Anisotropic long-range spin systems

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We consider anisotropic long-range interacting spin systems in d dimensions. The interaction between the spins decays with the distance as a power law with different exponents in different directions: We consider an exponent $d_1 + \sigma_1$ in d_1 directions and another exponent $d_2 + \sigma_2$ in the remaining $d_2 \equiv d - d_1$ ones. We introduce a low energy effective action with nonanalytic power of the momenta. As a function of the two exponents σ_1 and σ_2 we show the system to have three different regimes at criticality, two where it is actually anisotropic and one where the isotropy is finally restored. We determine the phase diagram and provide estimates of the critical exponents as a function of the parameters of the system, in particular considering the case where one of the two σ's is fixed and the other varying. A discussion of the physical relevance of our results is also presented.

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I. INTRODUCTION

Anisotropic interactions are present in a variety of physical systems. They are characterized by the property that the interaction energy V among two constituents of the system located in $\vec{r}_1$ and $\vec{r}_2$ depends on the relative distance $\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$ so that $V(\vec{r}_{12})$ assumes different values (possibly a different functional form) for $\vec{r}_{12}$ in different directions. A typical instance is provided by dipolar interactions (see [1] in [2]). For example, with a fixed direction of the dipoles, say $\hat{z}$, as it happens for ultracold dipolar gases [3], there is repulsion if the two dipoles have $\vec{r}_{12}$ in the x-y plane and attraction if $\vec{r}_{12}$ is parallel to $\hat{z}$, with $V(\vec{r}_{12}) \propto 1 - 3\cos^2\theta$ and θ being the angle between $\vec{r}_{12}$ and $\hat{z}$.

Anisotropy is one of the fundamental features of molecular interactions, and it is responsible for phase transitions between tilted hexatic phases in liquid-crystal films [4]. Liquid crystals can be described using low energy theories [5], where the order parameter represents the bond angle between molecules. At particular points of the phase diagram liquid crystals are efficiently described by the so-called Lifshitz point effective action [6,7].

Another major example of anisotropic systems is provided by layered superconductors. The layered structure can be described by the Lawrence-Doniach model which has different masses in different directions [8] (typically $m_\parallel$ in the x-y plane and $m_\perp$ in the $\hat{z}$ direction). Layered systems can occur naturally or be artificially created. Examples of artificial structures are alternating layers of graphite and alkali metals [9] or samples with layers of different metals [10]. On the other hand layered superconductors range from compounds of transition-metal dichalcogenide layers intercalated with organic insulating molecules [11] to cuprates [8]. Vortex dynamics in magnetically coupled layered superconductors was studied [12] by a multilayer sine-Gordon type model [13]. Layered ultracold superfluids can be induced by using a deep optical lattice in one spatial direction for fermions [14] or bosons [15].

A simple way of studying the effect of layering (and anisotropy in general) is to consider statistical mechanics models with different couplings in different directions. A typical case is provided by the study of the XY model in three dimensions with a coupling between nearest neighbors sites i and j equal to $J_\parallel$ if i,j belong to the same x-y plane and to $J_\perp$ if i,j belong to nearest neighbor planes in the $\hat{z}$ direction [16]. This model has been studied also in relation to layered superconductors and cuprates [17]. Depending on the value of the ratio $J_\perp/J_\parallel$ the behavior of the system can pass from being $3D$ to effectively $2D$ [16].

The main point of these and similar studies of anisotropic spin systems with short-range (SR) couplings is that, far from the critical point, anisotropy induces a series of very interesting effects, but for general reasons at the critical point isotropy is restored and strictly speaking an isotropic critical point is found for any finite value of the $J_\perp/J_\parallel$ ratio (different is the case of a finite number of $2D$ systems). This is a consequence of the divergence of the correlation length, so that the system does not see any longer the anisotropy at criticality. As another example, for fermions in the BCS-BEC crossover [18] in the presence of layering the anisotropy is strongly depressed at the unitary limit [14] even though there is no phase transition, but the system is scale invariant due to the divergence of the scattering length.

Therefore a general interesting question is to study the conditions under which one can have genuinely anisotropic critical points. A main observation of this paper is that, in the presence of anisotropic long-range (LR) interactions, the interplay between the divergence of the correlations and the LR nature of the couplings may induce such anisotropic critical behavior.

The interest in the statistical physics of systems with LR interactions is in general motivated by a large number of possible applications, ranging from plasma physics to astrophysics and cosmology [2,19]. The shape of LR interactions is typically considered as decaying as a power law of the distance $r^{-d-\sigma}$, where r is the distance between two elementary components of the system, d is the dimensionality, and σ is a real parameter determining the range of the interactions. Simple considerations show that for $\sigma < 0$ the mean-field interaction energy diverges and the system is ill defined. It is still possible to study this case using the so-called Kac rescaling [20], leading to many interesting results such as ensembles inequivalence and inhomogeneous ground states [21,22].

For $\sigma > 0$ the thermodynamics is well defined and spin systems may present in general a phase transition at a certain critical temperature T_c. In the isotropic case, as a function of the parameter σ, three regions are found [23]. For $\sigma \leqslant \frac{d}{2}$ the universal behavior is the one obtained at mean-field level, for σ larger than a critical value σ^* the system behaves as a SR one at criticality and for $d/2 < \sigma \leqslant \sigma^*$ the system has peculiar non-mean-field critical exponents. The precise determination of σ^* has been the subject of perduring interest [24–26]. Moreover, recent results on conformal invariance in LR systems are also available [27]. The theoretical interest for these systems is also supported by the recent exciting progresses in the experimental realization of quantum systems with tunable LR interactions [28–34].

The goal of the present paper is to introduce and study anisotropic spin models with LR interactions having different decay exponents in different directions: σ_1 in d_1 dimensions and σ_2 in the remaining $d_2 \equiv d - d_1$ ones. The SR limit is provided by such decay exponents going to infinite. Clearly, when both σ_1 and σ_2 go to infinity the isotropic SR limit is retrieved, while when only one of the two—say σ_2—is diverging the model is SR in the corresponding d_2 directions. It is expected that when one of the two exponents, σ_1 or σ_2, is larger than some threshold value, say σ_1^* or σ_2^*, the corresponding directions behave as if only SR interactions were present at criticality.

Apart from the already mentioned interest in investigating anisotropic fixed points, three other motivations underly our work. From one side we think it is interesting to study a problem in which rotational invariance is broken at criticality due to the division of the system in two subspaces, which is somehow the simplest global form in which such rotational invariance can be broken. From the other side in a natural way quantum systems with LR couplings are an example of the systems under study: Indeed, if one considers a quantum model in D dimensions with LR interactions or couplings, then at criticality one can map it on an anisotropic classical system in dimension $d = D + 1$, with the interactions along the imaginary time direction being of SR type [35]. This is of course the generalization of what happens for SR quantum systems: As an example in which the mapping can be worked out explicitly [36,37] we mention the mapping of the SR Ising chain in a transverse field on the classical SR Ising model, with the second dimension corresponding to the imaginary time. Therefore generically a D-dimensional quantum spin system with LR interactions can be seen as an example of an anisotropic classical system where the interaction is LR in D dimensions and SR in the remaining one. A similar situation would occur for LR quantum systems in the models in which two extra-time dimensions are added and the time can be regarded as a complex variable [38]. Finally, experiments of quantum systems with tunable LR interactions provide an experimental counterpart to implement and test the results we present in the following.

To study anisotropic LR spin systems we introduce a model, whose low energy behavior is well described by an anisotropic Lifshitz point effective action with nonanalytic momentum terms in the propagator. At variance with the usual Lifshitz point case in our system a standard second order phase transition is found, and there is no additional external field to tune in order to reach criticality.

Using functional renormalization group (RG) methods we study in the following the critical behavior of anisotropic LR spin systems determining the independent critical exponents and depicting the phase diagram in the parameter space of σ_1 and σ_2, mostly focusing on the case $\sigma_1,\sigma_2 \leqslant 2$.

II. THE MODEL

The model we consider is a lattice spin system in dimension d, with an arbitrary number of spin components N. The spins are classical but comments on quantum spin systems with LR interactions will also be presented.

The interactions among the spins are LR with different exponents depending on the spatial directions. The system is divided into two subspaces of dimension d_1 and d_2 with $d_1 + d_2 = d$. In the first subspace the interaction between the spins decays with the distance as a power law with exponent $d_1 + \sigma_1$, while in the other subspace it decays with exponent $d_2 + \sigma_2$.

This formally amounts to write the position of a spin, $\vec{r} = (r_1, \cdots, r_d)$, as $\vec{r} \equiv \vec{r}_\parallel + \vec{r}_\perp$ with $\vec{r}_\parallel = (r_1, \cdots, r_{d_1}, 0, \cdots, 0)$ and $\vec{r}_\perp = (0, \cdots, 0, r_{d_1+1}, \cdots, r_d)$. The ith spin is located in $\vec{r}_i = (r_{1,i}, \cdots, r_{d,i})$, so that $\vec{r}_{\parallel,i} = (r_{1,i}, \cdots, r_{d_1,i}, 0, \cdots, 0)$ and $\vec{r}_{\perp,i} = (0, \cdots, 0, r_{d_1+1,i}, \cdots, r_{d,i})$ with $d = d_1 + d_2$.

Given the two spins in $\vec{r}_i$ and $\vec{r}_j$ we define $\vec{r}_{ij}$ as $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$ and similarly we put $\vec{r}_{\parallel,ij} = \vec{r}_{\parallel,i} - \vec{r}_{\parallel,j}$ and $\vec{r}_{\perp,ij} = \vec{r}_{\perp,i} - \vec{r}_{\perp,j}$. The couplings between two spins in $\vec{r}_i$ and $\vec{r}_j$ decay with power law exponent $d_1 + \sigma_1$ if $\vec{r}_{ij}$ is parallel to $\vec{r}_{\parallel,ij}$ and with power law exponent $d_2 + \sigma_2$ if $\vec{r}_{ij}$ is parallel to the $\vec{r}_{\perp,ij}$ direction.

The model we consider then reads

$$H = -\sum_{i\neq j} \frac{J_\parallel}{2} \frac{\vec{S}_i \cdot \vec{S}_j}{r_{\parallel,ij}^{d_1+\sigma_1}} \delta(\vec{r}_{\perp,ij}) - \sum_{i\neq j} \frac{J_\perp}{2} \frac{\vec{S}_i \cdot \vec{S}_j}{r_{\perp,ij}^{d_2+\sigma_2}} \delta(\vec{r}_{\parallel,ij}), \quad (1)$$

where the $\vec{S}_i$ are classical N component vectors (normalized to 1). The distance $r_{\parallel,ij}$ is calculated on the d_1-dimensional subspace, to which both spins $\vec{S}_i$ and $\vec{S}_j$ belong, as ensured by the presence of the $\delta(\vec{r}_{\perp,ij})$. On the same ground $r_{\perp,ij}$ measures the distance between two spins i,j belonging to the same d_2-dimensional subspace. Thus any spin of the model interacts only with the spins sitting on the same subspace. For example, given an Ising model in two dimensions for variables $S_i = \pm 1$, setting $i \equiv (i_1,i_2)$ we are considering couplings nonvanishing only if $i_1 = j_1$ (and interactions decaying as $|i_2 - j_2|^{-d_2-\sigma_2}$, with $d_2 = 1$, in the same column) and if $i_2 = j_2$ (and interactions decaying as $|i_1 - j_1|^{-d_1-\sigma_1}$, with $d_1 = 1$, in the same row).

When one of the two exponents goes to infinite the interaction becomes SR in the corresponding subspace. However, in analogy with the isotropic LR case, two threshold values σ_1^* and σ_2^* exist such that for $\sigma_1 > \sigma_1^*$ or $\sigma_2 > \sigma_2^*$ the systems behaves as if only SR interactions were present at criticality in, respectively, the d_1 or d_2 dimensional subspace.

In (1) we disregard for simplicity interactions between spins if their relative distance $\vec{r}_{ij}$ is not perpendicular or parallel to $\vec{r}_{\perp,ij}$ (or $\vec{r}_{\parallel,ij}$). Notice that, although it is chosen as a simplifying assumption, this is the case for a d_1

dimensional quantum spin system with LR interactions, e.g., of transverse Ising type, when mapped to a classical system (couplings along the imaginary time are among same column discretized points). Additional finite-range interactions for spins of different columns or rows do not qualitatively affect our results. The assumption of absence of LR interactions between spins belonging to different subspaces is very useful to simplify the treatment of anisotropy, otherwise it would be necessary to introduce explicit angular dependences. We expect that, if the multisubspace interactions are well peaked around the intrasubspace interactions, then only quantitative modifications are found. When all the angles give a significant contribution to the energy the treatment is considerably more involved, also for SR systems, and as a future work one could perform a study of the RG flow also of the angular dependence of the couplings. Nevertheless our model already well shows how the presence of anisotropy can radically modify the standard behavior of LR interactions.

To discuss a specific example, we consider the ferromagnetic quantum Ising model in dimension D in the presence of LR interactions

$$H = -\frac{J}{2}\sum_{i\neq j}\frac{\sigma_i^{(z)}\sigma_j^{(z)}}{|i-j|^{d_1+\sigma_1}} - h\sum_i \sigma_i^{(x)}, \tag{2}$$

where $\sigma^{(z),(x)}$ are the z,x components of the quantum spin $\vec{\sigma}$ and J is a positive magnetic coupling. In the thermodynamic limit a quantum spin system can be mapped onto a classical analog [35,39,40]. The quantum phase transition at zero temperature of a SR quantum spin system in dimension D lies in the same universality class of a classical system in dimension $d = D + 1$. Then we can map a quantum Ising model on a classical analog in $d = D + 1$. A similar result is generally also valid with LR interactions and the mapping is between the quantum Ising model described in (2) and the anisotropic classical model (1) with $d_1 = D$, $d_2 = 1$, and SR interactions along the d_2 direction, which amounts to taking for our purposes $\sigma_2 > \sigma_2^*$. Also for $N > 1$ we expect in general that a quantum spin system in a dimension $d_1 \equiv D$ with LR interactions decaying with exponent σ_1 has a phase transition which lies in the same universality class as the one found in the classical system (1) with $d_1 + d_2 > D$ and $\sigma_2 > 2$. To this respect we point out that in our treatment d_1 and d_2 may be considered continuous variables.

III. EFFECTIVE FIELD THEORY

In order to study the critical behavior of anisotropic LR $O(N)$ models, we introduce the following low energy effective field theory:

$$S[\phi] = -\int d^dx\big(Z_{\sigma_1}\phi_i(x)\Delta_{\parallel}^{\frac{\sigma_1}{2}}\phi_i(x) + Z_{\sigma_2}\phi_i(x)\Delta_{\perp}^{\frac{\sigma_2}{2}}\phi_i(x) - U(\rho)\big), \tag{3}$$

where $\rho = \phi_i\phi_i/2$ and the summation over the index $i \in [1,2,\cdots,N]$ is implicit. The effective field theory in equation (3) is obtained by the low momentum expansion of the bare propagator of Hamiltonian (1). The higher order analytic terms $\Delta_\parallel$ and $\Delta_\perp$ were neglected and this expansion is valid only as long as $\sigma_1 \leqslant 2$ and $\sigma_2 \leqslant 2$.

In the following we mostly choose the convention $\sigma_2 < \sigma_1$ and $d_1 \leqslant d_2$, if not differently stated. To make the presentation of the results more compact we will also adopt the symbol $\vee$ standing for "or" or, according to the context, "or respectively."

It is worth noting that along different spatial directions physical properties essentially differ and this difference cannot be removed by a simple rescaling of the theory. Accordingly, the d-dimensional coordinate space is split into two subspaces $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$. Each position vector $x \equiv (x_1,x_2) \in \mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$ has a d_1-dimensional parallel component x_1 and d_2-dimensional perpendicular ones, x_2.

The laplacian operators $\Delta_\parallel$ and $\Delta_\perp$ act respectively in $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$. When the dimension of one of the subspaces, say $d_1 \vee d_2$ [i.e., d_1 or, respectively, d_2] shrinks to zero we retrieve an isotropic LR $O(N)$ model in dimension $d_2 \vee d_1$ [i.e., d_2 or, respectively, d_1] with the upper critical dimension $d^*_{2,1} = 2\sigma_{2,1}$ [i.e., $d_2^* = 2\sigma_2$ or, respectively, $d_1^* = 2\sigma_1$] and the critical behavior described in Refs. [25,26].

In the following we derive general results which are valid for every value of d_1, d_2, σ_1, and σ_2, but more attention will be paid to the special cases $d_2 = 1$ and $\sigma_2 > \sigma_2^*$ which is the interesting case for quantum spin chains with LR interactions. Using the notation $\vee$, in the special case $\sigma_1 \vee \sigma_2 = 2$ and $\sigma_2 \vee \sigma_1 = 4$, expression (3) reduces to the fixed point effective action of a $d_1 \vee d_2$ axial anisotropic Lifshitz point. However, in the standard Lifshitz point case, the SR analytic terms cannot be neglected, outside the fixed point, as in effective action (3) since they are relevant with respect to the $\sigma_2 \vee \sigma_1 = 4$ kinetic term. Thus the usual Lifshitz point behavior is only found in multicritical universality classes, where diverse fields are at their critical value. On the other hand the critical behavior described by the low energy action (3) is a standard second order one, and it is found in anisotropic LR systems for some critical value of the temperature.

IV. DIMENSIONAL ANALYSIS

The scaling hypothesis for the Green function in the asymptotic long wavelength limit reads

$$G(q_1,q_2) = q_1^{-\sigma_1+\delta\eta_1}G\big(1,q_2q_1^{-\theta}\big) = q_2^{-\sigma_2+\delta\eta_2}G\big(q_1q_2^{-\frac{1}{\theta}},1\big), \tag{4}$$

where the anisotropy index

$$\theta = \frac{\sigma_1 - \delta\eta_1}{\sigma_2 - \delta\eta_2}$$

has been defined. In latter formula we introduced the quantities $\delta\eta_1$ and $\delta\eta_2$, i.e., the eventual anomalous dimension corrections, with respect to the mean field result, to the low energy behavior of the system propagator. Through the paper we will often refer to the two anomalous dimensions η_1 and η_2 calculated with respect to the mean field behavior of the SR propagator, according the relations $\eta_1 \equiv 2 - \sigma_1 + \delta\eta_1$ and $\eta_2 \equiv 2 - \sigma_2 + \delta\eta_2$.

The system possesses two different correlation lengths ξ_1 and ξ_2, both diverging at the same critical temperature T_c, but following in general two different scaling laws:

$$\xi_1 \propto (T - T_c)^{-\nu_1}, \tag{5}$$

$$\xi_2 \propto (T - T_c)^{-\nu_2}. \tag{6}$$

The latter equations also define the correlation length exponents ν_1 and ν_2.

One could expect to have four independent critical exponents ($\delta\eta_1, \delta\eta_2, \nu_1, \nu_2$). However in analogy with the standard anisotropic Lifshitz point treatment [41], we can derive the following scaling relation

$$\frac{\sigma_1 - \delta\eta_1}{\sigma_2 - \delta\eta_2} = \frac{\nu_2}{\nu_1} = \theta \tag{7}$$

which leaves us with only three independent exponents. Relation (7) can be obtained generalizing the standard argument for the Fisher scaling law [42] to the scaling hypothesis introduced in equation (4): By equating the results for the critical exponent γ by the scalings of q_1 and q_2 one gets (7).

Due to spatial anisotropy, we define two momentum scales in our renormalization procedure [43,44]

$$[x_1] = k_1^{-1} \tag{8}$$

$$[x_2] = k_2^{-1}, \tag{9}$$

and both these scales must vanish in order to reach the thermodynamic limit. As it will become clear in the following, in order to enforce scale invariance at the critical point we must require both kinetic terms in effective action (18) to have the same scaling dimension. Consequently the following relation between the two momentum scales emerges

$$k_2 = k_1^{\theta} = k^{\theta}, \tag{10}$$

where $k \equiv k_1$. The choice $k \equiv k_1$ is arbitrary but consistent with the former choice of θ. All the physical results in this model are evidently invariant under the simultaneous exchange of dimensions and exponents $d_1 \to d_2$ and $\sigma_1 \to \sigma_2$. The last operation is equivalent to exchanging the definitions of θ and k ($k = k_2$ and $\theta \to \theta^{-1}$).

It is possible to develop the local potential as

$$U(\rho) = \sum_i \lambda_i \rho^i, \tag{11}$$

where the latter equation defines the couplings λ_i. The scaling dimensions of the field and the couplings are expressed in terms of the general scale k as

$$\phi = k^{D_\phi}\tilde{\phi} \tag{12}$$

$$\lambda_i = k^{D_{\lambda_i}}\tilde{\lambda}_i, \tag{13}$$

with the scaling dimensions

$$D_\phi = \frac{d_1 + \theta d_2 - \sigma_1 + \eta_{\sigma_1}}{2}, \tag{14}$$

$$D_{\lambda_i} = d_1 + \theta d_2 - i(d_1 + \theta d_2 - \sigma_1), \tag{15}$$

In order to draw the phase diagram of the system we can rely on canonical dimension arguments, studying the relevance of the coupling at bare level. This is equivalent to using the Ginzburg criterion to predict the range of validity of the mean-field approximation [45]. We then impose $\delta\eta_1 = \delta\eta_2 = 0$ and the system develops a nontrivial ith-order critical point when the coupling λ_i is relevant (i.e. diverges) in the infrared limit ($k \to 0$). From the condition $D_{\lambda_i} < 0$ we obtain

$$\frac{d_1}{\sigma_1} + \frac{d_2}{\sigma_2} < \frac{i}{i-1}. \tag{16}$$

When this condition is fulfilled the system presents $i-1$ universality classes, with the ith-order universality class describing an i phases coexistence critical point [46–48]. Since each new fixed point branches from the Gaussian one, the assumption of vanishing anomalous dimension is consistent and the existence of multicritical anisotropic LR $O(N)$ universality classes can be extrapolated to be valid in the full theory.

In the following we will focus only on the Wilson-Fisher (WF) universality class which appears in ϕ^4 theories. We then consider the case $i = 2$,

$$\frac{d_1}{\sigma_1} + \frac{d_2}{\sigma_2} < 2, \tag{17}$$

which is the condition for having a non-mean-field second order phase transition.

When $\sigma_1 = \sigma_2 = 2$ we recover the usual lower critical dimension of the Ising and $O(N)$ models in dimension d, i.e., 4. At variance the case $d_2 = 0$ reproduces the result for a d_1 dimensional LR $O(N)$ model ($d_1 < 2\sigma_1$). It is worth noting that while the numerical results we report in the following are calculated in the specific $i = 2$ case, most of the analytic results are valid also in the general i case.

V. EFFECTIVE ACTION AND RG APPROACH

To further proceed with the analysis of the critical behavior of LR anisotropic $O(N)$ models we use the functional RG approach [49,50]. We should choose a reasonable ansatz for our effective action in such a way that we can project the exact Wetterich equation [51,52]. We then consider the same functional form of action (3) including also highest order analytic kinetic terms in order to efficiently describe the boundary regions:

$$\Gamma_k[\phi] = -\int d^d x \big(Z_{\sigma_1}\phi_i(x)\Delta_\parallel^{\frac{\sigma_1}{2}}\phi_i(x) + \phi_i(x)\Delta_\parallel\phi_i(x) + Z_{\sigma_2}\phi_i(x)\Delta_\perp^{\frac{\sigma_2}{2}}\phi_i(x) + \phi_i(x)\Delta_\perp\phi_i(x) - U_k(\rho)\big), \tag{18}$$

where the summation over repeated indices is again assumed. The two wave-function renormalization terms Z_{σ_1,σ_2} are running and we are considering anomalous dimension effects for the analytic momentum powers, including them directly into the field scaling dimension, as in Ref. [52].

As already discussed in Ref. [26], the two wave-function renormalization flows vanish, since the RG evolution of the propagator does not contain any nonanalytic term:

$$k\partial_k Z_{\sigma_1} = 0, \tag{19}$$

$$k\partial_k Z_{\sigma_2} = 0, \tag{20}$$

where k is the isotropic scale already introduced in equation (12).

In order to extract the critical behavior of the system, we study the functional RG equations in terms of the scaled variables. We then define the scaled wave functions $\tilde{Z}_{\sigma_1}$ and $\tilde{Z}_{\sigma_2}$, as it was done for the field and the couplings in equations (12) and (14).

Transforming equations (19) and (20) to scaled variables, the flow of the scaled wave functions is an eigendirection of the RG evolution

$$k\partial_k \tilde{Z}_{\sigma_1,\sigma_2} = D_{\sigma_1,\sigma_2}\tilde{Z}_{\sigma_1,\sigma_2}. \tag{21}$$

In order to explicitly calculate the scaling dimension of the two wave functions it is necessary to define the dimension of the field. In the case of expression (18) we choose the analytic kinetic terms as reference for the field dimension rather than the nonanalytic terms we considered in the bare action (3). The dimension of the field becomes

$$D_\phi = \frac{d_1 + \theta d_2 - 2 + \eta_1}{2}, \tag{22}$$

where $\theta = \frac{2-\eta_1}{2-\eta_2}$ and η_1, η_2 are, respectively, the anomalous dimensions of the analytic terms in the $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ subspaces. The assumption of two different anomalous dimensions is the obvious consequence of anisotropy.

At the fixed point all the β functions of the scaled couplings vanish. We thus impose

$$D_{\sigma_1} = 2 - \sigma_1 - \eta_1 = 0 \quad \text{or} \quad \tilde{Z}_{\sigma_1} = 0, \tag{23}$$

$$D_{\sigma_2} = 2 - \sigma_2 - \eta_2 = 0 \quad \text{or} \quad \tilde{Z}_{\sigma_2} = 0, \tag{24}$$

where one of the conditions (23) shall be true to enforce the vanishing of $k\partial_k \tilde{Z}_{\sigma_1}$, while the same shall occur in conditions (24) to ensure $k\partial_k \tilde{Z}_{\sigma_2} = 0$.

From the two equations (23) and (24) we derive the existence of two thresholds values σ_1^* and σ_2^*. For $\sigma_1 < \sigma_1^* \vee \sigma_2 < \sigma_2^*$ we have $\eta_1 = 2 - \sigma_1 \vee \eta_2 = 2 - \sigma_2$ and the left condition in (23) $\vee$ (24) is fulfilled, conversely for $\sigma_1 > \sigma_1^* \vee \sigma_2 > \sigma_2^*$ we have to impose $\tilde{Z}_{\sigma_1} = 0 \vee \tilde{Z}_{\sigma_2} = 0$. The two conditions are independent; then four regimes exist in the system, obtained by the four possible combinations of σ_1 smaller or larger than σ_1^* and σ_2 smaller or larger than σ_2^*.

At mean-field level (see Appendix A for additional details) we have the following results for the critical exponents of the system

$$\begin{aligned} \delta\eta_1 &= 0, \quad \delta\eta_2 = 0,\\ \eta_1 &= 2 - \sigma_1, \quad \eta_2 = 2 - \sigma_2,\\ \nu_1 &= \frac{1}{\sigma_1}, \quad \nu_2 = \frac{1}{\sigma_2}. \end{aligned} \tag{25}$$

The threshold values at this approximation level are then simply $\sigma_1^* = \sigma_2^* = 2$, as shown in Appendix A.

However, when the inequality (17) holds, we know that the latter results do not reproduce the correct critical exponents, and we shall then consider renormalization effects. The complete phase diagram has therefore the structure reported in Fig. 1. Region *I* ($\sigma_1 < \sigma_1^*, \sigma_2 < \sigma_2^*$) is the pure anisotropic LR region, where the saddle point of the effective action (18) is valid. In regions $II_{A\vee B}$ the exponent $\sigma_1 \vee \sigma_2$ is larger than $\sigma_1^* \vee \sigma_2^*$ and the correct effective field theory is given by expression (3) in the absence of the corresponding nonanalytic term. In region *III* both kinetic terms are irrelevant compared to the SR kinetic terms, and the model becomes equivalent to a $d = d_1 + d_2$ dimensional isotropic SR system. The shaded

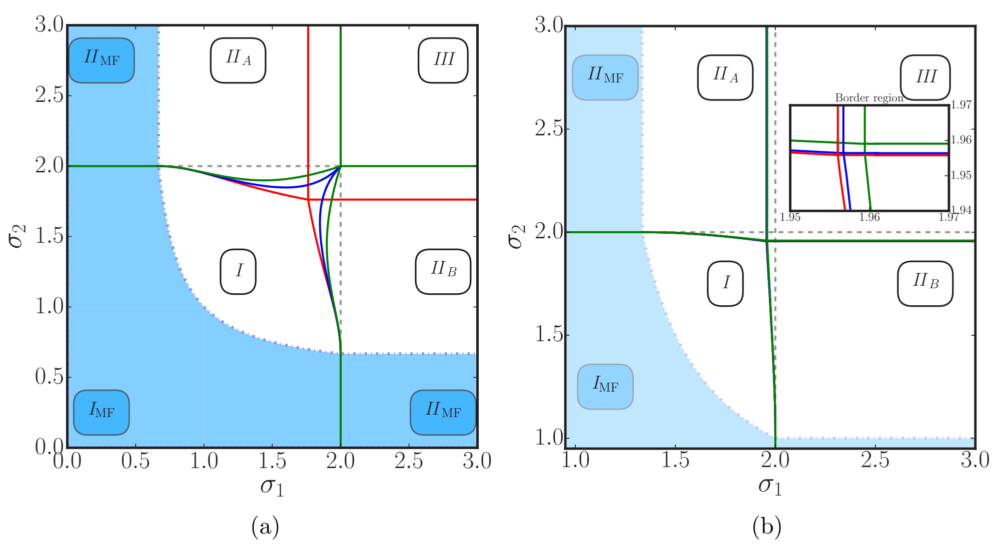


FIG. 1. In panel (a) we plot the parameter space of a anisotropic LR spin system for $d_1 = d_2 = 1$. In the cyan shaded area fluctuations are unimportant and the universal quantities are correctly reproduced by the mean-field approximation. The solid curves are the boundary regions σ_1^* and σ_2^* where the nonanalytic kinetic term becomes irrelevant. We show results for $N = 1,2,3$, respectively, in red, blue, green. The dashed lines are the mean-field results for the boundary curves. In panel (b) we show the parameter space of the model in $d_1 = 2$ and $d_2 = 1$. In the light cyan shaded area fluctuations are unimportant and the universal quantities are again correctly reproduced by mean-field approximation. The solid curves are the boundary regions σ_1^* and σ_2^*. In the inset we show the boundaries in an enlarged scale.

areas in Fig. 1 correspond to the region where inequality (17) is fulfilled only for $i = 1$ and then mean field is valid, hence the mean-field subscript MF. We finally observe that for comparison the mean-field phase diagram is reported in Appendix A as Fig. 8.

The obtained regions have the same structure obtained in Sec. IV with naive scaling arguments, see the mean-field Fig. 8 in Appendix A. However when we are focusing on nontrivial fixed points the competition between the renormalized couplings of different kinetic terms is ruled by the dressed value of the scaling dimension. It is then necessary to consider renormalized values also for the boundary lines. These lines will not be at $\sigma_1^* = \sigma_2^* = 2$, as in the mean-field Fig. 8, but they are now one-dimensional curves with nontrivial shape $\sigma_1^*(\sigma_2) = 2 - \eta_1(\sigma_2)$ and $\sigma_2^*(\sigma_1) = 2 - \eta_2(\sigma_1)$.

VI. THE PURE NONANALYTIC REGION

The values of σ_1^* and σ_2^* and their actual location is in general different from the mean-field values $\sigma_1^* = \sigma_2^* = 2$, as it happens for isotropic LR systems [23]. For the discussions in this section the precise values of σ_1^* and σ_2^* are not essential and we defer the study of σ_1^* and σ_2^* to Sec. VII.

Let us focus on the case $\sigma_1 < \sigma_1^*$ and $\sigma_2 < \sigma_2^*$ where the dominant kinetic terms are nonanalytic. The two conditions (23) and (24) are both satisfied on their left side. We thus have $\eta_1 = 2 - \sigma_1$ and $\eta_2 = 2 - \sigma_2$.

At a renormalized level the two analytic kinetic terms become equal to the nonanalytic ones, as happens in the usual isotropic LR case [26]. Eventually analytic terms give only small contributions to the numerical value of the universal quantities and will be discarded in this section.

We focus on the pure nonanalytic effective action:

$$\Gamma_k[\phi] = -\int d^d x \big(Z_{\sigma_1}\phi_i(x)\Delta_{\parallel}^{\frac{\sigma_1}{2}}\phi_i(x) + Z_{\sigma_2}\phi_i(x)\Delta_{\perp}^{\frac{\sigma_2}{2}}\phi_i(x) - U_k(\rho)\big)\,. \tag{26}$$

To proceed with the functional RG calculation we introduce an infrared cutoff function $R_k(q_1,q_2)$, which plays the role of a momentum dependent mass of the excitations [51,52]. This artificial mass should be vanishing for excitations with momentum $q_1 \vee q_2 \ll k$, while it should prevent the propagation of low momentum $q \vee q_2 \gg k$ ones. We then introduce the function

$$R_k(q_1,q_2) = \big(Z_{\sigma_1}\big(k_1^{\sigma_1} - q^{\sigma_1}\big) + Z_{\sigma_2}\big(k_1^{\sigma_2} - q^{\sigma_2}\big)\big) \times \theta\big(Z_{\sigma_1}\big(k_1^{\sigma_1} - q^{\sigma_1}\big) + Z_{\sigma_2}\big(k_1^{\sigma_2} - q^{\sigma_2}\big)\big), \tag{27}$$

obtained by generalizing the so-called optimized cutoff [50].

With this explicit choice for the cutoff we can explicitly evaluate the form of the potential flow equation

$$\partial_t \bar{U}_k = (d_1 + \theta d_2)\bar{U}_k(\bar{\rho}) - (d_1 + \theta d_2 - \sigma_1)\bar{\rho}\,\bar{U}_k'(\bar{\rho}) - \frac{\sigma_1}{2}(N-1)\frac{1}{1+\bar{U}_k'(\bar{\rho})} - \frac{\sigma_1}{2}\frac{1}{1+\bar{U}_k'(\bar{\rho}) + 2\bar{\rho}\,\bar{U}_k''(\bar{\rho})}\,, \tag{28}$$

where $t = -\log(k/k_0)$ is the RG time and k_0 is some ultraviolet scale. The flow equation (28) has been obtained using functional RG techniques, the procedure being outlined in Appendix B. For convenience sake we removed a geometric coefficient using scaling invariance of the field [52]. The wave functions still obey equations (19) and (20), but, in the absence of SR terms, they are dimensionless and then they do not have any flow.

A. Effective dimension

Comparing expression (28) with the one reported in Ref. [26] we have an equivalence between the ν_1 exponent of this model and the one of an isotropic LR model in dimension

$$d_{\rm eff} = d_1 + \theta d_2. \tag{29}$$

From ν_1 we can calculate ν_2 using scaling relation (7), with the anisotropic index which is stuck to its bare value $\theta = \frac{\sigma_1}{\sigma_2}$.

Similar effective dimension results already appeared in different treatments of the isotropic LR O(N) models [24–26,53] and can be recovered using standard scaling arguments. Using functional RG approach such effective dimension relations naturally appear without further assumptions, but they are found to be valid only within our approximations [26]. Anyway effective dimension arguments proved able to provide very good quantitative agreement with numerical simulations [25,26]. We can thus rely on them to calculate the correlations length exponents ν_1 and ν_2 as a function of the two parameters σ_1 and σ_2.

Since the wave-function renormalization terms are not running we have $\delta\eta_1 = \delta\eta_2 = 0$, and the momentum dependence of the propagator is the same at the bare and at the renormalized level. This result is evident at this approximation level, but it is conjectured to be valid also in the full theory as it happens for the usual LR case. In the latter case this result was verified at higher approximation levels both in the perturbative and nonperturbative RG approaches [54,55] (in agreement with very recent numerical simulations [56]). We are thus able to derive all the critical exponents in the pure LR region (region I in Fig. 1), but since we do not know exactly the threshold values σ_1^* and σ_2^* we have to extend our analysis to the mixed analytic nonanalytic kinetic terms ranges (regions $II_{A\vee B}$).

B. The $N = \infty$ limit

For isotropic interactions the spherical model is obtained in the large components limit $N \to \infty$ of the $O(N)$ spin systems. This model is exactly solvable [53], and in this limit the approximated flow equation (28) provides exact universal quantities.

The results for the critical exponents with anisotropic LR couplings are the following:

$$\nu_1 = \frac{\sigma_2}{\sigma_2 d_1 + \sigma_1 d_2 - \sigma_2\sigma_1}, \tag{30}$$

$$\nu_2 = \frac{\sigma_1}{\sigma_2 d_1 + \sigma_1 d_2 - \sigma_2\sigma_1}. \tag{31}$$

In the $d_2 \to 0 \vee d_1 \to 0$ limit the exponent $\nu_1 \vee \nu_2$ reduces to the one of the spherical LR model in dimension d_1 [53], $\nu_1 = \frac{1}{d_1-\sigma_1} \vee \nu_2 = \frac{1}{d_2-\sigma_2}$, while $\nu_2 = \theta\nu_1 \vee \nu_2 = \frac{\nu_1}{\theta}$ looses any significance. Also in the $\sigma_1 = \sigma_2 = 2$ limit the expressions become equal to the exact SR case.

Due to vanishing anomalous dimension in the spherical model limit we can apply the results of this section even to the case of higher analytic powers in the kinetic term of, say, the $\mathbb{R}^{d_2}$ subspace. This is the case $\sigma_2 = 2L$ with $L \in \mathbb{N}^+$: we notice that our results for the Ising and $O(N)$ models (with finite N) are in general not valid in the case $L \neq 1$, which is the case of the anisotropic next nearest neighbor (ANNNI) model.

In the case of the ANNNI model the fixed point is the usual axial anisotropic Lifshitz point. It is different from the case depicted in this work, since it is a multicritical fixed point. Indeed next nearest neighbors interaction is subleading with respect to the usual SR interaction and needs an additional external field to act on the system to become relevant.

However, we are interested only in the fixed point quantities of the ANNNI model in order to make a consistency check of our $N \to \infty$ results. It is then sufficient to assume to be at the Lifshitz point and make the substitutions $\sigma_1 \to 2$ and $\sigma_2 \to 2L$ ignoring the presence of further more relevant kinetic terms. We then immediately retrieve the $N \to \infty$ ANNNI case [57]:

$$\nu_1 = \frac{L}{(d_1 - 2)L + d_2}, \tag{32}$$

$$\nu_2 = \frac{1}{(d_1 - 2)L + d_2}. \tag{33}$$

The ANNNI model is paradigmatic in the physics of spin systems, and it would be interesting to have results also in the $N < \infty$ case. This is however beyond the scope of present analysis, since we would need to explicitly consider SR analytic terms in our ansatz (18). This will be the subject of future work.

VII. THE MIXED REGIONS

When one of the two exponents overcomes its threshold, say $\sigma_1 > \sigma_1^* \vee \sigma_2 > \sigma_2^*$, the correspondent analytic term in (18) becomes relevant and condition (23)$\vee$(24) shall be satisfied on its right side. We have then $Z_{\sigma_1} = 0 \vee Z_{\sigma_2} = 0$, and the system is purely analytic in one of the two subspaces.

In this case it is necessary to use ansatz (28) without the nonanalytic term in the $\mathbb{R}^{d_1} \vee \mathbb{R}^{d_2}$ subspace, since it has become irrelevant with respect to the corresponding analytic term. In addition we also disregard the SR analytic term in the $\mathbb{R}^{d_2} \vee \mathbb{R}^{d_1}$ subspace since it is subleading with respect to the LR one. This will introduce a bias in our computation close to the boundary $\sigma_1 \simeq \sigma_1^* \vee \sigma_2 \simeq \sigma_2^*$ resulting in a discontinuity in the critical exponents. In the following it will be shown that such discontinuity is very small (of the same order of the error committed by the LPA approximation itself) and does not affect the accuracy of our result. Let us also note that the correct computation involving all the terms in ansatz (18) is possible, but would not lead to explicit expressions for the flow equations as the ones given in the following.

Due to the SR dominant term we have now finite anomalous dimension effects. Let us focus on the $\sigma_1 = 2$ case, since the $\sigma_2 = 2$ case can be obtained trivially exchanging the subspaces dimensions $d_1 \leftrightarrow d_2$. The flow equation for the potential becomes

$$\begin{aligned}\partial_t \bar{U}_k =& (d_1 + \theta d_2)\bar{U}_k(\bar{\rho}) - (d_1 + \theta d_2 - 2 + \eta_1)\bar{\rho}\,\bar{U}_k'(\bar{\rho}) \\ &- (N-1)\frac{1 - \frac{\eta_1}{d_1+2} - \frac{2\eta_1 d_2}{d_1\sigma_2 + 2(d_2+\sigma_2)}}{1 + \bar{U}_k'(\bar{\rho})} \\ &- \frac{1 - \frac{\eta_1}{d_1+2} - \frac{2\eta_1 d_2}{d_1\sigma_2 + 2(d_2+\sigma_2)}}{1 + \bar{U}_k'(\bar{\rho}) + 2\bar{\rho}\,\bar{U}_k''(\bar{\rho})}.\end{aligned} \tag{34}$$

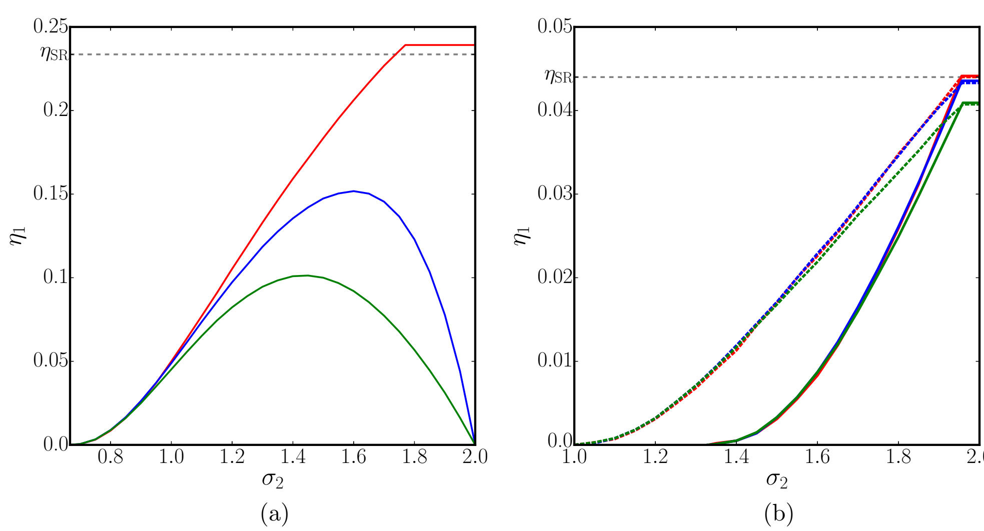


FIG. 2. In the left panel (a) we plot for $\sigma_1 > \sigma_1^*$ the anomalous dimension η_1 as a function of σ_2 in $d_1 = d_2 = 1$ for field component numbers $N = 1,2,3$, respectively, in red, blue, and green from the top. In the right panel (b) the anomalous dimension in $d_1 = 1 \vee 2, d_2 = 2 \vee 1$, again in the case $\sigma_1 > \sigma_1^*$, for the field component numbers $N = 1,2,3$, respectively, in red, blue, and green is shown (the solid $\vee$ dashed lines are for $d_1 = 2 \vee 1$ and $d_2 = 1 \vee 2$). In this case the lack of analytic term in the $\mathbb{R}^{d_2}$ subspace produces two different results for the isotropic limit $\sigma_2 \to \sigma_2^*$ between the two cases $d_2 = 2,1$ solid and dashed lines, respectively. The gray dashed lines represent the correct values of the anomalous dimension in the isotropic SR Ising model in the considered approximation for $d = 2$, panel (a), and $d = 3$, panel (b).

The anisotropy index is now given by $\theta = \frac{2-\eta_1}{\sigma_2}$. The anomalous dimension is then given by

$$\eta_1 = \frac{f(\tilde{\rho}_0,\tilde{U}^{(2)}(\tilde{\rho}_0))(\sigma_2 d_1 + 2d_2 + 2\sigma_2)}{2d_2 f(\tilde{\rho}_0,\tilde{U}^{(2)}(\tilde{\rho}_0)) + \sigma_2 d_1 + 2d_2 + 2\sigma_2}, \tag{35}$$

where the function $f(\tilde{\rho}_0,\tilde{U}^{(2)}(\tilde{\rho}_0))$ is the expression for the anomalous dimension of the correspondent SR range $O(N)$ model

$$f(\tilde{\rho}_0,\tilde{U}^{(2)}(\tilde{\rho}_0)) = \frac{4\tilde{\rho}_0\tilde{U}^{(2)}(\tilde{\rho}_0)^2}{(1+2\tilde{\rho}_0\tilde{U}^{(2)}(\tilde{\rho}_0))^2} \tag{36}$$

as is found in Ref. [47] after rescaling an unimportant geometric coefficient. Another possible definition of equation (36) is given in Ref. [58]. The two definitions are found depending on whether we calculate this quantity respectively from the Goldstone or the Higgs excitation propagator. In the following we always use result (36) in the numerical computation of the critical exponents.

One could be tempted to conclude that in regions $II_{A\vee B}$ the system is equivalent to a SR system in dimension $d_1+\theta d_2$ but this is not actually the case, since the value of the anomalous dimension η_1 is different from the one in the isotropic case.

The results for the anomalous dimension η_1 in region II_B as a function of σ_2 for the $d_1,d_2 = 1,1$ or 1,2 or 2,1 cases are reported in Figs. 2(a) and 2(b), respectively.

In $d = 2$ the SR system is exactly solvable for $N = 1$ and $\eta_{\rm SR} = \frac{1}{4}$, however at lowest order in derivative expansion the isotropic SR Ising approximated result is $\eta_{\rm SR} \approx 0.2336$, which is shown as a gray dashed line in Figs. 2(a) and 2(b). Our approximation level is, however, not able to recover this result, since for $\sigma_1 > \sigma_1^*$ we are not including for $\sigma_2 < \sigma_2^*$ any SR term in the $\mathbb{R}^{d_2}$ subspace. This is not a crucial issue of the method; indeed our result differs from the usual SR result by only 0.0058 which is smaller than the isotropic SR approximation error $|\eta_{\rm LPA}-\eta_{\rm exact}| \simeq 0.0164$, where $\eta_{\rm LPA}$ is the value of the anomalous dimension for the SR Ising model obtained using LPA′ (see appendix B). Thus the threshold

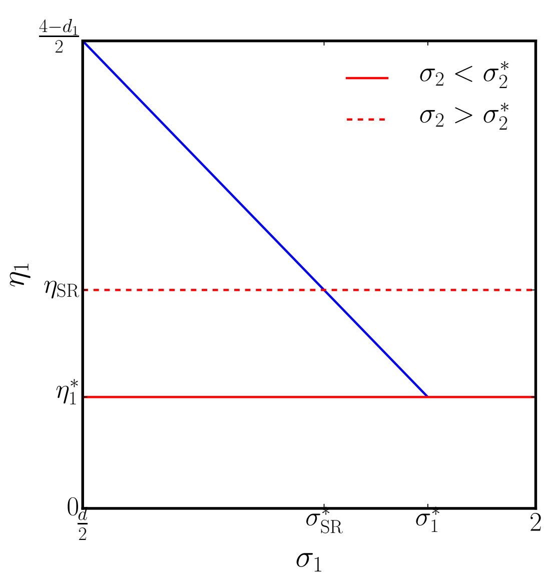


FIG. 3. Anomalous dimension η_1 as a function of σ_1.

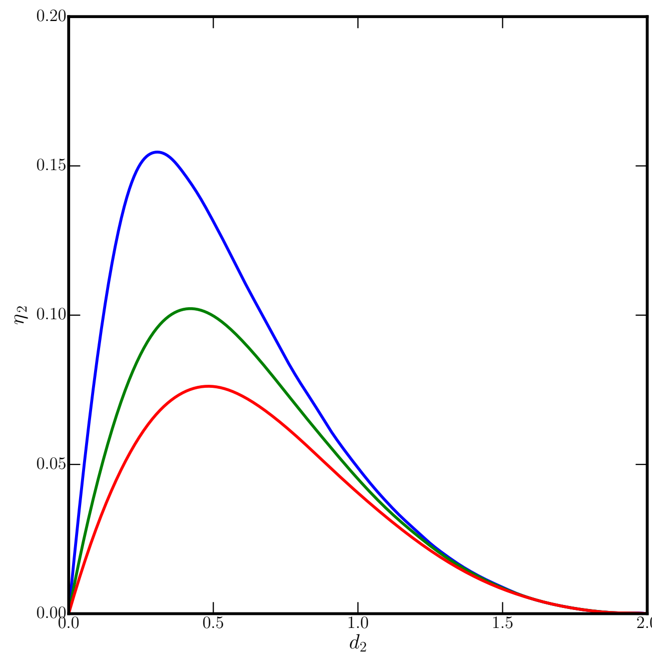


FIG. 4. Anomalous dimension η_2 for $d_1 = 1$ and general d_2 when $\sigma_1 = 1$ and $\sigma_2 > \sigma_2^*$ for field component numbers $N = 2,3,4$, respectively, in blue, green, and red from the top. These results may be used for studying phase transitions in quantum LR spin systems.

value $\sigma_2^* = 2-\eta_{\rm SR}$ does not directly appear in our treatment, since we do not include any SR correction to the nonanalytic term in the $\mathbb{R}^{d_2}$ subspace. However, for $\sigma_2 > \sigma_2^*$, the isotropy is restored and then the anomalous dimensions in both subspaces

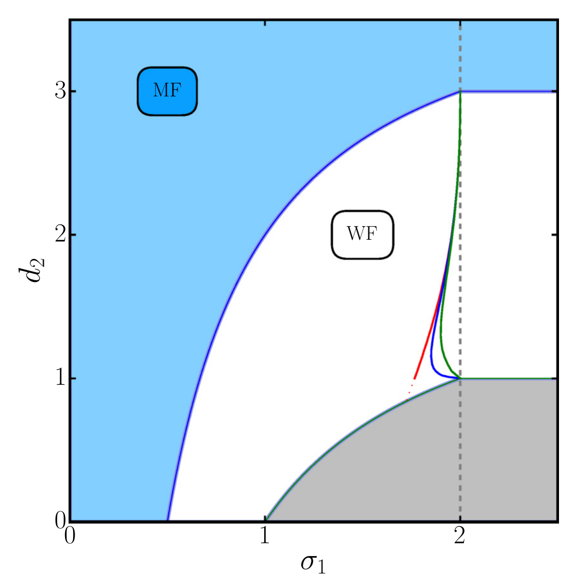
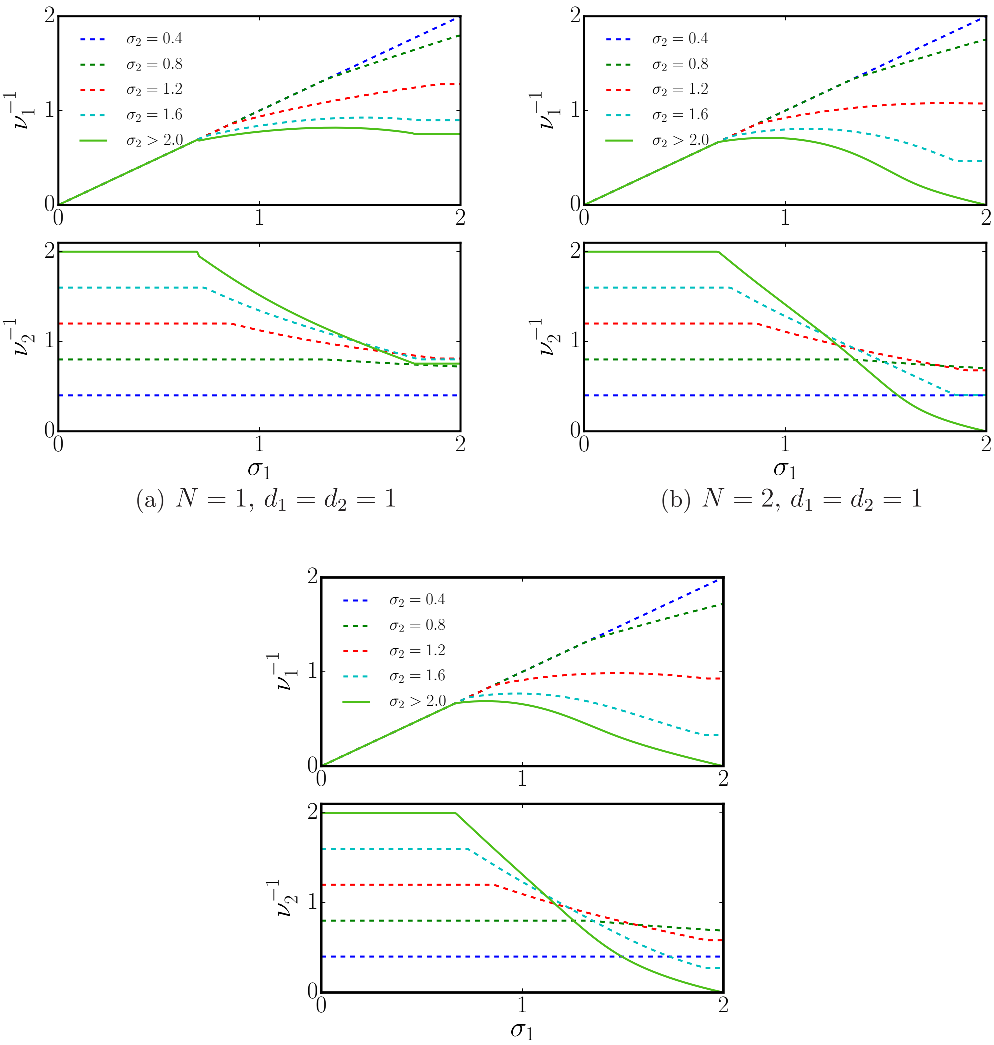


FIG. 5. The phase space of a LR anisotropic spin system with dimension $d_1 = 1$ as a function of d_2 with $\sigma_2 > \sigma_2^*$ for general σ_1. The cyan shaded region represents the mean-field validity region while in the white region WF type universality is found. The gray dashed line is the mean-field threshold above which SR behavior is recovered. The solid colored lines represent the dressed threshold values for the $N = 1,2,3$ cases in red, blue, and green, respectively. The gray area is the region where we expect the critical behavior to disappear and only a single phase is found.



(a) $N = 1$, $d_1 = d_2 = 1$ (b) $N = 2$, $d_1 = d_2 = 1$

(c) $N = 3$, $d_1 = d_2 = 1$

FIG. 6. In panel (a) the inverse of the correlation length exponents for the critical point of an anisotropic spin system for dimensions $d_1 = d_2 = 1$ are reported. The two exponents are shown for three values of the number of components $N = 1,2,3$ in panels (a), (b), and (c), respectively. For different values of σ_2 we report the behavior of the inverse exponents as a function of σ_1.

should coincide: $\eta_1 = \eta_2$. The threshold σ_1^* at a fixed $\sigma_2 < 2$ is then readily evaluated by the equation

$$\sigma_1^* = 2 - \eta_1(\sigma_1^*), \tag{37}$$

where $\eta_1(\sigma_1^*)$ is of course evaluated at the considered value of σ_2. A similar result holds for σ_2^*.

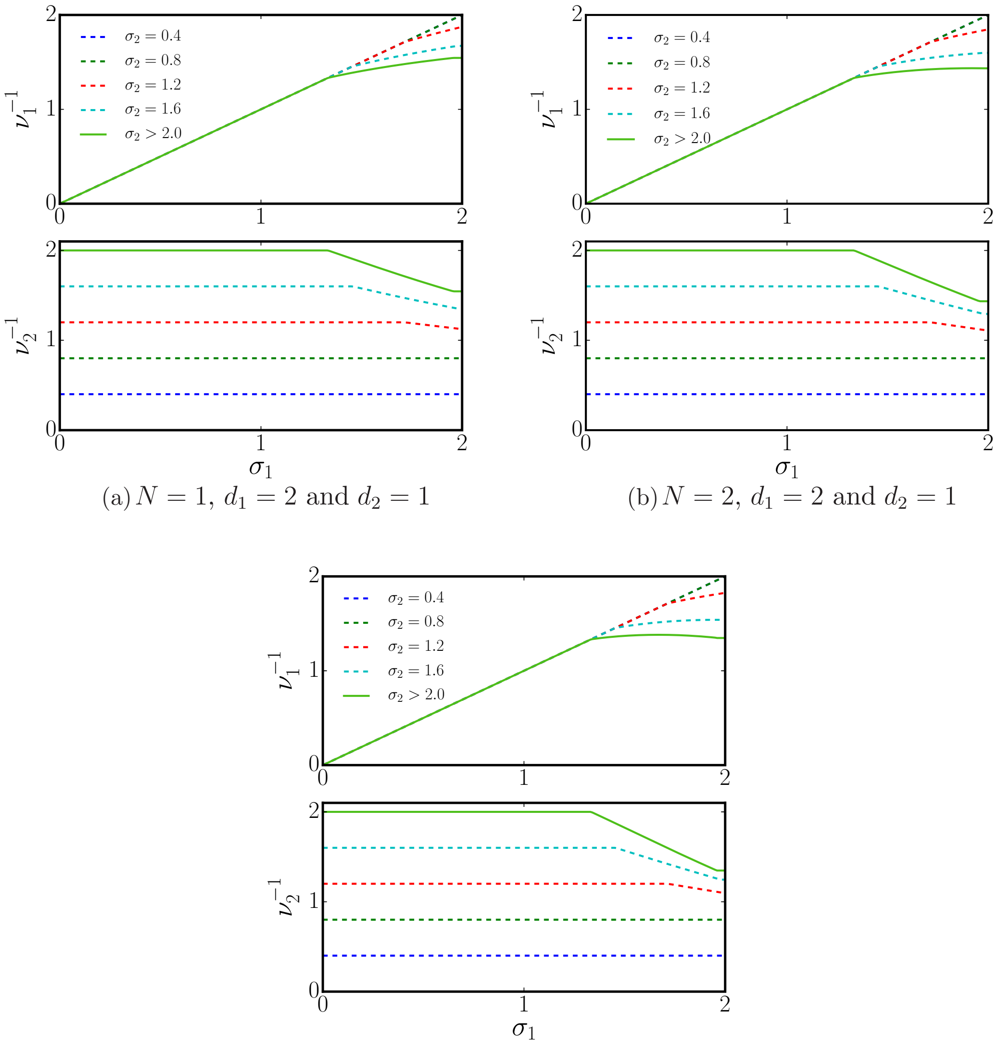
In order to clarify the above result we plot the value of the anomalous dimension η_1 in the $\mathbb{R}^{d_1}$ subspace as a function of σ_1, Fig. 3. For $\sigma_2 < \sigma_2^*$ the anomalous dimension has linear behavior $\eta_1 = 2 - \sigma_1$ until a boundary value $\sigma_1^* = 2 - \eta_1(\sigma_2)$ which depends on the value σ_2. Otherwise for $\sigma_2 > \sigma_2^*$ the boundary value is equal to $\sigma_1^* = 2 - \eta_{\rm SR}$ where $\eta_{\rm SR}$ is the anomalous dimension of the $d = d_1 + d_2$ dimensional SR system.

We observe that using a procedure based on Eq. (37) we do not exactly reproduce the expected boundary value in the mixed regions $\sigma_1^* = 2 - \eta_{SR}$, with η_{SR} the anomalous dimension of the SR isotropic case in $d = d_1 + d_2$ dimensions. However as explained in the caption of Fig. 2 the difference between the two results is small and the approximation of neglecting the analytic term in the $\mathbb{R}^{d_2}$ subspace appears to be well justified.

In Fig. 4 we report the result for the anomalous dimension η_2 as a function of d_2 for a one dimensional chain ($d_1 = 1$) with $\sigma_1 = 1$ and $\sigma_2 > \sigma_2^*$ (of course, for $\sigma_2 < \sigma_2^*$, η_2 would be just $2 - \sigma_2$ independent from d_2). In Fig. 5 we plot the phase diagram for $d_1 = 1$ in the d_2-σ_1 space with $\sigma_2 > \sigma_2^*$. The results in Figs. 4 and 5 could be used for a quantum LR spin chain.

A. The threshold values σ_1^* and σ_2^*

We have now all the information necessary to identify the correct values for the boundaries. Considering the results obtained both in the case of $\sigma_1 < \sigma_1^*$ and $\sigma_1 > \sigma_1^*$ we can deduce the existence of two fixed points in the full theory described by ansatz (18). One of these fixed points occurs at $Z_{\sigma_1} \neq 0$, while the other at $Z_{\sigma_1} = 0$. However this second fixed point is unstable in region I since any infinitesimal perturbation of the



(a) $N = 1,\ d_1 = 2$ and $d_2 = 1$

(b) $N = 2,\ d_1 = 2$ and $d_2 = 1$

(c) $N = 3,\ d_1 = 2$ and $d_2 = 1$

FIG. 7. Plot of the inverse of the correlation length exponents for the critical point of an anisotropic spin system with dimensions $d_1 = 2$ and $d_2 = 1$. The two exponents are shown for three values of the components number $N = 1,2,3$ in panels (a), (b), and (c), respectively. For different values of σ_2 we report the behavior of the inverse exponents as a function of σ_1.

Z_{σ_1} value around zero generates a nonvanishing flow which increases Z_{σ_1} itself.

Looking at condition (23) it is evident that this happens when $\sigma_1 < \sigma_1^*$, with σ_1^* defined by equation (37). However when $\sigma_1 > \sigma_1^*$ the nonanalytic term vanishes and, then, the value of η_1 is actually independent of σ_1. The value of η_1 is thus equal to its value in region II_B, i.e., $\eta_1 = \eta_1(\sigma_1^*)$.

As shown in equation (35) the value of η_1 in region II_B is actually a function of σ_2 and the boundary between region I and II_B is a curve in the (σ_1,σ_2) parameter space. Applying the same argument to the boundary between region I and II_A we can deduce that $\sigma_2^* = 2 - \eta_2(\sigma_2^*)$, where also here η_2 has to be evaluated at the considered value of σ_1. The final picture for the phase space of our theory is depicted in Fig. 1. For $d_1 = d_2 = 1$ and $N \geqslant 2$ the curves all terminate at the point $\sigma_1 = \sigma_2 = 2$, due to the presence of the Mermin-Wagner theorem, which prevents symmetry breaking for SR interactions and which is correctly described by functional RG truncations [59], as is shown in Fig. 1(a). For $N = 1$ the system shows discrete symmetry and the anisotropic region terminates at the point $\sigma_1^* = \sigma_2^* = 2 - \eta_{SR}$. In Fig. 1(b) we show results for $N = 1,2,3$ with $d_1 = 2$ and $d_2 = 1$, respectively, in red, blue, and green. In this case the boundaries are different from 2 even at the intersection where the system behaves as an isotropic classical SR system in dimension $d = d_1 + d_2$. The difference between the anomalous dimensions in the cases $N = 1,2,3$ is so small that the different boundaries cannot be distinguished.

We are now able to compute the correlation length exponents of the system for different values of σ_1 and σ_2, using the procedure outlined in Ref. [48]. In region I we can rely on the effective dimension relation (29) to compute them. Indeed, the correlation length exponent ν_1 is the same of an isotropic LR system of exponent σ_1 in dimension (29). The correlation length exponent ν_2 is determined from ν_1 using the scaling relation (7) with $\theta = \frac{\sigma_1}{\sigma_2}$.

In the regions $II_{A\vee B}$ the effective dimension is strictly not valid and one should in principle compute the correlation length exponent ν_1 by studying the stability equation around the fixed points, as described in Ref. [48]. It is still possible to reintroduce the effective dimension (29) neglecting the anomalous dimension terms in equation (28).

The procedure of neglecting the anomalous dimension in the potential flow is commonly employed to solve functional RG equations [49]. Indeed the dependence of the potential equation of anomalous dimension is only due to small cutoff dependent coefficients, which have a small effect on the universal quantities, at least at this approximation level.

Once these coefficients are neglected we can impose the fixed point condition $\partial_t \tilde{U}_k = 0$, and dividing equation (28) by θ one obtains

$$(d_2 + \theta' d_1)\bar{U}_k(\bar{\rho}) - (d_2 + \theta' d_1 - \sigma_2)\bar{\rho}\,\bar{U}_k'(\bar{\rho}) - (N-1)\frac{\sigma_2}{2 + 2\bar{U}_k'(\bar{\rho})} - \frac{\sigma_2}{2 + 2\bar{U}_k'(\bar{\rho}) + 4\bar{\rho}\,\bar{U}_k''(\bar{\rho})} = 0, \tag{38}$$

where $\theta' = \theta^{-1} = \frac{\sigma_2}{2-\eta_1}$ in the region II_B.

It is worth noting that regions $\mathrm{II}_{A\vee B}$ are interesting for possible applications to quantum LR systems since they represent the case of anisotropic SR plus LR interactions as it happens, e.g., for the quantum LR Ising model.

B. Correlation length exponent

In Figs. 6 and 7 we show the results for the correlation length exponents for various values of σ_1 as a function of the exponent σ_2 in dimensions $d_2 = 1$ (Fig. 6) and $d_2 = 2$ (Fig. 7) with $d_1 = 1$ in both cases in the trivial region, equation (17); the relevant exponents in each subspace are independent of the presence of the other subspace and are $\nu_1 = \sigma_1^{-1}$ and $\nu_2 = \sigma_2^{-1}$. Then in the pure LR region the exponents become nontrivial curves as a function of σ_1. For some value of σ_1 we will cross the boundary region $\sigma_1^*(\sigma_2)$ which is a function σ_2. For $\sigma_1 > \sigma_1^*$ the exponents both become constant. When $\sigma_2 > 2$ we are in the region where SR interactions are dominant in the subspace $\mathbb{R}^{d_2}$ (this is the relevant case for LR quantum rotor models), and the exponents are shown by a solid line. In this case the exponents are nontrivial functions of σ_1 for $\sigma_1 < \sigma_1^* = 2 - \eta_{SR}$, where η_{SR} is the anomalous dimension of the isotropic SR system in dimension $d_1 + d_2$, while they become constant for $\sigma_1 > \sigma_1^*$ and both equal to the correlation length exponent of the isotropic SR systems $\nu_1 = \nu_2 = \nu_{\mathrm{SR}}$. These results, together with the anomalous dimensions in the regions $\mathrm{II}_{A\vee B}$, complete the characterization of the phase diagram of LR anisotropic spin system.

VIII. CONCLUSIONS

Anisotropic long-range (LR) spin systems have a rich phase diagram as a function of the two exponents σ_1 and σ_2 and of the two dimensions d_1,d_2. In the $\sigma_1 - \sigma_2$ plane two boundary curves exist, namely $\sigma_1^* = \sigma_1^*(\sigma_2)$ and $\sigma_2^* = \sigma_2^*(\sigma_1)$, where the LR interactions in the subspaces $\mathbb{R}^{d_2}$ and $\mathbb{R}^{d_1}$ become irrelevant. At mean-field level the two boundaries are straight lines, $\sigma_1^* = \sigma_2^* = 2$, as shown in Fig. 8. Beyond mean field these boundaries become nontrivial curves, see Fig. 1. At the intersection between the boundaries the system recovers both short-range (SR) and isotropic behaviors and then the intersection point is simply given by $\sigma_1 = \sigma_2 = 2 - \eta_{SR}$, with η_{SR} the anomalous dimension of an isotropic SR system in dimension $d_1 + d_2$, as is found for isotropic LR systems [23–26].

In the pure LR region, denoted by I in Fig. 1, the low energy behavior can be described by the effective action (18). The field dynamics is characterized by two nonanalytic powers of the momentum excitations with, respectively, real exponents σ_1 and σ_2 in the two subspaces $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$. In this case the system universality class is equivalent to an isotropic LR system in an effective dimension $d_{\mathrm{eff}} = d_1 + \theta d_2$, defined in equation (29).

When one of the two exponents $\sigma_1 \vee \sigma_2$ becomes larger than its threshold value $\sigma_1^* \vee \sigma_2^*$ the corresponding nonanalytic kinetic term in the effective action (18) becomes subleading with respect to the analytic term, and LR interactions lie in the same universality of SR ones. The system enters then in the mixed regions $II_{A\vee B}$ where the subspace $\mathbb{R}^{d_1\vee d_2}$ effectively behaves as if only SR interactions were present.

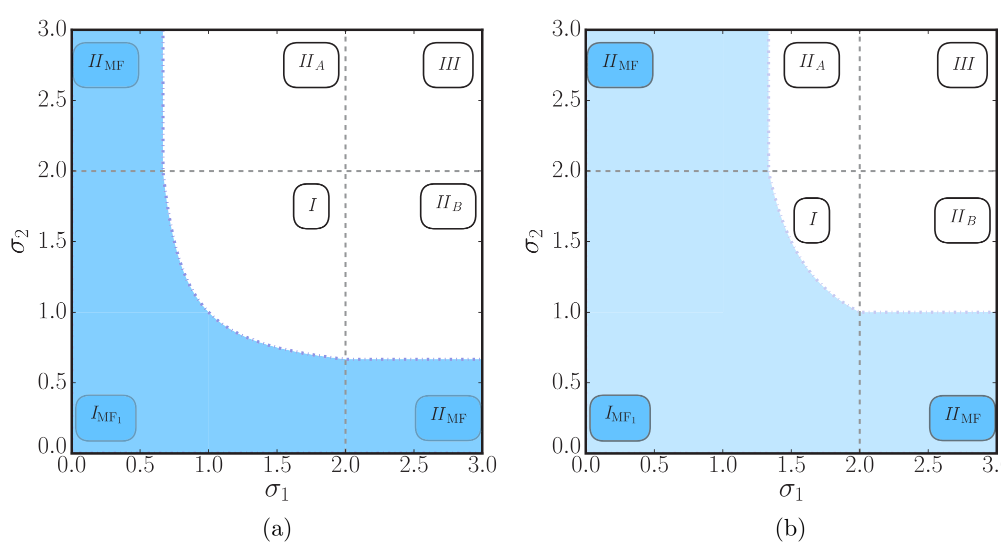


FIG. 8. The mean-field parameter space of a LR anisotropic spin system with dimensions $d_1 = d_2 = 1$, panel (a), and $d_1 = 2$,$d_2 = 1$, panel (b).

In regions $II_{A\vee B}$ the system is described by the effective action (18) with $\sigma_1 \vee \sigma_2 = 2$. We observe that in such regions the system is not equivalent to a SR system in dimension $d_1 + \theta d_2$, since the value of the anomalous dimension $\eta_{1\vee 2}$ is different from the one in the isotropic case. In these regions we can study the model with equation (34) and the anomalous dimension defined by (35). The result for the anomalous dimension in regions $II_{A\vee B}$ is given in Fig. 2. Once the anomalous dimension of the analytic term in the presence of nonanalytic anisotropic terms is known, we can calculate the threshold curves, which are $\sigma_2^*(\sigma_1) = 2 - \eta_2(\sigma_2^*,\sigma_1)$ and $\sigma_1^*(\sigma_2) = 2 - \eta_1(\sigma_1^*,\sigma_2)$, as given in equation (37) and depicted in Fig. 1.

Regions $II_{A\vee B}$ are relevant also because the quantum critical points at zero temperature of a quantum spin system with LR couplings lie in these regions. In particular the effective action (18) describes the universality of a quantum system in dimension $d_1 = D$, when one of the two subspaces has dimension d_1 with power-law decay exponent σ_1 and the other subspace, with the dimension $d_2 = 1$, contains only SR interactions.

Anisotropic LR systems have two different correlation length exponents which are connected by scaling relation (7). The exponent ν_1 can be obtained by studying the stability around the fixed points of equation (28) in region I or of equation (34) in regions $II_{A\vee B}$. On the other hand ν_1 is also equal to the correlation length exponent of an isotropic LR system in dimension d_{eff}, equation (29). In regions $II_{A\vee B}$ the effective dimension relation (29) is not strictly valid, but we can reintroduce it neglecting small anomalous dimension terms in equation (34).

Using the effective dimension relations (29) it is then possible to compute the critical exponents for the anisotropic LR $O(N)$ models for general values of the dimensions d_1 and d_2 and for different values of the field components N. An interesting case is the one with a one dimensional subspace ($d_1 \vee d_2 = 1$). The results are reported in Figs. 6 and 7.

The analysis of ansatz (18) also leads to exact results in the $N \to \infty$ limit, where only the correlation length exponents are different from zero in all the regions, see equations (30) and (33). The validity of ansatz (18) in the $N \to \infty$ limit also resulted in the reproduction of the correct result for the ANNNI models, equations (32) and (33).

This work provides a step forward in the comprehension of LR interaction effects in the critical behavior of spin systems. Since anisotropic interactions are widely present in condensed matter systems, it would be interesting to investigate whether anisotropic LR critical behavior could be responsible for various phase transitions occurring in the presence of multiaxial anisotropy. Our results can also be useful for the study of quantum LR systems via the quantum-to-classical correspondence. It would be interesting to extend our results to LR quantum spin chains and also to fermionic and bosonic models with LR interactions, in particular to determine the effects of anisotropy in the presence of LR couplings for which a nontrivial topology of the Fermi surface occurs [60]. Our results also call for further investigations of the critical behavior of anisotropic LR systems both in the numerical simulations and in experiments, in order to confirm the reliability of field theory description used in this paper.

Finally it is worth noting that we focused only on the sharp anisotropy case, where the interaction between two spins occurs only when their distance is parallel to one of the main system axes, and we did not consider the case of angular dependent interactions. Moreover we do not address the description of the higher order phase transitions occurring in these models for $\sigma_{1\vee 2} > 2$ as in the standard Lifshitz point critical behavior. Both of these very interesting studies are left for future work.

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APPENDIX A: MEAN-FIELD RESULTS

In this appendix we discuss the mean-field treatment of LR anisotropic systems. For any value of $\sigma_1 \vee \sigma_2$ larger than 2, the behavior reduces to the case of only SR interactions in the subspace $\mathbb{R}^{d_1} \vee \mathbb{R}^{d_2}$. Thus the (σ_1,σ_2) parameter space can be divided into four areas, as shown in Fig. 1. Notice that the mean-field phase diagram in Fig. 8 and discussed in the following should be compared with the one presented in Fig. 1 and obtained by the RG analysis of Sec. V.

At the mean-field level one has two thresholds (dashed lines) at $\sigma_1 = 2$ and $\sigma_2 = 2$, dividing the parameter space into four regions. The region I ($\sigma_1 < 2$, $\sigma_2 < 2$) is the pure anisotropic LR region, where the saddle point of effective action (3) is valid. In regions $II_{A\vee B}$ the exponent $\sigma_1 \vee \sigma_2$ is larger than two and the correct effective field theory is given by expression (18) with $\sigma_1 = 2 \vee \sigma_2 = 2$. In region III both kinetic terms are irrelevant compared to the SR kinetic terms and the model becomes equivalent to a $d = d_1 + d_2$ dimensional isotropic SR system. The shaded areas correspond to the region where inequality (17) is fulfilled only for $i = 1$ and then mean field is valid; here the region names the mean-field subscript MF. In region I ($\sigma_1,\sigma_2 < 2$) the system is LR in both subspaces. The cyan shaded area in Fig. 1(a) is the gaussian region in $d_1 = d_2 = 1$ and light cyan in Fig. 1(b) is for $d_1 = 1$ and $d_2 = 2$. In region $II_{A\vee B}$ the system is SR in the subspace of dimension $d_1 \vee d_2$ and LR in the other. Summarizing, when mean field is valid, one has the following results for the critical exponents of the system

$$\eta_{\sigma_1} = 0, \quad \eta_{\sigma_2} = 0, \qquad \nu_1 = \frac{1}{\sigma_1}, \quad \nu_2 = \frac{1}{\sigma_2}. \tag{A1}$$

It should be noted that for the $d_1 = d_2 = 1$ case, shown in Fig. 1(a), regions $II_{A\vee B}$ are completely equivalent since the system is invariant under the exchange of the two exponents. This is not true in the case $d_1 \neq d_2$, Fig. 1(b), where $d_1 = 1$ and $d_2 = 2$. Finally in region $III(\sigma_1,\sigma_2 > 2)$ the system is in the same universality class of an isotropic SR system.

As discussed in the main text, the previous analysis is valid at mean-field level, but, when fluctuations are relevant, one has to take into account the competition between analytic and nonanalytic momentum terms close to the boundaries $\sigma_1 \vee \sigma_2 \approx 2$. Indeed, while nonanalytic terms do not develop anomalous dimensions, the SR analytic terms normally do, and at the renormalized level the boundaries of the nonanalytic regions σ_1^* and σ_2^* could be different from the canonical dimension result $\sigma_1^* = \sigma_2^* = 2$, as happens in usual LR systems [23–26].

Regarding the case of quantum rotor Hamiltonians it is possible to use mean-field arguments to dig out the nontrivial phase transition region. Denoting the dimension of the quantum system by D and the exponent of the decay of the coupling by $D + \sigma_1$, we should then substitute $d_1 = D$ and $\sigma_2 = 2$ into relation (17) for general d_2 to obtain

$$d_2 < 4 - \frac{2D}{\sigma_1}, \tag{A2}$$

where one has to take $d_2 = 1$. Then, a quantum spin system in dimension D with LR interactions decaying with exponent σ_1 develops a nontrivial phase transition when equation (A2) is satisfied. This region is reported with the WF label in Fig. 5 for the $d_1 = 1$ case.

APPENDIX B: FUNCTIONAL RENORMALIZATION GROUP APPROACH

In this appendix we summarize the main results of functional RG needed in the main text. We start from the effective action for a N component vector field with $O(N)$ symmetry:

$$\Gamma[\phi] = -\int d^d x \{ Z_\sigma \phi_i(x) \Delta_1^{\frac{\sigma}{2}} \phi_i(x) + Z_\tau \phi_i(x) \Delta_1^{\frac{\tau}{2}} \phi_i(x) + Z_2 \phi_i(x) \Delta_1 \phi_i(x) - U_t(\rho) \}. \tag{B1}$$

An ansatz with a similar shape for the effective action is often called a local potential approximation (LPA) ansatz. Indeed equation (B1) has a local potential term $U(\rho)$. In the following we will introduce expressions containing two field independent wave-function renormalization terms Z_σ and Z_τ, providing for the considered case the counterpart of the so-called LPA$'$ ansatz.

The propagator in this scheme is the inverse second derivative of the action with respect to the fields. This derivative is diagonal in the field indexes when evaluated in a constant state of the field; it reads

$$\Gamma_t^{(2)}(-p,p) = Z_\sigma p_1^\sigma + Z_\tau p_2^\tau + Z_2 p_1^2 + \mu_{g,m}, \tag{B2}$$

where μ is the mass of the excitation,

$$\mu_m = U^{(1)}(\rho) + 2\rho U^{(2)}(\rho), \tag{B3}$$

$$\mu_g = U^{(1)}(\rho), \tag{B4}$$

and

$$q_1 = \left(\sum_{\mu=1}^{d_1} q_\mu^2 \right)^{\frac{1}{2}} \tag{B5}$$

$$q_2 = \left(\sum_{\mu=d_1+1}^{d} q_\mu^2 \right)^{\frac{1}{2}} \tag{B6}$$

are the moduli of the momentum in each subspace. Obviously the index (g,m) depends if the derivation is taken in the component on the field which has zero or nonzero average. The propagator of the theory where a smooth momentum cutoff has been inserted can be written as

$$G(q) = (\Gamma(q,-q) + R_t(q))^{-1}, \tag{B7}$$

where the inverse as to be intended in the matrix sense. The time derivative of the two point function is

$$\partial_t \Gamma_t^{(2)}(-p,p) = \partial_t Z_\sigma p_1^\sigma + \partial_t Z_\tau p_2^\tau + \partial_t Z_2 p_1^2 + \partial_t \mu_t, \tag{B8}$$

where the value of the μ depends on the index of the field with respect, which we are taking the derivation of in the above expression. Using the Wetterich equation for the LPA$'$ truncation level we obtain

$$\partial_t \Gamma_t^{(2)}(-p,p) = 2 \int \frac{d^d q}{(2\pi)^d} \partial_t R_t(q) G_g(q)^2 \Gamma^{(3)}(\rho)^2 G_m(p+q); \tag{B9}$$

for the flow of the two wave-function renormalizations we use the definitions,

$$\partial_t Z_\sigma = \lim_{p\to 0} \frac{d}{dp_1^\sigma} \partial_t \Gamma_t^{(2)}(-p,p). \tag{B10}$$

$$\partial_t Z_\tau = \lim_{p\to 0} \frac{d}{dp_2^\tau} \partial_t \Gamma_t^{(2)}(-p,p). \tag{B11}$$

$$\partial_t Z_\tau = \frac{1}{2} \lim_{p\to 0} \frac{d^2}{dp_1^2} \partial_t \Gamma_t^{(2)}(-p,p). \tag{B12}$$

When we apply the derivatives on the right end side of equation (B9), they go under the integral sign and act in the only part of the integrand which depends on p, i.e., $G(p+q)$, thus we get,

$$\partial_t Z_\sigma = \int \frac{d^d q}{(2\pi)^d} \partial_t R_t(q) G_g(q)^2 \Gamma^{(3)}(\rho)^2 \frac{d}{dp_1^\sigma} G_m(p+q) \Big|_{p=0}, \tag{B13}$$

$$\partial_t Z_\tau = \int \frac{d^d q}{(2\pi)^d} \partial_t R_t(q) G_g(q)^2 \Gamma^{(3)}(\rho)^2 \frac{d}{dp^\tau} G_m(p+q) \Big|_{p=0}, \tag{B14}$$

$$\partial_t Z_2 = \int \frac{d^d q}{(2\pi)^d} \partial_t R_t(q) G_g(q)^2 \Gamma^{(3)}(\rho)^2 \frac{d^2}{2dp^2} G_m(p+q) \Big|_{p=0}. \tag{B15}$$

Introducing into latter formulas the explicit expression derived from ansatz (B1) we obtain the flow equations (19), (20), (34), and (35) reported in the main text. The anomalous dimension has been obtained solving differential equation (34) iteratively until self consistency with equation (35) is reached. Substituting into the flow equations for the effective potential (28) or (34) the expression $\tilde{U}_k(\tilde{\rho}) = \tilde{U}^*(\tilde{\rho}) + u_k(\tilde{\rho}) e^{yt}$, obtaining an equation for the linear perturbation $u_k(\tilde{\rho})$ and the eigenvalues spectrum y as is done in Ref. [48]. By solving the equation for the perturbation around the fixed point solution $\tilde{U}^*(\tilde{\rho})$ the correlation length exponent ν is readily obtained following the procedure outlined in Ref. [48].

[1] S. T. Bramwell, *Dipolar Effects in Condensed Matter* in Ref. [2], p. 549.
[2] *Long-Range Interacting Systems*, edited by T. Dauxois, S. Ruffo, and L. F. Cugliandolo, Lecture Notes of the Les Houches Summer School, session XC, 4–29 August 2008 (Oxford University Press, Oxford, 2010).
[3] T. Lahaye, C. Menotti, L. Santos, M. Lewenstein, and T. Pfau, Rep. Prog. Phys. **72**, 126401 (2009).
[4] P. C. W. Holdsworth, M. J. P. Gingras, B. Bergersen, and E. Chan, Phys. Scr. **39**, 613 (1989).
[5] J. V. Selinger and D. R. Nelson, Phys. Rev. A **39**, 3135 (1989).
[6] A. Michelson, Phys. Rev. Lett. **39**, 464 (1977).
[7] S. B. Rananavare, V. G. K. M. Pisipati, and E. W. Wong, Phys. Rev. Lett. **72**, 3558 (1994).
[8] M. Tinkham, *Introduction to Superconductivity* (McGraw-Hill, New York, 1996), Chap. 9.
[9] N. B. Hannay, T. H. Geballe, B. T. Matthias, K. Andres, P. Schmidt, and D. MacNair, Phys. Rev. Lett. **14**, 225 (1965).
[10] S. T. Ruggiero, T. W. Barbee, Jr., and M. R. Beasley, Phys. Rev. Lett. **45**, 1299 (1980).
[11] F. R. Gamble, F. J. Disalvo, R. A. Klemm, and T. H. Geballe, Science **168**, 568 (1970).
[12] I. Nandori, K. Vad, S. Meszaros, U. D. Jentschura, S. Nagy, and K. Sailer, J. Phys.: Condens. Matter **19**, 496211 (2007).
[13] U. D. Jentschura, I. Nándori, and J. Zinn-Justin, Ann. Phys. **321**, 2647 (2006).
[14] M. Iazzi, S. Fantoni, and A. Trombettoni, Europhys. Lett. **100**, 36007 (2012).
[15] M. A. Cazalilla, A. Iucci, and T. Giamarchi, Phys. Rev. A **75**, 051603(R) (2007).
[16] B. Chattopadhyay and S. R. Shenoy, Phys. Rev. Lett. **72**, 400 (1994).
[17] T. Schneider and J. M. Singer, *Phase Transition Approach to High Temperature Superconductivity* (Lausanne-Dorigny, Université de Lausanne, 1998).
[18] *The BCS-BEC Crossover and the Unitary Fermi Gas*, edited by W. Zwerger (Heidelberg, Springer, 2012).
[19] A. Campa, T. Dauxois, D. Fanelli, and S. Ruffo, *Physics of Long-Range Interacting Systems* (Oxford University Press, Oxford, 2014).
[20] M. Kac, G. Uhlenbeck, and P. Hemmer, J. Math. Phys. **4**, 216 (1963).
[21] J. Barré, F. Bouchet, T. Dauxois, and S. Ruffo, J. Stat. Phys. **119**, 677 (2005).
[22] T. Mori, Phys. Rev. E **82**, 060103(R) (2010); J. Stat. Mech. (2013) P10003.
[23] J. Sak, Phys. Rev. B **8**, 281 (1973).
[24] E. Brezin, G. Parisi, and F. Ricci-Tersenghi, J. Stat. Phys. **157**, 855 (2014).
[25] M. C. Angelini, G. Parisi, and F. Ricci-Tersenghi, Phys. Rev. E **89**, 062120 (2014).
[26] N. Defenu, A. Trombettoni, and A. Codello, Phys. Rev. E **92**, 052113 (2015).
[27] M. F. Paulos, S. Rychkov, B. C. van Rees, and B. Zan, Nucl. Phys. B **902**, 246 (2016).
[28] J. W. Britton, B. C. Sawyer, A. C. Keith, C. C. Wang, J. K. Freericks, H. Uys, M. J. Biercuk, and J. J. Bollinger, Nature (London) **484**, 489 (2012).
[29] P. Schauß, M. Cheneau, M. Endres, T. Fukuhara, S. Hild, A. Omran, T. Pohl, C. Gross, S. Kuhr, and I. Bloch, Nature (London) **491**, 87 (2012).
[30] B. Yan, S. A. Moses, B. Gadway, J. P. Covey, K. R. A. Hazzard, A. M. Rey, D. S. Jin, and J. Ye, Nature (London) **501**, 521 (2013).
[31] O. Firstenberg, T. Peyronel, Q.-Y. Liang, A. V. Gorshkov, M. D. Lukin, and V. Vuletic, Nature (London) **502**, 71 (2013).
[32] R. Islam, C. Senko, W. C. Campbell, S. Korenblit, J. Smith, A. Lee, E. E. Edwards, C.-C. J. Wang, J. K. Freericks, and C. Monroe, Science **340**, 583 (2013).
[33] P. Jurcevic, B. P. Lanyon, P. Hauke, C. Hempel, P. Zoller, R. Blatt, and C. F. Roos, Nature (London) **511**, 202 (2014).
[34] H. Schempp, G. Günter, S. Wüster, M. Weidemüller, and S. Whitlock, Phys. Rev. Lett. **115**, 093002 (2015).
[35] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, 2011).
[36] T. D. Schultz, D. C. Mattis, and E. H. Lieb, Rev. Mod. Phys. **36**, 856 (1964).
[37] G. Mussardo, *Statistical Field Theory: An Introduction to Exactly Solved Models in Statistical Physics* (Oxford University Press, Oxford, 2010).
[38] I. Bars, C. Deliduman, and O. Andreev, Phys. Rev. D **58**, 066004 (1998).
[39] E. H. Lieb, Comm. Math. Phys. **31**, 327 (1973).
[40] D. Podolsky, E. Shimshoni, P. Silvi, S. Montangero, T. Calarco, G. Morigi, and S. Fishman, Phys. Rev. B **89**, 214408 (2014).
[41] R. M. Hornreich, M. Luban, and S. Shtrikman, Phys. Rev. Lett. **35**, 1678 (1975).
[42] M. E. Fisher, J. Math. Phys. **4**, 124 (1963).
[43] C. Mergulhaõ Jr. and C. E. I. Carneiro, Phys. Rev. B **58**, 6047 (1998).
[44] H. W. Diehl and M. Shpot, Phys. Rev. B **62**, 12338 (2000).
[45] J. Als-Nielsen and R. J. Birgeneau, Am. J. Phys. **45**, 554 (1977).
[46] I. D. Lawrie and S. Sarbach, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz, Vol. 9, Chap. 1 (Academic Press, London, 1984).
[47] A. Codello and G. D'Odorico, Phys. Rev. Lett. **110**, 141601 (2013).
[48] A. Codello, N. Defenu, and G. D'Odorico, Phys. Rev. D **91**, 105003 (2015).
[49] J. Berges, N. Tetradis, and C. Wetterich, Phys. Rep. **363**, 223 (2002).
[50] B. Delamotte, in *Order, Disorder and Criticality: Advanced Problems of Phase Transition Theory*, edited by Yu. Holovatch (World Scientific, Singapore, 2007).
[51] C. Wetterich, Phys. Lett. B **301**, 90 (1993).
[52] T. R. Morris, Phys. Lett. B **334**, 355 (1994).
[53] G. S. Joyce, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic Press, London, 1972), Vol. 2, Chap. 10.
[54] M. E. Fisher, S. K. Ma, and B. G. Nickel, Phys. Rev. Lett. **29**, 917 (1972).
[55] M. Baczyk, M. Tissier, G. Tarjus, and Y. Sakamoto, Phys. Rev. B **88**, 014204 (2013).

[56] T. Horita, H. Suwa, and S. Todo, arXiv:1605.09496.
[57] L. Frachebourg and M. Henkel, Physica A **195**, 577 (1993).
[58] A. Codello, J. Phys. A **45**, 465006 (2012).
[59] N. Defenu, P. Mati, I. G. Márián, I. Nándori, and A. Trombettoni, J. High Energy Phys. 05 (2015) 141.
[60] G. Gori, S. Paganelli, A. Sharma, P. Sodano, and A. Trombettoni, Phys. Rev. B **91**, 245138 (2015).