Universality classes of order parameters composed of many-body bound states

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This theoretical paper discusses microscopic models giving rise to special types of order in which conduction electrons are bound together with localized spins to create composite order parameters. It is shown that composite order is related to the formation of a spin liquid with gapped excitations carrying quantum numbers that are a fraction of those of an electron. These spin liquids are special in the sense that their formation necessarily involves spin degrees of freedom of both the conduction and the localized electrons and can be characterized by nonlocal order parameters. A detailed description of such spin-liquid states is presented with a special care given to a demonstration of their robustness against local perturbations preserving the Lie group symmetry and the translational invariance.

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I. INTRODUCTION

This paper puts forward a theoretical description of composite order parameters (COPs). Such order parameters emerge as a result of condensation of many-body bound states of conduction electrons with collective modes of interacting magnetic moments. On the formal level, the COPs are expressed as products of spin operators of localized electrons and various density operators of conduction electrons. The latter may include multiple products of charge, spin, and pair densities. Being related to many-body bound states, a formation of the COP's requires strong correlations and their study will take us to hitherto unexplored areas of physics. The first example of such COP was found in the Kondo-Heisenberg chain model by Zohar and the author [1]; it included a bound state of the staggered magnetization with the pair density, an analog of the Fulde-Ferell-Ovchinnikov-Larkin (FFLO) state, but created without a magnetic field. It also included the bound state with a charge density wave with a wave vector proportional to the total electron density (including the density of localized electrons). Although one-dimensional models can support only quasi-long-range order, a real order is possible in arrays of chains provided one manages to couple the corresponding COPs. This may be a problem since they usually carry finite wave vectors, so that a coupling between COPs with different wave vectors is suppressed due to the momentum conservation. This suppression mechanism was invoked in Ref. [2] to explain the exotic two-dimensional superconductivity found in a layered compound $La_{1.875}Ba_{0.125}CuO_4$ [3,4]. It has been suggested that the superconducting order parameters belong to the staggered pair density COPs and the COPs from neighboring layers do not couple since their wave vectors are perpendicular to each other.

Although the concept of composite order is a general one, to achieve a better understanding, we need to consider models that allow reliable and controlled calculations. I suggest that Kondo-Heisenberg models provide ideal platforms for these kind of studies. The core physics of Refs. [1,5] is the following. The Kondo-Heisenberg model brings together conduction electrons in the form of one-dimensional electron gas (1DEG) and antiferromagnetically correlated localized spins. Taken by themselves both electron and spin subsystems are quantum critical. In a quasi-one-dimensional setting, this means that (i) the low-energy modes are chiral and (ii) the spin and charge degrees of freedom of the 1DEGs are separated. These two facts suggest a possibility of a highly entangled state where right moving spinons from the 1DEG pair to left moving ones from the antiferromagnet and vice versa. As a result, two independent spin liquids are formed, each one uniting spin degrees of freedom of opposite chirality from the 1DEG and the spin chain; the charge sector of the 1DEG is left gapless and is populated by the Goldstone modes. Such a state has a hidden order associated with pairing of spinons from different chains and hence is robust with respect to local perturbations. The realization of such a spin liquid is possible when the band filling of the 1DEG is far from 1/2 so that the Kondo exchange cannot generate backscattering. Then the relevant coupling is between the spin currents of opposite chirality from the 1DEG and the spin chain. As I have said, the resulting spin liquid is a sum of two liquids formed by spinons of opposite chirality hence being mirror images of each other. The corresponding excitations carry fractional quantum numbers. Since such pairing takes place not between electrons, but between the spinons, this process cannot be treated perturbatively or via any kind of mean field making it even more interesting.

In Refs. [1,5] we studied the simplest version of the Kondo-Heisenberg (KH) model where the localized spins have magnitude $1/2$ and there is one electronic chain per each spin one. In this paper, I demonstrate that this is just one possibility out of many. One can construct entire universality classes of KH models corresponding to different representations of Lie groups with different topological orders, different gapped excitations and different COPs and generically a particular model may have several COPs.

Below, I consider two types of models. Both of them describe arrays of one-dimensional Kondo-Heisenberg wires. In the models of the first type, the wires are arranged in "cables," such that each one-dimensional unit consists of a chain of localized spins $S = n/2$ surrounded by a bunch of n conducting chains with incommensurate band fillings. In fact, to call this spin chain the Heisenberg one is an abuse of the term since the spin-spin interaction I consider includes higher powers of $(\mathbf{S}_n\mathbf{S}_{n+1})$. It would be more appropriate to call it generalized Heisenberg chain, but I will not do it for the sake of brevity.

In the suggested construction, the gapped fractionalized excitations are able to propagate only along a single cable even when the chains are connected into arrays. I will argue, however, that the three-dimensional coupling does not destroy these excitations, although it creates their bound states which carry quantum numbers of electrons and can propagate between the cables. A similar construction has been recently used in Ref. [6] in the context of fractional quantum Hall effect. The models of another type are SU(N) generalizations of the Kondo-Heisenberg ladders considered in Ref. [5].

The paper is organized as follows. In Secs. I–IV, I will consider one-dimensional models. In Sec. V, I will discuss their arrays. In Sec. II, I derive the continuum limit description for both types of models mentioned above. This continuum description is given by integrable field theories whose spin sector is gapped and has fractionalized excitations. For the cable model $n \geqslant 2$, their statistics is non-Abelian, for the SU(N)-symmetric ladder model it is Abelian. In Sec. III, I will construct the composite order parameter operators. There is a separate universality class for each symmetry group representation. The construction can be easily generalized for nonunitary Lie groups. In Sec. IV, it will be shown that the composite order and fractionalized excitations are robust with respect to group symmetry preserving perturbations around the integrable point. I will analyze in detail the perturbations breaking the symmetry between the exchange couplings and the perturbations driving the spin chain away from the criticality. In Sec. V, I will consider the physics of arrays of the KH cables. The paper has Conclusion and Acknowledgment sections and several appendices.

II. THE CORE MODELS

The core models of the present paper are of two kinds. One type of the model called the Kondo-Heisenberg cable (KHC) consists of a critical antiferromagnetic spin $S = n/2$ Takhtajan-Babujian chain (TBC) coupled by an antiferromagnetic exchange interaction to n conducting chains containing a one-dimensional electron gas (1DEG):

$$H = \sum_k \sum_{a=1}^{n} \epsilon_a(k)\psi^+_{k,a\sigma}\psi_{k,a\sigma} + \frac{1}{2}\sum_{k,q} J^{ab}\psi^+_{k+q,a\alpha}\boldsymbol{\sigma}_{\alpha\beta}\psi_{k,b\beta}\mathbf{S}_q + J_H\sum_l \mathcal{P}_n(\mathbf{S}_l\mathbf{S}_{l+1}), \tag{1}$$

where ψ^+_a,ψ_a are creation and annihilation operators of the 1DEG on chains $a = 1,\dots,n$, σ^b are the Pauli matrices, $\mathbf{S}_l$ is the spin $S = n/2$ operator on site j, and $\mathbf{S}_q$ is its Fourier transform. $\mathcal{P}_n(x)$ is the polynomial of degree n whose exact form is fixed by the integrability requirements [7,8]. For instance, $\mathcal{P}_1(x) = x, \mathcal{P}_2(x) = x - x^2$, etc. It is assumed that $J^{ab} \ll J_H$ and the 1DEGs have band fillings incommensurate with the TBC: $|2k_{F,a}a_0 - \pi| \sim 1$. Under these assumptions one can formulate the low-energy description of (1), taking into account that the backscattering processes between excitations in the TBC and the 1DEGs are suppressed by the above incommensurability. The effective theory is valid for energies much smaller than both the average Fermi energy $\epsilon_{F,a}$ and the exchange interaction J_H of the model (1).

The reader should not remain under the impression that the obtained results require a fine tuning of the spin sector to the integrable point. It will be shown in Sec. IV that they remain robust against those perturbations around the integrable point, which preserve the translational and the SU(2) symmetry.

Another core model is a SU(N) symmetric generalization of the Kondo-Heisenberg chain:

$$H = \sum_k \sum_{a=1}^{N} \epsilon(k)\psi^+_{k,a}\psi_{k,a} + \frac{1}{2}\sum_{k,q} J^l\psi^+_{k+q,a}\tau^l_{ab}\psi_{k,b}T^l_q + J_H\sum_n \left(T^l_{n+1}T^l_{n+1}\right), \tag{2}$$

where $T^l,(l = 1,\dots,N^2-1)$ are generators of the su(N) algebra in the single-box representation.

The KHC model (1) is a one-dimensional version of the spin fermion (SF) model frequently used to study violations of the Landau-Fermi liquid theory in the vicinity of quantum critical points. As has been demonstrated in Ref. [5], an array of KH chains can be used as a quasi-1D SF model. Here the electrons also interact with a critical insulating subsystem. However, the interacting regime I am going to study is different from what is usually assumed. In the standard treatment of the SF model, the quantum character of the spin fluctuations is not important, it is suggested that the quantum features are generated by the conduction electrons. In the quasi-1D version of the SF model considered here, this is not the case: the quantum nature of the spins is responsible for creation of the spin gap and the formation of the spin liquid.

A. Continuum limit of model (1)

As usual I start with the linearization of the spectrum of the 1DEG:

$$\epsilon_a(k) \approx \pm v_{F,a}(k \mp k_{F,a}), \tag{3}$$

and introduce the right- and the left-moving fermions R and L:

$$\psi_a(x) = e^{-ik_{F,a}x}R_a(x) + e^{ik_{F,a}x}L_a(x). \tag{4}$$

In the rest of my paper, I will employ the formalism of non-Abelian bosonization most adequate for the task. Although this version of bosonization is not as widely known as the Abelian one, it has a venerable history and has been discussed in literature. The most recent review can be found in Ref. [6].

The continuum limit of the TBC chain is described by the SU$_n$(2) Wess-Zumino-Novikov-Witten (WZNW) model. This is a critical theory whose primary fields transform in the spin $S \leqslant n/2$ representations of the SU(2) group. The excitations are gapless with linear spectrum $\omega = v_H|q|$, $v_H = \pi J_H/2$. In the continuum limit, the spin operators are approximated as [9,10]

$$\mathbf{S}_l = [\mathbf{j}_R(x) + \mathbf{j}_L(x)] + i(-1)^l\mathrm{Tr}[\boldsymbol{\sigma}(h - h^+)] + \dots, \tag{5}$$

($x = la_0$) where the dots stand for less relevant operators, a_0 is the lattice distance, h is the WZNW SU(2) matrix field. The current operators j^a_L, j^a_R satisfy the SU$_n$(2) Kac-Moody algebra:

$$\left[j_R^a(x), j_R^b(x')\right] = \mathrm{i}\epsilon^{abc} j_R^c(x)\delta(x-x') + \frac{\mathrm{i}n}{4\pi}\delta_{ab}\delta'(x-x'), \tag{6}$$

with the same commutation relations for the left currents j_L^a. The electron spin $\mathbf{F}_R = \frac{1}{2}\sum_{a=1}^n R_a^+ \boldsymbol{\sigma} R_a$, $\mathbf{F}_L = \frac{1}{2}\sum_{a=1}^n L_a^+ \boldsymbol{\sigma} L_a$ satisfy the same algebra. The remarkable fact is that the WZNW Hamiltonian describing the low-energy part of the TBC can be expressed solely in terms of the currents:

$$H_{\rm WZNW} = \frac{2\pi v_H}{n+2}\int dx(:\mathbf{j}_R\mathbf{j}_R: + :\mathbf{j}_L\mathbf{j}_L:), \tag{7}$$

The double dots denote normal ordering.

Due to the incommensurability of the Fermi wave vectors in the continuum limit the Kondo term in (1) is reduced to the interaction of the currents [11] (see also Ref. [12]):

$$V_{ex} = \frac{J_K}{2}\int dx(\mathbf{j}_R + \mathbf{j}_L)(R_a^+ \boldsymbol{\sigma} R_a + L_a^+ \boldsymbol{\sigma} L_a), \tag{8}$$

where $J_K = J^{aa}$ (all diagonal elements are taken to be equal). At $v_{F,1} = v_{F,2}$ the sum of the electronic currents adds up to a single $\mathrm{SU}_n(2)$ current

$$\mathbf{F}_R = \sum_{a=1}^n R_a^+ \boldsymbol{\sigma} R_a, \quad \mathbf{F}_L = \sum_{a=1}^n L_a^+ \boldsymbol{\sigma} L_a. \tag{9}$$

Further simplification comes from the fact that the relevant part of (8) contains only products of the currents of different chirality so that the marginal interaction $V_{\rm marg} = J_K(\mathbf{F}_R\mathbf{j}_R + \mathbf{F}_L\mathbf{j}_L)$ can be dropped as the first approximation. Hence only the $\mathrm{SU}_n(2)$ part of the 1DEGs is involved in the interaction.

Below, I will use the fact that the Hamiltonian of n copies of spin-1/2 noninteracting fermions with identical Fermi velocities (I will assume this to simplify the calculations) can be written as a sum of the U(1) Gaussian model and $\mathrm{SU}_2(n)$ and $\mathrm{SU}_n(2)$ WZNW models [12,13]. The resulting Hamiltonian is

$$\mathcal{H}_{\rm eff} = \mathcal{H}_{\rm orb} + \frac{v_F}{2}[(\partial_x\Theta_c)^2 + (\partial_x\Phi_c)^2] + \mathcal{H}_{\rm spin}, \tag{10}$$

$$\mathcal{H}_{\rm orb} = \frac{2\pi v_F}{n+2}\sum_{A=1}^{n^2-1}(:I_R^A I_R^A: + :I_L^A I_L^A:), \tag{11}$$

$$\mathcal{H}_{\rm spin} = \frac{2\pi v_F}{n+2}(:\mathbf{F}_R\mathbf{F}_R: + :\mathbf{F}_L\mathbf{F}_L:), \tag{12}$$

where $I^A, A = 1,\ldots,n^2-1$ are $\mathrm{SU}_2(n)$ currents. Φ and Θ are mutually dual bosonic fields. Both the charge part and model (11) are critical, the spectrum is linear: $\omega = v_F|k|$.

Now, we will put the relevant part of (8), (12), and (7) together and rearrange the terms in such a way to obtain two commuting Hamiltonians:

$$\mathcal{H}_{\rm spin} + V_{ex} + \mathcal{H}_{\rm WZNW} = \mathcal{H}_s^{(Rl)} + \mathcal{H}_s^{(Lr)}, \tag{13}$$

$$\mathcal{H}_s^{(Rl)} = \frac{2\pi v_F}{n+2}:\mathbf{F}_R\mathbf{F}_R: + \frac{2\pi v_H}{n+2}:\mathbf{j}_L\mathbf{j}_L: + J_K\mathbf{F}_R\mathbf{j}_L, \tag{14}$$

$$\mathcal{H}_s^{(Lr)} = \frac{2\pi v_F}{n+2}:\mathbf{F}_L\mathbf{F}_L: + \frac{2\pi v_H}{n+2}:\mathbf{j}_R\mathbf{j}_R: + J_K\mathbf{F}_L\mathbf{j}_R. \tag{15}$$

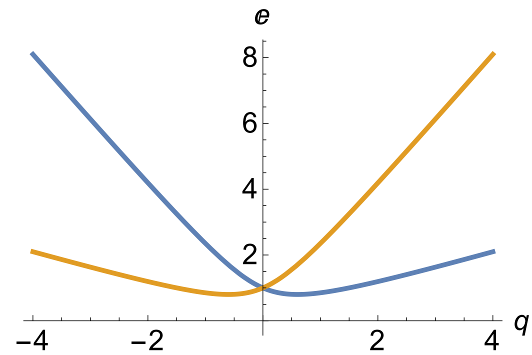

FIG. 1. The dispersion of the solitons in the KH chain (16). $e = E/\Delta$, $q = k_x(v_H v_F)^{1/2}/\Delta$, and $v_F/v_H = 1/4$.

Here, $v_F, v_H = \pi J_H/2$ are the Fermi velocity of the 1DEGs and the spinon velocity of the TBC, respectively. This factorization of the spin sector into two part with one being a mirror image of another is the key feature of the KHC model (1) from which everything else will follow. Such factorization can be easily generalized for any other Lie group symmetry besides SU(2) and any other representation of the spin operators.

Models (14) and (15) are strongly interacting and integrable [14]. These are WZNW models perturbed by a marginally relevant current-current interaction. Their Bethe ansatz solution has many common features with the solution of the multichannel Kondo model [15–17] with the difference that in the case when the spins are represented by a single impurity the spectral gaps cannot be formed. At $J_K > 0$, the spectrum consists of gapped non-Abelian solitons. Each soliton carries a zero mode of Z_n parafermion. The further details are provided in Sec. IV B and Appendix A. The dispersion relations $E(k)_{Lr} = E(-k)_{Rl} = E(k)$ are (see Fig. 1)

$$E(k) = k(v_H - v_F)/2 + \sqrt{k^2(v_F+v_H)^2/4 + \Delta^2}, \tag{16}$$

where $\Delta = \Lambda g\exp(-\pi/g)$, $g = J_K/(v_F + v_H)$, with Λ being the ultraviolet cutoff, is the spin gap. Hence they describe spin liquids. Since these models are mirror images of each other, under open boundary conditions the many-body wave functions of the two copies coincide at the boundaries.

B. Continuum limit of model (2)

The derivation here is very similar to the one given in the previous subsection. Therefore I will concentrate on differences. The first one is that the continuum limit of the SU(N) spin chain is now given by the $\mathrm{SU}_1(N)$ WZNW model. The SU(N) spin operators are expressed as

$$T_n^l = \left[j_R^l(x) + j_L^l(x)\right] + \sum_{q=1}^{N-1} \mathrm{e}^{2\pi nq/N}\mathrm{Tr}(\tau^l :h^q:) + \ldots, \tag{17}$$

where h is the SU(N) matrix field of the $\mathrm{SU}_1(N)$ WZNW model and τ^l are generators of the su(N) algebra. The primary fields : h^q : are obtained by fusion of the fundamental one their scaling dimensions are $d_q = q(N-q)/N$.

The resulting continuum limit Hamiltonian density is

$$\mathcal{H} = \frac{v_F}{2}[(\partial_x\Theta_c)^2 + (\partial_x\Phi_c)^2] + \mathcal{H}_s^{(Rl)} + \mathcal{H}_s^{(Lr)}, \quad (18)$$

$$\mathcal{H}_s^{(Rl)} = \frac{2\pi v_F}{N+1} : F_R^l F_R^l : + \frac{2\pi v_H}{N+1} : j^l L j_L^l : + J_K^l F_R^l j_L^l, \quad (19)$$

$$\mathcal{H}_s^{(Lr)} = \frac{2\pi v_F}{N+1} : F_L^l F_L^l : + \frac{2\pi v_H}{N+1} : j_R^l j_R^l : + J_K^l F_L^l j_R^l. \quad (20)$$

Models (19) and (20) can be written is a more familiar fermionic form. One can take advantage of the fact that $SU_1(N)$ currents can be written in terms of fermionic bilinears and write the currents of the spin chain in terms of the auxiliary right- and left-moving fermions ρ,λ. The resulting model constitutes the spin sector of the $SU(N)$ chiral Gross-Neveu model so that for (19), we have

$$\mathcal{H}_s^{(Rl)} = -\mathrm{i}v_F R_a^+\partial_x R_a + \mathrm{i}v_H\lambda_a^+\partial_x\lambda_a + J_K^l(R^+\tau^l R)(\lambda^+\tau^l\lambda), \quad (21)$$

where $a = 1,2,\ldots,N$ with a similar expression with $x \to -x$ and R replaced by L and ρ replaced by λ.

When all coupling constants are equal $J_K^l = J_K$ models (19) and (20) are integrable [18], but now the spectrum contains $N-1$ branches of gapped excitations with spectral gaps Δ_j:

$$\Delta_j = \Delta_1\frac{\sin(\pi j/N)}{\sin(\pi/N)}, \quad j = 1,2,\ldots,N-1,$$
$$\Delta_1 \sim \exp[-\pi(v_F + v_H)/NJ_K]. \quad (22)$$

These excitations transform according to single column j-box irreducible representations of the $SU(N)$ group. So they are fractional number particles. The spectrum of each branch is given by (16) with Δ replaced by Δ_j. When J_K^l are different the picture remains qualitatively the same since the SU(N) symmetry is restored in the low-energy limit [19].

III. COMPOSITE ORDER PARAMETERS

In $D = 1$, critical points are located at $T = 0$ and there is only quasi long range order. Hence by order parameter (OP) operators I mean the operators whose susceptibilities diverge at $T = 0$. At $T = 0$, their correlation functions have a power-law decay in space and time, at $T = 0$, they decay exponentially with the correlation length $\sim 1/T$. In the core cable models, such quasi-long-range order is expressed in term of OPs, which include delocalized and localized fermions—composite order parameters (COPs). Ones 1D cables are arranged in a three-dimensional array real long-range order will be established. This will be discussed in more detail in Sec. V.

A. COPs in model (1)

For simplicity, I will consider the case $n = 2$ in detail and discuss other cases briefly. I will use the remarkable fact established in Ref. [20] that two noninteracting 1DEG with equal Fermi velocities can be described by the theory of eight Majorana fermions with O(8) symmetry. At the same time, the $SU_2(2)$ WZNW is equivalent to the theory of three noninteracting Majoranas. So the O(8) theory can be factorized into 5+3 Majoranas: $O_1(8) = O_1(5) \oplus O_1(3)$.

Remarkable properties of the $SU_2(2)$ WZWN model has been first studied in Ref. [21]. The reader can also find details in Refs. [12,13]. The $SU_2(2)$ currents can be represented as products of Majorana fermions:

$$j^a = \frac{\mathrm{i}}{2}\epsilon^{abc}\kappa^b\kappa^c, \quad F^a = \frac{\mathrm{i}}{2}\epsilon^{abc}\chi^b\chi^c. \quad (23)$$

As a consequence, the two independent Gross-Neveu models (14,15) become the O(3) Gross-Neveu models of Majorana fermions:

$$\mathcal{H}_s^{(Rl)} = -\frac{\mathrm{i}v_F}{2}\chi_R^a\partial_x\chi_R^a + \frac{\mathrm{i}v_H}{2}\kappa_L^a\partial_x\kappa_L^a + J_K\sum_{a>b}\left(\kappa_L^a\chi_R^a\right)\left(\kappa_L^b\chi_R^b\right), \quad (24)$$

$$\mathcal{H}_s^{(Lr)} = \frac{\mathrm{i}v_F}{2}\chi_L^a\partial_x\chi_L^a - \frac{\mathrm{i}v_H}{2}\kappa_R^a\partial_x\kappa_R^a + J_K\sum_{a>b}\left(\kappa_R^a\chi_l^a\right)\left(\kappa_R^b\chi_l^b\right). \quad (25)$$

The gapless sector given by the sum of (11) and the U(1) Gaussian model can be described as a model of five gapless Majoranas:

$$\mathcal{H}_{\text{charge-orb}} = \frac{\mathrm{i}}{2}\sum_{a=1}^{5}\left(-\eta_R^a\partial_x\eta_R^a + \eta_L^a\partial_x\eta_L^a\right). \quad (26)$$

For convenience we can group these fermions as follows: $\eta^{1,2}$ will correspond to fermionization of the charge sector, the other three η's will describe the orbital sector.

At criticality, the $SU_2(2)$ WZNW model can also be represented as a sum of three critical quantum Ising models. This representation is particularly useful since the spin $S = 1/2$ primary field (the matrix h) can be expressed in terms of order σ_a and disorder μ_a parameter fields of the Ising models [21]:

$$\hat{h} = \hat{\tau}^0\sigma_1\sigma_2\sigma_3 + \mathrm{i}(\hat{\tau}_1\mu_1\sigma_2\sigma_3 + \hat{\tau}^2\sigma_1\mu_2\sigma_3 + \hat{\tau}^3\sigma_1\sigma_2\mu_3). \quad (27)$$

Here, τ^a, $a = 0,1,\ldots,3$ are unit and Pauli matrices.

As is clear from (14) and (15), the spectral gaps are generated by pairing of Majoranas of a given chirality from the 1DEG with their partners of opposite chirality from the TBC. To clarify this, it is instructive to do the Hubbard-Stratonovich transformation for, for instance, model (24). For $J_K > 0$, the interaction is decoupled as

$$J_K\sum_{a>b}\left(\kappa_L^a\chi_R^a\right)\left(\kappa_L^b\chi_R^b\right) \to \frac{\Delta^2}{2J_K} + \mathrm{i}\Delta\left(\kappa_L^a\chi_R^a\right). \quad (28)$$

Integration over the fermions creates a double-well potential for field Δ. The minima of the potential correspond to degenerate vacua for the Majorana fermions where $\langle(\kappa_L^a\chi_R^a)\rangle \neq 0$. As far as the operators of the original model (1) are concerned, the structure of the vacuum is more subtle since the local operators of this model are expressed not just in terms of the Majorana fermion bilinears, but also in terms of Ising model operators (see Appendix B). A vacuum with one sign of Δ corresponds to the disordered phase of the Ising models where $\langle\sigma^a\rangle = 0$, the other one corresponds to the ordered phase where $\langle\sigma^a\rangle \neq 0$ and may have any sign. Therefore the vacuum has a triple degeneracy. I will talk more about it in Sec. IV.

The important point is that since the Majoranas from the 1DEGs do not pair to each other, there are no order parameters formed solely from the electronic operators or spin operators. Instead, there are composite order parameters (COPs) whose correlation functions have a power law decay.

As a preliminary step towards formulation of the COPs, I will organize the fermions into Nambu spinors:

$$\Psi_{a\sigma} = \begin{pmatrix} \psi_{\sigma,a} \\ \epsilon_{\sigma\sigma'}\psi^+_{\sigma',a} \end{pmatrix}. \tag{29}$$

This reflects the orthogonal symmetry of the low-energy sector. The spinor has 8 components; their quantum numbers include charge $q = \pm 1$, spin $\sigma = \pm 1$ and chain index $p = \pm 1$. Products of the Nambu spinor components with the right and left chiralities give rise to 8×8 real matrix with 64 entries:

$$\Delta_{(q,p,\sigma),(q',p',\sigma')} = \bar{r}_{(q,p,\sigma)} l_{(q',p',\sigma')}. \tag{30}$$

Fusing it with the 4×4 h-matrix spin field of the WZNW model one is left with the matrix COP containing 16 real entries:

$$\mathcal{O}_{(q,p),(q',p')} = \bar{r}_{(q,p,\sigma)} h_{\sigma\sigma'} l_{(q',p',\sigma')}. \tag{31}$$

As it is discussed in Appendix B, this operator can be factorized into the part which condenses, acquiring a finite vacuum expectation value, and the part which fluctuates. The former one constitutes an amplitude of the fluctuating COP. The fluctuating part is a primary field of the critical $O_1(5)$ theory with a scaling dimension 5/8. COP (31) contains charge density wave ($q = -q'$) and superconducting ($q = q'$) components. A given matrix element carries the wave vector

$$Q_{(q,p),(q',p')} = qk_{F,p} - q'k_{F,p'} + \pi/a_0. \tag{32}$$

Operators (31) constitute a reducible representation of the SO(5) group. This representation consists of an SO(5) scalar, vector, and antisymmetric tensor representations. To obtain the latter representations, one has to define five Dirac Γ^a ($a = 1,\dots,5$) matrices, for instance,

$$\Gamma^1 = \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}, \quad \Gamma^{2,3,4} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}, \quad \Gamma^5 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}, \tag{33}$$

where unit and Pauli matrices $I,\boldsymbol{\sigma}$ act on the chain indices. Then the ten SO(5) generators are defined as $\Gamma^{ab} = -\frac{1}{2}[\Gamma^a,\Gamma^b]$. The corresponding COPs are defined as $\mathrm{Tr}\mathcal{O}$ (with wave vector π/a_0), $\mathrm{Tr}\Gamma^a\mathcal{O}$, and $\mathrm{Tr}\Gamma^{ab}\mathcal{O}$. Notice that besides the scalar COP, which carries wave vector π/a_0, all others contain components with different wave vectors. The vector components with $a = 1,5$ contain CDW order parameters with wave vectors $\pm[\pi/a_0 + 2(k_{F,1} + k_{F,-1})]$ (mod$[2\pi/a_0]$) corresponding to the total electron density (which includes the density of localized electrons). The same vector multiplet contains the $a = 2$ component corresponding to the SC COP with π/a_0 wave vector and $a = 3,4$ components corresponding to CDWs with incommensurate wave vectors $\pm[k_{F,1} - k_{F,-1} + \pi/a_0]$.

The above COPs (31) are not the only ones. One can make COPs by fusing products of fermionic bilinears with the higher spin primary fields of the $SU_n(2)$ WZNW. For general n, these are the fields with spin $J \leqslant n/2$. For $n = 2$, there are two such primary fields with $J = 1/2,1$ and hence there is only one extra operator:

$$\Phi_{ab} = i\kappa^a_R\kappa^b_L. \tag{34}$$

As I have pointed out, the trace of this operator describes a smooth part of $(\mathbf{S}_l\mathbf{S}_{l+1})$ lattice field. One can fuse (34) with either of the two operators

$$\begin{aligned}(R^+_1\boldsymbol{\sigma}L_1)(R^+_{-1}\boldsymbol{\sigma}L_{-1}) &\sim e^{i\sqrt{4\pi}\Phi_c}\left(\xi^a_R\xi^a_L - 3\eta^5_R\eta^5_L\right),\\ (R^+_1L_1)(R^+_{-1}R_{-1}) &\sim e^{i\sqrt{4\pi}\Phi_c}\left(\xi^a_R\xi^a_L + \eta^5_R\eta^5_L\right)\end{aligned} \tag{35}$$

to get

$$\mathcal{O}_{\mathrm{CDW}}[2(k_{F,1}+k_{F,2})] = (\mathbf{S}_l\mathbf{S}_{l+1})(\psi^+_1\boldsymbol{\sigma}\psi_1)(\psi^+_{-1}\boldsymbol{\sigma}\psi_{-1}) \times e^{-2i(k_{F,1}+k_{F,2})x} \sim e^{i\sqrt{4\pi}\Phi_c}, \tag{36}$$

or with the product of two SC order parameter operators

$$(R_1\sigma^yL_1)(R_{-1}\sigma^yL_{-1}) \sim e^{i\sqrt{4\pi}\Theta_c}\left(\xi^a_R\xi^a_L + \eta^5_R\eta^5_L\right), \tag{37}$$

to get a charge-4 "bipairing" operator

$$\mathcal{O}_{\mathrm{SC}} = (\psi_1\sigma^y\psi_1)(\psi_{-1}\sigma^y\psi_{-1})(\mathbf{S}_l\mathbf{S}_{l+1}) \sim e^{i\sqrt{4\pi}\Theta_c}, \tag{38}$$

which existence of in four-leg ladders was discussed in Refs. [22,23]. This operator carries zero momentum. To get other products one fuse, for instance,

$$(R^+_1\boldsymbol{\sigma}L_{-1})(L^+_{-1}\boldsymbol{\sigma}R_{-1}),\ (R^+_1L_1)(L^+_{-1}R_{-1}) \sim e^{i\sqrt{4\pi}\Phi_f}\left(\xi^a_R\xi^a_L + \eta^5_R\eta^5_L\right). \tag{39}$$

This operator carries zero charge and momentum $2(k_{F,1} - k_{F,-1})$.

All operators (36), (38), and (39) have scaling dimension 1. They are components of the SO(5) symmetric tensor representation; in the Majorana language they are biproducts of right and left Majorana fermions $\eta^a_R\eta^b_L$.

For higher n, one can fuse $2J$ fermionic bilinears with $J \leqslant n/2$-spin primary field of the spin chain to get operators with scaling dimension

$$d_J = 2\left[J - \frac{J(J+1)}{n+2}\right], \tag{40}$$

some of which will carry charge $Q = 4J$. However, for $J > 1$, these operators have nonsingular susceptibilities.

B. COPs in model (2)

Below, I will discuss only the case $N > 2$, since the case $N = 2$ is discussed at length in [5]. The primary fields of the $SU_1(N)$ WZNW model are tensors in the antisymmetric representations described by a single column Young tableau with $q \leqslant N$ boxes. They can be considered as products of fermion bilinears with the charge sector being factored out:

$$\Phi^{(q)} = \rho^+_{a_1}\dots\rho^+_{a_q}\lambda_{b_q}\dots\lambda_{b_1}e^{q\sqrt{4\pi/N}\psi}, \tag{41}$$

where ρ,λ are right- and left-moving Dirac fermions with velocity v_H and ψ is a real Gaussian bosonic field. Its correlation functions cancel the correlators of the charge field of the fermions. The scaling dimensions of (41) are

$$d_q = \frac{q(N-q)}{N}, \tag{42}$$

and they carry wave vectors $Q_q = \pm 2\pi q/Na_0$.

The COPs are SU(N) singlets and carry wave vectors $Q = (2k_F + 2\pi/Na_0)q$:

$$\mathcal{O}_q = \left(R^+_{a_1}L_{b_1}\right)\dots\left(R^+_{a_q}L_{b_q}\right)\Phi^{(q)}_{a_1,\dots,a_q;b_1,\dots,b_q}$$
$$= \left[\left(R^+_{a_1}\lambda_{a_1}\right)\left(\rho^+_{b_1}L_{b_1}\right)\right]^q e^{q\sqrt{4\pi}\psi} = A e^{iq\sqrt{4\pi/N}\Phi_c}. \tag{43}$$

The wave vector $(2k_F + 2\pi/Na_0)$ includes the density of localized and delocalized electrons in agreement with the Oshikawa theorem [24]. Ones the spin gaps are formed the amplitude A is finite. The scaling dimensions are

$$d_q = q^2/N. \tag{44}$$

Notice that for $N > 2$, the COPs are of the charge density wave type and do not include the superconducting ones.

IV. ROBUSTNESS AGAINST PERTURBATIONS

The spin-liquid states described above represent only a part of the Hilbert space of the original models (1) and (2). The rest of it belongs to gapless excitations. Hence the current models describe conducting states. Nevertheless, since the spin sector is decoupled from the gapless modes [the charge and orbital ones for (1) and the charge one for (2)], it is instructive to find out how robust are its fractionalized excitations against various perturbations. Below I will consider several perturbations concentrating mostly on model (1) and show that the fractionalized gapped excitations are robust against perturbations, which do not violate the SU(2) [for model (1)] and the SU(N) [for model (2)] symmetry of the spin chain and do not break the translational invariance.

Such perturbations fall into several categories which will be considered below. First, there are electron-electron interactions of the band electrons. Away from half filling they generate only current-current interactions. In Sec. IV A, it will be demonstrated that such interactions together with current-current interactions in the spin sector will need to exceed some critical value to radically modify the spin-liquid state. Second, there are perturbations in the spin chain which would destroy the SU$_n$(2) critical point of an isolated chain. They will be analysed in Sec. IV B. If not too strong such perturbations are ineffective since ones the spin liquid is formed its stability is protected by the spectral gap. Third, there are perturbations corresponding to channel anisotropy $J^{11} \neq J^{22}$ which will be discussed in Sec. IV C. At last, there is external magnetic field, but the spin liquid is protected against it by the spin gap. These four categories exhaust the list of the symmetry preserving perturbations.

A. Electron-electron interactions of the band electrons

It is instructive to find out whether the gapped state described in the previous section can be adiabatically connected to a topologically trivial state of decoupled band electrons and a gapped TBC. To show this, I introduce a deformation of the original model adding to it the additional interaction

$$V = \gamma(\mathbf{F}_R\mathbf{F}_L + \mathbf{j}_R\mathbf{j}_L), \tag{45}$$

and consider a trajectory in the J_K-γ plane from $(J_K,0)$ to $(0,\gamma)$. Since the charge-orbital sector remains decoupled, the trajectory lies entirely inside of the spin sector which remains gapped except, as we will see, at one critical point separating the two phases. One of those is the phase of interest and the other one is phase where TBC and 1DEGs are disconnected. The spin excitations are gapped; at $J_K = 0$ and $\gamma > 0$, both the band electrons and the TBC are perturbed by the marginally relevant products of the currents. These are integrable perturbations of the same kind as in (14) and (15); they generate spectral gaps. For the spin chain, there is also OP local in the spin operators:

$$\mathcal{O} = \langle(\mathbf{S}_j\mathbf{S}_{j+1})\rangle \sim \langle \mathrm{Tr}h\mathrm{Tr}h^+\rangle, \tag{46}$$

which describes a spontaneously generated deviation from the integrable point. On the other hand, the phase $\gamma = 0$ has no local OPs, there is only a quasi-long-range order (see Sec. III). As we will see, the two phases are separated by a quantum critical point.

For simplicity, I set $v_F = v_H$. Let us introduce new operators:

$$\mathbf{J} = \mathbf{F} + \mathbf{j}, \quad \mathbf{K} = \mathbf{F} - \mathbf{j}. \tag{47}$$

The operators $\mathbf{J}_{R,L}$ are SU$_{2n}$(2) Kac-Moody currents. Then the total interaction becomes

$$V_{ex} + V = \tfrac{1}{2}(J_K + \gamma)\mathbf{J}_R\mathbf{J}_L + \tfrac{1}{2}(\gamma - J_K)\mathbf{K}_R\mathbf{K}_L. \tag{48}$$

The part of the Hamiltonian describing the critical point can be represented as the sum of the SU$_{2n}$(2) WZNW and the SU$_n$(2) × SU$_n$(2)/SU$_{2n}$(2) coset theory. At $\gamma = J_K$, the product of the $\mathbf{K}$ operators vanishes and the latter theory decouples and becomes critical; so the entire theory has a critical point. At this point, only SU$_{2n}$(2) part of the spin Hilbert space is gapped, the remaining SU$_n$(2) × SU$_n$(2)/SU$_{2n}$(2) one is gapless. Hence the phase with small γ is separated from the topologically trivial phase with $J_K = 0$ by a quantum critical point described by the SU$_n$(2) × SU$_n$(2)/SU$_{2n}$(2) coset theory.

In my opinion, it is possible that the gapped spin state described above is topologically nontrivial. Indeed, it has nonlocal OP of the string type and is likely to have zero modes located on a boundary with the topologically trivial phase $\gamma > J_K$. However, in order to determine a place of this model in the general classification of topological phases [25], I have to consider the edge zero-energy modes. I leave this problem for future studies.

B. Deviations from the SU$_n$(2) critical point

In this section, I demonstrate that the deviations of the spin chain from the TBC integrable point do not confine the non-Abelian massive excitations. For simplicity, I do it for the $n = 2$ KHC model. If the perturbation is not too strong, it just creates bound states of the non-Abelian solitons, but these particles still remain in the spectrum.

I start with the unperturbed model for $n = 2$. The spin sector is described by a sum of two copies of the O(3) Gross-Neveu model (24) and (25). The exact solution of the O(3) GN model was first found in Ref. [26] as a particular limit of the supersymmetric sine-Gordon model and was later analyzed in detail in Refs. [27,28]. The reader can find an excellent and pedagogical analysis of (1+1)-dimensional supersymmetric theories in a recent paper by Mussardo [29]. As I have stated above [see the text around (16)], the excitations are massive and non-Abelian. Their nature can be visualized with a help of Hubbard-Stratonovich transformation (28). Then, as I have mentioned above, the integration over the Majorana fermions creates a double-well potential for field Δ. The Majorana fermions have zero-energy modes on the kinks of Δ field; the kinks with attached zero modes constitute excitations of the O(3) GN model, the so-called Bohomol'nyi-Prasad-Sommerfield (BPS) solitons [27]. A multikink state is a highly entangled one and cannot be factorized into a product of states even when the kinks are far from each other. This becomes clear when one considers a Hilbert space of Majorana zero modes. These modes γ_a obey the Clifford algebra

$$\{\gamma_a,\gamma_b\} = \delta_{ab}, \tag{49}$$

and the Hilbert space of N kinks have $2^{[3N/2]}$ states.

According to the exact solution [26–28], the excitation spectrum does not contain vector particles. Hence the Majorana fermions themselves do not survive as coherent excitations; this fact is important for the survival of the fractionalized particles.

For the following analysis, it will be convenient to use the relativistic parametrization of the soliton spectrum (16):

$$\begin{aligned} E_p &= \frac{\Delta}{2}(\sqrt{v_F/v_H}\mathrm{e}^{p\theta} + \sqrt{v_H/v_F}\mathrm{e}^{-p\theta}),\\ P &= \Delta(v_H v_F)^{-1/2}\sinh\theta, \end{aligned} \tag{50}$$

where parameter θ is called rapidity. Different signs correspond to different copies of the O(3) GN model: $p = +$ for (24) and $p = -$ for (25).

In description of the excitations I will follow Ref. [30]. The ground state of a single O(3) GN model is triple degenerate. The excitations are solitons interpolating between different vacua. A soliton with rapidity θ interpolating between the vacua a and b is created by operator $K^{\sigma}_{ab}(p,\theta)$ with $\sigma = \pm 1/2$ for soliton and antisoliton ($s^z = \pm 1/2$), respectively. The vacuum indices a,b take values 0, 1/2, and 1 with $|a-b| = 1/2$. The latter restriction is responsible for the fact that a multisoliton state cannot be disentangled into a product of single-particle states even if solitons are far from each other. Multisoliton state of a given model with total spin projection $S^z = \sum_j \sigma_j$ is given by

$$\big|K^{\sigma_1}_{a_0a_1}(p,\theta_1)K^{\sigma_2}_{a_1a_2}(p,\theta_2)\dots K^{\sigma_N}_{a_{N-1}a_N}(p,\theta_N)\big|0_{a_N}\big\rangle, \tag{51}$$

where $\theta_1 > \theta_2 > \ldots > \theta_N$ for an *in* and $\theta_1 < \theta_2 < \cdots < \theta_N$ for an *out* state. The two-particle scattering process

$$K^{\sigma_1}_{ab}(p,\theta_1) + K^{\sigma_2}_{bc}(p,\theta_2) \to K^{\sigma_2'}_{ad}(p,\theta_2) + K^{\sigma_1'}_{dc}(p,\theta_1), \tag{52}$$

is described by the scattering matrix

$$S_{\rm SUSY}\begin{pmatrix} a & d\\ b & c\end{pmatrix}\!\!\Bigg|\theta_1-\theta_2\Bigg) \times S^{\sigma_1',\sigma_2'}_{\sigma_1,\sigma_2}(\theta_1-\theta_2), \tag{53}$$

where $S_{\rm SUSY}$ is described in Ref. [30] and the other S matrix is the one of the SU(2) Thirring model:

$$\begin{aligned} S^{\sigma_1',\sigma_2'}_{\sigma_1,\sigma_2}(\theta) &= -S_0(\theta)\frac{\big(\theta\delta_{\sigma_1,\sigma_1'}\delta_{\sigma_2,\sigma_2'} + \mathrm{i}\pi\delta_{\sigma_1,\sigma_2'}\delta_{\sigma_2,\sigma_1'}\big)}{\theta+\mathrm{i}\pi},\\ S_0(\theta) &= \frac{\Gamma(1/2-\theta/2\pi)\Gamma(1+\mathrm{i}\theta/2\pi)}{\Gamma(1/2+\theta/2\pi)\Gamma(1-\mathrm{i}\theta/2\pi)}. \end{aligned} \tag{54}$$

The relevant operators of the $SU_2(2)$ WZNW model include spin $S = 1/2,1$ primary fields and the product of the left and right currents. As is obvious from (5), the $S = 1/2$ operator breaks the translational invariance. If we do not allow this, the most relevant perturbation is the $S = 1$ primary field which is local in the Majorana fermions:

$$V_{\rm pert} = \mathrm{i}m\kappa^a_R\kappa^a_L. \tag{55}$$

For $J_K = 0$, this perturbation would lead to a confinement of the fractionalized excitations of the TBC [31,32]. However, as I am going to show, for finite $J_K > 0$, this is no longer the case provided $|m| \ll \Delta$.

For the following, we will need to obtain some information about matrix elements of the perturbing operator (55). Leaving a complete calculation for the future, I will just establish the properties necessary to resolve the problem of confinement. This can be done on the basis of Lorentz invariance and crossing symmetry.

From the exact solution, we know that the Majoranas are not coherent particles. Hence operator κ_R (κ_L) has matrix elements between a vacuum and states of even number of solitons of model (25) [respectively of (24)]. The minimal matrix elements corresponding to annihilation of two solitons are

$$\begin{aligned} &\langle 0_a|\kappa^l_R(\tau,x)|K^{\sigma_1}_{ab}(-,\theta_1)K^{\sigma_2}_{ba}(-,\theta_2)|0_a\rangle\\ &\quad= \exp\{-\tau[E_{Lr}(\theta_1)+E_{Lr}(\theta_2)] - \mathrm{i}x[P(\theta_1)+P(\theta_2)]\}\\ &\qquad\times \Delta^{1/2}\mathrm{e}^{(\theta_1+\theta_2)/4}g_a(\theta_1-\theta_2)C^l_{\sigma_1\sigma_2}, \end{aligned} \tag{56}$$

$$\begin{aligned} &\langle 0_a|\kappa^l_L(\tau,x)|K^{\sigma_1}_{ab}(+,\theta_1)K^{\sigma_2}_{ba}(+,\theta_2)|0_a\rangle\\ &\quad= \exp\{-\tau[E_{Rl}(\theta_1)+E_{Rl}(\theta_2)] - \mathrm{i}x[P(\theta_1)+P(\theta_2)]\}\\ &\qquad\times \Delta^{1/2}\mathrm{e}^{-(\theta_1+\theta_2)/4}g_a(\theta_1-\theta_2)C^l_{\sigma_1\sigma_2}, \end{aligned} \tag{57}$$

where C is the Klebsh-Gordon factor and $g_a(\theta)$ is a dimensionless function to be determined. This form is dictated by the fact that (i) κ^l has spin 1 under the SU(2) group and the solitons have spin 1/2, (ii) $\kappa_{R,L}$ are components of a spinor, that is, they have Lorentz spin $\pm 1/2$. The latter fact explains the presence of the exponential factors: under a Lorentz boost $\theta_i \to \theta_i + \alpha$ the matrix elements must acquire a factor $\mathrm{e}^{\pm\alpha/2}$.

We can extract more specific information about the matrix elements from the crossing symmetry. It allows one to extract another matrix element:

$$\begin{aligned} &\langle 0_a|K^{-\sigma_1}_{ab}(p;\theta_2)\big|\kappa^l_R(0,0)\big|K^{\sigma_2}_{ab}(p;\theta_1)|0_a\rangle\\ &\quad= \Delta^{1/2}\mathrm{i}\mathrm{e}^{p(\theta_1+\theta_2)/4}g_a(\mathrm{i}\pi-\theta_1+\theta_2)C^l_{\sigma_1\sigma_2}. \end{aligned} \tag{58}$$

As we shall see, the issue of the soliton confinement is decided by the behavior of this matrix element at $\theta_1 \to \theta_2$. The solitons are confined if the function $g(\theta)$ has a pole at $\theta = \mathrm{i}\pi$. In that case, the effective potential between the kinks grows with distance (see below). However, according to the general theorem (see, for instance, Ref. [29]) at the pole, we have

$$g_a(\mathrm{i}\pi-\theta)\sim\frac{\langle 0_a|\kappa^l_{R,L}|0_a\rangle}{\theta}, \tag{59}$$

and the residue is zero since $\kappa_{R,L}$ are fermion operators and cannot have a nonzero vacuum average. This conclusion is also supported by the semiclassical calculation for the supersymmetric sine-Gordon model done in Ref. [29] [see Eqs. (43) and (44)], which gives an explicit expression for $g_a(\theta)$.

Now we can use all this accumulated information to write down the Schrödinger equation for two solitons belonging to the sectors with different parity. Their wave function is

$$B_{\sigma_1,\sigma_2}\int d\theta_1 d\theta_2 \Psi_{ab;cd}(\theta_1,\theta_2)K^{\sigma_1}_{ab}(+;\theta_1)K^{\sigma_2}_{cd}(-;\theta_2)|0_b,0_d\rangle.$$

Acting on this state by (55), we create a two-soliton state plus multisoliton states. Since the latter ones lay higher in energy, we can neglect them when $m\ll\Delta$.

In the reference frame with the total zero momentum, we have

$$(-E+2M\cosh\theta)\Psi(\theta,-\theta)+(m\Delta/M)\hat{P}_{S=1}\times\int\frac{du}{\cosh u}\mathrm{e}^{(\theta+u)/2}|g(\mathrm{i}\pi+\theta-u)|^2\Psi(u,-u)=0, \tag{60}$$

where $M=\Delta(v_H+v_F)/\sqrt{v_Hv_F}$ and $\hat{P}_{S=1}$ is a projector to spin $S=1$ space which sets the spins of the solitons into a triplet configuration.

For $m\ll M$, one can expand the kernels in small rapidities and obtain the Schrödinger equation:

$$\left[-E+2M-\frac{1}{4M}\frac{\partial^2}{\partial x^2}+V(x)\hat{P}_{S=1}\right]\tilde{\Psi}(x)=0,$$
$$V(x)=(m\Delta/M)\int_{-\infty}^{\infty}\frac{du}{2\pi}\mathrm{e}^{2\mathrm{i}Mux}|g(u+\mathrm{i}\pi)|^2,$$
$$\tilde{\Psi}(x)=\int_{-\infty}^{\infty}\frac{du}{2\pi}\mathrm{e}^{2\mathrm{i}Mux}\Psi(u,-u). \tag{61}$$

The potential $V(x)$ decays at large distances unless function $g(\theta)$ has a pole at $\mathrm{i}\pi$. As we have already established, there is no pole. For $m<0$, the potential is attractive and there is at least one bound state below the two-particle continuum. Bound states do not kill the fractionalized excitations, they remain in the spectrum. The topologically nontrivial state survives.

C. Asymmetry of the Kondo exchange

The asymmetry of the Kondo couplings is a marginally irrelevant perturbation which dies out under renormalization. This is the case for both types of models. The renormalization group dynamics of the KHC model (1) is identical to the one of the Kondo impurity. For the impurity problem, it has been well known from the late 70s (see [33]) that the stable exchange configuration is the one when the impurity spin is completely screened. This means that when the impurity spin S interacts with several screening channels with different exchange integrals, the renormalization selects $2S$ channels with strongest couplings, which become identical under the RG flow and suppresses all weaker ones. Likewise, the SU(N) symmetry is restored in strong coupling limit for model (2), as was shown in Ref. [19].

V. THE KONDO-HEISENBERG ARRAYS

In this section, I will discuss a generalization of the "wire construction" of Kondo-Heisenberg arrays developed in my previous publication [5]. Namely, I briefly consider an array of parallel KHC models connected by interchain electron tunneling and exchange interactions.

As it was discussed in Ref. [5], the effect of these interactions is twofold. First, they couple the COPs which eventually leads to a real long range order. Second, the interchain tunneling and exchange create coherent excitations. In particular, the tunneling create bound states of holons and spinons (quasiparticles) whose dispersion is located inside of the spinon gap. When the tunneling matrix elements are sufficiently large (of order of the spinon gap) the quasiparticle dispersion crosses the chemical potential and a Fermi surface appears in the form of electron and hole pockets [5] and formation of Fermi liquid. Likewise, the interchain exchange interaction leads to the creation of bound states from fractionalized spin excitations which can propagate in the bulk. The fractionalized particles themselves remain confined to the chains, at least in the model I consider. I will not discuss these subjects further not to distract attention from the main subject of this paper which is composite order. Instead, I will construct several possible Ginzburg-Landau (GL) functionals for the COPs.

Model (2) provides the simplest example due to the simplicity of the order parameters (43). Here they are just bosonic exponents with U(1) symmetry. Hence the GL Hamiltonian for the array of (2) cables is

$$H=\sum_r\int dx\left\{\frac{v_F}{2}[(\partial_x\Theta_r)^2+(\partial_x\Phi_r)^2]+\sum_q\sum_{r'}J^{(q)}_{r,r'}\cos[q\sqrt{4\pi/N}(\Phi_r-\Phi'_r)]\right\}, \tag{62}$$

where indices r,r' mark positions of different chains. As was noticed in Ref. [5], the interchain couplings J are generated not just by the electron hopping, but also by the interchain spin exchange. This feature may lead to some interesting consequences as far as the ordering is concerned. For example, due to the composite nature of the OP it will not be so easy to pin the phase by disorder. Indeed, the pinning operator must simultaneously act on the spins located on the central chains and on the electrons located on their own chains.

For model (2), the situation is reacher due to the presence of the orbital degrees of freedom related to spatial position of the 1DEGs around the central spin chain. Hence the coupling of the cables will in general break all symmetries except of the U(1) charge.

It would be too tedious to discuss here all possible Ginzburg-Landau theories. I will just discuss one which is sufficiently exotic and interesting, namely, the theory of charge-4 "biparing" superconductivity related to condensation of COP (38). This is the only relevant COP which has zero momentum. To suppress coupling between all other COPs, one should arrange the cables in such a way that their mutual positions prevent a coupling of COPs with finite wave vectors. An example of such arrangement is a pyroclore lattice where all chains intersect each other at finite angles. The resulting GL Hamiltonian is

$$H = \sum_r \int dx \Big\{ \frac{v_F}{2}[(\partial_x \Theta_r)^2 + (\partial_x \Phi_r)^2] + \sum_{r'} J_{r,r'} \cos[\sqrt{4\pi}(\Theta_r - \Theta_{r'})] \Big\}, \quad (63)$$

where Θ_r is $4e$ charged phase field on chain r.

A. Influence of perturbations on the ordering

The presence of various perturbations can generate additional couplings between COPs from different cables. For instance, since the fusion of operator (55) with four conduction electron operators gives rise to COPs [(36) and (38)], the deviation from the quantum critical point of the spin model helps to establish an interchain coupling of the quartic operators. In the presence of such perturbation, these COPs couple by the interchain hopping alone. On the other hand, the perturbations which break the translational invariance of the spin chain generate operators $\mathrm{Tr}(h + h^+)$ or $\mathrm{iTr}[\boldsymbol{\sigma}(h - h^+)]$ and hence through (31) generate a coupling between the conventional CDW and superconducting order parameters. Here it is again enough to have the interchain hopping to generate the coupling.

VI. CONCLUSIONS

In this paper, I have shown that the models which combine conduction and localized electrons provide a platform for very intricate types of order where the conduction electrons bind to slow collective modes of the spin subsystem. As a result, the localized spins and the conduction electrons together create spin liquids with gapped fractionalized excitations. The local order parameters (COPs) include bound states of more than two electrons and are not amenable to analysis based on perturbative methods. The discussion has rotated around quasi-one-dimensional models (the ones I dubbed Kondo-Heisenberg cable arrays) where these fractionalized excitations remain one-dimensional even when different KH cables are coupled in $D > 1$ array.

Composite orders naturally give rise to rich order parameter manifolds, which include various types of density waves, including those of pairs and quartets of electrons. The formation of the spin liquid is accompanied by a simultaneous formation of the order parameter amplitudes, but the phase coherence is established only by three-dimensional interactions in the cable array. As a consequence, the magnitudes of the transition temperatures are not related to the spin gaps.

If the electron hopping matrix elements between different cables exceed the spin gap, pockets of quasiparticle Fermi surface appear. As it has been pointed out in Ref. [5], KHC model reproduces many features found in the pseudogap phase of the cuprates.

It remains to be seen whether the present ideas can be generalized for isotropic models in $D > 1$. As we know from the literature on spin liquids, to propagate in $D > 1$ dimensions fractional particles need to have companions in the form of visons. For instance, in the exactly solvable Kitaev model [34] of spin liquid, the role of visons for propagating Majorana fermions is played by static Z_2 gauge field fluxes. They facilitate a propagation of the Majorana fermions in all lattice directions. As far as I can see there are no visons in the present construction and the fractional particles remain one-dimensional.

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APPENDIX A: EXACT SOLUTION OF THE WZNW MODEL PERTURBED BY THE CURRENT-CURRENT INTERACTION

The exact solution of the WZNW model perturbed by the current-current interaction can be derived from the relativistic limit of the fermion model

$$H = \int dx \Big(-\frac{1}{2m} \psi^+_{j\alpha} \partial_x^2 \psi_{j\alpha} - \mu \psi^+_{j\alpha} \psi_{j\alpha} - U \psi^+_{j\alpha} \psi^+_{j\beta} \psi_{i\beta} \psi_{i\alpha} \Big), \quad (A1)$$

where $j,i = 1,\dots,n$, $\alpha,\beta = 1,2$, and $U > 0$. This model is exactly solvable by the Bethe ansatz [14]. The relativistic limit is obtained by the spectrum linearization (3) and (4). The interaction then becomes

$$- U \psi^+_{j\alpha} \psi^+_{j\beta} \psi_{i\beta} \psi_{i\alpha} \to U \mathbf{F}_R \mathbf{F}_L - U J^a_R J^a_L, \quad (A2)$$

where F^a_R, F^a_L are $SU_n(2)$ and J^a_R, J^a_L $(a = 1,\dots,n^2 - 1)$ are $SU_2(n)$ currents. The latter interaction is marginally irrelevant and can be discarded. As a result the only gapped sector is the one described by the $SU_n(2)$ WZNW perturbed by the current-current interaction.

The solution can be also extracted from the following Bethe ansatz equations:

$$[e_n(u_a - v_F/J_K)]^L [e_n(u_a + v_H/J_K)]^L = \prod_{b=1}^{M} e_2(u_a - u_b), \quad S^z = nL/2 - M,$$

$$E = \frac{1}{2\mathrm{i}} \sum_a [v_F \ln e_n(u_a - v_F/J_K) - v_H \ln e_n(u_a + v_H/J_K)], \quad (A3)$$

where

$$e_j(x) = \frac{x - \mathrm{i}j/2}{x + \mathrm{i}j/2}.$$

The thermodynamic Bethe ansatz equations in the continuum limit are [14]

$$\epsilon_j(\theta) = Ts * \ln[1 + e^{\epsilon_{j-1}(\theta)/T}][1 + e^{\epsilon_{j+1}(\theta)/T}] - \delta_{j,n}E(\theta),\quad j = 1,2,\ldots, \tag{A4}$$

$$F/L = E_0 - T\int \frac{dP(\theta)}{2\pi}\ln[1 + e^{\epsilon_n(\theta)/T}], \tag{A5}$$

$$s * f(x) = \int \frac{dy}{\pi\cosh(x-y)} f(y), \tag{A6}$$

where $E(\theta)$ and $P(\theta)$ are given by Eq. (50). Expanding at $T \ll \Delta$, one obtains

$$F/L = -T\mathcal{Q}\int \frac{dP}{2\pi}e^{-E(\theta)/T},\quad \mathcal{Q} = 2\cos\left(\frac{\pi}{n+2}\right), \tag{A7}$$

with $\mathcal{Q}$ being the so-called quantum dimension. This number indicates that the state of $N \gg 1$ particles with energy E is degenerate and the degeneracy is approximately $\mathcal{Q}^N$. The fact that $\mathcal{Q}$ is not an integer is an indication of the non-Abelian nature of the excitations.

APPENDIX B: USEFUL FACTS ABOUT $n = 2$ KHC MODEL

In classification of fermionic fields of the $n = 2$ 1DEG, I follow the scheme described in [20]. Namely, one introduce four bosonic holomorphic fields

$$\varphi_c,\ \varphi_f,\ \varphi_s,\ \varphi_{sf}, \tag{B1}$$

with their antiholomorphic counterparts $\bar{\varphi}_a$ $(a = c,f,s,sf)$ to bosonize the fermions:

$$R_{p,\sigma} = \frac{\lambda_{p\sigma}}{\sqrt{2\pi a_0}}\exp[i\sqrt{\pi}(\varphi_c + p\varphi_f + \sigma\varphi_s + p\sigma\varphi_{sf})],$$
$$L_{p,\sigma} = \frac{\lambda_{p\sigma}}{\sqrt{2\pi a_0}}\exp[-i\sqrt{\pi}(\bar{\varphi}_c + p\bar{\varphi}_f + \sigma\bar{\varphi}_s + p\sigma\bar{\varphi}_{sf})], \tag{B2}$$

where $\lambda_{p\sigma}$ are anticommuting Klein factors. Then right-moving the Majorana fermions are

$$\eta_1 = \frac{\xi_c}{\sqrt{2\pi a_0}}\cos(\sqrt{4\pi}\varphi_c),\quad \eta_2 = \frac{\xi_c}{\sqrt{2\pi a_0}}\sin(\sqrt{4\pi}\varphi_c),$$
$$\eta_3 = \frac{\xi_f}{\sqrt{2\pi a_0}}\cos(\sqrt{4\pi}\varphi_f),\quad \eta_4 = \frac{\xi_f}{\sqrt{2\pi a_0}}\sin(\sqrt{4\pi}\varphi_f),$$
$$\eta_5 = \frac{\xi_{sf}}{\sqrt{2\pi a_0}}\cos(\sqrt{4\pi}\varphi_{sf}), \tag{B3}$$

where ξ_a are anticommuting Klein factors and

$$\chi_1 = \frac{\xi_s}{\sqrt{2\pi a_0}}\cos(\sqrt{4\pi}\varphi_s),\quad \chi_2 = \frac{\xi_s}{\sqrt{2\pi a_0}}\sin(\sqrt{4\pi}\varphi_s),$$
$$\chi_3 = \frac{\xi_{sf}}{\sqrt{2\pi a_0}}\sin(\sqrt{4\pi}\varphi_{sf}). \tag{B4}$$

It is assumed that the bosonic fields are governed by the Gaussian action.

The Ising order and disorder parameters are related to $\Phi = \varphi + \bar{\varphi}$, $\Theta = \varphi - \bar{\varphi}$ fields. If one takes two copies of the critical Ising model, I have [35]

$$\sigma_1\sigma_2 = \frac{1}{(\pi a_0)^{1/4}}\sin(\sqrt{\pi}\Phi),\quad \mu_1\mu_2 = \frac{1}{(\pi a_0)^{1/4}}\cos(\sqrt{\pi}\Phi),$$
$$\sigma_1\mu_2 = \frac{1}{(\pi a_0)^{1/4}}\sin(\sqrt{\pi}\Theta),\quad \mu_1\sigma_2 = \frac{1}{(\pi a_0)^{1/4}}\cos(\sqrt{\pi}\Theta). \tag{B5}$$

The most convenient and economic way to establish a correspondence between different representations of the two-leg problem is to use the $SU_2^s(2) \times SU_2^f(2)$ basis and employ the non-Abelian bosonization. One SU(2) group represents rotations generated by currents of total spin and the other by chain currents. Transition from chain to band representation can be viewed as a rotation basis in $SU^f(2)$ space. To make sure this approach is sound, I will make a crosscheck with the Abelian bosonization.

There is one subtlety discussed in Ref. [13]. Namely, the group we are dealing with is not really SU(2), but its complexification SU(2,C).

Below, there are examples of OPs which are spin singlets. Being fused with the spin matrix h, they will leave

$$e^{\pm i\sqrt{\pi}\Phi}G,\quad e^{\pm i\sqrt{\pi}\Theta}G \tag{B6}$$

as the fluctuating COPs. All this can be expressed as products of five Ising fields, which constitutes the 16-dimensional spinor representation of the SO(5) group. An important qualitative difference with the single chain case is that the COPs include pairs with momentum $\neq \pi$.

Using the standard bosonization rules I derived the following formulas. The s-wave CDW order parameter. In the chain representation, we have

$$\begin{aligned}
&R^+_{1\sigma}L_{1\sigma} + R^+_{2\sigma}L_{2\sigma}\\
&= 2ie^{i\sqrt{\pi}\Phi_c}[e^{i\sqrt{\pi}\Phi_f}\cos\sqrt{\pi}(\Phi_s + \Phi_{sf})\\
&\quad + e^{-i\sqrt{\pi}\Phi_f}\cos\sqrt{\pi}(\Phi_s - \Phi_{sf})]\\
&= 4ie^{i\sqrt{\pi}\Phi_c}(\cos\sqrt{\pi}\Phi_f\cos\sqrt{\pi}\Phi_s\cos\sqrt{\pi}\Phi_{sf}\\
&\quad + i\sin\sqrt{\pi}\Phi_f\sin\sqrt{\pi}\Phi_s\sin\sqrt{\pi}\Phi_{sf})\\
&= -4e^{i\sqrt{\pi}\Phi_c}(M_1M_2M_3\mu_1\mu_2\mu_3 + i\Sigma_1\Sigma_2\Sigma_3\sigma_1\sigma_2\sigma_3)\\
&= -\tfrac{1}{4}e^{i\sqrt{\pi}\Phi_c}[\mathrm{Tr}(G + G^+)\mathrm{Tr}(g + g^+)\\
&\quad - i\mathrm{Tr}(G - G^+)\mathrm{Tr}(g - g^+)].
\end{aligned} \tag{B7}$$

Here, M,μ are disorder and Σ,σ order parameter fields of the Ising models describing the flavor and spin sectors, G and g are matrices from the flavor and spin sectors, respectively. In the band representation, the expression in terms of fermions looks the same with chain indices 1,2 being replaced by band indices a,b. Naturally, the expression in terms of matrices looks the same, as it should be.

Now let us consider a more general CDW OP:

$$\mathcal{O}^a_{\rm CDW} = R^+_{j\sigma}\tau^a_{jk}L_{k\sigma}. \tag{B8}$$

The simplest member of this family is

$$\begin{aligned}\mathcal{O}^3 &= 2\mathrm{i}e^{\mathrm{i}\sqrt{\pi}\Phi_c}[e^{\mathrm{i}\sqrt{\pi}\Phi_f}\cos\sqrt{\pi}(\Phi_s+\Phi_{sf})\\ &\quad - e^{-\mathrm{i}\sqrt{\pi}\Phi_f}\cos\sqrt{\pi}(\Phi_s-\Phi_{sf})]\\ &= 4e^{\mathrm{i}\sqrt{\pi}\Phi_c}(-\sin\sqrt{\pi}\Phi_f\cos\sqrt{\pi}\Phi_s\cos\sqrt{\pi}\Phi_{sf}\\ &\quad - \mathrm{i}\cos\sqrt{\pi}\Phi_f\sin\sqrt{\pi}\Phi_s\sin\sqrt{\pi}\Phi_{sf})\\ &= -4e^{\mathrm{i}\sqrt{\pi}\Phi_c}(\Sigma_1\Sigma_2M_3\mu_1\mu_2\mu_3 + \mathrm{i}M_1M_2\Sigma_3\sigma_1\sigma_2\sigma_3)\\ &= \frac{1}{4}e^{\mathrm{i}\sqrt{\pi}\Phi_c}\{\mathrm{Tr}(g-g^+)\mathrm{Tr}[\tau^3(G-G^+)]\\ &\quad + \mathrm{i}\mathrm{Tr}(g+g^+)\mathrm{Tr}[\tau^3(G+G^+)]\}.\end{aligned} \tag{B9}$$

The superconducting SCd ($\lambda_{2\uparrow}\lambda_{1\downarrow} = \lambda_{1\uparrow}\lambda_{2\downarrow} = \mathrm{i}$):

$$\begin{aligned}\Delta_d &= R_{1\uparrow}L_{2\downarrow} + R_{2\uparrow}L_{1\downarrow} - (\uparrow\to\downarrow)\\ &= \lambda_{1\uparrow}\lambda_{2\downarrow}[e^{\mathrm{i}\sqrt{4\pi}(\varphi_{1\uparrow}-\bar\varphi_{2\downarrow})} + e^{\mathrm{i}\sqrt{4\pi}(\varphi_{2\uparrow}-\bar\varphi_{1\downarrow})}]\\ &\quad + \lambda_{2\uparrow}\lambda_{1\downarrow}[e^{\mathrm{i}\sqrt{4\pi}(\varphi_{1\downarrow}-\bar\varphi_{2\uparrow})} + e^{\mathrm{i}\sqrt{4\pi}(\varphi_{2\downarrow}-\bar\varphi_{1\uparrow})}]\\ &= 2e^{\mathrm{i}\sqrt{\pi}\Theta_c}[e^{\mathrm{i}\sqrt{\pi}\Phi_f}\cos\sqrt{\pi}(\Phi_s+\Theta_{sf})\\ &\quad + e^{-\mathrm{i}\sqrt{\pi}\Phi_f}\cos\sqrt{\pi}(\Phi_s-\Theta_{sf})]\\ &= 4e^{\mathrm{i}\sqrt{\pi}\Theta_c}(M_1M_2\Sigma_3\mu_1\mu_2\mu_3 + \mathrm{i}\Sigma_1\Sigma_2M_3\sigma_1\sigma_2\sigma_3)\\ &\quad\times\frac{1}{4}e^{\mathrm{i}\sqrt{\pi}\Theta_c}\{\mathrm{Tr}(g+g^+)\mathrm{Tr}[\tau^3(G-G^+)]\\ &\quad + \mathrm{i}\mathrm{Tr}(g-g^+)\mathrm{Tr}[\tau^3(G+G^+)]\}.\end{aligned} \tag{B10}$$

APPENDIX C: THE DETAILED DESCRIPTION OF THE COMPOSITE OPs

In this appendix, I discuss the formation of the simplest COP (31) for the case $n = 2$. As the first step of the proof, I recast the products of the OPs of the 1DEGs and the TBC in terms of the operators of the GN models (24) and (25). More precisely, we have to express the order and disorder parameters of the band fermions and the TBC antiferromagnet (they carry labels F and H, respectively) in terms of the corresponding operators of models (24) and (25) labeled R and L. I will use the Abelian bosonization formulas (B5). Consider, for instance, the product

$$\begin{aligned}(\sigma_1\sigma_2)_F(\sigma_1\sigma_2)_H &\sim 2\sin(\sqrt{\pi}\Phi_F)\sin(\sqrt{\pi}\Phi_H)\\ &= \cos[\sqrt{\pi}(\varphi_F+\bar\varphi_F-\varphi_H-\bar\varphi_H)]\\ &\quad - \cos[\sqrt{\pi}(\varphi_F+\bar\varphi_F+\varphi_H+\bar\varphi_H)]\\ &= \cos[\sqrt{\pi}(\Theta_L-\Theta_R)] - \cos[\sqrt{\pi}(\Phi_L+\Phi_R)]\\ &= (\mu_1\sigma_2)_L(\mu_1\sigma_2)_R - (\sigma_1\mu_2)_L(\sigma_1\mu_2)_R - (\mu_1\mu_2)_L(\mu_1\mu_2)_R\\ &\quad - (\sigma_1\sigma_2)_L(\sigma_1\sigma_2)_R.\end{aligned} \tag{C1}$$

Hence it is plausible that the product of F and H OPs contains products

$$(\mu_1\mu_2\mu_3)_L(\mu_1\mu_2\mu_3)_R,\ (\sigma_1\sigma_2\sigma_3)_L(\sigma_1\sigma_2\sigma_3)_R. \tag{C2}$$

Such products have nonzero expectation values at least in some of the degenerate vacua of (14) and (15). These expectation values may have a different sign in different vacua, but this does not affect the correlation functions of the COPs, since the two-point functions contains only squares of the amplitudes:

$$\langle 0_j|(\sigma_1\sigma_2\sigma_3)_L(1)(\sigma_1\sigma_2\sigma_3)_L(2)|0_j\rangle = [\langle 0_j|(\sigma_1\sigma_2\sigma_3)_L(0)|0_j\rangle]^2. \tag{C3}$$

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