. The plaquette Ising model, for example, commutes with the product of τ^x_i along a plane, and the ground state has subextensive, classical degeneracy since there are $O(L)$ independent planes along which a spin flip may be performed.

From the operator dictionary for the *F-S* duality, the dual representation of this spin-flip operator is given by a product of nexus charges A_i along the same subsystem Σ. Furthermore, each interaction term $\mathcal{O}_i^{(a)}$ is dual to a single-spin operator $\sigma^z_{i,a}$. Since the *F-S* duality preserves the commutation relations between operators, we conclude that due to the subsystem symmetry of H_{spin}, the operators in the dual theory satisfy

$$\left[\sigma^z_{i,a}, \prod_{i \in \Sigma} A_i \right] = 0 \qquad (14)$$

for all i, a. This commutation relation can be satisfied only if the product of nexus charges along Σ yields the identity, so that $\prod_{i \in \Sigma} A_i = 1$. This relation implies that not all of the nexus charge operators are independent on the torus. Each of the $O(L)$ independent subsystems associated with the subsystem symmetry of H_{spin} reduces the number of independent nexus

TABLE II. Correspondence between operators in the Hilbert spaces of the Ising matter fields and the nexus spins, obtained from the *F-S* duality.

H_{spin}	H_{nexus}	Explanation
$\mathcal{O}_i^{(a)}$	$\sigma_{i,a}^z$	A classical configuration of Ising spins τ may be specified by spins σ placed at the centers of each of the interaction terms $\mathcal{O}_i^{(a)}$ for the Ising matter.
τ_i^x	A_i	The dual representation of τ_i^x is given by the nexus charge A_i, defined as the product of σ_i^x terms that correspond to the interactions flipped by τ_i^x.
$\prod\limits_{(i,a)\in Q(j)} \mathcal{O}_i^{(a)} = 1$	$B_j^{(1)}\lvert\Psi\rangle = \lvert\Psi\rangle$	A local product of interactions for the Ising matter fields that yields the identity corresponds to a constraint on the dual Hilbert space. The constraint restricts the Hilbert space of $\{\sigma\}$ to that of *domain wall* configurations in the ordered phase of H_0.
$\widetilde{W} \equiv \prod\limits_{(i,a)\in\Sigma} \mathcal{O}_i^{(a)}$	$W \equiv \prod\limits_{(i,a)\in\Sigma} \sigma_{i,a}^z$	A product of σ^z operators is dual to a product of interaction terms for the matter fields. As a result, the nexus charge is a fracton only if there is no product of interaction terms that can create an isolated pair of spin flips.

charge operators by one. We denote the total number of such independent subsystems by k_A.

In the ground state of the fracton Hamiltonian, the 2^N-dimensional Hilbert space of N nexus spins is constrained by the M nexus charge and monopole operators that appear in the Hamiltonian. However, only $M - k$ of the operators are independent on the torus, where $k = k_A + k_B$ is the number of "dependency relations" on both the nexus charge and monopole operators. The topological ground-state degeneracy on the torus is given by $D = 2^{k+(N-M)}$. When the number of interactions appearing in H_{fracton} is identical to the total number of nexus spins ($N = M$), as is the case for all of the fracton models considered in this work, the topological degeneracy is precisely $D = 2^k$. In this case, the subextensive degeneracy of the $h = 0$ ground state of the spin model H_{spin} provides a lower bound on the topological degeneracy of H_{fracton}. For example, the checkerboard spin model has topological ground-state degeneracy $\log_2 D = 6L - 6$ on the length-L three-torus, as we compute in the Supplemental Material [30], while the tetrahedral Ising model has only classical degeneracy $\log_2 D_c \sim O(3L)$ since the model has subsystem symmetries along three orthogonal planes.

In addition to the subextensive, topological degeneracy of H_{fracton}, we also wish to show that that there is no degeneracy in the spectrum of the Hamiltonian due to the presence of *local* observables. In the absence of local observables, the local reduced density matrix will be identical for any degenerate states in the spectrum of the Hamiltonian, and the topological degeneracy, as computed by constraint counting, will be stable to local perturbations [39]. As we demonstrate in the Supplemental Material [30], the ground states of H_{fracton} are guaranteed to be locally indistinguishable, provided that the classical spin system H_0 has no lower-dimensional symmetries along subsystems of dimension $d_s < 2$ (e.g., linelike symmetries). We prove this by using an algebraic representation of H_{fracton} and also argue this as a consequence of the *F-S* duality.

Having demonstrated that H_{fracton} exhibits subextensive topological degeneracy and that the degenerate ground states are locally indistinguishable, we now demonstrate that the nexus charge is indeed a fracton excitation, provided that the

spin model (12) satisfies a simple condition. Consider acting on the ground state of H_{fracton} with the operator

$$W \equiv \prod_{(i,a)\in\Sigma} \sigma_{i,a}^z, \tag{15}$$

where Σ is some subset of the lattice. The operator W will create nexus charge excitations by anticommuting with a collection of A_i operators. Invoking the *F-S* duality, we observe that the pattern of excitations created by W is precisely given by the location of spin flips created by the dual operator $\widetilde{W} \equiv \prod_{(i,a)\in\Sigma} \mathcal{O}_i^{(a)}[\tau^z]$ when acting on the paramagnetic state $\lvert\Psi_{\text{para}}\rangle \equiv \lvert\rightarrow \cdots \rightarrow\rangle$, as shown schematically in Fig. 3. The spectrum of H_{fracton} contains fractons only if there is *no operator* of the form $\widetilde{W}$ that can create a single pair of spin-flip excitations. If such an operator did exist, then it would be possible to move a single nexus charge without any energy cost, and the charge would be mobile.

Our condition for the existence of fracton excitations is simple to demonstrate for the plaquette Ising and tetrahedral Ising Hamiltonians. Here, it is evident that any product of the four-spin interactions shown in Table I creates *at least* four spin-flip excitations when acting on the paramagnetic state $\lvert\Psi\rangle$. Therefore, the nexus charge for each of the corresponding H_{fracton} Hamiltonians is a fracton. In fact, if any Ising Hamiltonian H_0 has a subsystem symmetry along three orthogonal planes and two or more independent interactions per lattice site, then the quantum dual will always exhibit fracton topological order, as we demonstrate in the Supplemental Material [30]. All type-I fracton topological phases that have been discovered thus far fit into this framework.

We now summarize the precise conditions on the classical spin system H_0, as derived in the Supplemental Material [30] using the algebraic representation of the classical spin system, that guarantee that H_{fracton} exhibits fracton topological order:

(1) H_{spin} contains more than one independent interaction term per lattice site.

(2) No product of the interaction terms $\mathcal{O}_i^{(a)}$ can generate an isolated pair of spin flips.
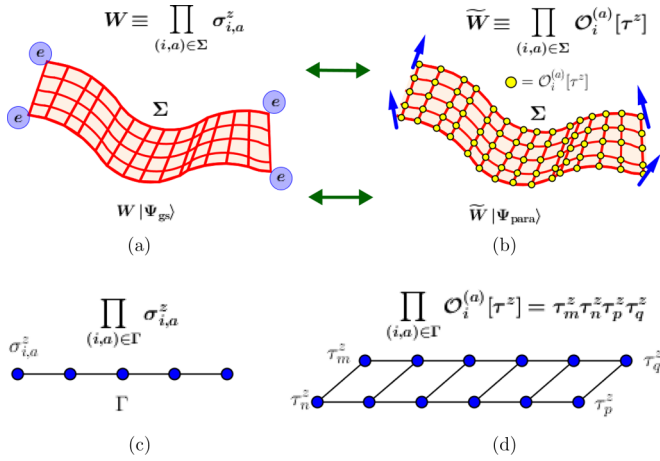
FIG. 3. The nexus charge is a fracton only if there is no operator W that can create an isolated pair of excitations when acting on the ground state of H_{fracton}, as in (a). Equivalently, a dual representation of the operator, given as a product of the interaction terms in the quantum dual as shown in (b), cannot create an isolated pair of spin flips when acting on the paramagnetic state $|\Psi_{\text{para}}\rangle \equiv |\rightarrow \cdots \rightarrow\rangle$. An example is given in (c) and (d); a straight Wilson line acting on the ground state of $H_{\text{X-Cube}}$ in (c) admits a dual representation as a product of four-spin plaquette interactions along a line, as shown in (d). No product of interaction terms in the plaquette Ising model can produce an isolated pair of spin flips. As a result, the nexus charge in $H_{\text{X-Cube}}$ must be a fracton.

IV. PHASE DIAGRAM

We discuss the phase diagram of the spin-nexus Hamiltonian (6). In the limit $t \ll h$, the nexus field decouples from the Ising matter; τ_i^x is set to 1, and the effective Hamiltonian for the nexus field is

$$H_{t \ll h} = -\tilde{t} \sum_{i,k} B_i^{(k)} - J \sum_i \sigma_i^x. \tag{16}$$

The generalized Gauss's law becomes $A_i = 1$. Here, $\tilde{t}$ is some power of t as $B_i^{(k)}$ is obtained from perturbation theory. When $J \ll \tilde{t}$, the nexus field forms a fracton phase that is described by Hamiltonian H_{fracton}. The topologically ordered fracton phase and the Ising paramagnet survive up to a finite t/h and J/h, as both phases are gapped and stable to perturbations [39].

Confinement. From the topologically ordered fracton phase, we may proceed in two directions. First, we consider increasing J/h while keeping $t/h \ll 1$ a constant. Above a critical value $(J/h) \geqslant (J/h)_*$, the ground state will be a condensate of nexus flux excitations, and the fracton topological order will be destroyed. The nature of the transition between the fracton phase and the trivial (confined) phase is currently unknown.

"Higgs" phase. We now consider the region of the phase diagram where $t \gg h$, keeping $J \ll h$ at a fixed constant. Here, the matter fields enter an ordered state with $\langle \sigma_i^z \mathcal{O}_i^{(a)}[\tau^z] \rangle = 1$. This may be seen as the analog of a Higgs phase, as the Ising order gives the nexus field a mass $m \sim O(t)$ that destroys the fracton topological phase. The ground state in this region of the phase diagram is nondegenerate, even though the
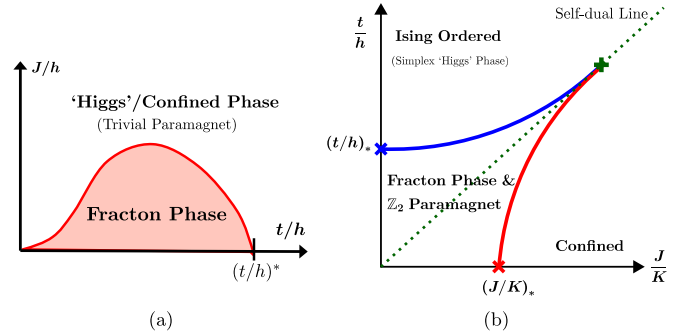

FIG. 4. (a) Schematic phase diagram of the spin-nexus Hamiltonian (6). The Higgs phase for the nexus field is smoothly connected to the phase reached by condensing the nexus charge. The checkerboard model coupled to Ising matter fields admits an additional *self-duality* under the exchange of the nexus charge and flux; as a result, the phase diagram is as shown in (b).

ordered phase of the pure spin model (12) has subextensive degeneracy. We may demonstrate this by observing that in the gauge-fixed Hamiltonian (8), increasing t/h destroys the fracton topological order by condensing the nexus charge and produces a nondegenerate ground state. We also observe from H_{eff} that the confined and Higgs regions of the spin-nexus phase diagram are smoothly connected, as in the Ising lattice gauge theory. We summarize our schematic phase diagram in Fig. 4(a).

In passing, we observe that the checkerboard fracton Hamiltonian in Table I in the presence of two transverse fields $H = -K \sum_c B_c - h \sum_c A_c - \sum_{i,j} (J \sigma_{i,a}^x + t \sigma_{i,a}^z)$ has a symmetry under $\sigma^z \Leftrightarrow \sigma^x$. This implies that the phase diagram should be symmetric under $K \leftrightarrow h$ and $J \leftrightarrow t$. The confinement and Higgs transition must be dual to each other, and the line of phase transitions must meet at a self-dual point of the phase diagram. It is unknown whether any of the indicated phase transitions in Fig. 4 are continuous.

V. CONCLUDING REMARKS

Translationally invariant, commuting Hamiltonians built from interacting qubits [36] and fermions [18] admit a convenient algebraic representation as a collection of polynomials over a finite field. The translation group of the lattice is $\mathbb{Z}^D$, whose group algebra happens to be the polynomial algebra. The polynomials conveniently keep track of the support of various operators. Remarkably, this algebraic characterization of the Hamiltonian terms enables us to decide whether the given Hamiltonians are commuting, degenerate, and topologically ordered and the nature of the excitations [18,36]. Also, it gives a unique method to calculate the ground-state degeneracy of our exotic models. In the context of our nexus theory, the polynomial representation has a natural physical interpretation, as it precisely specifies the generalized Gauss's law G_n that defines the spin-nexus Hamiltonian (6). In this way, the polynomial representation encodes the local symmetry that defines a fracton topological phase. We elaborate on these methods in the Supplemental Material [30], which we intend to be pedagogical.

With the identification of a generalized Gauss's law that characterizes a fracton topological phase, our work provides an important step towards searching for material realizations of fracton topological order. Such a local conservation law can, in principle, appear in physical systems such as frustrated magnets, where our generalized gauge theory can emerge as an effective description at low energies, leading to fracton topological order.

[1] F. Wegner, Duality in Generalized Ising Models and Phase Transitions without Local Order Parameters, J. Math. Phys. **12**, 2259 (1971).

[2] E. Fradkin and S. H. Shenker, Phase diagrams of lattice gauge theories with Higgs fields, Phys. Rev. D **19**, 3682 (1979).

[3] S. C. Zhang, T. H. Hansson, and S. Kivelson, Effective-Field-Theory Model for the Fractional Quantum Hall Effect, Phys. Rev. Lett. **62**, 82 (1989).

[4] X. G. Wen and Q. Niu, Ground-state degeneracy of the fractional quantum Hall states in the presence of a random potential and on high-genus Riemann surfaces, Phys. Rev. B **41**, 9377 (1990).

[5] N. Read and S. Sachdev, Large-N Expansion for Frustrated Quantum Antiferromagnets, Phys. Rev. Lett. **66**, 1773 (1991).

[6] X. G. Wen, Mean-field theory of spin-liquid states with finite energy gap and topological orders, Phys. Rev. B **44**, 2664 (1991).

[7] T. Senthil and M. P. A. Fisher, Z_2 gauge theory of electron fractionalization in strongly correlated systems, Phys. Rev. B **62**, 7850 (2000).

[8] R. Moessner and S. L. Sondhi, Resonating Valence Bond Phase in the Triangular Lattice Quantum Dimer Model, Phys. Rev. Lett. **86**, 1881 (2001).

[9] A. Yu. Kitaev, Fault-tolerant quantum computation by anyons, Ann. Phys. (N.Y.) **303**, 2 (2003).

[10] M. Hermele, M. P. A. Fisher, and L. Balents, Pyrochlore photons: The $U(1)$ spin liquid in a $S = \frac{1}{2}$ three-dimensional frustrated magnet, Phys. Rev. B **69**, 064404 (2004).

[11] X. G. Wen, *Quantum Field Theory of Many-Body Systems* (Oxford University Press, Oxford, 2004).

[12] B. I. Halperin, Statistics of Quasiparticles and the Hierarchy of Fractional Quantized Hall States, Phys. Rev. Lett. **52**, 1583 (1984).

[13] D. Arovas, J. R. Schrieffer, and F. Wilczek, Fractional Statistics and the Quantum Hall Effect, Phys. Rev. Lett. **53**, 722 (1984).

[14] C. Chamon, Quantum Glassiness in Strongly Correlated Clean Systems: An Example of Topological Overprotection, Phys. Rev. Lett. **94**, 040402 (2005).

[15] S. Bravyi, B. Leemhuis, and B. Terhal, Topological order in an exactly solvable 3D spin model, Ann. Phys. (N.Y.) **326**, 839 (2011).

[16] J. Haah, Local stabilizer codes in three dimensions without string logical operators, Phys. Rev. A **83**, 042330 (2011).

[17] B. Yoshida, Exotic topological order in fractal spin liquids, Phys. Rev. B **88**, 125122 (2013).

[18] S. Vijay, J. Haah and L. Fu, A New Kind of Topological Quantum Order: A Dimensional Hierarchy of Quasiparticles Built from Stationary Excitations, Phys. Rev. B **92**, 235136 (2015).

[19] Other models *without* any immobile excitations (fractons) but with fractal operators have been studied [41].

[20] B. M. Terhal, Quantum error correction for quantum memories, Rev. Mod. Phys. **87**, 307 (2015).

[21] I. H. Kim and J. Haah, Localization from Superselection Rules in Translationally Invariant Systems, Phys. Rev. Lett. **116**, 027202 (2016).

[22] S. Bravyi and J. Haah, Quantum Self-Correction in the 3D Cubic Code Model, Phys. Rev. Lett. **111**, 200501 (2013).

[23] G. K. Savvidy and F. J. Wegner, Geometrical string and spin systems, Nucl. Phys. B **413**, 605 (1994).

[24] G. K. Savvidy and K. G. Savvidy, Self-avoiding surfaces and spin systems, Phys. Lett. B **324**, 72 (1994).

[25] G. K. Savvidy and K. G. Savvidy, Interaction hierarchy: String and quantum gravity, Mod. Phys. Lett. A **11**, 1379 (1996).

[26] R. Pietig and F. J. Wegner, Low temperature expansion of the gonihedric Ising model, Nucl. Phys. B **525**, 549 (1998).

[27] A. Lipowski, D. Johnston, and D. Espriu, Slow dynamics of Ising models with energy barriers, Phys. Rev. E **62**, 3404 (2000).

[28] P. Dimopoulos, D. Espriu, E. Jane, and A. Prats, Slow dynamics in the three-dimensional gonihedric model, Phys. Rev. E **66**, 056112 (2002).

[29] More precisely, a configuration of Ising spins may be specified exactly by the eigenvalues of interaction terms in the Hamiltonian, only up to spin flips performed along subsystems that leave all of the interaction terms invariant.

[30] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.94.235157 for a pedagogical discussion of the algebraic techniques used to prove the statements regarding the emergence of fracton topological order in the quantum dual of interacting systems with subsystem symmetries. We also apply our generalized gauging prescription to further characterize fracton models such as Haah's code and the CBLT model. Finally, we determine the topological ground-state degeneracy for the checkerboard and X-cube models on the three-torus.

[31] C. D. Batista and Z. Nussinov, Phys. Rev. B **72**, 045137 (2005).

[32] Z. Nussinov and E. Fradkin, Discrete Sliding Symmetries, Dualities, and Self-Dualities of Quantum Orbital Compass Models and $(p + ip)$ Superconducting Arrays, Phys. Rev. B **71**, 195120 (2005).

[33] N. Read, Topological phases and quasiparticle braiding, Phys. Today **65**, 38 (2012).

[34] A. E. Lipstein and R. A. Reid-Edwards, Lattice Gerbe Theory, J. High Energy Phys. **09**, (2014).

[35] D. A. Johnston, Z_2 Lattice Gerbe Theory, Phys. Rev. D **90**, 107701 (2014).

[36] J. Haah, Commuting Pauli Hamiltonians as maps between free modules, Commun. Math. Phys. **324**, 351 (2013).

[37] D. Eisenbud, *Commutative Algebra with a View toward Algebraic Geometry*, Gradaute Texts in Mathematics Vol. 150 (Springer, New York, 1994).

[38] D. A. Buchsbaum and D. Eisenbud, What Makes a Complex Exact?, J. Algebra **25**, 259 (1973).

[39] S. Bravyi, M. Hastings, and S. Michalakis, Topological quantum order: Stability under local perturbations, J. Math. Phys. **51**, 093512 (2010).

[40] D. Williamson (private communication).

[41] C. Castelnovo and C. Chamon, Topological Quantum Glassiness, Philos. Mag. **29**, 1 (2011).