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The "parity" anomaly on an unorientable manifold

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The "parity" anomaly—more accurately described as an anomaly in time-reversal or reflection symmetry—arises in certain theories of fermions coupled to gauge fields and/or gravity in a spacetime of odd dimension. This anomaly has traditionally been studied on orientable manifolds only, but recent developments involving topological superconductors have made it clear that one can get more information by asking what happens on an unorientable manifold. In this paper, we give a full description of the "parity" anomaly for fermions coupled to gauge fields and gravity in $2 + 1$ dimensions on a possibly unorientable spacetime. We consider an application to topological superconductors and another application to M theory. The application to topological superconductors involves using knowledge of the "parity" anomaly as an ingredient in constructing gapped boundary states of these systems and in particular in gapping the boundary of a $\nu = 16$ system in a topologically trivial fashion. The application to M theory involves showing the consistency of the path integral of an M theory membrane on a possibly unorientable worldvolume. In the past, this has been done only in the orientable case.

I. INTRODUCTION

A U(1) gauge theory in three spacetime dimensions, coupled to a single massless Dirac fermion χ of charge 1, is invariant at the classical level under time-reversal and reflection symmetry, which we will call T and R. However, quantum mechanically there is an anomaly: To quantize this theory in a gauge-invariant fashion, one must give up T and R symmetry [1–3]. This anomaly is commonly called a "parity" anomaly, but this terminology is somewhat misleading as parity (a spatial inversion, acting as -1 on all spatial coordinates) is in the connected component of the rotation group in $2 + 1$ dimensions. The anomaly is better understood as an anomaly in T and R.

In this paper, we consider a refinement of the usual "parity" anomaly. We will make this analysis for arbitrary gauge groups, but for gauge group U(1), what we will learn can be stated as follows. The usual statement of the anomaly is that a U(1) gauge theory in $2 + 1$ dimensions coupled to a single Dirac fermion χ of charge 1 cannot be quantized in a gauge-invariant way, consistent with R and T symmetry. If one has two such Dirac fermions, both of charge 1, then the usual "parity" anomaly is absent and, on an orientable manifold, the theory can be quantized in a way that preserves U(1) gauge-invariance as well as R and T. If R and T were ordinary global symmetries (not acting on spacetime), one would now ask: Can these symmetries be gauged? In general, for ordinary global symmetries, the answer to such a question can be "No": Even if the symmetries are valid as global symmetries, there may be an anomaly (sometimes called an 't Hooft anomaly) that would obstruct gauging them. For R and T, the closest analog of gauging the symmetry is to use these symmetries to formulate a theory on an unorientable manifold.[1] Therefore, to get a refined version of the usual "parity" anomaly, we can consider a U(1) gauge theory with a general number y

of Dirac fermions of charge 1 and ask for what values of y can this theory be placed on an unorientable manifold without violating gauge invariance. We shall answer this question and find that y must be a multiple of 4. Thus, while the usual "parity" anomaly is a mod 2 effect, its refinement in which one "gauges" R and T symmetry is really a mod 4 effect.

The existence of this refinement should not come as a surprise, since the purely gravitational analog is already known.[2] Consider a theory of ν Majorana fermions coupled to gravity only. As long as one is on an orientable manifold, the fermion path integral of this system is real in Euclidean signature (because the Dirac operator is Hermitian). One should worry about a possible problem in defining the *sign* of the fermion path integral. However, as long as ν is even—to ensure that the path integral is positive—the theory of ν Majorana fermions is completely well defined and R and T invariant on an orientable manifold.[3] Can the R and T symmetries be gauged, or, more precisely, is the theory well defined when formulated on an unorientable manifold? The answer to this question is that when the theory of ν Majorana fermions is formulated on an unorientable manifold, one runs

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[1] If a theory with R or T symmetry can be formulated on an unorientable manifold, we say that R or T is gaugeable, while actually gauging the symmetry in the context of quantum gravity would mean summing over unorientable manifolds.

[2] Also, it is already known from another point of view (see Sec. III C of [4] and Appendix C of [5]) that a purely $2 + 1$-dimensional T-invariant U(1) gauge theory with $4n + 2$ Dirac fermions of charge 1 is anomalous from the standpoint of condensed-matter physics. The basic monopole operator of this theory, defined by a monopole singularity of charge 1, is a bosonic operator that transforms under T as a Kramers doublet. This is not possible in a purely $2 + 1$-dimensional system that is built microscopically from electrons and nuclei. We discuss this issue in Sec. III B 2.

[3] In this introduction, we take all ν fermions $\psi_1, \ldots, \psi_\nu$ to transform in the same way under T or R. Specifically, we define T by $T\psi_i(t, \vec{x}) = \gamma_0 \psi_i(-t, \vec{x})$, $i = 1, \ldots, \nu$, where here γ_μ, $\mu = 0, \ldots, 2$ are the Dirac γ matrices, and similarly for R. A fermion transforming with the opposite sign—as $T\psi(t, \vec{x}) = -\gamma_0 \psi(-t, \vec{x})$—would make a contribution to the anomaly with the opposite sign. In general, ν should be defined as the number of fermions that transform with a $+$ sign minus the number that transform with a $-$ sign. For more on this, see [7], as well as Sec. II below.

into an anomaly that involves an 8th root of 1 in the simplest calculations [6] and a 16th root of 1 in a more systematic analysis (see Sec. 4 of [7]).

The mod 16 anomaly for $2+1$-dimensional Majorana fermions coupled to gravity only is intimately related to the theory of a topological superconductor in $3+1$ dimensions. A topological superconductor in $3+1$ dimensions is characterized by an invariant usually called ν; in Euclidean signature, it has a bulk partition function $\exp(-\nu\pi i\eta/2)$, where η is the eta invariant of the four-dimensional Dirac operator. One can think of this factor, which was suggested in Ref. [8], as coming from integrating out ν bands[4] of gapped bulk fermions [7,9]. Here $\exp(-\pi i\eta/2)$ is a topological invariant if the worldvolume X of the topological superconductor has no boundary; it is the partition function of a topological field theory defined on closed four-dimensional spin (or more precisely pin$^+$) manifolds. This invariant is, in general, an arbitrary 16th root of 1, so $\exp(-\nu\pi i\eta/2)$ is trivial (for all X) if and only if ν is divisible by 16. However, what happens if X has a boundary? Then, if ν is not a multiple of 16, $\exp(-\nu\pi i\eta/2)$ cannot be defined as a topological invariant. However, if we include ν massless Majorana fermions on the boundary of X (and transforming under T as in footnote 3), the anomaly of the fermions compensates for the ill-definedness of the bulk partition function and the combined system is well-defined and anomaly-free. In this paper, we will analyze the generalization of this construction to include gauge fields, either gauge fields that propagate in $3+1$ dimensions or gauge fields that propagate only on the boundary of a four-manifold.

In band theory, and as we have defined it above, ν is an integer, but when fermion interactions in the bulk theory are taken into account, it is known from several points of view [4,5,10–12] that ν is only an invariant mod 16. One way to understand this is by the fact that the partition function of the bulk theory and the anomaly of the boundary theory depend only on the value of ν mod 16.

This paper is organized as follows. We analyze the general "parity" anomaly for gauge fields on an unorientable manifold[5] in Sec. II. In the remainder of the paper, we describe two applications of the result, one in condensed-matter physics and one in string/M theory.

In Sec. III, we use our results to construct gapped, symmetry-preserving boundary states of a topological superconductor. Such states and analogous ones for topological insulators and for certain bosonic states of matter have been analyzed in the literature from various points of view [4,5,13–21]. We use here the same methods as in Ref. [22] and our basic examples were already analyzed there. However, having a more complete knowledge of which models are anomaly-free enables us to be more precise on some points, and in particular, it will help us to construct in a straightforward way a gapped and topologically trivial boundary state at $\nu=16$. (If one incorrectly believed that the "parity" anomaly is a mod 2 effect

even on an unorientable manifold, one would incorrectly infer the existence of a gapped and topologically trivial boundary state at $\nu=8$.) We primarily consider models that are gapped at the perturbative level, but in Sec. III B 6, we consider models that are gapped with the help of nonperturbative effects. Here we give two related constructions, both at $\nu=16$, one of which uses the mass gap of SU(2) gauge theory and one of which uses the Polyakov model [23] of compact QED in $2+1$ dimensions. (An interpolation between different bulk states at $\nu=16$ using nonperturbative properties of four-dimensional gauge theory has been described in Ref. [12].)

In Sec. IV, we apply some of the same ideas to a problem in string/M theory. It has been known [24] that the "parity" anomaly in $2+1$ dimensions has implications for M2-branes (membranes with $2+1$-dimensional worldvolume that arise in M theory). In fact, the fermions that propagate on the M2-brane worldvolume do have an anomaly, which has to be canceled by a version of anomaly inflow [25] involving a shifted quantization law of the three-form field of M theory. In the past, this analysis has been made only for the case that the M2-brane worldvolume is orientable. Here we analyze the unorientable case, showing that, with a slightly refined statement of the shifted quantization law, anomalies still cancel. (Previously, the path integral on an unorientable string worldsheet has been analyzed in Refs. [26,27].) A short summary of the paper can be found in Sec. V.

We conclude this introduction with a general comment. Apart from technical details, the analysis in this paper differs in the following way from more familiar discussions of the "parity" anomaly. On an orientable manifold, the usual "parity" anomaly is a conflict between T or R symmetry and gauge invariance. However, when we use R symmetry to formulate a theory on an unorientable manifold (we work in Euclidean signature so R is more relevant here than T), that symmetry is built in and it is too late to give it up. So the anomalies we compute on unorientable manifolds represent breakdowns of gauge symmetry or diffeomorphism symmetry rather than conflicts between those local symmetries and T or R symmetry.

II. ANALYSIS OF THE ANOMALY

A. Preliminaries

In this paper, we will consider fermions on a $2+1$-dimensional manifold W interacting with gauge fields of some gauge group G and with gravity. In the main condensed-matter application, W will be the boundary of a four-manifold X that will be the worldvolume of a topological superconductor and G will be a group of emergent gauge fields that propagate only along W (and not X). In the string/M theory example, W will be the worldvolume of an M2-brane, and G will be the structure group of the normal bundle to W in the full higher-dimensional spacetime.

In either case, we consider gauge fields a that transform under time-reversal T in the natural-looking way

$$\mathsf{T}(a_0(x^0,\vec{x})) = -a_0(-x^0,\vec{x}),$$
$$\mathsf{T}(a_i(x^0,\vec{x})) = a_i(-x^0,\vec{x}), \quad i=1,2. \qquad (2.1)$$

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[4]Or, more generally, any number of bands with a net invariant of ν.

[5]Mathematical background to this rather technical analysis can be found in Ref. [7]. The applications in the rest of the paper can be understood without a detailed reading of Sec. II, provided the reader is willing to take a few statements on faith.

In terms of differential forms, if τ is the time-reversal transformation of spacetime $x^0, \vec{x} \rightarrow -x^0, \vec{x}$, Eq. (2.1) would be written $\mathsf{T}(a) = \tau^*(a)$. These conditions imply that the gauge charge (which couples to a_0) is odd under T, and the "magnetic" flux (the integral of $F_{12} = \partial_1 a_2 - \partial_2 a_1$) is even. This behavior under T will be assumed for all gauge fields considered in the present paper.

The choice in Eq. (2.1) really requires some explanation. In nature, there is also an operation C of charge conjugation, so there are two possible time-reversal operations, namely T and CT. The usual convention in particle physics and condensed-matter physics is that the gauge field A of electromagnetism transforms under T with an opposite sign to Eq. (2.1); thus, $\mathsf{T}(A) = -\tau^*(A)$, as opposed to $\mathsf{CT}(A) = +\tau^*(A)$. This definition ensures that electric charge is T-even. By contrast, the choice (2.1) means that the conserved charge q that couples to a_0 is T-odd.

A rationale for the choice (2.1) is that in the present paper we will consider problems in which there is no natural operation corresponding to C. Since C reverses the sign of the electric charge, and electric charge is not conserved in a superconductor,[6] there is no natural C operation in the theory of a topological superconductor. Likewise, in the string/M theory problem, there is no operation corresponding to C.

Since there will be no analog of C in the problems that we will consider, we will simply refer to the transformation (2.1) as T. Actually, to analyze anomalies, which will be our goal in the present section, it is convenient to work in Euclidean signature. Here we consider a spatial reflection rather than time reversal. We take R to act by reflection of one coordinate:

$$\mathsf{R}(a_1(x_1, x_2, x_3)) = - a_1(-x_1, x_2, x_3),$$
$$\mathsf{R}(a_i(x_1, x_2, x_3)) = a_i(-x_1, x_2, x_3), \quad i > 1. \quad (2.2)$$

Equivalently, $\mathsf{R}(a) = \rho^*(a)$, where ρ is the spatial reflection $x_1, x_2, x_3 \rightarrow -x_1, x_2, x_3$.

Thus, in going to an unorientable manifold, we will consider gauge fields a that transform under an orientation-reversing diffeomorphism ρ as $a \rightarrow \rho^*(a)$. As we will see, this leads to a richer structure than the familiar "parity" anomaly that is seen on an orientable manifold. By contrast, if one uses CR to go to an unorientable manifold [for example, by considering a U(1) gauge field a that transforms under an orientation-reversing diffeomorphism as $a \rightarrow -\rho^*(a)$], then one finds only the standard "parity" anomaly. This has been explained in Sec. 4.7

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[6] To be more precise, in a superconductor electric charge is conserved mod 2, meaning that the sign of the charge is irrelevant and thus there is no natural C operation. To say it differently, a superconductor has fermionic quasiparticles but there is not a meaningful sense in which their electric charge is positive or negative. In terms of the electromagnetic gauge field A, one would say that in a superconductor, as U(1) is broken to $\mathbb{Z}_2$, A becomes a $\mathbb{Z}_2$ gauge field, so that $2A$ is gauge equivalent to 0 at long distances and the operation $\mathsf{C} : A \rightarrow -A$ is irrelevant. From a macroscopic point of view, this is true even in the presence of flux lines. A fermionic excitation propagates around a flux line with a change of sign regardless of whether the flux line has flux π or $-\pi$.

of [7]. That is why the considerations of the present paper are more directly relevant to topological superconductors (and M theory) rather than to topological insulators.

We reiterate that in this paper we consider theories that have no natural C operation at the microscopic level so the discrete symmetries are T and R. Should we contemplate the possibility of C as an emergent symmetry? In general, *exact* emergent symmetries are gauge symmetries; an emergent C symmetry, if it is an exact symmetry, would be part of the emergent gauge group and could be incorporated as such in the analysis that follows. It is certainly possible to have *approximate* emergent global symmetries, but an approximate C symmetry would not be relevant in the present paper; for example, we cannot use CR in putting a theory on an unorientable manifold unless CR is an exact symmetry.

A final comment is that what we call T and R in this paper correspond more closely to what are usually called CT and R, respectively, when electric charge conservation is relevant and in theories with a microscopic C symmetry.[7] In a theory that does have a microscopic C symmetry, analytic continuation from Lorentz signature to Euclidean signature relates CT to R and T to CR.

B. The fermion path integral

In general, there is a choice of sign in the transformation of a $2 + 1$-dimensional fermion field ψ under T:

$$\mathsf{T}(\psi(t, \vec{x})) = \pm \gamma_0 \psi(-t, \vec{x}). \quad (2.3)$$

We note that for either choice of sign, one has $\mathsf{T}^2 = (-1)^{\mathsf{F}}$ [here $(-1)^{\mathsf{F}}$ is the operator that counts the number of fermions mod 2]. However, the two choices are physically inequivalent; no linear transformation of ψ would reverse this sign.

There is a similar sign choice in the transformation under a spatial reflection. In Euclidean signature,

$$\mathsf{R}(\psi(x_1, x_2, x_3)) = \pm \gamma_1 \psi(-x_1, x_2, x_3). \quad (2.4)$$

In a relativistic theory, the signs in Eqs. (2.3) and (2.4) can be equated, as the CRT theorem[8] ensures that a theory has a T symmetry with a given choice of sign if and only if it has an R symmetry with the same choice of sign.

We consider a system of $2 + 1$-dimensional massless Majorana fermions $\Psi = (\psi_1, \psi_2, \ldots, \psi_n)$ transforming in an n-dimensional real[9] representation R of a gauge group G and

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[7] For example, what we call T was called CT in Secs. 2–5 of [22], which are devoted to a topological insulator (in which electric charge is conserved). It was called T_{sc} in Sec. 6 of that paper, which is devoted to a topological superconductor.

[8] Here C can be dropped if gauge fields transform under T and R as stated in Eqs. (2.1) and (2.2). What we are calling T is called CT in the usual statement of the theorem, so what is usually called CRT reduces to RT with our choices. As explained in Sec. II A, we are using conventions that are more natural in a context (such as the theory of a topological superconductor) in which there is no natural C operation.

[9] The representation is real because the global symmetry group of n massless Majorana fermions in $2 + 1$ dimensions is O(n). G is necessarily a subgroup of O(n) and the representation R is the vector

all transforming under T (or R) with a + sign. We want to know whether this theory makes sense as a purely $2 + 1$-dimensional theory and to measure the anomaly if it does not. We will schematically call that anomaly α_R. One can think of this anomaly as the change in phase of the path integral under a diffeomorphism and/or gauge transformation, though we also come later to a more powerful way to think about it involving topological field theory.

Alternatively, we could assume that the fermions Ψ transform under T and R with a minus sign. This complex conjugates the path integral, so it changes the sign of the anomaly, replacing α_R with $-\alpha_R$.

More generally, we could have one set of fermions $\Psi = (\psi_1, \psi_2, \ldots, \psi_n)$ transforming in the representation R of G, and transforming under T with a + sign, and a second set $\Psi' = (\psi'_1, \psi'_2, \ldots, \psi'_{n'})$ transforming in some possibly different representation R' of G, and transforming under T with a − sign. The net anomaly is then $\alpha_R - \alpha_{R'}$. Clearly, then, to understand the anomaly for an arbitrary system of fermions, it suffices to compute α_R; that is, it suffices to consider a system of fermions Ψ that all transform under T with a + sign.

Formally, the path integral Z_Ψ of a system of fermions, with Dirac operator $\mathcal{D} = i\slashed{D}$, is the Pfaffian $\text{Pf}(\mathcal{D})$. Our approach to computing the anomaly will be based on the Dai-Freed theorem [28]. This theorem[10] gives a recipe (in Euclidean signature) to define Z_Ψ in a gauge-invariant fashion, compatible with general physical principles such as unitarity, once some auxiliary choices are made. One then has to check whether the result does depend on the choices. If it does not, then one has succeeded in finding a satisfactory definition of Z_Ψ and the $2 + 1$-dimensional theory is anomaly-free. If Z_Ψ as defined by the theorem does depend on the auxiliary choices, then the theory is anomalous, and its anomaly can be measured by how the theory depends on the choices.

Let W be a compact three-manifold endowed with a pin$^+$ structure[11] and a background gauge field. Suppose that we wish to calculate the path integral Z_Ψ of the fermions Ψ on a possibly unorientable three-manifold W of Euclidean signature. The Dai-Freed theorem gives us a definition of Z_Ψ once we are given a four-dimensional manifold X with boundary W such that the pin$^+$ structure and the metric and gauge field on W all extend over X. The definition is

$$Z_\Psi = |\text{Pf}(\mathcal{D})| \exp(-\pi i \eta_R/2). \tag{2.5}$$

Here η_R is the Atiyah-Patodi-Singer [30] (APS) eta invariant of the Dirac operator acting on four-dimensional Majorana fermions in the representation R of G, and $|\text{Pf}(\mathcal{D})|$ is simply the absolute value of $\text{Pf}(\mathcal{D})$ (this absolute value is always anomaly-free for an arbitrary fermion system).

Mathematically, the justification for the formula is as follows. First, the right-hand side is well defined and gauge

invariant. Second, the variation of the right-hand side is as expected; that is, its variation in a change in the background metric or gauge field on X depends only on the fields along W and is given by the (regularized) one-point function of the stress tensor or current in the fermion theory on W, as expected. An important consequence of the second point is that the right-hand side vanishes precisely when Z_Ψ is expected to vanish, namely for metrics and gauge fields such that $\mathcal{D}$ has a zero mode; and moreover, it varies smoothly near such a point in field space [this is required in order for the formula to be physically sensible and is nontrivial as neither $|\text{Pf}(\mathcal{D})|$ nor $\exp(-\pi i \eta_R/2)$ varies smoothly near such a point]. Finally, in view of the gluing theorem for the eta invariant [28], which will be described below, the right-hand side has a behavior under cutting and pasting that is compatible with physical principles such as unitarity.

Physically, one might interpret the formula (2.5) as follows, by analogy with standard constructions in topological band theory [31–33]. Consider on X a gapped system with G symmetry consisting of massive Majorana fermions Υ transforming in the representation R of G. For one sign[12] of the Υ mass parameter, this theory is topologically trivial and can be given a boundary condition such that the theory remains gapped along the boundary. For the opposite sign of the Υ mass parameter, the bulk theory is topologically nontrivial, and gapless $2 + 1$-dimensional Majorana fermions—with the properties of the fields that we have called Ψ—appear on the boundary. The formula (2.5) for Z_Ψ should be understood as the partition function for the combined system comprising the gapped fermions Υ in bulk and the gapless fermions Ψ on the boundary.

On a compact four-manifold X without boundary, the Υ path integral (after removing nonuniversal local terms) would be $Z_\Upsilon = \exp(-\pi i \eta_R/2)$. This fact goes back in essence to [3] and has been exploited in several recent papers [7,9]. The right-hand side is a topological invariant (invariant, that is, under continuous variation of the metric and gauge field on X). Indeed, the APS index theorem [30] implies[13] that η_R

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[10]Some aspects of the Dai-Freed theorem were recently described for physicists in Ref. [29].

[11]A pin$^+$ structure is the generalization of a spin structure on a possibly unorientable manifold for fermions with $T^2 = (-1)^F$. See Appendix A of [7] for an introduction.

representation of $O(n)$, restricted to G. This is a real representation because the vector representation of $O(n)$ is real.

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[12]In topological band theory, the description in terms of a massive Majorana fermion is valid only near a point in the Brillouin zone at which the band gap—controlled in relativistic terminology by a fermion mass parameter—is small. Which sign of the mass parameter corresponds to a trivial theory and which corresponds to a nontrivial theory depends on the topology of the rest of the band. In a relativistic theory, defined with Pauli-Villars regularization, the massive fermions Υ are accompanied by regulator fields $\widetilde{\Upsilon}$ of opposite statistics. The trivial case is that Υ and $\widetilde{\Upsilon}$ have mass parameters of the same sign, and the nontrivial case is that the mass parameters have opposite signs.

[13]See, for example, [7], especially Sec. 2 and Appendix A, for an explanation of assertions made here. In brief, in even dimensions, η_R is a topological invariant except that it jumps by ± 2 when an eigenvalue of the Dirac operator passes through 0. In four dimensions, for a real representation R (which we have here since the $2 + 1$-dimensional fermions that we started with are always in a real representation of G), the eigenvalues of the Dirac operator have even multiplicity because of a version of Kramers doubling. So the jumps are by ± 4 and η_R is a topological invariant mod 4.

is a topological invariant mod 4, so that $\exp(-\pi i\eta_R/2)$ is a topological invariant. In fact, $\exp(-\pi i\eta_R/2)$ is not just a topological invariant, but a cobordism invariant (it equals 1 if X is the boundary of a five-dimensional pin$^+$ manifold over which the G bundle over X extends). This again follows from the APS index theorem. Cobordism invariance implies[14] [34] that $\exp(-\pi i\eta_R/2)$ is the partition function of a topological field theory (defined on a four-dimensional pin$^+$ manifold endowed with a G bundle). Cobordism invariance of the eta invariant in four dimensions motivated the conjecture in Ref. [8] that this function would be relevant to fermionic symmetry-protected topological (SPT) phases.

However, if X is a manifold with nonempty boundary W, then it is not possible to define η_R to make $\exp(-\pi i\eta_R/2)$ a topological invariant. This means that there is a problem in defining on a manifold with boundary the topological field theory that on a manifold without boundary has partition function $\exp(-\pi i\eta_R/2)$. This problem does not have a unique solution; the topological field theory in question has different possible boundary states. However, it always has one simple gapless boundary state, in which Majorana fermions in the representation R propagate on the boundary. $Z_\Psi = |\mathrm{Pf}(\mathcal{D})| \exp(-\pi i\eta_R/2)$ is the partition function of the theory on X with this boundary state along W. Here η_R is defined using APS boundary conditions. On a four-manifold without boundary, $\exp(-\pi i\eta_R/2)$ is a topological invariant; on a four-manifold with boundary, it is not a topological invariant, but varies in such a way that it makes sense in conjunction with the boundary fermions in the combination $|\mathrm{Pf}(\mathcal{D})| \exp(-\pi i\eta_R/2)$.

Now we can discuss whether the theory of the $2+1$-dimensional massless fermions Ψ on W makes sense as a purely $2+1$-dimensional theory. In the present framework, this amounts to asking whether the formula $Z_\Psi = |\mathrm{Pf}(\mathcal{D})| \exp(-\pi i\eta_R/2)$ depends on the choices that were made in defining η_R. There is a standard way to answer this sort of question. Let X and X' be two different four-manifolds with boundary W, and with choices of extension over X and over X' of the pin$^+$ structure of W and the G bundle over W. Let η_R^X and $\eta_R^{X'}$ be the eta invariants computed on X or on X'. To decide whether the formula for Z_Ψ depends on X, we need to know whether $\exp(-\pi i\eta_R^X/2)$ is always equal to $\exp(-\pi i\eta_R^{X'}/2)$.

The gluing theorem for the eta invariant [28] gives a convenient way to answer this question. To motivate the statement of this theorem, let us first recall the following basic fact about local effective actions in quantum field theory. Let $I_{\mathrm{eff}}^Y = \int_Y d^4x \sqrt{g}\mathcal{L}$ be any local effective action on a manifold Y. Let Y and Y' be two manifolds with boundary such that W is a boundary component of each. Suppose further that Y and Y' can be glued together along W to make a manifold[15] Y^*, as in Fig. 1. We assume that in this gluing, all structures on Y and Y' that are used in defining the effective action are compatible.

FIG. 1. Two manifolds Y and Y' glued along a component of their common boundary to make a manifold Y^* that itself may have a boundary.

Thus, if Y and Y' are orientable (and the definition of the action depends on the orientation), the orientations must agree in the gluing; if they are spin manifolds, the spin structures must agree, and if they are pin$^+$ manifolds, the pin$^+$ structures must agree. Under these conditions, for any local effective action, we have the gluing or factorization property

$$\exp\left(i I_{\mathrm{eff}}^Y\right) \exp\left(i I_{\mathrm{eff}}^{Y'}\right) = \exp\left(i I_{\mathrm{eff}}^{Y^*}\right), \tag{2.6}$$

which is intimately connected with locality and unitarity in physics. The gluing theorem for the eta invariant says that even though η_R is not the integral of a local effective action, $\exp(-\pi i\eta_R/2)$ behaves as if it were:

$$\exp\left(-\pi i\eta_R^Y/2\right)\exp\left(-\pi i\eta_R^{Y'}/2\right) = \exp\left(-i\pi\eta_R^{Y^*}/2\right). \tag{2.7}$$

It is because of this formula that it is physically sensible to have a topological field theory with G symmetry that—when defined on a manifold Y endowed with a G bundle—has $\exp(-\pi i\eta_R^Y/2)$ as its partition function.

To apply this theorem to our problem, we have to be careful with a minus sign. We start with two manifolds X and X' that each have the same boundary W (and an extension of the same gauge bundle) as in Fig. 2(a). When we say that X and X' have the same boundary W, this is in a convention in which they are on the "same side" of W, meaning (if they are orientable) that they are oriented in the same way, and in general in the context of our problem that their pin$^+$ structures restrict to the same pin$^+$ structure on W. To glue them to make a manifold X^* without boundary, we first "flip over" X', which means that we reverse its orientation if it is orientable, and in general we

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[14]This assertion is closely related to the gluing theorem [28] for the eta invariant, which we state shortly.

[15]W may be only one component of the boundary of Y and Y', so that Y^* may be itself a manifold with boundary, as in the figure. However, the boundary of Y^* will be empty in the application we make momentarily.
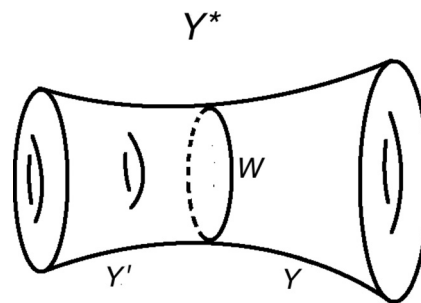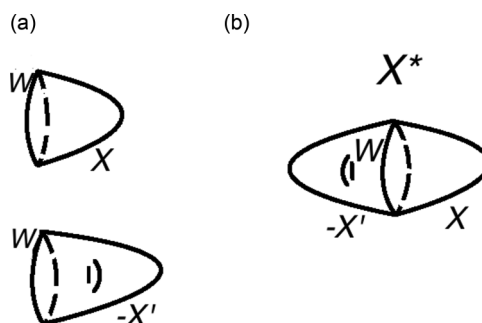

FIG. 2. Two manifolds X and X' with common boundary W are glued together along their common boundary to build a compact manifold X^* without boundary.

replace its pin$^+$ structure by the complementary one.[16] Let us write $-X'$ for the flipped version of X'. The gluing involves joining X to $-X'$, as in Fig. 2(b). "Flipping" X' reverses the sign of its eta invariant. So in this context, the gluing formula reads

$$\frac{\exp\left(-\pi i \eta_R^X/2\right)}{\exp\left(-\pi i \eta_R^{X'}/2\right)} = \exp\left(-i\pi \eta_R^{X^*}/2\right). \qquad (2.8)$$

Clearly, then, the left-hand side of Eq. (2.8) is always 1 if and only if the right-hand side is always 1. So Z_Ψ as defined in Eq. (2.5) is independent of the choice of X if and only if $\exp(-\pi i \eta_R^{X^*}/2)$ is 1 whenever X^* has no boundary.

The physical interpretation is straightforward: The boundary fermions make a consistent $2+1$-dimensional theory by themselves if and only if the bulk topological field theory with partition function $\exp(-\pi i \eta_R^X/2)$ is trivial.

C. A more precise question, and a strategy to answer it

The question that we have stated so far is a little too crude for some of our applications. Even if the emergent gauge group G is absent, the theory of n Majorana fermions in $2+1$ dimensions (all transforming under T with a $+$ sign) is anomalous unless n is a multiple of 16. In this paper, we are not mainly interested in this purely gravitational anomaly, which is familiar from several points of view and has been studied from the present point of view in Ref. [7] (following a calculation in Ref. [6] in a particular example).

We can always reduce to the case that n is a multiple of 16 by adding gauge-singlet fermions. This will eliminate the purely gravitational anomaly and focus attention on the dependence of the anomaly on the gauge bundle. Instead of doing this, let us just agree to refine the question we ask. Instead of asking whether $\exp(-\pi i \eta_R/2)$ is always 1 (on a four-manifold X without boundary), we will ask whether it has a nontrivial dependence on the choice of a G bundle over X. A "no" answer to this question means that the anomaly is purely gravitational and does not depend on the gauge fields at all.

In our condensed-matter application, we will want to know when it is possible to couple boundary fermions to emergent gauge fields without any new contribution to the anomaly. For this, we will need to know precisely when $\exp(-\pi i \eta_R/2)$ does not depend on the G bundle. Knowing how to analyze this question will also provide the starting point for our application to string/M theory.

For our purposes, a very simple criterion will be sufficient. Let $V_R \to X$ be the real vector bundle associated to the representation R of X. (Thus, R-valued fermions are sections of $P \otimes V_R$, where P is the pin$^+$ bundle of X. Mathematically, V_R is called the vector bundle associated with the underlying G bundle over X in the representation R.) Since $\exp(-\pi i \eta_R/2)$ can depend on a choice of gauge bundle only via the topology of V_R, if V_R is always trivial, then $\exp(-\pi i \eta_R/2)$ is certainly independent of the choice of G bundle.

We can generalize this criterion slightly. Let $\mathfrak{o}$ be a trivial real line bundle over X, and let $\mathfrak{o}^{\oplus k}$ be a trivial rank k bundle constructed as the direct sum of k copies of $\mathfrak{o}$. It can happen that V_R is not always trivial but there is some k such that $V_R \oplus \mathfrak{o}^{\oplus k}$ is always trivial. This means that the field Ψ, when supplemented with k neutral Majorana fermions (coupled to gravity only), has only a gravitational anomaly. However, the neutral fermions that we had to add to get this result do not couple to the gauge fields at all, so they only contributed to the purely gravitational part of the anomaly. Hence, also under these more general conditions, the anomaly does not depend on the gauge bundle.

As a matter of terminology, if $V_R \oplus \mathfrak{o}^{\oplus k}$ is trivial for some k, we say that V_R is stably trivial. Thus, a fermion system such that V_R is always stably trivial has only a gravitational anomaly.[17]

We will use this criterion in Sec. II E to show that some fermion systems in $2+1$ dimensions have only a gravitational anomaly, and in particular become entirely anomaly-free if a suitable number of gauge-singlet fermions is added to cancel the gravitational anomaly. These are the models that we will use in Sec. III to construct gapped boundary states of a topological superconductor. (The application to string/M theory in Sec. IV involves similar but slightly different considerations.) All models we study in which V_R is not stably trivial have the opposite property; for these models, the anomaly does depend on the gauge bundle.

D. Some additional facts

A few more facts will greatly simplify the study of examples in Sec. II E. As a preliminary, we ask the following question. In what sense does the anomaly that we have described involving the eta invariant generalize the more familiar "parity" anomaly that can be seen on an orientable manifold?

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[16]This notion is described in Ref. [7]. If P is a pin$^+$ structure over X, then there is a complementary pin$^+$ structure P' such that the monodromies of P' around any orientation-reversing loop in X are the negatives of those of P. "Flipping" X' exchanges P and P', and this maps $\exp(-\pi i \eta_R^{X'}/2)$ to its inverse, as stated in the text. To understand why the flip exchanges P and P', let ψ be a Fermi field on X and ψ' a Fermi field on X'. In gluing X to X' along their common boundary W, one sets $\psi'|_W = \not{n}\psi|_W$, where $\not{n} = \vec{\gamma} \cdot \vec{n}$, $\vec{n}$ being the unit normal vector to W in X or X' and $\vec{\gamma}$ the Dirac γ matrices. Since $\not{n}$ anticommutes with the γ matrices of W, acting with $\not{n}$ reverses the pin$^+$ structure of W. (A monodromy around an orientation-reversing loop in W is represented by a γ matrix of W, so acting with $\not{n}$ changes the sign of this monodromy.) Thus, for the gluing relation $\psi'|_W = \not{n}\psi|_W$ to make sense, X and X' must have opposite pin$^+$ structures along W. Reversing the pin$^+$ structure can be accomplished by changing the sign of all gamma matrices. This changes the sign of the Dirac operator, so it changes the sign of the eta invariant, and replaces $\exp(-\pi i \eta_R^{X'}/2)$ with its inverse.

[17]In general, two vector bundles V and V' over X are said to be stably equivalent if $V \oplus \mathfrak{o}^k$ is equivalent to $V' \oplus \mathfrak{o}^k$ for some k. Thus, for V to be stably trivial means that it is stably equivalent to a trivial bundle. If V and V' are stably equivalent and have a rank greater than the dimension of X, then they are actually equivalent. (This can be proved using a triangulation of X; V and V' are certainly isomorphic on the 0-skeleton of a triangulation and one proves inductively that an isomorphism over the r skeleton can be extended over the $r+1$ skeleton.) We will not need to know this in our analysis of the anomaly, but it will be useful later in the paper.

This question may be answered as follows (see Secs. 2 and 4 of [7] for more detail). If X is an orientable four-manifold, and $\mathcal{D}_R$ is the Dirac operator acting on R-valued spinors on X, then[18] the eta invariant η_R of $\mathcal{D}_R$ receives contributions only from zero modes of $\mathcal{D}_R$ and is congruent mod 4 to the index $\mathcal{I}$ of $\mathcal{D}_R$. This means that $\exp(-\pi i \eta/2)$ can be replaced by $\exp(-\pi i \mathcal{I}/2)$. Since $\mathcal{I}$ is an even integer, $\exp(-\pi i \mathcal{I}/2)$ can be written in a manifestly real (and thus T invariant) form as $(-1)^{\mathcal{I}/2}$. The anomaly is thus a mod 2 question, involving evenness or oddness of the integer $\mathcal{I}/2$, just as in more traditional treatments of the "parity" anomaly. The Atiyah-Singer index theorem can be used to write the integer $\mathcal{I}/2$ as a linear combination of gauge and gravitational instanton numbers. The anomaly $(-1)^{\mathcal{I}/2}$ depends on the gauge bundle if and only if (on a four-manifold X without boundary) the gauge theory instanton number can make an odd contribution to $\mathcal{I}/2$. Now let us compare this to the usual "parity" anomaly in three dimensions. This anomaly can be computed in terms of the spectral flow of a one-parameter family of three-dimensional Dirac operators, and that spectral flow is equal[19] to $\mathcal{I}/2$ on a four-manifold (the mapping torus) constructed from the given one-parameter family. So there is a gauge theory contribution to the "parity" anomaly in three dimensions if and only if there is a gauge theory contribution to $(-1)^{\mathcal{I}/2}$ in four dimensions.

So we can only get something beyond the standard "parity" anomaly if X is unorientable. This actually leads to a useful simplification. A real vector bundle V_R over an unorientable four-manifold X is stably trivial if and only if its Stieffel-Whitney classes vanish:

$$w_1(V_R) = w_2(V_R) = w_4(V_R) = 0. \tag{2.9}$$

(We need not consider w_3, since it vanishes whenever $w_1 = w_2 = 0$.) Necessity of this criterion is clear, since the Stieffel-Whitney classes are stable characteristic classes of a real vector bundle. Sufficiency is proved as follows starting with a triangulation of X. Vanishing of $w_1(V_R)$ ensures that V_R is stably trivial on the 1-skeleton of X; vanishing of $w_2(V_R)$ ensures that it is stably trivial on the 2-skeleton and [since $w_3(V_R) = 0$] the 3-skeleton; and finally, vanishing of $w_4(V_R)$ ensures that V_R is stably trivial on the 4-skeleton and thus on all of X. In the last step, we use the fact that X is unorientable. If X were orientable, then the integer-valued instanton number of V_R would be an obstruction to stable triviality of V_R on the 4-skeleton of X. However, with X unorientable, which ensures that $H^4(X, \mathbb{Z}) = \mathbb{Z}_2$, the instanton number of a real vector bundle V_R is only a $\mathbb{Z}_2$-valued invariant and can be measured by $w_4(V_R)$. (Instanton number is only a $\mathbb{Z}_2$-valued invariant on an unorientable manifold, because an instanton, when transported around an orientation-reversing loop, comes back as an anti-instanton.)

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[18]In brief, on an orientable manifold, the spectrum of $\mathcal{D}_R$ has a symmetry $\lambda \leftrightarrow -\lambda$ that follows from considerations of chirality and ensures that the nonzero modes make no net contribution to η_R. Let n_+ and n_- be the numbers of zero modes of $\mathcal{D}_R$ of positive or negative chirality; n_+ and n_- are both even because of a version of Kramers doubling. Moreover, $\eta_R = n_+ + n_-$, while $\mathcal{I} = n_+ - n_-$, so η_R is congruent to $\mathcal{I}$ mod 4.

[19]See Sec. 2 of [7].

E. Examples

For our first example, we take $G = \mathrm{U}(1)$ or equivalently SO(2).

Let $R_n^{(c)}$ be a complex one-dimensional representation of U(1) with charge n. A one-dimensional complex vector space can be viewed as a two-dimensional real vector space. So $R_n^{(c)}$ can be regarded as a real two-dimensional representation of SO(2); when we do this, we denote it as R_n.

A U(1) gauge field on a four-manifold X is a connection on a complex line bundle $\mathcal{L}$ over X. A field of charge n is a section of $\mathcal{L}^n$. $\mathcal{L}^n$ can be regarded as a rank 2 real vector bundle V_n over X. V_n is the real vector bundle associated to the representation R_n of SO(2). Now let us specialize to the case that X is unorientable and ask if V_n is stably trivial. As explained in Sec. II D, this is so if and only if the Stieffel-Whitney classes of V_n vanish.

In general, if $\mathcal{U}$ is any complex line bundle and V is the corresponding rank 2 real vector bundle, then the only nonzero Stieffel-Whitney class[20] of V is the mod 2 reduction of $c_1(\mathcal{U})$, the first Chern class of $\mathcal{U}$:

$$w_1(V) = 0 = w_i(V), \quad i > 2,$$
$$w_2(V) = c_1(\mathcal{U}) \bmod 2. \tag{2.10}$$

[Here $w_i(V) = 0$ for $i > 2$ since, in general, the Stieffel-Whitney classes $w_s(V)$ of a rank r real vector bundle V vanish for $s > r$.] In our case, this means that the only nonzero Stieffel-Whitney class of V_n is

$$w_2(V_n) = c_1(\mathcal{L}^n) = n c_1(\mathcal{L}) \bmod 2. \tag{2.11}$$

If n is even, then since the Stieffel-Whitney classes are mod 2 classes, this result implies that $w_2(V_n) = 0$. Hence, V_n is stably trivial, in general (that is, on any four-manifold X and for any gauge bundle), if n is even. For odd n, this is not true, since, in general, $n c_1(\mathcal{L})$ does not vanish mod 2.

From this we learn that a U(1) or SO(2) gauge theory in $2+1$ dimensions with a single fermion multiplet of even charge n is completely anomaly-free, even on an unorientable manifold. For odd n, this is not true; in fact, for odd n, this theory possesses the standard "parity" anomaly. (The "parity" anomaly is most often considered for charge 1, but it is a mod 2 effect and only depends on the charge mod 2; see Sec. 3.6 of [22] for a detailed explanation.)

More generally, let us consider a theory with gauge group U(1) or SO(2) and with Dirac fermions $\chi_1, \ldots, \chi_k$ of charges $n_1, \ldots, n_k$. Here the n_i are integers that (after possibly replacing some of the χ's with their charge conjugates) we can take to be non-negative. The corresponding real representation of SO(2) is $R = \oplus_{i=1}^k R_{n_i}$, and the corresponding real vector bundle is $V_R = \oplus_{i=1}^k V_{n_i}$. To compute the Stieffel-Whitney classes of V_R, we need the Whitney sum formula. If V is a real vector bundle over a d-dimensional manifold X, the total Stieffel-Whitney class of V is defined as $w(V) = 1 + w_1(V) + w_2(V) + \cdots + w_d(V)$. The Whitney sum formula says that if V and V' are two such bundles, then

$$w(V \oplus V') = w(V)w(V'). \tag{2.12}$$

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[20]Apart from the trivial class $w_0(V) = 1$.

This determines the Stieffel-Whitney classes of the direct sum $V \oplus V'$ in terms of those of V and V'. Repeated application of the formula determines the Stieffel-Whitney classes of a direct sum with more than two summands, such as $V_R = \oplus_{i=1}^{k} V_{n_i}$. We find

$$w_2(V_R) = \sum_i n_i c_1(\mathcal{L}) \bmod 2,$$

$$w_4(V_R) = \sum_{i<j} n_i n_j c_1(\mathcal{L})^2 \bmod 2. \quad (2.13)$$

In general, neither $c_1(\mathcal{L})$ nor $c_1(\mathcal{L})^2$ vanishes mod 2. So the condition for V_R to be stably trivial is

$$\sum_{i=1}^{k} n_i = 0 \bmod 2,$$

$$\sum_{1\leqslant i<j\leqslant k} n_i n_j = 0 \bmod 2. \quad (2.14)$$

Since these are mod 2 conditions, they receive contributions only from those n_i that are odd. Let y be the number of odd n_i's. The first condition in Eq. (2.14) says that y is even, and the second says that $y(y-1)/2$ is even. Taken together, these conditions say that y is divisible by 4:

$$y = 0 \bmod 4. \quad (2.15)$$

V_R is stably trivial if and only if this is true.

That y should be even to avoid a gauge anomaly is what one would expect from the standard "parity" anomaly. The requirement that y should be divisible by 4 is a stronger condition that goes beyond what one finds on an orientable manifold. The above argument shows only that a U(1) or SO(2) gauge theory is completely anomaly-free if y is divisible by 4. However, the converse—there is an anomaly if y is not divisible by 4—can be shown by an explicit example that is described momentarily. Thus, in trying to define a U(1) gauge theory on an unorientable manifold, we really do find a condition stronger than the standard "parity" anomaly: The number of fermion multiplets of odd charge must be not just even but divisible by 4.

To demonstrate an anomaly when y is not divisible by 4, we should find a four-manifold X and a U(1) bundle $\mathcal{L}$ over X such that, for a representation R with y congruent to 2 mod 4, $\exp(-\pi i\eta_R/2)$ is different from what it would be if $\mathcal{L}$ is replaced by a trivial line bundle. For this, we take $X = \mathbb{RP}^4$ (see Appendix C of [7] for background to the following). On X, there are two pin$^+$ structures, say P and P'. They can be distinguished by the eta invariant of the Dirac operator (coupled to gravity only), which satisfies $\exp(-\pi i\eta/2) = \exp(\pm 2\pi i/16)$, where the sign depends on the pin$^+$ structure; we choose P to be the pin$^+$ structure with $\exp(-\pi i\eta/2) = \exp(-2\pi i/16)$. Since $\pi_1(\mathbb{RP}^4) = \mathbb{Z}_2$, there is over $\mathbb{RP}^4$ a nontrivial real line bundle ε, and the relation between P' and P is $P' = P \otimes \varepsilon$. The Dirac operator $\mathcal{D}_\varepsilon$ for a Majorana fermion, still with pin$^+$ structure P, but coupled to ε, is the same as the Dirac operator with pin$^+$ structure P', but coupled to gravity only. So, writing η_ε for the eta invariant of $\mathcal{D}_\varepsilon$, it satisfies

$$\exp(-\pi i\eta_\varepsilon/2) = \exp(+2\pi i/16). \quad (2.16)$$

If we take the gauge bundle $\mathcal{L}$ to be trivial, then V_R is a trivial bundle of rank $2k$ and, hence, for pin$^+$ structure P, $\exp(-\pi i\eta_R) = \exp(-2k \cdot 2\pi i/16)$. Now, again using the fact that $\pi_1(\mathbb{RP}^4) = \mathbb{Z}_2$, take $\mathcal{L}$ to be a flat but nontrivial complex line bundle over $\mathbb{RP}^4$. Then $\mathcal{L}^2$ is trivial and so $\mathcal{L}^n$ is trivial or isomorphic to $\mathcal{L}$ for even or odd n. Likewise, the corresponding rank 2 real vector bundle V_n is a trivial bundle of rank 2 or a copy of $\varepsilon \oplus \varepsilon$ depending on whether n is even or odd. So $V_R = \oplus_{i=1}^{k} V_{n_i}$ is the direct sum of a trivial real bundle of rank $2k - 2y$ and $2y$ copies of ε. Now using (2.16), for the gauge bundle $\mathcal{L}$ we get $\exp(-\pi i\eta_R/2) = \exp[-(2k - 4y)2\pi i/16]$. The condition for this to coincide with the result $\exp(-2k \cdot 2\pi i/16)$ that we get when $\mathcal{L}$ is trivial is simply that y should be divisible by 4. So there is indeed an anomaly when that condition is not satisfied.

In Sec. III, we will also want to know what happens for $G = \mathbb{Z}_2$. In this case, there is only one nontrivial irreducible representation, a one-dimensional real representation R_0 on which the nontrivial element of $\mathbb{Z}_2$ acts as -1. A $\mathbb{Z}_2$ bundle over a four-manifold X associates to R_0 a real line bundle that we will call α. Now let the representation R consist of the direct sum of k copies of R_0. The corresponding vector bundle V_R is the direct sum of k copies of α. Its total Stieffel-Whitney class is

$$w(V_R) = w(\alpha)^k = [1 + w_1(\alpha)]^k$$
$$= 1 + kw_1(\alpha) + \frac{k(k-1)}{2}w_1(\alpha)^2$$
$$+ \frac{k(k-1)(k-2)}{3!}w_1(\alpha)^3$$
$$+ \frac{k(k-1)(k-2)(k-3)}{4!}w_1(\alpha)^4. \quad (2.17)$$

This equals 1 for all X and all α if and only if the coefficients of positive powers of $w_1(\alpha)$ are all even. This is so precisely if k is a multiple of 8. We thus learn that V_R is stably trivial if and only if k is a multiple of 8. If k is a multiple of 8, it follows that the theory has no gauge anomaly. If k is not a multiple of 8, there can be a gauge anomaly, as one can see again from the example of $X = \mathbb{RP}^4$. In this example, with P as before, $\exp(-\pi i\eta_R/2)$ is equal to $\exp(-2k\pi i/16)$ if α is trivial, but it equals $\exp(+2k\pi i/16)$ if $\alpha = \varepsilon$. These are only equal if k is a multiple of 8.

In Sec. III B 6, we will need further examples of the following sort. Let G be any subgroup of SO(p), for some p. We view the vector representation **p** of SO(p) as a real representation R_0 of G. Because G is a subgroup of the connected group SO(p) [rather than O(p)], the gauge bundle V_{R_0} associated to this representation satisfies

$$w_1(V_{R_0}) = 0. \quad (2.18)$$

Its higher Stieffel-Whitney classes may not vanish, and Majorana fermions valued in the representation R_0 may be anomalous. However, let $R = R_0 \oplus R_0 \oplus R_0 \oplus R_0$ be the direct sum of four copies of R_0. The corresponding real vector bundle is likewise a fourfold direct sum: $V_R = V_{R_0} \oplus V_{R_0} \oplus V_{R_0} \oplus V_{R_0}$. Using (2.18) and the Whitney sum formula, one finds that the Stieffel-Whitney classes of V_R vanish in four dimensions. Accordingly, V_R is always stably trivial and

$2 + 1$-dimensional fermions valued in the representation R have no gauge anomaly.

In Sec. III B 6, we will make use of two cases of this construction. For the first case, we take $G = \mathrm{U}(2)$, and we take R_0 to be the two-dimensional complex representation of $\mathrm{U}(2)$, viewed as a four-dimensional real representation. For the second case, we take G to be the maximal torus $\mathrm{U}(1) \times \mathrm{U}(1)$ of $\mathrm{U}(2)$, with the same R_0.

In either of these examples, if we take only two copies of R_0, we would have an anomaly. In fact, we would have an anomaly even if we restrict the gauge group to the $\mathrm{U}(1)$ subgroup of $\mathrm{U}(2)$ (or of its maximal torus) embedded as

$$\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.19}$$

Under this subgroup, Majorana fermions valued in R_0 are equivalent to a pair of complex fermions of respective charges 1 and 0. So it we take two copies of R_0, the charges are 1, 1, 0, and 0. This is a $\mathrm{U}(1)$ theory with $y = 2$, and it is anomalous, as we have seen.

F. The partition function

What we have found can potentially be applied in condensed-matter physics in two ways.

If G and R are such that $\exp(-\pi i \eta_R/2)$ depends on the gauge bundle in a nontrivial way, then it is the partition function of a four-dimensional topological field theory with G symmetry (defined on pin^+ manifolds and coupled to a background gauge field). If this theory is formulated on a manifold X with boundary W, then it has a boundary state in which Majorana fermions in the representation R propagate on W. The partition function of the combined system consisting of the boundary fermions and the bulk topological field theory is, as we have explained,

$$Z_X = |\mathrm{Pf}(\mathcal{D})| \exp\left(-\pi i \eta_R^X/2\right). \tag{2.20}$$

(For a reason that will be clear in a moment, we here denote the eta invariant as η_R^X and not just as η_R.)

If instead $\exp(-\pi i \eta_R^X/2)$ is always independent of the gauge bundle, then a system of Majorana fermions in $2 + 1$ dimensions, transforming in the representation R of G (and transforming with a $+$ sign under time reversal), has only the usual gravitational anomaly that the same number of neutral fermions would have. It is a conceivable gapless boundary state of a topological superconductor (with $\nu = \dim R$, the real dimension of R) and can be the starting point in constructing gapped symmetry-preserving boundary states, as we will discuss in Sec. III. If the dimension of R is divisible by 16, the gravitational anomaly is also absent, and this system is a completely consistent and anomaly-free $2 + 1$-dimensional field theory, even on unorientable manifolds.

If the gauge bundle on W can be extended over X, then the same formula (2.20) serves as the partition function for the combined system consisting of a bulk topological superconductor and the boundary fermions. The fact that $\exp(-\pi i \eta_R^X/2)$ is independent of the gauge bundle on a manifold without boundary means, by virtue of the gluing formula (2.8), that the choice of how the gauge bundle is extended over X does not matter.

If the gauge bundle on W cannot be extended over X, how can we describe the partition function of the combined system consisting of the bulk topological superconductor on X coupled to boundary fermions on W? It is easier to explain what to do if there is *some* four-manifold X' over which the gauge bundle and pin^+ structure of W can be extended, so we begin with this case. Let $\eta_R^{X'}$ be the eta invariant of the Dirac operator on X'. For a further simplification, assume first that the dimension of R is divisible by 16, so that the fermion system on W is completely anomaly-free. Then X plays no role, and since the partition function depends only on W, we denote it as Z_W. For Z_W we can use the same formula as in Eq. (2.20), but with η_R^X replaced by $\eta_R^{X'}$ on the right-hand side:

$$Z_W = |\mathrm{Pf}(\mathcal{D})| \exp\left(-\pi i \eta_R^{X'}/2\right). \tag{2.21}$$

The usual arguments show that this does not depend on the choice of the pin^+ manifold X' and the extension of the gauge bundle over X'.

What if $\nu = \dim R$ is not a multiple of 16? Then the bulk topological superconductor on X is nontrivial, and (2.21) cannot be correct; it describes a topological superconductor with boundary W whose worldvolume is X' rather than X! To fix the situation, let us recall the four-manifold X^* without boundary that is built by gluing X and X' along their common boundary (Fig. 2). Let $\eta_0^{X^*}$ be the eta invariant of the Dirac operator on X^* coupled to gravity only. Then the appropriate generalization of (2.21) for the partition function is

$$Z_{W;X} = |\mathrm{Pf}(\mathcal{D})| \exp\left(-\pi i \eta_R^{X'}/2\right) \exp\left(-\nu \pi i \eta_0^{X^*}/2\right). \tag{2.22}$$

We denote this partition function as $Z_{W;X}$ because—since the bulk topological superconductor on X is now nontrivial—it does depend on X, not just on W. The gluing theorem for the eta invariant can be used to show that this formula does not depend on the choice of the pin^+ manifold X' and the extension over X' of the gauge bundle and pin^+ structure. [Equation (2.22) is analogous to Eq. (3.41) in Ref. [22], where X was assumed to be orientable and a coupling to electromagnetism was incorporated.]

However, it may happen that a suitable X' does not exist, because W, with its G bundle, may represent a nontrivial element of the group Γ that classifies up to cobordism three-dimensional pin^+ manifolds W endowed with a G bundle. Note that W is always trivial in cobordism if we ignore the G bundle (since it is given as the boundary of the worldvolume X of the topological superconductor) but it may be that W together with its G bundle is not a boundary.[21] The physical meaning of Γ is that [34] $\mathrm{Hom}(\Gamma, \mathrm{U}(1))$ parametrizes a family of invertible three-dimensional topological field theories defined on a pin^+ manifold W with a G bundle. Concretely, suppose we are given a homomorphism $\varphi : \Gamma \to \mathrm{U}(1)$. The associated topological field theory $\mathcal{T}$ may be described as follows. Let W be a three-manifold with pin^+ structure and a G bundle. Then W has a class $[W]$ in Γ and the partition function of the theory $\mathcal{T}$ on W is $\varphi([W])$. This is relevant because when such theories exist, we should not expect to get a unique

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[21] An obstruction is given by the mod 2 index of the Dirac operator on W with values in any real representation of G.

answer for the theory of fermions on W in the representation R of G coupled to the topological superconductor on X. We could always modify any possible answer by tensoring by a three-dimensional topological field theory that depends only on the boundary data.

Actually, since in our application W is a boundary if we forget the G bundle, Γ is not quite the right group to consider. We should replace Γ by its subgroup Γ' that classifies up to cobordism pairs consisting of a pin$^+$ manifold W with a G bundle, such that W is a boundary if we forget the G bundle. To explain how to proceed, suppose, for example, that $\Gamma' = \mathbb{Z}_2$. Let W_0, together with some G bundle, generate Γ'. There is then a three-dimensional topological field theory $\mathcal{T}$, defined on pin$^+$ manifolds W with G bundle, such that W is a boundary if we forget the G bundle, that assigns the value -1 to W_0 and is trivial if W with its G bundle is a boundary.

Pick a pin$^+$ manifold X_0 with boundary W_0. (By hypothesis, the G bundle over W_0 does not extend over X_0.) In this situation, we should not expect to get a unique answer for $Z_{W_0;X_0}$, because any answer could be modified by tensoring with $\mathcal{T}$. However, let $X_0 \sqcup X_0$ be the disjoint union of two copies of X_0, with boundary $W_0 \sqcup W_0$. Then $W_0 \sqcup W_0$, together with its gauge bundle, is a boundary (we have assumed that $\Gamma' = \mathbb{Z}_2$ so two copies of W_0 make a boundary), so the formula (2.22) can be used to determine the partition function $Z_{W_0 \sqcup W_0;X_0 \sqcup X_0}$. We interpret this partition function as $Z^2_{W_0;X_0}$ (the square of $Z_{W_0;X_0}$) and we make some choice of the square root to determine $Z_{W_0;X_0}$. Now let W be any three-dimensional pin$^+$ manifold with a G bundle. Then since W_0 generates Γ', either W or $W \sqcup W_0$ is (together with its G bundle) the boundary of some X'. Hence, Eq. (2.22) can be used to determine either $Z_{W;X}$ or $Z_{W \sqcup W_0;X' \sqcup X_0}$, which we interpret as the product $Z_{W;X} Z_{W_0;X_0}$. $Z_{W_0;X_0}$ has already been determined, so in any case we arrive at a result for $Z_{W;X}$. Thus, this gives a complete description of the partition function in general.

In this construction, we made an arbitrary choice of sign of $Z_{W_0;X_0}$. The two possible choices of sign will give results that differ by tensoring with the purely $2 + 1$-dimensional invertible topological field theory $\mathcal{T}$. We also assumed that $\Gamma' = \mathbb{Z}_2$. In general, Γ' is always a finite Abelian group. The above reasoning can be modified in a fairly obvious way for any Γ'.

III. GAPPED T-INVARIANT BOUNDARY STATES OF A TOPOLOGICAL SUPERCONDUCTOR

In this section, we describe gapped T-invariant boundary states of a topological superconductor with even ν. (The methods we use do not suffice to construct such states for odd ν.) The class of models we consider and most of the methods for analyzing them were already presented in Ref. [22] (boundary states for topological superconductors are treated in Sec. 6 of that paper, following a similar treatment of topological insulators earlier in the paper). We can add something here because of having a more complete knowledge of which models are anomaly-free. However, we will be brief because many points were already explained in Ref. [22]. See also related analysis in Refs. [4,5,13–21].

We begin with an overview and then describe some properties of models for different values of ν. As in Ref. [22],

it turns out that the main results for all values of ν can be described using considerations of weak coupling.

A. Overview

We consider a bulk topological superconductor with a given even ν. The standard boundary state would consist of ν massless Majorana fermions, all transforming with a $+$ sign under T. As before, we denote the worldvolume of the topological superconductor as X and its boundary as W.

To construct a new type of boundary state, we postulate the appearance on W of an emergent gauge symmetry with gauge group G. We also assume that on W there propagate ν Majorana fermions, transforming with a $+$ sign under T and in some ν-dimensional real representation R of G. The representation is chosen to have no gauge anomaly according to a criterion explained in Sec. II. Thus, the only anomaly is the standard gravitational anomaly of ν Majorana fermions that transform the same way under T. This is then a conceivable (gapless) boundary state of a topological superconductor with the given value of ν.

More generally, we could pick a positive integer s and a pair of real representations R and R' of G of respective dimensions $\nu + s$ and s. Then we introduce Majorana fermions Ψ and Ψ' transforming respectively with a $+$ sign or a $-$ sign under T and respectively in the representations R and R' of G. If the anomalies cancel between[22] R and R', then this gives another conceivable gapless boundary state of the same topological superconductor. In the absence of the gauge symmetry, pairs of Majorana fermions transforming oppositely under T can combine and get a mass, but R and R' can be chosen so that the gauge symmetry prevents this.

Now we introduce two sets of charged scalar fields w and ϕ, chosen to make possible two phases with the following properties.

In one phase, the fields w have expectation values that completely break the emergent gauge symmetry while preserving T symmetry. For example, if $G = U(1)$, w can consist of a single complex scalar field of charge 1, with a suitable transformation law under T. In the phase in which w has an expectation value, the emergent gauge symmetry is completely broken and disappears at low energy. Since we also assume that ϕ is massive in this phase, it follows that in this phase, the only gapless modes are ν Majorana fermions transforming with a $+$ sign under T (or $\nu + s$ transforming with a $+$ sign and s transforming with a $-$ sign, for some s; once the gauge symmetry is completely broken, such extra modes can combine and get masses). Thus, this phase reduces at low energies to the standard boundary state of a topological superconductor.

The second phase, in which ϕ gets an expectation value, is supposed to be gapped and T conserving. Assuming that the gap should be visible semiclassically, this means that the expectation value of ϕ has to break the gauge symmetry

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[22]According to the analysis in Sec. II, this cancellation means that if X is a four-dimensional pin$^+$ manifold without boundary, and η_R and $\eta_{R'}$ are the eta invariants of the R- or R'-valued Dirac operator on X, then $\exp[-\pi i(\eta_R - \eta_{R'})/2]$ is independent of the choice of G bundle over X.

down to a finite subgroup. An unbroken subgroup of positive dimension would lead to a gapless spectrum in perturbation theory. The possibility of an unbroken subgroup of positive dimension that nevertheless leads to a gapped spectrum through nonperturbative effects is the topic of Sec. III B 6. In the present section, we assume that $\langle \phi \rangle$ breaks G down to a finite subgroup $\mathcal{K}$. It will be clear in a moment that this subgroup cannot be trivial.

To get a gapped state, ϕ must also have Yukawa couplings to the fermions, so that the fermions acquire mass from the expectation value of ϕ. A linear Yukawa coupling of scalar fields ϕ_a, $a = 1, \ldots, t$ to Majorana fermions ψ_i, $i = 1, \ldots, \nu$ takes the general form

$$L_{\text{Yuk}} = \sum_{i,j=1}^{\nu} \sum_{a=1}^{t} m_{ij}^a \overline{\psi}_i \psi_j \phi_a, \tag{3.1}$$

for some constants m_{ij}^a. The transformations of ϕ_a under G and under T and the constants m_{ij}^a must be chosen so that this coupling (along with possible nonlinear couplings of ϕ to the fermions) preserves the symmetries and gives mass to all fermions in the phase with $\langle \phi \rangle \neq 0$.

One general comment is that in a phase in which the fermions get a mass from the expectation value of ϕ, that expectation value can never be invariant under the microscopic time-reversal transformation that acts on the fermions as

$$\mathsf{T}(\psi(t,\vec{x})) = \gamma_0 \psi(-t,\vec{x}), \tag{3.2}$$

regardless of how this symmetry is assumed to act on ϕ. Indeed, any fermion bare mass term is odd under T acting as in Eq. (3.2). Since L_{Yuk} is by hypothesis T invariant, the expectation value of ϕ will necessarily be also odd under T. The time-reversal transformation $\mathcal{T}$ that is a symmetry of the gapped phase is therefore not the microscopic T that acts on the fermions as in Eq. (3.2), but the combination of T with a gauge symmetry, which we call $\mathsf{K}^{1/2}$:

$$\mathcal{T} = \mathsf{T}\mathsf{K}^{1/2}. \tag{3.3}$$

For $\langle \phi \rangle$ to be $\mathcal{T}$ invariant, given that it is odd under T, it must be odd under $\mathsf{K}^{1/2}$.

From a microscopic point of view, one has $\mathsf{T}^2 = (-1)^\mathsf{F}$. Therefore,

$$\mathcal{T}^2 = (-1)^\mathsf{F}\mathsf{K}. \tag{3.4}$$

Note that K is an unbroken symmetry, since it leaves fixed the expectation value of ϕ (by contrast, $\mathsf{K}^{1/2}$ is not an unbroken gauge symmetry as the expectation value of ϕ is odd under $\mathsf{K}^{1/2}$). We should note that, since K is an element of the group G of emergent gauge symmetries, any state of a compact sample has $\mathsf{K} = 1$ and hence satisfies the standard relation $\mathcal{T}^2 = (-1)^\mathsf{F}$, as one would expect microscopically. However, individual quasiparticles can have $\mathsf{K} \neq 1$, and the relation (3.4) is important in understanding their properties.

The definition of $\mathcal{T}$ is not unique, because in Eq. (3.3), we could multiply $\mathsf{K}^{1/2}$ by any unbroken gauge symmetry S (that is, any $\mathsf{S} \in G$ that leaves fixed $\langle \phi \rangle$) without changing the fact that $\mathcal{T}$ is a symmetry of the phase with $\langle \phi \rangle \neq 0$. However, it turns out that no matter what choice of $\mathcal{T}$ we make, K is always nontrivial if the bulk topological superconductor is nontrivial, that is, if ν is not a multiple of 16. This is not immediately

obvious because, as K leaves fixed the expectation value of ϕ, one can ask if it is possible to have $\mathsf{K} = 1$. We may argue as follows. If $\mathsf{K} = 1$, then $\mathsf{K}^{1/2}$ generates a $\mathbb{Z}_2$ subgroup of the gauge group G. All fermion fields are therefore either even or odd under $\mathsf{K}^{1/2}$. Let ϕ_0 be the component of ϕ that has an expectation value; thus, ϕ_0 is odd under $\mathsf{K}^{1/2}$. A $\mathsf{K}^{1/2}$-invariant Yukawa coupling that involves ϕ_0 has to take the form

$$\psi_e \psi_o \phi_0, \tag{3.5}$$

where ψ_e is even under $\mathsf{K}^{1/2}$ and ψ_o is odd. If all fermions are supposed to gain mass in this way, the number of fermions even under $\mathsf{K}^{1/2}$ must be the same as the number of fermions that are odd. However, we have assumed that the full collection of fermions has no gauge anomaly under the full G symmetry, so certainly there is no anomaly under the $\mathbb{Z}_2$ subgroup of G generated by $\mathsf{K}^{1/2}$. According to the analysis of anomaly-free $\mathbb{Z}_2$ gauge theories in Sec. II E, to avoid a gauge anomaly, the number of fermions odd under $\mathbb{Z}_2$ must be a multiple of 8; allowing for an equal number of $\mathbb{Z}_2$-even fermions, the total number of fermions is a multiple of 16 and the bulk topological superconductor is trivial.

The fact that the unbroken gauge symmetry K is nontrivial (if the bulk topological superconductor is nontrivial) means that the boundary state, though gapped, is topologically nontrivial. It supports a group $\mathcal{K}$ of emergent gauge symmetries that at least contains a cyclic subgroup generated by K.

In bulk, a topological superconductor has a symmetry group $\mathcal{T}$ generated by T; it is isomorphic to $\mathbb{Z}_4$, the four elements being T, $\mathsf{T}^2 = (-1)^\mathsf{F}$, $\mathsf{T}^3 = \mathsf{T}(-1)^\mathsf{F}$, and $\mathsf{T}^4 = 1$. The boundary state that we have described has a group $\mathcal{K}$ of emergent unbroken gauge symmetries. However (assuming that the bulk superconductor is topologically nontrivial), the full symmetry group is not a product $\mathcal{H} = \mathcal{T} \times \mathcal{K}$. Rather, it is a nontrivial extension

$$1 \to \mathcal{K} \to \mathcal{H} \to \mathcal{T} \to 1. \tag{3.6}$$

The extension is nontrivial because we have shown above that no matter how we lift $\mathsf{T} \in \mathcal{T}$ to $\mathcal{T} \in \mathcal{H}$, we get $\mathcal{T}^2 = (-1)^\mathsf{F}\mathsf{K}$ with K a nontrivial element of $\mathcal{K}$. In other words, there is no way to lift $\mathcal{T}$ to a subgroup of $\mathcal{H}$.

B. Examples based on U(1)

1. Generalities

Following [22], to construct concrete examples, we will take $G = \mathrm{U}(1)$ or equivalently $\mathrm{SO}(2)$. We denote the $\mathrm{U}(1)$ gauge field as a and its field strength as $f = \mathrm{d}a$. We consider only the simplest examples for different values of ν. It is convenient to work with a basis of Dirac fermions; that is, instead of considering a pair ψ_1, ψ_2 of Majorana fermions transforming in a two-dimensional real representation of $\mathrm{SO}(2)$, we combine them to a Dirac fermion $\chi = (\psi_1 + i\psi_2)/\sqrt{2}$ and its adjoint $\overline{\chi}$. They have equal and opposite $\mathrm{U}(1)$ charge. Reversing the sign of ψ_2 would exchange χ and $\overline{\chi}$, so we can assume that χ has non-negative $\mathrm{U}(1)$ charge.

As in Ref. [22], one does want not the macroscopic theory to have an exact symmetry that has no microscopic origin, so one does not want $j_\mu = \epsilon_{\mu\nu\lambda} f^{\nu\lambda}/4\pi$ (where $\epsilon_{\mu\nu\lambda}$ is the Levi-Civita tensor of the boundary) to be a conserved current. This is prevented by adding a monopole operator to the Lagrangian

or Hamiltonian. In all models that we will consider, there is a symmetry-preserving monopole operator of unit magnetic charge that can be added.

For $G = \mathrm{U}(1)$, we can take w to consist of a single complex scalar field of charge 1, whose expectation value would completely break the gauge theory. It will also suffice, as in Ref. [22], to take ϕ to consist of a single complex scalar field. Expanding w and ϕ in terms of real scalar fields as $w = (w_1 + i w_2)/\sqrt{2}$, $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$, we take[23] T to act by $\mathsf{T}(w_i) = w_i$, $\mathsf{T}(\phi_i) = -\phi_i$. We describe this by saying that w is T-even and ϕ is T-odd. The fact that the w_i are T-even means that the expectation value $\langle w \rangle$ can break the emergent $\mathrm{U}(1)$ gauge symmetry completely while conserving T. The fact that the ϕ_i are T-odd ensures T-invariance of the Yukawa coupling (3.1).

Now suppose that ϕ has charge p under $\mathrm{U}(1)$. Then to ensure that the expectation value of ϕ is invariant under $T = \mathsf{TK}^{1/2}$, we can take $\mathsf{K}^{1/2}$ to be a gauge transformation by $\exp(2\pi \mathrm{i}/2p)$. K is then a gauge transformation by $\exp(2\pi \mathrm{i}/p)$ and generates the group $\mathcal{K} \cong \mathbb{Z}_p$ of unbroken gauge symmetries.

If p is even and χ is a Dirac fermion of charge $p/2$, then a Yukawa coupling

$$\sum_{a,b=1}^{2} \epsilon^{ab} \chi_a \chi_b \overline{\phi} + \text{H.c.} \tag{3.7}$$

(here a and b are spinor indices) is $\mathrm{U}(1)$ invariant and T invariant and gives χ a mass. Similarly, if χ and χ' are two Dirac fermions with charges t and $p - t$, then they can receive a mass from a coupling

$$\sum_{a,b=1}^{2} \epsilon^{ab} \chi_a \chi_b' \overline{\phi} + \text{H.c.} \tag{3.8}$$

We will construct models by combining these ingredients. It is also possible to introduce pairs of fermions that receive masses from couplings cubic or higher order in ϕ.

If ϕ has charge p, then a vortex in which the phase of ϕ twists by 2π has a flux $\int f/2\pi = 1/p$. A monopole operator shifts the flux by an integer, so after adding a monopole operator of unit charge to the action, the vorticity is conserved mod p. In fact, conservation of vorticity mod p is a feature of low-energy $\mathbb{Z}_p$ gauge theory.

2. Monopole operators

The assertion that vorticity is conserved precisely mod p depends on existence of a charge 1 monopole operator that is a Kramers singlet boson of spin 0 and hence can be added to the action or Hamiltonian. Such an operator exists for any model that lacks the generalized "parity" anomaly of Sec. II E, but the point is somewhat subtle. This question has been addressed in Refs. [4,5], in a slightly different language, for Dirac fermions

of charge 1, and in Ref. [22] for fermions of even charge as well as some other cases (the latter analysis was made mainly in the context of a topological insulator rather than superconductor).

We first consider a Dirac fermion χ of even charge $2s$, which makes no contribution to the anomaly so it can be considered by itself. We determine the quantum numbers of a charge 1 monopole operator in a standard way via radial quantization on S^2 [35]. In a sector with $\int_{S^2} f/2\pi = 1$, the field χ has $2s$ zero modes ζ_a, $a = 1, \ldots, 2s$ of spin $(2s-1)/2$ and $\mathrm{U}(1)$ charge $2s$, while its adjoint $\chi^{\dagger}$ has the same number of zero modes $\zeta_a^{\dagger}$ of the same spin and charge $-2s$. One can choose $4s$ real linear combinations of these modes, but those real linear combinations are not T invariant. That is because of the factor of γ_0 in the transformation of a Majorana fermion under time reversal, which we have taken to be

$$\mathsf{T}\psi(t,\vec{x}) = \gamma_0 \psi(-t,\vec{x}). \tag{3.9}$$

However, it is possible to define a modified T transformation under which the real zero modes are all invariant. This is

$$T' = \mathsf{T} \exp(-2\pi \mathrm{i} \mathsf{q}/8s), \tag{3.10}$$

where q is the $\mathrm{U}(1)$ charge generator. The basis for this statement is as follows. The zero modes of χ and $\chi^{\dagger}$ have respectively positive and negative chirality along S^2. The chirality in the 2d sense is measured by the operator $\mathrm{i}\gamma_1\gamma_2$. In $2+1$ dimensions, one can take the Dirac γ matrices to be 2×2 real matrices that obey $\gamma_0\gamma_1\gamma_2 = 1$, so actually $\mathrm{i}\gamma_1\gamma_2 = -\mathrm{i}\gamma_0$. Thus, the γ_0 that appears in the transformation law (3.9) acts as i on zero modes of χ and as $-\mathrm{i}$ on zero modes of $\chi^{\dagger}$. The factor $\exp(-2\pi \mathrm{i} \mathsf{q}/8s)$ in the definition of T' compensates for this and ensures that T' acts on fermion zero modes only by complex conjugation, leaving real zero modes invariant.

The total number of χ and $\chi^{\dagger}$ zero modes is $4s$. Quantizing $4s$ real fermion modes gives a Hilbert space H of dimension 2^{2s}. An antiunitary operator T' that commutes with all of the real fermion modes acts on this space as

$$(T')^2 = (-1)^s. \tag{3.11}$$

This reflects the fact that the spinor representation of $\mathrm{Spin}(4s)$ is pseudoreal or real for odd or even s.

The state in H of lowest $\mathrm{U}(1)$ charge is a state $|\downarrow\rangle$ that is annihilated by the $\zeta_i^{\dagger}$; its $\mathrm{U}(1)$ charge is $-2s^2$ (as in a classic analysis [36], this follows from T invariance, which implies that the state $|\uparrow\rangle = \zeta_1\zeta_2\cdots\zeta_{2s}|\downarrow\rangle$ of highest $\mathrm{U}(1)$ charge must have opposite charge from $|\downarrow\rangle$). The state $|\downarrow\rangle$ is the unique state in H of lowest charge, so it transforms under rotations of S^2 with spin zero. The $\mathrm{U}(1)$-invariant states in H are obtained by acting s times with the modes ζ_a: They have the form $|a_1 a_2 \cdots a_s\rangle = \zeta_{a_1}\zeta_{a_2}\cdots\zeta_{a_s}|\downarrow\rangle$. Since the modes ζ_a have spin $(2s-1)/2$, these states have spin $s/2$ mod 1. For a $\mathrm{U}(1)$-invariant state, T' coincides with T, so Eq. (3.11) implies that these states obey $\mathsf{T}^2 = (-1)^s$. If s is even, these states are Kramers singlet bosons, and a real linear combination of the corresponding monopole operators can be added to the action or Hamiltonian. If s is odd, the monopole operators $M_{a_1 a_2 \cdots a_s}$ corresponding to these states are Kramers doublet fermions. However, we can get a charge 1 monopole operator that is a Kramers singlet boson by multiplying $M_{a_1 a_2 \cdots a_s}$ by

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[23]The following transformations are the transformations under CT given in Eqs. (3.3)–(3.6) of [22]. What we call T in the present paper was called CT in Ref. [22], because that paper focused primarily on topological insulators rather than superconductors.

an ordinary gauge-invariant local operator that is a Kramers doublet fermion, such as $\overline{w}^{2s}\chi$. So in all cases, there is a charge 1 monopole operator that is a boson and a Kramers singlet.

This argument shows that one or more Dirac fermions of even U(1) charge produce no anomaly in the quantization of the monopole operators, in the sense that those operators have the same quantum numbers as ordinary local operators. So it suffices to consider the case of Dirac fermions $\chi_1, \ldots, \chi_y$ that all have odd charges $n_1, \ldots, n_y$. We cannot expect to make the argument in quite the same way, because individual Dirac fermions with odd charge do contribute anomalies, and we have to combine them to demonstrate anomaly cancellation. In imitating the above argument, one's first thought might be to replace $2s$ in the definition of T' by the fermion charge n_i. The trouble with this is that, in general, n_i depends on i. In demonstrating anomaly cancellation between different χ_i, one really would like to use the same T' for all of them. It turns out that there is a choice that works nicely, given that the n_i are all odd:

$$T' = T \exp(-2\pi i q/4). \tag{3.12}$$

With this choice, the zero modes of χ_i are T'-even if n_i is congruent to 1 mod 4, and T'-odd if n_i is congruent to 3 mod 4. So let us rearrange the n_i as integers $u_1, \ldots, u_p$ that are congruent to 1 mod 4 and integers $v_1, \ldots, v_{p'}$ that are congruent to 3 mod 4. (Thus, $p + p' = y$.) The total number of real fermion zero modes that are T'-even is $a_+ = 2\sum_i u_i$, and the number that are T'-odd is $a_- = 2\sum_j v_j$. When we quantize all the zero modes, we get a Hilbert space H of dimension $2^{\sum_i u_i + \sum_j v_j}$. The state of lowest U(1) charge in this Hilbert space is a state $|\downarrow\rangle$ annihilated by all modes of the $\chi_i^\dagger$; its charge is

$$q = -\frac{1}{2}\left(\sum_i u_i^2 + \sum_j v_j^2\right). \tag{3.13}$$

This is an integer if and only if the total number y of Dirac fermions is even. If y is odd, then q is a half-integer and there is no U(1)-invariant state in the whole Hilbert space (boson and fermion modes of nonzero energy have integer q and do not help). This is a manifestation of the usual "parity" anomaly. Let us assume henceforth that y is even. This implies that $a_+ + a_-$ and $a_+ - a_-$ are multiples of 4. When we quantize a_+ T'-even real fermion modes and a_- T'-odd ones, with both of these numbers multiples of 4, we get a Hilbert space H on which[24]

$$(T')^2 = (-1)^t, \tag{3.14}$$

where

$$t = \frac{a_+ - a_-}{4} = \frac{1}{2}\left(\sum_i u_i - \sum_j v_j\right). \tag{3.15}$$

Now we have to find a U(1)-invariant state of monopole charge 1. For completely general odd integers $n_1, \ldots, n_y$, there is no such state in the Hilbert space H obtained by quantizing the fermion zero modes. However, by using the charge 1 scalar field w, we can make the U(1)-invariant state $|\Psi\rangle = w^{-q}|\downarrow\rangle$. In a field of monopole charge 1, w has spin-$\frac{1}{2}$, so the state $|\Psi\rangle$ has spin $q/2$ mod 1. Since w is a Kramers singlet under T and has charge 1, it is a Kramers doublet of T'. From Eq. (3.14), $(T')^2|\downarrow\rangle = (-1)^t|\downarrow\rangle$, so $(T')^2|\Psi\rangle = (-1)^{t-q}|\Psi\rangle$. However, on the U(1)-invariant state Ψ, T' is the same as T:

$$T^2|\Psi\rangle = (-1)^{t-q}|\Psi\rangle. \tag{3.16}$$

So $|\Psi\rangle$ is a gauge-invariant state of spin $q/2$ mod 1 that transforms under T^2 with a sign $(-1)^{t-q}$. The condition that this state is either a Kramers singlet boson or a Kramers doublet fermion is that t should be even. However, the formula (3.15) shows that any of the χ_i of odd electric charge, whether its charge is congruent to 1 or 3 mod 4, makes a contribution to t that is $1/2$ mod 2. Hence, requiring t to be even is the same as requiring that the total number y of Dirac fermions of odd charge is a multiple of 4. This was the condition found in Sec. II E for the theory to be completely free of gauge anomaly on an unorientable manifold.

As in our discussion of the even charge case, once we have found a monopole operator M that is either a Kramers singlet boson or a Kramers doublet fermion, by acting if necessary with a gauge-invariant Kramers doublet fermion $\overline{w}^{n_i}\chi_i$, we get a monopole operator M' that is a Kramers singlet boson. So such an operator exists whenever the generalized "parity" anomaly is absent.

We turn next to examples for various values of ν.

3. ν Congruent to 2 mod 4

The most minimal model with $\nu = 2$ was analyzed in detail in Ref. [22]. One simply introduces one χ field with some charge r, and to make a mass term possible, one takes ϕ to have charge $2r$. One chooses r to be an even number $r = 2s$ to avoid the "parity" anomaly. As we know from Sec. II E, this gives a theory with no gauge anomaly, not even the more subtle anomaly that can only be detected on an unorientable manifold.

Since ϕ has charge $4s$, the expectation value of ϕ reduces the gauge symmetry from U(1) to $\mathbb{Z}_{4s}$. The low-energy theory, however, is not just a $\mathbb{Z}_{4s}$ gauge theory. In a vortex field, there is a single Majorana fermion zero mode. As a result, the vortices have non-Abelian statistics, and, as in the composite fermion description [37] of the Moore-Read state [38], the theory can be described, at least on an orientable manifold, as the quotient by $\mathbb{Z}_2$ of a $\mathbb{Z}_{4s}$ gauge theory times an Ising topological field theory. For details, see [22].

For any even $\nu = 2t$, one simple approach is to introduce t χ fields, all with the same even charge and coupled in the same way to ϕ. This family of models was discussed in Sec. 6.3 of [22] for odd $t = 1, 3, 5, 7$, corresponding to $\nu = 2, 6, 10, 14$.

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[24]One way to see this is to observe that if ζ and ζ' are real fermion modes that transform with opposite signs under T', then on the Hilbert space H, we can define the T'-invariant "Hamiltonian" $h = i\zeta\zeta'$. Reducing to ground states of h, we can eliminate ζ and ζ' and replace H with a subspace of one-half the dimension on which $(T')^2$ acts with the same sign as before. Continuing in this way, we reduce to the case that a_+ or a_- is 0. If, say, $a_- = 0$, to determine the sign we must need to know if the spinor representation of $\mathrm{Spin}(a_+)$ is real or pseudoreal, and this leads to Eq. (3.14).

As long as one is on an orientable manifold, the resulting topological field theories are the same, but they have four different actions of T (mainly because of subtleties in the quantization of the vortices). On an unorientable manifold, these models are inequivalent at low energies, as is clear both from the different values of ν and from the different actions of T.

For ν congruent to 2 mod 4, the unbroken gauge symmetry is always $\mathbb{Z}_{4s}$ for some integer s, even if the χ fields do not all have the same charge. This follows from the constraints of avoiding the "parity" anomaly and ensuring that all fermions gain mass from the gauge symmetry breaking. (If ϕ has charge $4s + 2$ for some integer s, then, to give masses to all χ fields, the number of χ fields with odd charge will have to be odd, leading to a "parity" anomaly.) In general, if formulated on an orientable manifold, the $\mathbb{Z}_{4s}$ gauge theory has a Chern-Simons (or Dijkgraaf-Witten [39]) term, as analyzed in Ref. [22].

4. ν divisible by 4

For $\nu = 4$, we need two χ fields. Naively, they can both have odd charge, avoiding the usual "parity" anomaly, but we learned in Sec. II E that this will actually lead to a more subtle gauge anomaly. So we take them to have even charge.

Two Dirac fermion fields χ_1, χ_2 of charges $2r_1$ and $2r_2$ can gain mass from a Yukawa coupling to a ϕ field of charge $2(r_1 \pm r_2)$. (The relevant Yukawa coupling is $\chi_1\chi_2\overline{\phi}$ or $\chi_1\overline{\chi}_2\phi$.) Since $r_1 \pm r_2$ can be any integer r, the low-energy gauge symmetry can be $\mathbb{Z}_{2r}$ for any r.

As soon as ν is a multiple of 4, the number of χ fields is even and likewise the number of fermion zero modes in a vortex field is even, leading to Abelian statistics only. If formulated on an orientable manifold, the low-energy theory can be understood as a $\mathbb{Z}_{2r}$ gauge theory with Chern-Simons term, by arguments explained in Ref. [22].

For ν any multiple of 4, similar models are possible. The first case in which we can take all of the χ fields to have odd charge is $\nu = 8$, which requires four χ fields. For example, for $\nu = 8$, we can introduce four χ fields all of charge 1, and take ϕ to have charge 2. The low-energy theory is a $\mathbb{Z}_2$ gauge theory (with no Chern-Simons term, as it turns out).

5. ν divisible by 16

$\nu = 16$ is the first case in which the emergent gauge symmetry can be completely broken, as we learned in Sec. III A. Accordingly, it is the first case that it might be possible to obtain a gapped phase that is also topologically trivial. This is consistent with the fact that the anomaly of the bulk topological superconductor is trivial at $\nu = 16$. Several constructions of T-invariant, gapped, topologically trivial boundary states are known at $\nu = 16$.

In the present approach, at $\nu = 16$, we can take four Dirac fermion fields χ_i of charge 0 and four such fields χ_i' of charge 1. We also take ϕ to have charge 1. The expectation value of ϕ completely breaks the gauge symmetry and a Yukawa coupling $\sum_{i=1}^{4} \chi_i \chi_i' \overline{\phi}$ can give mass to all fermions. A monopole operator that can be added to the action or Hamiltonian completely destroys the conservation of vorticity. The resulting low-energy theory is gapped, symmetry-preserving, and topologically trivial.

6. Models at $\nu = 16$ that are gapped via nonperturbative effects

All of the models described so far can be understood entirely in the framework of weak coupling. The only effect beyond conventional perturbation theory was to add to the action a monopole operator with a small coefficient. The only role of this operator was to explicitly break a symmetry, so it can itself be treated perturbatively.

It is interesting to also consider models that become gapped with the aid of nonperturbative effects.

For a first example, we take the gauge group to be $G = U(2)$. The center of G is thus a copy of $U(1)$, embedded as $\mathrm{diag}(e^{i\alpha}, e^{i\alpha})$. We introduce four doublets of Dirac fermions $\chi_1, \ldots, \chi_4$, each transforming in the **2** of U(2). The **2** is a two-dimensional complex representation or equivalently a four-dimensional real representation. We have shown in Sec. II E that a model with fermions comprising four copies of the **2** has no gauge anomaly. Since it has $\nu = 4 \times 4 = 16$, it also has no gravitational anomaly and is completely anomaly-free.

To make possible complete Higgsing of U(2) to the standard boundary state of the topological superconductor, we introduce w fields consisting of two or more (T-even) doublets of U(2). To make possible a flow to a symmetry-preserving, gapped, topologically trivial state, we add another (T-odd) Higgs field ϕ that is invariant under the subgroup $SU(2) \subset G$ and transforms with charge 2 under the center of G. This has been chosen to make possible gauge-invariant and T-conserving Yukawa couplings:

$$\overline{\phi}\epsilon_{AB}\epsilon_{ab}\left(\chi_1^{Aa}\chi_2^{Bb} + \chi_3^{Aa}\chi_4^{Bb}\right) + \text{H.c.} \qquad (3.17)$$

[Here $A, B = 1, 2$ and $a, b = 1, 2$ are, respectively, U(2) indices and spinor indices.] Once ϕ gets an expectation value, the fermions become massive and the gauge symmetry is spontaneously broken from U(2) to SU(2). In perturbation theory, therefore, one sees at low energy a pure SU(2) gauge theory, with no massless charged fields. In the infrared, though there is no explicit proof (except from numerical simulations), it is strongly believed that this theory flows to a confining, gapped, T-conserving, and topologically trivial phase. If so, this gives another way to gap a $\nu = 16$ system in a T-conserving way.

The reason that in this construction we had to start with an emergent U(2) gauge symmetry rather than just the center U(1) of U(2) is the following. Had we gauged only the central U(1), then after the charge 2 Higgs field ϕ acquired an expectation value, U(1) would have been spontaneously broken to $\mathbb{Z}_2$. The low-energy physics would have been a $\mathbb{Z}_2$ gauge theory, rather like other examples that were discussed earlier. The low-energy time-reversal symmetry would satisfy the familiar relation

$$T^2 = (-1)^F K, \qquad (3.18)$$

where K is the nontrivial element of $\mathbb{Z}_2$.

How might we get rid of the $\mathbb{Z}_2$ gauge symmetry? One idea would be to let a scalar carrying $\mathbb{Z}_2$ charge get an expectation value. However, this would necessarily break time-reversal symmetry, because Eq. (3.18) shows that a scalar field that is odd under K cannot be T invariant.

An alternative might be to confine the $\mathbb{Z}_2$ gauge symmetry rather than Higgsing it.[25] The trouble is that although there is no problem to see confinement of a discrete gauge symmetry such as $\mathbb{Z}_2$ in a lattice theory, it is hard to interpret this idea in a continuum field theory. One could take the emergent gauge symmetry G to be defined on a lattice, but then how would we couple it to the fermions? Unfortunately, it is very difficult to explicitly construct a T-invariant lattice fermion theory with nonzero ν. Because of the anomaly, one expects that it is only possible to do this (if at all) if ν is a multiple of 16, and any way to do it would probably itself lead immediately to a way to see that the $\nu = 16$ model is topologically trivial. Because of issues such as this, one approach to gapping the $\nu = 16$ problem in a T-invariant way has actually involved coupling lattice bosons to continuum fermions [11]. It is conceivable that this could be adapted to the present context.

Instead of that, in the above construction, we took the emergent gauge symmetry G to be U(2) rather than its center U(1). The effect of this was that the expectation value of ϕ broke G not to the finite group $\mathbb{Z}_2$ but to the connected Lie group SU(2), of which the $\mathbb{Z}_2$ in question is the center. Now to gap the system, we want confinement in pure SU(2) gauge theory and this involves no problem, except that it is hard to understand.

There is a variant of this in which the emergent gauge group G is taken to be not U(2) but U(1) × U(1), the maximal torus of U(2). The fermion representation is taken to be the same, viewed as a representation of the smaller gauge group, and the same Higgs fields w and ϕ are introduced, with the same couplings. The expectation value of ϕ now breaks G not to SU(2) but to the maximal torus of SU(2), which is the subgroup $\mathrm{U}(1)' = \mathrm{diag}(e^{\mathrm{i}\alpha}, e^{-\mathrm{i}\alpha})$ of U(2). The low-energy theory is a pure $\mathrm{U}(1)'$ gauge theory. If we add to the Hamiltonian a monopole operator of unit magnetic charge with a small coefficient, then in the infrared we get the Polyakov model [23] of compact QED. The infrared theory becomes confining, gapped, and topologically trivial.

This claim depends on assuming that the monopole operator has magnetic flux 1 with respect to $\mathrm{U}(1)'$. Otherwise, after confinement, we will be left over with a nontrivial "magnetic" gauge group. The monopole operator must also satisfy Dirac quantization with respect to the full underlying gauge symmetry G. To satisfy these conditions, we simply take a monopole operator of charges (1,0) [or (0,1)] in the original U(1) × U(1) gauge theory.

In this approach, the $\mathbb{Z}_2$ that potentially is left unbroken by the expectation value of a charge 2 Higgs field ϕ has been embedded in the connected group $\mathrm{U}(1)'$ and then has been confined.

C. Bosonic analog

1. Goal

An important part of the above story is the extension of symmetry groups

$$1 \to \mathcal{K} \to \mathcal{H} \to \mathcal{T} \to 1 \qquad (3.19)$$

that is implied by the relation $\mathsf{T}^2 = (-1)^\mathsf{F}\mathsf{K}$. Here, as explained in Sec. III A, $\mathcal{T}$ is the bulk symmetry group generated by T, $\mathcal{K}$ is the group of emergent gauge symmetries on the boundary, and $\mathcal{H}$ is a nontrivial extension of $\mathcal{T}$ by $\mathcal{K}$. This group extension does not explain everything about these models, since, for example, it does not account for the non-Abelian statistics that occur when ν is congruent to 2 mod 4. However, it accounts for a great deal, especially when ν is a multiple of 4.

It is therefore of interest that this part of the story has an analog for purely bosonic SPT phases. Here we consider a purely bosonic theory in D spacetime dimensions with a global symmetry group $\mathcal{F}$. To keep things simple, we will assume that $\mathcal{F}$ is a finite group of purely internal symmetries, not including reflection or time-reversal symmetries. We will describe boundary states for such theories constructed using group extensions analogous to (3.19).

2. Review

A unified description of a large class of SPT phases was presented in Ref. [40] in terms of group cohomology. These are theories with a global symmetry $\mathcal{F}$ that can be gauged[26] (as opposed to theories with a global symmetry $\mathcal{F}$ that cannot be gauged because of an anomaly). Gapped phases with a gaugeable $\mathcal{F}$ symmetry are then classified by their response to being gauged. What this means is that one specifies the partition function the theory has if formulated on a compact, oriented D-manifold X, endowed with an $\mathcal{F}$ bundle (which is automatically flat, since $\mathcal{F}$ is assumed to be a finite group).

To specify the partition function in the presence of a background gauge field, the starting point is a cohomology class in $H^D(\mathcal{F}, \mathrm{U}(1))$ that we will write as $e^{\mathrm{i}\gamma}$. For any X as above endowed with an $\mathcal{F}$ bundle, $e^{\mathrm{i}\gamma}$ can be "pulled back" to a class $e^{\mathrm{i}\beta} \in H^D(X, \mathrm{U}(1))$. [A slightly abstract explanation of this pullback operation is presented in the paragraph preceding Eq. (3.23).] The class $e^{\mathrm{i}\beta}$ can be "integrated" over X, and this integral is supposed to be the partition function.[27] We write the integral evocatively as $Z_\beta = \exp(\mathrm{i}\int_X \beta)$. The motivation

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[25]The necessary dynamics is sometimes discussed under the name of "vortex condensation." Note that in the present context, there is no local operator that creates a vortex of unit vorticity [in which the phase of ϕ changes by 2π and the flux of the broken U(1) gauge field is π]. This would be a half-monopole operator, which does not exist. The phrase "vortex condensation" can be understood as a reference to confinement of the gauge symmetry.

[26]In the framework of [40], $\mathcal{F}$ is assumed to be an "on-site" symmetry, which ensures that it can be gauged. There is a somewhat similar reason that T in a topological superconductor can be gauged, in the sense that the effective description makes sense on an unorientable spacetime. The Schrödinger equation of electrons and nuclei makes sense on an unorientable spatial manifold, so a T- and R-invariant phase of matter derived from it that has an emergent relativistic symmetry that puts space and time on the same footing will always make sense on an unorientable spacetime.

[27]Gauge theories of a finite gauge group in which the contribution of a particular flat bundle to the partition function is the integral of $e^{\mathrm{i}\beta}$ (divided by the order of the automorphism group of the bundle) were constructed in Ref. [39], without envisaging the application to condensed-matter physics.

for the notation $e^{i \int_X \beta}$ is that although what we are formally writing as $\int_X \beta$ is not the integral over X of a gauge-invariant local effective action,[28] Z_β has the same formal properties as if it were. For example, after making some gauge choices or introducing a more abstract language to avoid having to do this,[29] one can extend the definition of $e^{i \int_X \beta}$ to the case that the manifold X has a boundary, and then it satisfies a gluing law precisely of the form of the usual gluing relation for the integrand of the Feynman path integral [Eq. (2.6)], which as we have discussed also has a counterpart for the eta invariant [Eq. (2.7)]. The gluing formula says that if an oriented D-manifold Y^* is built by gluing together two such manifolds Y and Y' along a component W of their common boundary, as in Fig. 1 of Sec. II B, then

$$\exp\left(i \int_Y \beta\right) \exp\left(i \int_{Y'} \beta\right) = \exp\left(i \int_{Y^*} \beta\right). \qquad (3.20)$$

The gluing formula is one way to understand the fact that $Z_\beta = \exp(i \int_X \beta)$ is the partition function of a topological field theory.

Likewise,[30] $\exp(i \int_X \beta)$ is a cobordism invariant (of an oriented D-manifold with an $\mathcal{F}$ bundle) rather as in even dimensions $\exp(-\pi i \eta_R / 2)$ is a cobordism invariant (of a pin$^+$ D-manifold with a gauge bundle). This cobordism property of $\exp(-\pi i \eta_R / 2)$ has not been made explicit in the present paper; it can be deduced from the APS index theorem [30], as is explained in Ref. [7], Sec. 4.2. In general, a U(1)-valued cobordism invariant is the partition function of an invertible topological field theory (see [41], Sec. 5.5, and [34]), and, in particular, cobordism invariance implies that $\exp(i \int_X \beta)$ and $\exp(-\pi i \eta_R / 2)$ are the partition functions of suitable topological field theories. That $\exp(-\pi i \eta_R / 2)$ is the analog of $\exp(i \int_X \beta)$ for a T-invariant theory with fermions satisfying $\mathsf{T}^2 = (-1)^{\mathsf{F}}$ was the main idea[31] in Ref. [7] (this was inspired

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[28]To be more exact, there is no general, natural procedure to write it that way. In a specific case, one may be able to add additional fields and gauge invariances and write β as some sort of local integral.

[29]The more abstract language involves introducing a one-dimensional vector space H_W of "physical states" for every boundary component W of X (H_W is one-dimensional because we are describing an SPT phase with no intrinsic topological order) and interpreting $\exp(i \int_Y \beta)$ as a quantum mechanical transition amplitude between initial and final states rather than a complex number. Thus, this more abstract language is actually essential in developing the theory more fully. Nevertheless, we will not develop that language here.

[30]Cobordism invariance of $\exp(i \int_X \beta)$ means that if the oriented manifold X, equipped with an $\mathcal{F}$ bundle, is the boundary of an oriented manifold Z, over which the given $\mathcal{F}$ bundle extends, then $\exp(i \int_X \beta) = 1$. This amounts to Stokes's theorem together with $d\beta = 0$. (To be more exact, it follows from facts valid in any cohomology theory that reduce for differential forms to Stokes's theorem and $d\beta = 0$.)

[31]As explained there, in some important examples, a more elementary description is possible in which one uses an ordinary index or a mod 2 index of the Dirac operator instead of the eta invariant. The eta invariant provides a general framework to understand all possible examples.

by an earlier remark [8] concerning cobordism invariance of the eta invariant in four dimensions).

3. Gapped boundaries for bosonic SPT states

To pursue the analogy, we will describe a simple class of gapped, symmetry-preserving boundary states for bosonic SPT phases that are analogous to the fermionic states described in Sec. III B. We consider a worldvolume X of D spacetime dimensions with a boundary W of one dimension less. We assume that we are given a nontrivial group extension

$$1 \to \mathcal{K} \to \mathcal{H} \to \mathcal{F} \to 1, \qquad (3.21)$$

where, as before, $\mathcal{F}$ is a (gaugeable) finite group of global symmetries of a system in D spacetime dimensions and $\mathcal{K}$ is a finite group of emergent gauge symmetries that appear on W. The reason that we assume that the symmetries in $\mathcal{K}$ are *gauge* symmetries is the following. The symmetries in $\mathcal{K}$ have to be exact symmetries or the construction that follows will not make sense. In condensed-matter physics, it is natural for *approximate* global symmetries to emerge in the infrared; they are explicitly violated by interactions that are irrelevant in the renormalization group sense. For gauge symmetries, the situation is the opposite; it does not make much sense to have an emergent gauge symmetry in the infrared that is explicitly broken by irrelevant interactions. On the contrary, exact emergent gauge symmetries are often postulated in models of condensed-matter systems, for example, in models of the fractional quantum Hall effect.

How can we use the group extension (3.21) to construct a gapped boundary state? Given a homomorphism $\mathcal{H} \to \mathcal{F}$, the class $e^{i\gamma} \in H^D(\mathcal{F}, \mathrm{U}(1))$ can be pulled back to a class $e^{i\hat{\gamma}} \in H^D(\mathcal{H}, \mathrm{U}(1))$. [See the discussion of Eq. (3.23) for a slightly abstract explanation.] If $e^{i\hat{\gamma}}$ is trivial, then any trivialization of this class can be used to construct a gapped boundary state, as we will explain. Note, however, that if $e^{i\gamma}$ is nontrivial (so that the bulk SPT phase for which we are trying to construct a gapped boundary state is nontrivial), then to make $e^{i\hat{\gamma}}$ trivial, the group extension (3.21) will have to be nontrivial. (If the full symmetry group is just a product $\mathcal{K} \times \mathcal{F}$, the existence of $\mathcal{K}$ will not help us trivialize a cohomology class of $\mathcal{F}$.) So to construct a gapped, symmetry-preserving boundary state in this way requires a nontrivial group extension at the starting point. For a simple concrete example, take $\mathcal{F} = \mathcal{K} = \mathbb{Z}_2$, $\mathcal{H} = \mathbb{Z}_4$. These fit in a nontrivial extension:

$$0 \to \mathbb{Z}_2 \xrightarrow{2} \mathbb{Z}_4 \xrightarrow{r} \mathbb{Z}_2 \to 0. \qquad (3.22)$$

The first map is multiplication by 2 and the second is reduction mod 2. [Because the groups involved are all Abelian, we have used an additive notation rather than the multiplicative notation in Eq. (3.21).] This particular example is interesting in $D = 3$ spacetime dimensions, since $H^3(\mathcal{F}, \mathrm{U}(1)) \cong \mathbb{Z}_2$, and the nontrivial class $e^{i\gamma}$ in $H^3(\mathcal{F}, \mathrm{U}(1))$ is trivial when pulled back to $e^{i\hat{\gamma}} \in H^3(\mathcal{H}, \mathrm{U}(1))$. [In this example, $H^2(\mathcal{K}, \mathrm{U}(1)) = 0$, so the trivialization of $e^{i\hat{\gamma}}$ is essentially unique, and, accordingly, the following construction will give only one gapped boundary state.]

If $e^{i\hat{\gamma}}$ is trivial, then its trivializations form topological classes that differ by elements of $H^{D-1}(\mathcal{H}, \mathrm{U}(1))$. As we will see, any class of trivialization will give a gapped

symmetry-preserving boundary state for the topological field theory with partition function $e^{i\beta}$. So the number of such boundary states that can be constructed in this way [once the group extension (3.21) is chosen] is simply the order of the finite group $H^{D-1}(\mathcal{H}, U(1))$. The physical meaning of this is straightforward. An element of $H^{D-1}(\mathcal{H}, U(1))$ determines a purely $D-1$-dimensional invertible topological field theory $\mathcal{T}$ with symmetry group $\mathcal{H}$. Any boundary state we define can be modified by tensoring it with $\mathcal{T}$. So if we can make any boundary states at all, the number of such boundary states that we can make will be the order of $H^{D-1}(\mathcal{H}, U(1))$. What we have just said can be compared with what was explained in Sec. II F. Triviality of $\exp(i\widehat{\gamma})$ is the analog of triviality of $\exp(-\pi i \eta_R/2)$. $H^{D-1}(\mathcal{H}, U(1))$ is the analog of the finite group $\mathrm{Hom}(\Gamma', U(1))$ that appeared in Sec. II F. Given that $\exp(i\widehat{\gamma})$ or $\exp(-\pi i \eta_R/2)$ is trivial, the number of symmetry-preserving gapped boundary states that can be constructed using this fact is the order of $H^{D-1}(\mathcal{H}, U(1))$ or of $\mathrm{Hom}(\Gamma', U(1))$.

We will now explain a few mathematical points that have been omitted so far. One way to describe group cohomology is that it is cohomology of the classifying space. The classifying space $B\mathcal{F}$ of a group $\mathcal{F}$ is defined as the quotient $E\mathcal{F}/\mathcal{F}$, where $E\mathcal{F}$ is a contractible space with a free action of $\mathcal{F}$. (Such a space always exists, and any two choices are homotopic, so it does not matter which one is chosen.) Then, by definition, $H^D(\mathcal{F}, U(1)) = H^D(B\mathcal{F}, U(1))$. The space $B\mathcal{F}$ is endowed with a principal $\mathcal{F}$ bundle, which is simply the total space of the fibration $E\mathcal{F} \to B\mathcal{F}$. This $\mathcal{F}$ bundle is "universal" in the sense that if X is any topological space with an $\mathcal{F}$ bundle, then that $\mathcal{F}$ bundle is the pullback to X of the universal $\mathcal{F}$ bundle over $B\mathcal{F}$ by some map $\Phi : X \to B\mathcal{F}$ (defined up to homotopy and called the classifying map). Accordingly, if we are given a class $e^{i\gamma} \in H^D(\mathcal{F}, U(1)) = H^D(B\mathcal{F}, U(1))$, we can pull this class back to $e^{i\beta} \in H^D(X, U(1))$. This is the pullback operation that we used in Sec. III C 2 in the initial construction of an SPT phase based on the cohomology class $e^{i\gamma}$. On an oriented manifold X without boundary, the partition function of the SPT phase associated with $e^{i\gamma} \in H^D(\mathcal{F}, U(1))$ is then $\exp(i\int_X \beta)$.

Now let us consider the group extension (3.21). By definition, $\mathcal{H}$ acts freely on the contractible space $E\mathcal{H}$. Since the group extension $\mathcal{H}$ entails a homomorphism $\mathcal{H} \to \mathcal{F}$, $\mathcal{H}$ also acts on any space that $\mathcal{F}$ acts on, and in particular $\mathcal{H}$ acts (but not freely) on $E\mathcal{F}$. Consequently, $\mathcal{H}$ acts on $E\mathcal{H} \times E\mathcal{F}$, and this action is free because it is free on the first factor. Hence, we can define $B\mathcal{H} = (E\mathcal{H} \times E\mathcal{F})/\mathcal{H}$. By forgetting the first sector, $B\mathcal{H}$ projects to $E\mathcal{F}/\mathcal{H} = E\mathcal{F}/\mathcal{F} = B\mathcal{F}$. The fiber is $E\mathcal{H}/\mathcal{K}$, where $\mathcal{K}$ (like all of $\mathcal{H}$) acts freely on $E\mathcal{H}$, so we can set $E\mathcal{H}/\mathcal{K} = B\mathcal{K}$. Thus, there is a fibration

$$B\mathcal{K} \xrightarrow{i} B\mathcal{H} \quad\quad\quad (3.23)$$
$$\downarrow{\pi}$$
$$B\mathcal{F}.$$

Now we can describe the "pullback" operation that is part of the definition of the boundary state. Given $e^{i\gamma} \in H^D(B\mathcal{F}, U(1))$, we simply pull it back to $e^{i\widehat{\gamma}} = \pi^*(e^{i\gamma}) \in H^D(B\mathcal{H}, U(1))$.

We are almost ready to define the boundary state. To describe a boundary state for a theory whose partition function

would be $e^{i\int_X \beta}$ if X has no boundary, we have to describe what is meant by $e^{i\int_X \beta}$ when X has a boundary. There is no symmetry-preserving definition without introducing some new variables on the boundary. On what sort of boundary data will $e^{i\int_X \beta}$ depend? The bulk physics on X has $\mathcal{F}$ symmetry, and we assume it has been coupled to some background $\mathcal{F}$ bundle (possibly trivial). This means that the bulk theory on X has been coupled to a background gauge field of the finite group $\mathcal{F}$. However, on W we postulate the existence of fields whose role is to "lift" the background $\mathcal{F}$ bundle (or more precisely its restriction to W) to an $\mathcal{H}$ bundle. (These fields are analogous to ϕ in the fermionic models that were introduced in Sec. III A above.) Accordingly, we assume that on W we are given an $\mathcal{H}$ bundle which, under the projection $\mathcal{H} \to \mathcal{F}$ that is part of the extension (3.21), is mapped to (the restriction to W of) the background $\mathcal{F}$ bundle. The definition of $\exp(i\int_X \beta)$ will depend on this $\mathcal{H}$ bundle.

A given $\mathcal{F}$ bundle over X, when restricted to W, may not lift to $\mathcal{H}$ at all. Such a bundle contributes 0 to the path integral for the boundary state that we will define; it is not compatible with this boundary state. Alternatively, a given $\mathcal{F}$ bundle, restricted to W, may lift to $\mathcal{H}$ in more than one way. If $\mathcal{H}$ is Abelian, then any liftable $\mathcal{F}$ bundle can be lifted in precisely N ways, where N is the order of the finite group $H^1(W, \mathcal{K})$. There is no equally simple statement if $\mathcal{H}$ is not Abelian. Regardless, assuming that the background $\mathcal{F}$ bundle over X is liftable to $\mathcal{H}$, we compute the partition function, in the presence of the boundary state on W, by summing over the possible lifts, with the contribution of each lift being $\frac{1}{n}e^{i\int_X \beta}$; here n is the order[32] of the subgroup of $\mathcal{K}$ that acts by automorphisms on a given lift, and we explain next what is meant by $e^{i\int_X \beta}$. That will complete the definition of the boundary state.

To integrate the cohomology class $e^{i\beta}$ over a manifold X with boundary, we need a trivialization of this class over $W = \partial X$. That is precisely what we have if we are given a trivialization of $e^{i\widehat{\gamma}}$. Indeed, given the background $\mathcal{F}$ bundle over X and a lift to $\mathcal{H}$ of the restriction of this bundle to W, we have the following commutative diagram:

$$
\begin{array}{ccc}
W & \xrightarrow{j} & X \\
\downarrow{\widehat{\Phi}} & & \downarrow{\Phi} \\
B\mathcal{H} & \xrightarrow{\pi} & B\mathcal{F}.
\end{array}
\quad\quad (3.24)
$$

Here j is the inclusion of W in X, π was described in Eq. (3.23), the background $\mathcal{F}$ bundle over X is the pullback by Φ of the universal $\mathcal{F}$ bundle over $B\mathcal{F}$, and the lifted $\mathcal{H}$ bundle over W is similarly a pullback by $\widehat{\Phi}$ of the universal $\mathcal{H}$ bundle over $B\mathcal{H}$. Since by definition $e^{i\beta} = \Phi^*(e^{i\gamma})$, the restriction to W of $e^{i\beta}$ is $j^*\Phi^*(e^{i\gamma})$. Commutativity of the diagram means that this is the same as $\widehat{\Phi}^*\pi^*(e^{i\gamma}) = \widehat{\Phi}^*(e^{i\widehat{\gamma}})$. Hence, any trivialization of $e^{i\widehat{\gamma}}$ gives a trivialization of the restriction to W of $e^{i\beta}$ and enables us to define $e^{i\int_X \beta}$.

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[32]The reason to divide by n is that $\mathcal{K}$ is a group of gauge symmetries. In quantizing a gauge symmetry, one always has to divide by the volume of the automorphism group.

4. The boundary topological field theory

This construction has at least one more important facet. Going back to the fibration (3.23), we see that $B\mathcal{K}$ is the fiber of the fibration $B\mathcal{H} \to B\mathcal{F}$, and thus we can think of $B\mathcal{K}$ as the inverse image in $B\mathcal{H}$ of a point $p_0 \in B\mathcal{F}$. The class $e^{i\gamma} \in H^D(B\mathcal{F}, \mathrm{U}(1))$ is trivial when restricted to a point, and this gives it a natural trivialization when pulled back to $B\mathcal{K}$. In other words, the restriction to $B\mathcal{K}$ of $e^{i\widehat{\gamma}} \in H^D(B\mathcal{H}, \mathrm{U}(1))$ is naturally trivial, whether $e^{i\widehat{\gamma}}$ is trivial or not. On the other hand, if we are given a global trivialization of $e^{i\widehat{\gamma}}$, we can restrict this trivialization to $B\mathcal{K}$, and therefore we now have two trivializations of the restriction to $B\mathcal{K}$ of $e^{i\widehat{\gamma}}$. The ratio of two trivializations of $e^{i\widehat{\gamma}}|_{B\mathcal{K}} \in H^D(B\mathcal{K}, \mathrm{U}(1))$ is a class $e^{i\delta} \in H^{D-1}(B\mathcal{K}, \mathrm{U}(1))$.

Such a class is precisely what we need to describe a version of $\mathcal{K}$ gauge theory on a $D-1$-dimensional spacetime manifold such as W. This theory is a nontrivial topological field theory, and we claim that it is realized along W in the boundary state that we have described. The justification for this statement is as follows. Let us take the background $\mathcal{F}$ bundle over X to be trivial. Then its restriction to W is, of course, also trivial. For a lift of the trivial $\mathcal{F}$ bundle to $\mathcal{H}$, we can simply take any $\mathcal{K}$ bundle over W. In this situation, we can take $\Phi : X \to B\mathcal{F}$ to map X to a point, which we can take to be p_0. Then $e^{i\beta} = \Phi^*(e^{i\gamma})$ is naturally trivial along X, and in particular along W, since a pullback of a class of positive degree by a constant map is always naturally trivial. This trivialization agrees along W with the one that comes from the fact that $e^{i\widehat{\gamma}}$ is naturally trivial when restricted to $B\mathcal{K}$. (They are both derived from the fact that $e^{i\gamma}$ is trivial when restricted to a point.) We can further take $\widehat{\Phi} : W \to B\mathcal{H}$ to be a map to $B\mathcal{K} \subset B\mathcal{H}$. If we use the trivialization of $e^{i\beta}$ along W that comes from constancy of Φ to define $e^{i \int_X \beta}$, we get the answer 1, because this trivialization extends over all of X. We are instead supposed to use along W the other trivialization of $e^{i\beta}$, the one that comes by restricting to $B\mathcal{K}$ the global trivialization of $e^{i\widehat{\gamma}}$ that was used to define the boundary state. By definition, the two differ by $e^{i\delta}$ and therefore $e^{i \int_X \beta}$ computed using the second trivialization of $e^{i\beta}$ along W is equal to $e^{i \int_W \delta}$.

So our boundary state can be defined by taking on W a $\mathcal{K}$ gauge theory defined with the cohomology class $e^{i\delta}$ and coupling it the global symmetry $\mathcal{F}$ in bulk. This coupling uses the class $e^{i\gamma} \in H^D(B\mathcal{F}, \mathrm{U}(1))$ and a trivialization of the pullback of this class to $B\mathcal{H}$. This coupling of the $D-1$-dimensional boundary gauge theory to the global symmetry $\mathcal{F}$ is anomalous in the sense that it cannot be defined in $D-1$ dimensions. A purely $D-1$-dimensional coupling of the boundary gauge theory to an $\mathcal{F}$ global symmetry would be made using a class $e^{i\gamma'} \in H^{D-1}(B\mathcal{H}, \mathrm{U}(1))$ whose restriction to $B\mathcal{K}$ is $e^{i\delta}$. This gives what is known as a symmetry-enriched topological (SET) phase of matter [42–45].[33] Obviously, the type of theory we have described is not a purely $D-1$-dimensional theory of

this kind. It was obtained by a sort of anomalous coupling of a topological field theory on the boundary of a D-manifold to an SPT phase in the bulk of that D-manifold, and the starting point was a class in $H^D(B\mathcal{F}, \mathrm{U}(1))$, not in $H^{D-1}(B\mathcal{H}, \mathrm{U}(1))$. (It has been suggested by J. Wang that the model in Ref. [46] may be an example of the class of states described here.)

We started this analysis with the bulk physics—controlled by a class $e^{i\gamma} \in H^D(B\mathcal{F}, \mathrm{U}(1))$—and not the boundary topological field theory—controlled by a class $e^{i\delta} \in H^{D-1}(B\mathcal{K}, \mathrm{U}(1))$—because this made the analysis much more straightforward.

IV. APPLICATION TO M2-BRANES

M theory is defined on an 11-dimensional spacetime manifold that we will call Y. Y is not necessarily orientable; in general,[34] $w_1(Y) \neq 0$. However, Y is a pin$^+$ manifold and, in particular,

$$w_2(Y) = 0. \qquad (4.1)$$

M theory also has "membranes," known as M2-branes. An M2-brane is a two-dimensional object; at a given time, it fills out a two-manifold Σ in space. (Σ has no boundary unless Y itself has a boundary or one considers M5-branes; we will not consider these possibilities here.) Taking the time into account, the worldvolume of an M2-brane is a three-manifold that we will call W. As usual, in analyzing anomalies, it is convenient to work in Euclidean signature and to take W to be compact. We write TW for the tangent bundle to W and N for its normal bundle in Y. N has rank $11 - 3 = 8$.

A. The classical picture

We will first describe M2-branes at the classical level, and then explain how this picture is modified by a quantum anomaly.

There is no restriction at all on the topology of W, which can be a completely general three-manifold. However, there is a topological restriction on how W is embedded in Y: Its normal bundle N must be oriented. Orientability of N is equivalent to

$$w_1(N) = 0. \qquad (4.2)$$

We stress, however, that N is not just orientable but oriented: It comes with a choice of orientation. The orientation of N determines whether what is wrapped on W is an M2-brane or an M2-antibrane.[35] (By convention, we will call the wrapped object an M2-brane.)

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[33]The idea behind the name is that a nontrivial topological field theory in $D-1$ dimensions—the $\mathcal{K}$ gauge theory defined with the cohomology class $e^{i\delta}$—has been enriched with a global symmetry $\mathcal{F}$. Note that since $\mathcal{K}$ is a group of gauge symmetries, the global symmetry group that acts on gauge-invariant excitations in a model

of this type is indeed precisely $\mathcal{F}$. An SET phase reduces to an SPT phase if the topological field theory is trivial.

[34]Stiefel-Whitney classes $w_k(V)$ (which entered our analysis in Secs. II D and II E) are defined for any real vector bundle V over a space Z. If Z is a manifold and TZ is its tangent bundle, one writes $w_k(Z)$ (or just w_k if the context is clear) for $w_k(TZ)$.

[35]If the C field, which we introduce momentarily, were an ordinary rather than twisted three-form field, then W would have to be orientable and the difference between wrapping an M2-brane or an M2-antibrane on W would involve a choice of orientation of W, which would determine the sign of $\int_W C$. Because C is a twisted three-form, it is the normal bundle to W that has to be orientable,

The tangent bundle of Y decomposes along W as $TY|_W = TW \oplus N$ (where the symbol $|_W$ represents restriction to W). Together with the Whitney sum formula (2.12) and $w_1(N) = w_2(Y) = 0$, this implies that

$$w_1(W) = w_1(Y)|_W \qquad (4.3)$$

and

$$w_2(N) = w_2(W). \qquad (4.4)$$

Equation (4.3) implies that W is orientable if and only if Y is. The orientation of W is reversed in going around a loop $\ell \subset W$ if and only if the orientation of Y is reversed in going around that loop. More specifically, since N is oriented (and not just orientable), a local orientation of Y determines a local orientation of W. A fancy way to describe the situation is to say that the orientation bundle of Y restricts along W to the orientation bundle[36] of W.

M theory also has a field C that, roughly speaking, is a three-form gauge field (twisted in a sense that we will discuss). We will first describe the properties of this field at the classical level and then describe how they are modified by a quantum anomaly.

The C field has a gauge invariance $C \to C + d\Lambda$, where Λ is a two-form, and it has a gauge-invariant four-form field strength $G = dC$. C is a three-form analog of an Abelian gauge field A, which locally is a one-form that has a gauge-invariance $A \to A + d\phi$ and that has a gauge-invariant two-form field strength $F = dA$. In addition, "big" gauge transformations are allowed that shift the periods of either A or C (i.e., their integrals over closed one-manifolds or three-manifolds) by integer multiples of 2π. [In the one-form case, allowing "big" gauge transformations amounts to saying that the gauge group is U(1) rather than $\mathbb{R}$. This has a three-form analog.]

A crucial difference is that C is a "twisted" three-form, twisted by the orientation bundle of Y. This means that a choice of a local orientation of Y turns C into an ordinary three-form, but in going around an orientation-reversing loop $\ell \subset Y$, C comes back with the opposite sign. An ordinary three-form on Y could be integrated on an oriented three-dimensional submanifold $Q \subset Y$, but a three-form that is twisted in the sense just described can be integrated instead on a submanifold W with the property stated just after Eq. (4.4) (a local orientation of Y determines a local orientation of W, so that C changes sign whenever the orientation of W is reversed). Thus,

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and whose orientation determines whether what is wrapped is an M2-brane or M2-antibrane.

[36]We can think of the orientation bundle as a principal $\mathbb{Z}_2$ bundle ζ over Y whose local sections correspond to local choices of orientation of Y. The holonomy of ζ around an orientation-reversing loop $\ell \subset Y$ is (in multiplicative notation) -1, while its holonomy around an orientation-preserving loop is $+1$. Later, we introduce a twisted sheaf of integers $\widetilde{\mathbb{Z}}$. Once one picks a local orientation, $\widetilde{\mathbb{Z}}$ is equivalent to the constant sheaf $\mathbb{Z}$ of integers, but reversal of orientation acts as -1 on $\widetilde{\mathbb{Z}}$. A fancy definition is $\widetilde{\mathbb{Z}} = \zeta \times_{\mathbb{Z}_2} \mathbb{Z}$, where $\mathbb{Z}_2$ acts on $\mathbb{Z}$ as multiplication by -1. If here one replaces $\mathbb{Z}$ with a trivial real line bundle $\mathfrak{o}$, again with $\mathbb{Z}_2$ acting on $\mathfrak{o}$ as multiplication by -1, one can define a real line bundle $\varepsilon = \zeta \times_{\mathbb{Z}_2} \mathfrak{o}$. It can be identified topologically with $\det TY$ and will play a role in Sec. IV E.
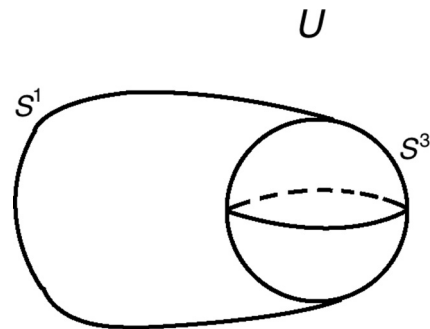

FIG. 3. Schematic depiction of a four-manifold $U = S^3 \times S^1$.

we can consider an integral $\int_W C$, where W is an M2-brane worldvolume. We call this (if W is compact) a "period" of C. However, as the period of C can be shifted by an integer multiple of 2π by a "big" gauge transformation, the natural quantity is really the exponential $\exp(i \int_W C)$. This exponential is analogous to the holonomy $\exp(i \oint_\ell A)$ of an Abelian gauge field A around a closed loop ℓ.

Because C is twisted, its curvature $G = dC$ is not an ordinary four-form but a twisted one, which changes sign in going around an orientation-reversing loop.

The M2-brane is "charged" with respect to W, meaning that it couples to C via a factor $\exp(i \int_W C)$. For this coupling to be well-defined in the absence of fermion anomalies, G must obey an analog of Dirac quantization of magnetic flux. This states that if $U \subset Y$ is any suitably twisted four-cycle,[37] then the flux on U of $G/2\pi$ must be an integer:

$$\int_U \frac{G}{2\pi} \in \mathbb{Z}. \qquad (4.5)$$

The reason for this condition is similar to the reason for Dirac quantization in Abelian gauge theory. Consider, for example, the case that $U = S^3 \times S^1$, as is schematically shown in Fig. 3. Let $W = S^3 \times p$, with p a point in S^1. We want the coupling of W to the C field, namely $\exp(i \int_W C)$, to be single valued when p makes a loop around S^1 and returns to the starting point. The condition for this is

$$\exp\left(i \int_{S^3 \times S^1} G\right) = 1, \qquad (4.6)$$

which is equivalent to the flux quantization condition (4.5).

The analogy between the three-form field C and an Abelian gauge field A can be further extended as follows. An Abelian gauge field on a manifold Y (or more accurately the complex line bundle on which the gauge field is a connection) has a characteristic class c_1 (the first Chern class), valued in $H^2(Y, \mathbb{Z})$. At the level of differential forms, c_1 is represented by the de Rham cohomology class of $F/2\pi$, but c_1 also contains torsion information that is not captured by de Rham cohomology. The C field has an analogous characteristic class that we will call x. Modulo torsion, x is represented by the de

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[37]That is, U must have the same property as W: Its normal bundle is oriented, so that the orientation bundle of Y pulls back along U to the orientation bundle of U.

Rham cohomology class of $G/2\pi$, which is integral by virtue of Dirac quantization. However, as G is a twisted four-form, x takes values not in the ordinary cohomology $H^4(Y, \mathbb{Z})$ but in a twisted cohomology group that we will call $H^4(Y, \widetilde{\mathbb{Z}})$. Here (see footnote 36) $\widetilde{\mathbb{Z}}$ is a twisted sheaf of integers over Y, twisted by the orientation bundle of Y. The assertion that x takes values in this twisted cohomology rather than in the ordinary cohomology of Y just reflects the fact that C and G are not ordinary differential forms but twisted ones.

The topological choice of an Abelian gauge field on a manifold Y is completely classified by the first Chern class, which can be an arbitrary element of $H^2(Y, \mathbb{Z})$, and likewise the topological choice of the M theory C field is completely classified, at the classical level, and ignoring the fermion anomaly, by the characteristic class x, which can be an arbitrary element of $H^4(Y, \widetilde{\mathbb{Z}})$.

B. Worldvolume fermions and the anomaly

The M2-brane volume W also supports fermions. Roughly speaking, these fermions are spin-$\frac{1}{2}$ fermions on W, with values in positive chirality spinors of the normal bundle N to W.

A more precise description is as follows. Since the full spacetime manifold Y is a pin$^+$ manifold, it is endowed with a pin$^+$ bundle $P \to Y$. In 11 dimensions, the rank of P is 32.

Since the normal bundle N to W in Y is oriented and of rank 8, it is endowed with an antisymmetric Levi-Civita tensor $\epsilon_{j_1 j_2 \cdots j_8}$. At a point $p \in Y$, there are Dirac γ matrices Γ_I, $I = 1, \ldots, 11$ acting on P. For $p \in W$, we can split the Γ_I into a triplet γ_i, $i = 1 \cdots 3$ that are tangent to W, and an additional eight γ matrices $\widetilde{\gamma}_j$, $j = 1, \ldots, 8$ that are normal to W. We introduce the chirality operator of N:

$$\widehat{\Gamma} = \frac{1}{8!} \epsilon^{j_1 \cdots j_8} \widetilde{\gamma}_{j_1} \widetilde{\gamma}_{j_2} \cdots \widetilde{\gamma}_{j_8}. \qquad (4.7)$$

The M2-brane worldvolume fermion field Ψ is a section of P, or more exactly of $P|_W$ (the restriction of P to W) that obeys

$$\widehat{\Gamma} \Psi = \Psi. \qquad (4.8)$$

This can be described roughly by saying that Ψ is a spinor on W with values in positive chirality spinors of N, but that is only a rough description since, in general, $w_2(W)$ and $w_2(N)$ are both nonzero [they are equal as in Eq. (4.4)], in which case there exist neither spinors of W nor spinors of N. However, given the pin$^+$ structure of Y and the orientation of N, there is never a problem in defining the bundle $P_+ \to W$ whose sections are sections of $P|_W$ that satisfy Eq. (4.8). The M2-brane fermions are sections of P_+. Note that the rank of P_+ is $\frac{1}{2} \cdot 32 = 16$. The Dirac operator that acts on the M2-brane fermions is the obvious $\mathcal{D} = i\slashed{D} = i\gamma^i D_i$, where D_i is the Riemannian connection on P, projected to P_+.

The reason that the classical picture of Sec. IV A requires modification is that the path integral of the fermion field Ψ on W has anomalies. To orient the reader, we will describe these anomalies in two opposite cases.

First, let us assume that $w_2(W) = 0$, but allow for the possibility that $w_1(W) \neq 0$. Since $w_2(W) = 0$, we have $w_1(N) = w_2(N) = 0$ [Eqs. (4.2) and (4.4)]. That means that

N is topologically trivial.[38] Let us then restrict to the special case that $Y = W \times \mathbb{R}^8$. In this case, the pin$^+$ structure on Y determines a pin$^+$ structure P_W over W, and the field Ψ just consists of eight Majorana fermions all valued in P_W.

This is a familiar system in the theory of topological superconductors. Consider a topological insulator with the value $\nu = 8$ of the usual mod 16 invariant. The standard boundary state of this system consists of 8 Majorana fermions, all taking values in the pin$^+$ bundle of the boundary. In general, for any value of ν, a system of ν Majorana fermions on a possibly unorientable three-dimensional pin$^+$ manifold W has a partition function that is only well defined up to a power of $\exp(-2\pi i\nu/16)$ [6,7]. For $\nu = 8$, this means that the partition function is only well defined up to sign. Something must cancel this anomaly. As will become clear, the anomaly cancellation mechanism is similar in spirit to what happens in a topological superconductor, though different in detail.

For an example of an opposite kind, suppose that Y and therefore also W is orientable and pick orientations. Then the pin$^+$ structure of Y becomes a spin structure. Suppose further that $w_2(W) = w_2(N) = 0$. One can then pick a spin structure on W, with spin bundle $\mathcal{S}$, and once this is done, the normal bundle N to W also acquires a spin structure. This means that one can define the spinor bundle of N, which we call $\mathcal{S}(N)$ [$\mathcal{S}(N)$ is the vector bundle over W that is associated to N via the spinor representation of Spin(8)]. In this situation, the spin bundle of Y, restricted to W, is $\mathcal{S} \otimes \mathcal{S}(N)$. Moreover, $\mathcal{S}(N)$ has a decomposition $\mathcal{S}(N) = \mathcal{S}_+(N) \oplus \mathcal{S}_-(N)$ in eigenspaces of the chirality operator $\widehat{\Gamma}$ [Eq. (4.7)]. Equation (4.8) now means that Ψ is valued in $\mathcal{S} \otimes \mathcal{S}_+(N)$, that is, it is a spinor on W valued in positive chirality spinors of the normal bundle N.

From the point of view of fermion anomalies, Ψ is simply a fermion field on W coupled to a Spin(8) gauge field. The Spin(8) gauge field is the Levi-Civita connection of the normal bundle N of W, lifted from SO(8) to Spin(8) via the choice of spin structure. The microscopic origin of the effective gauge field as part of the gravitational connection in higher dimensions does not affect the analysis of the fermion anomaly, so we can borrow standard results. The spinor representation of Spin(8) is subject to the standard "parity" anomaly, which is a problem in defining the sign of the fermion path integral. An anomaly cancellation mechanism is again needed.

A concrete example of this type in which the "parity" anomaly affects the sign of the M2-brane path integral is as follows. We return to the example of Fig. 3, with Y containing an embedded copy of $U = S^3 \times S^1$. Assume that Y is the total space of a rank 7 vector bundle Q over U. The structure group of Q is SO(7), which lifts to Spin(7) via the choices of spin structure, and we take it to have instanton number 1 (or more generally any odd instanton number). As before, we take W to be $S^3 \times p$, with p a point in S^1. As p is drawn in a loop around S^1 and returns to its starting point, the spectrum of the

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[38]As in the discussion of Eq. (2.9), a real vector bundle N over a three-manifold W is stably classified by $w_1(N)$ and $w_2(N)$. If the rank of N is large enough to be in the "stable range," then the stable classification of N is the same as a topological classification. For W of any dimension, a real vector bundle N is in the stable range if its rank exceeds the dimension of W.

Dirac operator on W undergoes a spectral flow and the fermion path integral on W changes sign. This sign change is the usual "parity" anomaly, whose description via spectral flow can be found in section 2 of [7].

It is notable that in the two examples we considered, the anomaly only affects the sign of the M2-brane fermion path integral. In general, a fermion path integral in a spacetime of odd dimension has only discrete, "global" anomalies. In dimension 3, the anomaly is in general a 16th root of 1. However, for the M2-brane, the anomaly, in general, is just a sign ± 1, as we will see.

C. Anomaly cancellation on an orientable manifold

In Ref. [24], a natural anomaly cancellation mechanism was described for the case that Y is orientable. This mechanism was also shown to solve several other problems (such as canceling an anomaly in the overall M theory path integral).

We will explain this mechanism in the context of the example of Fig. 3 and later generalize it. Anomaly cancellation depends on combining the fermion path integral, which we formally denote as $\mathrm{Pf}(\slashed{D})$ (where $\slashed{D}$ is the Dirac operator that acts on Ψ and Pf is the Pfaffian), with the classical coupling of W to the C field. Formally, the product of the two is

$$\mathrm{Pf}(\slashed{D})\exp\left(\mathrm{i}\int_W C\right). \qquad (4.9)$$

It is this product that must be well defined, rather than either factor separately. So if $\mathrm{Pf}(\slashed{D})$ changes sign when p is taken around a loop in S^1, we want $\exp(\mathrm{i}\int_W C)$ to likewise change sign. In Eq. (4.5), we imposed an integrality condition on $\int_{S^3 \times S^1} G/2\pi$ in order to ensure that $\exp(\mathrm{i}\int_W C)$ is well defined. However, now we want $\exp(\mathrm{i}\int_W C)$ to change sign when W goes around the loop precisely when the instanton number on U, which we will call ϱ, is odd. This will happen if $\int_U G/2\pi$ is a half integer when the instanton number is odd:

$$\int_U \frac{G}{2\pi} = \frac{\varrho}{2} \ \mathrm{mod}\ \mathbb{Z}. \qquad (4.10)$$

This condition can be reformulated as follows. Since Y is orientable in the example under discussion, the pin$^+$ structure of Y reduces to a spin structure. On a spin manifold Y, there is a four-dimensional characteristic class[39] λ that measures the instanton number of the tangent bundle, in the sense that for any oriented four-manifold $U \subset Y$ without boundary, the instanton number of the tangent bundle of Y, integrated over U, is $\int_U \lambda$. Thus, the condition (4.10) can be written

$$\int_U \frac{G}{2\pi} = \frac{1}{2}\int_U \lambda \ \mathrm{mod}\ \mathbb{Z}, \qquad (4.11)$$

and we impose on G this modified quantization condition (for any embedded four-manifold U without boundary). It is shown in Eq. (4.9) that for Y orientable, the M2-brane path integral

is well defined if Eq. (4.11) is satisfied. Our main goal in what follows is to generalize this analysis to the case that Y is not necessarily orientable. (We will also make some technical improvements in the derivation even in the orientable case.)

The shifted quantization condition (4.10) has a one-form analog that is possibly more familiar. If A is a U(1) gauge field with field strength $F = \mathrm{d}A$, then the periods of $F/2\pi$ are integers. Suppose instead that A is not an ordinary U(1) gauge field, but a spin$_c$ connection (a notion that was important in Ref. [22]). Then, on a manifold Z that is not necessarily a spin manifold, F obeys a shifted version of Dirac quantization. For any two-cycle $V \subset Z$,

$$\int_V \frac{F}{2\pi} = \frac{1}{2}\int_V w_2(Z) \ \mathrm{mod}\ \mathbb{Z}. \qquad (4.12)$$

To make the analogy even closer, we note that the mod 2 reduction of λ is the Stieffel-Whitney class w_4. Hence, we can reformulate (4.10) as follows:

$$\int_U \frac{G}{2\pi} = \frac{1}{2}\int_U w_4(Y) \ \mathrm{mod}\ \mathbb{Z}. \qquad (4.13)$$

Thus, the shifted quantization condition means that C is a sort of three-form analog of a spin$_c$ connection. The formulation (4.13) has the advantage of making sense on an unorientable manifold. [The condition (4.11) does not make much sense if Y is unorientable, because in that case λ can be integrated on an embedded submanifold $U \subset Y$ if U is oriented, while G can be integrated on U if U has oriented normal bundle.]

The notion of a spin$_c$ connection can be stated somewhat more precisely as follows. Suppose that A is a spin$_c$ connection. Then $2A$ is an ordinary U(1) gauge field, with field strength $2F$. In terms of $2F$, the condition (4.12) reads

$$\int_U \frac{2F}{2\pi} = \int_V w_2(X) \ \mathrm{mod}\ 2. \qquad (4.14)$$

This may be interpreted as follows. The gauge field $2A$ has a first Chern class that we will call $\widehat{c}_1$, represented in de Rham cohomology by $2F/2\pi$. We interpret Eq. (4.14) to mean that the mod 2 reduction of $\widehat{c}_1$ is w_2:

$$\widehat{c}_1 \cong w_2 \ \mathrm{mod}\ 2. \qquad (4.15)$$

This is often described by saying that $\widehat{c}_1$ is an integer lift of w_2. A spin$_c$ structure, for our purposes, is the topological class of a spin$_c$ connection. Similarly, if $G = \mathrm{d}C$ obeys the modified quantization law (4.13), then $2C$ is a conventional twisted three-form gauge field, with a characteristic class $\widehat{x}$, represented in de Rham cohomology by $2G/2\pi$. We interpret Eq. (4.13) to mean that the mod 2 reduction of $\widehat{x}$ is w_4:

$$\widehat{x} \cong w_4 \ \mathrm{mod}\ 2. \qquad (4.16)$$

So $\widehat{x}$ is a twisted integer lift of w_4 [a lift to the twisted integer cohomology $H^4(Y; \widetilde{\mathbb{Z}})$]. By analogy with the terminology "spin$_c$ connection," and for lack of any other name, we will refer to an M theory C field that obeys the condition (4.16) and enables one to define the M2-brane path integral as an m$_c$

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[39] λ does not coincide with $p_1(Y)$, the first Pontryagin class of Y. Rather, the relation between them is $p_1 = 2\lambda$. On a spin manifold, p_1 can be divided by 2 in a canonical way and we write λ for $p_1/2$. Modulo torsion, λ can be characterized by the condition on the instanton number that is mentioned shortly.

connection. By an m_c structure, we will mean the topological class of an m_c connection.[40]

Not every manifold admits a spin$_c$ structure or an m_c structure. In the spin$_c$ case, the obstruction is as follows. Consider the short exact sequence of groups

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{r} \mathbb{Z}_2 \to 0, \qquad (4.17)$$

where the first map is multiplication by 2 and the second is reduction mod 2. This leads to a long exact sequence of cohomology groups that reads, in part,

$$\cdots \to H^2(X;\mathbb{Z}) \xrightarrow{r} H^2(X;\mathbb{Z}_2) \xrightarrow{j} H^3(X;\mathbb{Z})$$
$$\xrightarrow{2} H^3(X;\mathbb{Z}) \to \cdots . \qquad (4.18)$$

Here j is called the connecting homomorphism, and $j(w_2(X))$ is called $W_3(X)$. The composition of successive maps in Eq. (4.18) vanishes, so $2W_3 = 2 \circ j(w_2) = 0$ and thus W_3 is a 2-torsion class. The exactness of the sequence (4.18) says that w_2 is the mod 2 reduction of some $\hat{c}_1 \in H^2(X,\mathbb{Z})$ if and only if $W_3 = 0$. This is the obstruction to existence of a spin$_c$ structure. (For a simple example of a manifold in which this obstruction is nontrivial, one may take a $\mathbb{CP}^2$ bundle over S^1 such that $\mathbb{CP}^2$ is complex conjugated in going around S^1.)

The obstruction to an m_c structure is similar. The exact sequence of (4.17), after being tensored with the orientation bundle of Y, leads to a long exact sequence of twisted cohomology groups of Y that reads, in part,

$$\cdots \to H^4(Y;\widetilde{\mathbb{Z}}) \xrightarrow{r} H^4(Y;\mathbb{Z}_2) \xrightarrow{j} H^5(Y;\widetilde{\mathbb{Z}})$$
$$\xrightarrow{2} H^5(Y;\widetilde{\mathbb{Z}}) \to \cdots . \qquad (4.19)$$

As before, we define $\widetilde{W}_5 = j(w_4)$; it is a 2-torsion element of $H^5(Y,\widetilde{\mathbb{Z}})$. Exactness of the sequence tells us that w_4 can be lifted to an integral class $\hat{x}$ if and only if $\widetilde{W}_5 = 0$. Thus, M theory can be formulated on a pin$^+$ manifold Y only if[41] $\widetilde{W}_5(Y) = 0$. For an example of a pin$^+$ manifold with $\widetilde{W}_5 \neq 0$, one may take $\mathbb{RP}^8$ or (to get an 11-dimensional example) $\mathbb{RP}^8 \times \mathbb{R}^3$. Indeed, $w_4(\mathbb{RP}^8)$ is the nonzero element of $H^4(\mathbb{RP}^8;\mathbb{Z}_2) \cong \mathbb{Z}_2$ and cannot be lifted to a twisted integral class, since $H^4(\mathbb{RP}^8;\widetilde{\mathbb{Z}}) = 0$.

D. Formula for the M2-brane path-integral measure

In this section, we will propose a formula for the path-integral measure of an M2-brane wrapped on $W \subset Y$. The consistency of this formula will be demonstrated in Sec. IV E.

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[40]What we have said so far is less than a full definition of an m_c connection, because specifying the curvature G and the characteristic class $\hat{x}$ do not specify C completely: One can still add to C a flat twisted three-form field. We implicitly complete the definition of an m_c connection in Sec. IV D by describing exactly how C couples to an M2-brane and verify the consistency of the picture in Sec. IV E. However, it would be desirable to have a more succinct definition of an m_c connection.

[41]This condition has apparently not been stated previously. However, a somewhat similar condition $W_7 = 0$ [where as in the previous examples $W_7 = j(w_6)$] was argued in Ref. [47].

Naively, as in Eq. (4.9), the path-integral measure is $\mathrm{Pf}(\slashed{D}) \exp(\mathrm{i} \int_W C)$. However, both factors have ambiguous signs: $\mathrm{Pf}(\slashed{D})$ has a sign problem because of the anomaly, and $\exp(\mathrm{i} \int_W C)$ has a sign problem because C is an m_c connection rather than an ordinary three-form gauge field. We aim to make sense of the product of these two factors.

To begin with, let us assume that W is the boundary of a submanifold $U \subset Y$ that has the same property as W: Its normal bundle is oriented. To be precise about this, note that once U is given, the normal bundle N to W in Y decomposes as $\mathfrak{o} \oplus N'|_W$, where $\mathfrak{o}$ is a trivial real line bundle (generated by the outward normal to W in U), N' is the normal bundle to U in Y, and $N'|_W$ is the restriction of N' to W. Because of this splitting, an orientation of N determines an orientation of $N'|_W$ and the condition we want on U is that this orientation of $N'|_W$ extends to an orientation of N'. This condition, along with $w_2(Y) = 0$, implies the obvious analog of Eqs. (4.2) and (4.4):

$$w_1(N') = 0, \quad w_2(N') = w_2(U). \qquad (4.20)$$

Given the choice of U, we will replace the ill-defined quantities $\exp(\mathrm{i} \int_W C)$ and $\mathrm{Pf}(\slashed{D})$ with well-defined expressions. Then we will argue that the product of these expressions does not depend on the choice of U.

Once U is picked, we can simply replace the ill-defined $\exp(\mathrm{i} \int_W C)$ with the well-defined $\exp(\mathrm{i} \int_U G)$. The motivation for this replacement is that the two would be equal if C were an ordinary three-form gauge field, with G obeying standard Dirac quantization. As it is, $\exp(\mathrm{i} \int_U G)$ is well-defined for given U, and is invariant under small displacements of U because $\mathrm{d}G = 0$. However, $\exp(\mathrm{i} \int_U G)$ does depend on the topological choice of U, because G obeys the shifted quantization condition (4.11).

As usual, to make sense of $\mathrm{Pf}(\slashed{D})$ once U is given, we invoke the Dai-Freed theorem [28]. This theorem is applicable if we are given on U a Dirac operator $\slashed{D}'$ that is related to $\slashed{D}$ in a suitable fashion along $W = \partial U$. In the present case, we take $\slashed{D}'$ to be the natural Dirac operator on U acting on $P|_U$ (that is, on the restriction to U of the pin$^+$ bundle P of Y). The Dai-Freed theorem suggests replacing $\mathrm{Pf}(\slashed{D})$ with $|\mathrm{Pf}(\slashed{D})| \exp(-\mathrm{i}\pi \eta/2)$, where here $\eta = \eta(\slashed{D}')$ is the eta invariant of the operator $\slashed{D}'$, with APS boundary conditions along $W = \partial U$. As usual, the justification for this step is that $|\mathrm{Pf}(\slashed{D})| \exp(-\mathrm{i}\pi \eta/2)$ is gauge invariant and well defined (once U is picked), depends on the metric only along W and not along U, and depends in a physically sensible[42] way on the metric of W.

The upshot of this is to replace $\mathrm{Pf}(\slashed{D}) \exp(\mathrm{i} \int_W C)$ with

$$\mathcal{Z}_U = |\mathrm{Pf}(\slashed{D})| \exp(-\mathrm{i}\pi \eta_U/2) \exp\left(\mathrm{i} \int_U G\right). \qquad (4.21)$$

Here we denote the eta invariant as η_U to emphasize its dependence on U. To show that the formula $\mathcal{Z}_U$ for the

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[42]"Physically sensible" means that when the metric of U is varied, the variation of $|\mathrm{Pf}(\slashed{D})| \exp(-\mathrm{i}\pi \eta/2)$ depends only on the variation of the metric of W. Moreover, this variation is given in the expected way by a matrix element of the stress tensor. An important consequence is that $|\mathrm{Pf}(\slashed{D})| \exp(-\mathrm{i}\pi \eta/2)$ varies smoothly with the expected simple zero when the metric variation is such that $\slashed{D}$ acquires a zero mode.

measure of the M2-brane path integral makes sense, we have to show that it is independent of the choice of U. There is a standard strategy to prove this, essentially illustrated in Fig. 2 of Sec. II B. We let U' be some other submanifold of Y with boundary W and let U^* be the closed manifold[43] obtained by gluing U to $-U'$. Because of the gluing law for the eta invariant, the desired relation $\mathcal{Z}_U = \mathcal{Z}_{U'}$ is equivalent to

$$\exp(-i\pi \eta_{U^*}/2)\exp\left(i\int_{U^*} G\right) = 1. \qquad (4.22)$$

This can be restated without reference to G using the flux quantization condition (4.13):

$$\exp(-i\pi \eta_{U^*}/2) = (-1)^{\int_{U^*} w_4(Y)}. \qquad (4.23)$$

This relation will be demonstrated in Sec. IV E.

At the outset of this section, we assumed that W is the boundary of some $U \subset Y$. If this is not so, we are in a situation somewhat like what was considered at the end of Sec. II F. If W is not a boundary, and thus represents a nontrivial element of the twisted homology $H_3(Y; \widetilde{\mathbb{Z}})$, then the path integral measure of an M2-brane wrapped on W cannot be determined from a knowledge of G only and not C. It would be possible to add to C a flat three-form gauge field with $\int_W C \neq 0$, leaving G unchanged but rotating by a constant phase the path-integral measure for an M2-brane wrapped on W. To understand how to proceed, suppose, for example, that $H_3(Y; \widetilde{\mathbb{Z}}) \cong \mathbb{Z}$ and pick a generator W_0. The phase of the path-integral measure for an M2-brane wrapped on W_0 can be changed in an arbitrary fashion by adding to C a flat three-form gauge field. Thus, we proceed by making an arbitrary choice of this phase, and then we attempt to use Eq. (4.21) to determine the path-integral measure for an M2-brane wrapped on any other $W \subseteq Y$. For this, we observe first that, since W_0 generates $H_3(Y; \widetilde{\mathbb{Z}})$, there is some integer n such that the disjoint union of W with n copies[44] of W_0 is the boundary of some $U \subset Y$. This means that the formula (4.21) determines the path-integral measure for a collection of M2-branes consisting of one brane wrapped on W and n others wrapped on (slightly separated copies of) W_0. Since the M2-brane path-integral measure is known for W_0, this determines it for W.

This procedure generalizes in a fairly obvious way for any $H_3(Y; \widetilde{\mathbb{Z}})$ and uniquely determines the M2-brane path-integral measure for any W, up to the possibility of twisting by a flat C field.

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[43]For dimensional reasons, one can assume that U and U' do not intersect away from W so that U^* is embedded in Y. However, the following would actually make sense without this assumption.

[44]We can displace these n copies slightly so that they do not intersect each other or W, and for dimensional reasons we can assume that U is embedded in Y. Once the phase of the path-integral measure is fixed for W_0, it can be determined for any small displacement of W_0 by evolving it continuously, relying on the fact that the M2-brane path integral suffers only from global anomalies. An alternative way to determine what happens in a small displacement of W_0 is to note that if W can be reached from W_0 by a small displacement, this means that the disjoint union of W and $-W_0$ is a boundary in Y; the argument in the text will then show that the path-integral measure for W is determined in terms of W_0.

E. Consistency of the formula

The statement (4.23) that we wish to justify depends only on the four-manifold $U^* \subset Y$ and the normal bundle N' to U^* in Y. Here Y is completely arbitrary except for the constraint that $w_2(Y) = 0$, and the only condition on N' is that it is oriented. The upshot is that Eq. (4.23) must hold with U^* replaced by absolutely any compact four-manifold X (without boundary) and N' replaced by any oriented rank 7 real vector bundle $B \to X$ that satisfies

$$w_1(B) = 0, \quad w_2(B) = w_2(X). \qquad (4.24)$$

For Y we can simply take the total space of the vector bundle B. We want to prove Eq. (4.23) in this generality. (In what follows, we write just η for $\eta_X = \eta_{U^*}$.)

The conditions (4.24) ensure that $w_2(TX \oplus B) = 0$, and therefore there is a corresponding pin$^+$ bundle P. On this pin$^+$ bundle P, there act γ matrices $\gamma_\mu, \mu = 1, \ldots, 4$ of TX and additional γ matrices $\widetilde{\gamma}_a, a = 1, \ldots, 7$ of B, obeying the Clifford algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}, \quad \{\widetilde{\gamma}_a, \widetilde{\gamma}_b\} = 2\delta_{ab}, \quad \{\gamma_\mu, \widetilde{\gamma}_a\} = 0. \quad (4.25)$$

Here $g_{\mu\nu}$ and δ_{ab} are, respectively, the Riemannian metric of X and the metric of B.

Assuming that fermions transform under spatial reflection by the four-dimensional analog of Eq. (2.4),

$$\mathsf{R}(\psi(x_1, x_2, x_3, x_4)) = \pm\gamma_1 \psi(-x_1, x_2, x_3, x_4), \qquad (4.26)$$

the standard Dirac operator $i\slashed{D} = i\sum_\mu \gamma^\mu D_\mu$ (where D_μ is the connection on P) is actually odd under a reflection, and the natural self-adjoint Dirac operator on a possibly unorientable four-manifold is not $i\slashed{D}$ but

$$\mathcal{D} = \overline{\gamma}\slashed{D}. \qquad (4.27)$$

(In going around an orientation-reversing loop in X, both $\overline{\gamma}$ and $\slashed{D}$ change sign so $\mathcal{D}$ is well defined.) On an orientable four-manifold, $\mathcal{D}$ is conjugate to $i\slashed{D}$ by $(1 + i\overline{\gamma})/\sqrt{2}$, so the choice between them does not matter, but $\mathcal{D}$ is the version that generalizes to an unorientable four-manifold. Informally, $\mathcal{D}$ is the Dirac operator on X with values in spinors of B, but that is a rough description as, in general, $w_2(TX)$ and $w_2(B)$ are nonzero, so that P exists but cannot be decomposed as the tensor product of a spin bundle of TX and one of B.

We have written Eq. (4.26) in four-dimensional terms, but from an 11-dimensional point of view, R may act as a reflection on some of the normal directions in B. For future reference, we will write the 11-dimensional version of the formula. If in local coordinates $x_1, x_2, \ldots, x_{11}$, a diffeomorphism ρ acts as -1 on some subset of coordinates $x_{I_1}, x_{I_2}, \ldots, x_{I_s}$, then a section Ψ of P transforms under ρ as

$$\rho\Psi(x) = \pm\Gamma_{I_1}\Gamma_{I_2}\cdots\Gamma_{I_s}\Psi(\rho(x)). \qquad (4.28)$$

Here $\Gamma_I, I = 1, \ldots, 11$ are the whole set of γ matrices $\gamma_1, \ldots, \gamma_4, \widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_7$. The undetermined sign $\pm$ is present, as usual, because the group that acts on spinors is a double cover of the orthogonal group.

Aiming to prove the identity (4.23), our first step is just to give, for any X, an example of a bundle B that obeys the conditions (4.24). This example may seem rather special, but it will turn out that studying it will give almost all we need. To

construct such an example, let TX be the tangent bundle of X, define the real line bundle[45] $\varepsilon = \det TX$, and let $\mathfrak{o}$ be a trivial real line bundle. Then an example of an allowed B is

$$B = \varepsilon \otimes TX \oplus \varepsilon \oplus \mathfrak{o} \oplus \mathfrak{o}. \qquad (4.29)$$

To show that $w_1(B) = 0$, $w_2(B) = w_2(TX)$, suppose first that $TX = \oplus_{i=1}^4 z_i$ is a direct sum of real line bundles z_i; this implies that $\varepsilon = \otimes_{i=1}^4 z_i$. In this case, one can verify the desired relations $w_1(B) = 0$, $w_2(B) = w_2(TX)$ using the Whitney sum formula (2.12) and the relation $w_1(\otimes_\alpha z_\alpha) = \sum_\alpha w_1(z_\alpha)$, valid for an arbitrary collection of real line bundles. The "splitting formula" for characteristic classes of vector bundles implies that the general result we want follows from this seemingly very special case.

For this particular B, the pin$^+$ bundle of $TX \oplus B$ can be described in a particularly simple and useful way. However, this will take a number of steps. Let us begin with the case that X is orientable, so (ignoring for now the action of orientation-reversing diffeomorphisms) we can set $\varepsilon = \mathfrak{o}$ and B becomes $B' \oplus \mathfrak{o}^{\oplus 3}$, with $B' = TX$. Moreover, to start with, let us omit the trivial summand $\mathfrak{o}^{\oplus 3}$ and just consider the spinors of $TX \oplus B' = TX \oplus TX$. Thus, the Clifford algebra of $TX \oplus B'$ in this special case is just the product of two copies of the Clifford algebra of TX. So we have two sets of γ matrices γ_μ and $\widetilde{\gamma}_\mu$, $\mu = 1, \ldots, 4$, and the algebra (4.25) becomes

$$\{\gamma_\mu, \gamma_\nu\} = \{\widetilde{\gamma}_\mu, \widetilde{\gamma}_\nu\} = 2g_{\mu\nu},$$
$$\{\gamma_\mu, \widetilde{\gamma}_\nu\} = 0. \qquad (4.30)$$

It is fairly well known that the doubled Clifford algebra (4.30) is related to differential forms on X, but there is a small twist that may come as a surprise. On differential forms on X, there acts a Clifford algebra that is generated, in a local coordinate system, by $\mathrm{d}x^\mu$ (that is, by the wedge product with the one-form $\mathrm{d}x^\mu$) and the corresponding contraction operator $\iota_{\mathrm{d}x^\mu}$. They obey

$$\{\mathrm{d}x^\mu, \mathrm{d}x^\nu\} = 0 = \{\iota_{\mathrm{d}x^\mu}, \iota_{\mathrm{d}x^\nu}\}, \quad \{\mathrm{d}x^\mu, \iota_{\mathrm{d}x^\nu}\} = \delta_\nu^\mu, \qquad (4.31)$$

and this algebra acts irreducibly on the differential forms $\Omega^*(X) = \oplus_{j=0}^4 \Omega^j(X)$ [here $\Omega^j(X)$ is the bundle of j forms on X]. We can reproduce this algebra from the doubled Clifford algebra (4.30) by setting

$$\mathrm{d}x^\mu = \frac{1}{\sqrt{2}}(\mathrm{i}\gamma^\mu + \widetilde{\gamma}^\mu), \quad \iota_{\mathrm{d}x^\nu} = \frac{1}{\sqrt{2}}(-\mathrm{i}\gamma_\nu + \widetilde{\gamma}_\nu). \qquad (4.32)$$

Thus, the doubled Clifford algebra (4.30) acts on the space $\Omega^*(X)$ of differential forms on X.

However, as the doubled Clifford algebra (4.30) and the corresponding algebra generated by $\mathrm{d}x^\mu$ and $\iota_{\mathrm{d}x^\nu}$ are both real, the factors of $\mathrm{i} = \sqrt{-1}$ that appear in Eq. (4.32) are unnatural. Technically, those factors mean that by this mapping, we can identify the complexification of $\Omega^*(X)$ with a complexification of a spin bundle of $TX \oplus TX$.

We can do better if we introduce an orientation of X and introduce the chirality operators $\overline{\gamma} = \gamma_1\gamma_2\gamma_3\gamma_4$, $\overline{\gamma}' = \widetilde{\gamma}_1\widetilde{\gamma}_2\widetilde{\gamma}_3\widetilde{\gamma}_4$

of the first and second summands of $TX \oplus TX$. We replace i with $\overline{\gamma}$ so that (4.32) becomes

$$\mathrm{d}x^\mu = \frac{1}{\sqrt{2}}(\overline{\gamma}\gamma^\mu + \widetilde{\gamma}^\mu), \quad \iota_{\mathrm{d}x^\nu} = \frac{1}{\sqrt{2}}(-\overline{\gamma}\gamma_\nu + \widetilde{\gamma}_\nu). \qquad (4.33)$$

This map is real and is simply an isomorphism between the doubled Clifford algebra of TX and the natural Clifford algebra that acts on Ω^*X.

Up to this point, there is a different isomorphism that we could have made instead:

$$\mathrm{d}x^\mu = \frac{1}{\sqrt{2}}(\gamma^\mu + \overline{\gamma}'\widetilde{\gamma}^\mu), \quad \iota_{\mathrm{d}x^\nu} = \frac{1}{\sqrt{2}}(\gamma_\nu - \overline{\gamma}'\widetilde{\gamma}_\nu). \qquad (4.34)$$

This also gives an isomorphism of the Clifford algebra of $TX \oplus TX$ with the algebra generated by $\mathrm{d}x^\mu$ and $\iota_{\mathrm{d}x^\nu}$. However, the mapping (4.33) is the one we want, because under this mapping, the Dirac operator $\mathcal{D}$ defined in Eq. (4.27) maps to the natural Dirac-like operator $\mathrm{d} + \mathrm{d}^* = (\mathrm{d}x^\mu - g^{\mu\nu}\iota_{\mathrm{d}x^\nu})D_\mu$ acting on $\Omega^*(X)$.

An important operator on $\Omega^*(X)$ is the operator $(-1)^{\mathsf{F}}$ that takes the value $(-1)^d$ on a differential form of degree d. This operator anticommutes with $\mathrm{d}x^\mu$ and $\iota_{\mathrm{d}x^\mu}$, and thus with all the γ matrices. So it is the overall chirality operator

$$\overline{\Gamma} = \overline{\gamma}\,\overline{\gamma}' = \gamma_1\gamma_2\gamma_2\gamma_4\gamma_1'\gamma_2'\gamma_3'\gamma_4'. \qquad (4.35)$$

[In a moment, it will be important that this operator can be defined globally even if X is unorientable. That is because $w_1(TX \oplus \varepsilon \otimes TX) = 0$.]

As a step toward considering what happens if X is unorientable, we will now analyze the effects of an orientation-reversing diffeomorphism of X in the above story. With this in view, we return to Eq. (4.29) and note that the first summand B' of B is really $\varepsilon \otimes TX$ rather than TX. Here ε is a trivial line bundle if X is orientable, so we have omitted it so far, but an orientation-reversing diffeomorphism of X acts as -1 on ε, so it will be important henceforth to include this factor. Also, it will be important to include the additional summand ε of B. Thus, when X is possibly unorientable, we will compare $\Omega^*(X)$ to a pin$^+$ bundle of $TX \oplus B''$, where $B'' = B' \oplus \varepsilon = \varepsilon \otimes TX \oplus \varepsilon$.

For the moment, however, we still assume that X is orientable and describe the spin bundle of $TX \oplus B'' = TX \oplus \varepsilon \otimes TX \oplus \varepsilon$. To do this, we need to extend the Clifford algebra that we used before by adding a ninth generator γ_ε, corresponding to the summand ε of B''. This ninth γ matrix must anticommute with the eight that we already have, so it must be[46] $\gamma_\varepsilon = \overline{\Gamma}$, where $\overline{\Gamma}$ was defined in Eq. (4.35). Note that we can add this new generator to the Clifford algebra without changing the bundle that the algebra acts on.

Next we compare the action of an orientation-reversing diffeomorphism on a spin bundle of $TX \oplus B''$ to its action on $\Omega^*(X)$. Let us consider an orientation-reversing reflection of X that in local coordinates acts as

$$\rho(x_1, x_2, x_3, x_4) = (-x_1, x_2, x_3, x_4), \qquad (4.36)$$

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[45] Here ε, defined in another way in footnote 36, is a trivial real line bundle on which an orientation-reversing symmetry acts as -1.

[46] We could reverse the sign of γ_ε, but we will be including both signs momentarily anyway.

and acts accordingly on B. By virtue of Eq. (4.28), the element of the Clifford algebra that will represent this reflection is $\rho = \gamma_1 \widetilde{\gamma}_2 \widetilde{\gamma}_3 \widetilde{\gamma}_4 \gamma_\varepsilon$. (Here we include γ_1 and not γ_i for $i = 2,3,4$ because only x_1 is being reflected; for the $\widetilde{\gamma}_\mu$, we make the opposite choice and include $\widetilde{\gamma}_i$, $i = 2,3,4$, because these are γ matrices of $\varepsilon \otimes TX$, where the reflection acts as -1 on ε; and likewise because ε is odd under the reflection, we include γ_ϵ.) With $\gamma_\varepsilon = \overline{\Gamma}$, we get $\rho = \gamma_2 \gamma_3 \gamma_4 \widetilde{\gamma}_1$. Using the definitions (4.34), this means that ρ anticommutes with $\mathrm{d}x^1$ and $\iota_{\mathrm{d}x^1}$ and commutes with $\mathrm{d}x^i$ and $\iota_{\mathrm{d}x^i}$ for $i > 1$. This is exactly the expected action of ρ on differential forms.

Thus, the identification between the spin bundle of $TX \oplus B''$ and $\Omega^*(X)$ maps the action of an orientation-reversing diffeomorphism on one side to the action of such a diffeomorphism on the other side. Now we are finally ready to state a similar identification for the case that X is not necessarily orientable.

Suppose that the unorientable manifold X is the quotient of an orientable manifold X' by a $\mathbb{Z}_2$ symmetry generated by a freely acting orientation-reversing diffeomorphism ρ. After identifying as before the spin bundle of $TX' \oplus \varepsilon \otimes TX' \oplus \varepsilon$ with $\Omega^*(X')$, and then dividing by ρ on both sides, we learn that, as promised, we can identify a pin$^+$ bundle of $TX \oplus B''$ with $\Omega^*(X)$.

To be more precise, this argument shows that there is a particular pin$^+$ bundle of $TX \oplus B''$ that can be identified with $\Omega^*(X)$. Any other pin$^+$ bundle is obtained by twisting this one with a real line bundle α over X. Thus, a general pin$^+$ bundle of $TX \oplus B''$ is what we might call $\Omega^*_\alpha(X)$, the differential forms on X twisted by the real line bundle α. For example, if as before we construct X as $X'/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is generated by ρ, then starting with the differential forms $\Omega^*(X')$ and dividing by $\mathbb{Z}_2$, the two choices of sign in the action of ρ on the pin$^+$ bundle $\Omega^*(X')$ will give us on the quotient either $\Omega^*(X)$ or $\Omega^*_\varepsilon(X)$, the differential forms on X twisted by ε.

Finally, let us discuss the effect of including the last two summands $\mathfrak{o} \oplus \mathfrak{o}$ in Eq. (4.29). To do this, we have to double the rank of the pin$^+$ bundle. There is a straightforward construction. Let Γ'_I, $I = 1, \ldots, 9$ be the whole set of γ matrices of $TX \oplus B''$ (thus, the Γ'_I are γ_μ and $\widetilde{\gamma}_\mu$ for $\mu = 1, \ldots, 4$ and γ_ε). For the full set of 11 γ matrices of $TX \oplus B$, we can take

$$\Gamma_I = \Gamma'_I \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$I = 1, \ldots, 9,$$
$$\Gamma_{10} = \mathbf{I} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\Gamma_{11} = \mathbf{I} \otimes \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \tag{4.37}$$

[Here $\mathbf{I}$ is the identity matrix. It is not possible to choose all 11 γ matrices to be real, as the spinors of Spin(11) are pseudoreal.] If we ignore the last two γ matrices, which do not enter the construction of the Dirac operator, this tells us that the pin$^+$ bundle of $TX \oplus B$ is the direct sum of two pin$^+$ bundles of $TX \oplus B''$ except that in one copy the γ matrices of $TX \oplus B'$ all have opposite sign. What is the effect of this sign reversal? Reversing the signs of γ_μ and $\widetilde{\gamma}_\nu$, $\mu, \nu = 1, \ldots, 4$, is not very consequential since one can compensate for this by

conjugating by γ_ε. However, changing the sign of γ_ε reverses the sign of the element of the Clifford algebra that represents an orientation-reversing symmetry. (For instance, in the example considered earlier, reversing the sign of γ_ε changes the sign of $\rho = \gamma_1 \widetilde{\gamma}_2 \widetilde{\gamma}_3 \widetilde{\gamma}_4 \gamma_\varepsilon$.) This has the effect of replacing $\Omega^*(X)$ by $\Omega^*_\varepsilon(X)$ and more generally for any real line bundle α of replacing $\Omega^*_\alpha(X)$ by $\Omega^*_{\alpha \otimes \varepsilon}(X)$.

The upshot then is that one particular pin$^+$ bundle P of $TX \oplus B$ is equivalent to $\Omega^*(X) \oplus \Omega^*_\varepsilon(X)$ and a general pin$^+$ bundle P_α of $TX \oplus B$, obtained by twisting P by a real line bundle α, is $\Omega^*_\alpha(X) \oplus \Omega^*_{\alpha \otimes \varepsilon}(X)$.

Now we can calculate the eta invariant and verify the identity (4.23). We use the fact that the Dirac operator $\mathcal{D}$ acting on P reduces to the operator $\mathrm{d} + \mathrm{d}^*$ acting on $\Omega^*(X) \oplus \Omega^*_\varepsilon(X)$ (or a version of this twisted by some α, as we discuss shortly). Moreover, $\mathrm{d} + \mathrm{d}^*$ anticommutes with the operator $(-1)^\mathsf{F}$ that measures the degree of a differential form mod 2. Accordingly, the spectrum of the operator $\mathrm{d} + \mathrm{d}^*$ is invariant under $\lambda \leftrightarrow -\lambda$. This implies that nonzero eigenvalues of $\mathcal{D}$ make no net contribution to the eta invariant. This invariant is therefore simply the number of zero modes of $\mathcal{D}$.

Let b_i, $i = 1, \ldots, 4$ be the ith Betti number of X [the number of i-form zero modes of $\mathrm{d} \oplus \mathrm{d}^*$ acting on $\Omega^*(X)$]. The total number of zero modes of $\mathrm{d} + \mathrm{d}^*$ is $\sum_{i=1}^4 b_i$, and this is the eta invariant of $\mathcal{D}$ acting on $\Omega^*(X)$. When $\mathrm{d} \oplus \mathrm{d}^*$ acts on $\Omega^*_\varepsilon(X)$, the number of i-form zero modes is[47] b_{4-i}, and the total number of zero modes is again $\sum_{i=1}^4 b_i$. Thus, the eta invariant of $\mathcal{D}$ acting on the pin$^+$ bundle $\Omega^*(X) \oplus \Omega^*_\varepsilon(X)$ is $\eta = 2 \sum_{i=1}^4 b_i$.

Let us compare this to the Euler characteristic of X, defined as $\chi = \sum_{i=0}^4 (-1)^i b_i$. We see immediately that

$$\eta = 2\chi \bmod 4. \tag{4.38}$$

Therefore, we can evaluate the left-hand side of the identity (4.23) that we are trying to verify:

$$\exp(-\pi \mathrm{i} \eta / 2) = (-1)^\chi. \tag{4.39}$$

To evaluate the right-hand side of this identity, we let $\widehat{\chi}$ be the Euler class of TX. The Euler characteristic of X is $\chi = \int_X \widehat{\chi}$, and, on the other hand, the mod 2 reduction of $\widehat{\chi}$ is the Stieffel-Whitney class w_4. So $(-1)^\chi = (-1)^{\int_X w_4}$ and we get the desired identity,

$$\exp(-\pi \mathrm{i} \eta / 2) = (-1)^{\int_X w_4}. \tag{4.40}$$

It is straightforward to extend this computation to the general case that $\mathcal{D} = \mathrm{d} + \mathrm{d}^*$ acts on $\Omega^*_\alpha(X) \oplus \Omega^*_{\alpha \otimes \varepsilon}$ for some α. Let $b_{i,\alpha}$ be the number of zero modes of $\mathrm{d} + \mathrm{d}^*$ acting on $\Omega^*_\alpha(X)$. One might call these the Betti numbers of the α-twisted cohomology. Then $b_{i,\alpha \otimes \varepsilon} = b_{4-i,\alpha}$ as in footnote 47. Just as before, $\eta = 2 \sum_i b_{i,\alpha}$ and hence is congruent mod 4 to the

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[47] On an orientable manifold, the Hodge $\star$ operator establishes an isomorphism between $\Omega^i(X)$ and $\Omega^{4-i}(X)$, leading to an identity $b_i = b_{4-i}$. However, because $\star$ is odd under reversal of orientation, on an unorientable manifold, it maps $\Omega^i(X)$ to $\Omega^{4-i}_\varepsilon(X)$, showing that the number of i-form zero modes of $\mathrm{d} + \mathrm{d}^*$ acting on $\Omega^*(X)$ is the number of $(4-i)$-form zero modes of $\mathrm{d} + \mathrm{d}^*$ acting on $\Omega^*_\varepsilon(X)$, and vice versa.

Euler characteristic of the α-twisted cohomology, which is defined as $\chi_\alpha = \sum_{i=0}^{4} (-1)^i b_{i,\alpha}$. However, the Gauss-Bonnet formula expressing the Euler characteristic as a curvature integral is valid for any α and shows that χ_α is independent of α. Hence, $\exp(-\pi i \eta / 2)$ is independent of α and the identity (4.40) holds for all α.

So far we have shown that for any X, there is some choice of B—described in Eq. (4.29)—such that the identity (4.40) holds. To complete the argument, we will now show the following: For a given X, the identity (4.40) holds for one choice of B if and only if it holds for any B. In proving this, we will use the fact that the left- and right-hand sides of Eq. (4.40) are both cobordism invariant. Cobordism invariance means that they equal 1 if X is the boundary of a five-manifold Z that is endowed with appropriate structure. Concerning the right-hand side of Eq. (4.40), no additional structure is needed; the cobordism invariance of $(-1)^{\int_X w_4}$ is a special case of cobordism invariance of Stieffel-Whitney numbers (integrals of products of Stieffel-Whitney classes). On the left-hand side, we are given X and B and a pin$^+$ structure of $TX \oplus B$. We have to require that the rank 7 real vector bundle B over X extends to a rank 7 real vector bundle B_Z over Z, such that there is a pin$^+$ structure of $TZ \oplus B_Z$ that restricts on the boundary to the given pin$^+$ structure of $TX \oplus B$. (These conditions are straightforwardly satisfied in the example considered in the next paragraph.) Under these circumstances, we can define a Dirac operator $\mathcal{D}_Z$ on Z that is related to the Dirac operator $\mathcal{D}$ of X in a way that will let us invoke the Atiyah-Patodi-Singer index theorem [30]. Let η be the eta invariant of the Dirac operator on X and $\mathcal{I}$ the index[48] of $\mathcal{D}_Z$. The APS index theorem says that $\eta/2 = \mathcal{I}$, but, on the other hand, $\mathcal{I}$ is even because of a version of Kramers doubling. So η is a multiple of 4 and $\exp(-\pi i \eta / 2) = 1$, establishing the cobordism invariance of the left-hand side of Eq. (4.40).

Now we will use cobordism invariance to show that for a given X the identity (4.40) is true for all B if it is true for one B. Given a triangulation of B, the constraints $w_1(B) = 0$, $w_2(B) = w_2(X)$, determine B on the two-skeleton of X. Since $\pi_2(SO(7)) = 0$, there is no further choice to be made on the three-skeleton of X. So any possible B differs from the particular choice in Eq. (4.29) at most by a local modification made near a point $p \in X$. The local modification is made by gluing in a certain number of instantons.[49] These are ordinary
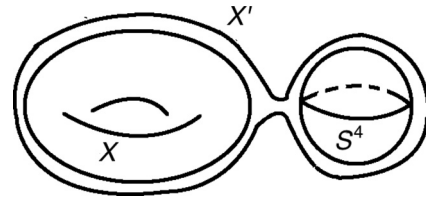

FIG. 4. Like peas in a pod, the disjoint union of a four-manifold X and a four-sphere S^4 is cobordant to their connected sum X'. The connected sum is defined by cutting an open ball out of each and gluing them together along their boundaries. The normal bundles are trivialized in the cutting and gluing region before this is done.

gauge theory instantons, classified by $\pi_3(SO(7)) = \mathbb{Z}$, where $SO(7)$ is the structure group of B. To use these facts, start with a copy of X, endowed with some normal bundle B for which we know that the identity (4.40) holds, and a copy of S^4, endowed with a normal bundle of some instanton number n. Since S^4 is orientable, the identity (4.40) holds for S^4 with any normal bundle by virtue of the computation in Ref. [24] (the basic idea was explained in Sec. IV C). Now we form the connected sum of X and S^4, by cutting a small open ball out of each and gluing them together along the resulting boundaries.[50] The connected sum is a manifold X' that is topologically equivalent to X, but with a local modification of B that shifts the instanton number by n. The identity we want is true for X and for S^4 and therefore for their disjoint union, and X' is cobordant to the disjoint union of X and S^4 (Fig. 4). So the identity is true for X'. Notice that this argument works for any choice of the pin$^+$ structure of $TX \oplus B$.

It is possible to make a similar argument using the gluing theorem for the eta invariant, rather than its cobordism invariance.

V. SUMMARY

In Sec. II of this paper, we extended the standard analysis of the "parity" anomaly for fermions in a spacetime of odd dimension to the case of a possibly unorientable manifold. We gave a general description of the anomaly, valid in any dimension, in terms of the eta invariant in one dimension more. In the important case of $2 + 1$ dimensions, we made this formula more concrete for gauge theories with gauge group $\mathbb{Z}_2$, U(1), or SU(2) and described methods by which this could be done for any gauge group.

A key result of Sec. II was to show that the usual "parity" anomaly on an orientable manifold is not the whole story, and that a theory which is anomaly-free if considered on an orientable manifold only may be anomalous if formulated on an unorientable manifold.

The rest of the paper was devoted to two applications. Section III was devoted to an application to topological

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[48]On a pin$^+$ manifold of odd dimension, the Dirac operator maps sections of one pin$^+$ bundle to sections of the complementary pin$^+$ bundle. Thus, it is not self-adjoint (that is why the anomaly studied in the three-dimensional case in Sec. II is possible) and, in particular, can have a nonzero index. This index vanishes on a compact manifold without boundary (this statement is a special case of the ordinary Atiyah-Singer index theorem), but on a manifold with boundary, with APS boundary conditions, the index can be nonzero. For example, see Appendixes B.4 and C of [7].

[49]If X is unorientable—which is the most relevant case as the identity of interest was already proved in Ref. [24] for orientable X—then the number of instantons that are glued in is only an invariant mod 2. (An instanton that is transported around an orientation-reversing loop in X comes back as an anti-instanton.) So for unorientable X, there are actually only two possible choices of B up to isomorphism:

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the one in Eq. (4.29) and one with a different value of w_4, reached by gluing in a single instanton. The proof given in the text does not require a knowledge of this fact.

[50]The normal bundle B to X in Y and the analogous normal bundle to S^4 are trivialized along these small balls before the cutting and gluing is done.

superconductors. Gapped boundary states of topological insulators and superconductors can be constructed by postulating suitable emergent gauge fields, matter fields, and symmetry-breaking patterns on the surface of a material. This approach was developed in Ref. [22], mainly but not entirely in the context of a topological insulator. To fully exploit this approach in the case of a topological superconductor requires a more complete understanding of the "parity" anomaly than was available previously. In Sec. III, we use the results of Sec. II to analyze more fully and precisely the gapped boundary states of a topological superconductor that can be constructed using these methods. A typical application is to show that the boundary of a topological superconductor with $\nu = 16$, but not one with $\nu = 8$, can be gapped in a topologically trivial way (that is, without introducing topological order on the boundary).

The existing literature on gapped boundary states of topological insulators and superconductors is extensive and varied; see, for example, [4,5,10,11,13–17,19–21]. One difference between the present paper (and the previous one [22]) and some other approaches is that our treatment is completely explicit, based on weak coupling, with no reliance on nonexplicit dynamics.[51]

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[51] There is one partial exception to this. In Secs. III B 5 and III B 6, we describe three ways to gap the boundary of a $\nu = 16$ topological superconductor in a without introducing topological order. One of the three approaches relies on standard but nonexplicit dynamics, namely confinement in SU(2) gauge theory without matter fields in $2 + 1$ dimensions.

The explicit nature of the dynamics has made it possible to see in Ref. [22] and in the present paper an important feature of these boundary states. This is a group extension: The bulk relation $\mathsf{T}^2 = (-1)^F$ is generalized along the boundary to $\mathcal{T}^2 = (-1)^F \mathsf{K}$, where K generates a discrete group of unbroken gauge symmetries. It is not clear to the author to what extent this has been understood in the previous literature. In Sec. III C, we show that an analogous group extension gives a unified way to construct gapped boundary states of bosonic SPT states. The presentation in this section is somewhat abstract, but this will hopefully be remedied in forthcoming work.

Section IV applies similar methods to a problem in M theory. M theory describes, among other things, membranes, usually called M2-branes, with $2 + 1$-dimensional worldvolume. It has been known for some time [24] that consistency of the M2-brane path integral depends on a fairly subtle cancellation of the "parity" anomaly. However, this cancellation has hitherto been understood only for the case that the M2-brane worldvolume is orientable. With the help of the tools developed in Sec. II, the unorientable case is treated in Sec. IV.

[1] A. N. Redlich, Parity violation and gauge noninvariance of the effective gauge field action in three-dimensions, Phys. Rev. D **29**, 2366 (1984).

[2] A. J. Niemi and G. W. Semenoff, Axial-Anomaly-Induced Fermion Fractionization and Effective Gauge-Theory Actions in Odd-Dimensional Space-Times, Phys. Rev. Lett. **51**, 2077 (1983).

[3] L. Alvarez-Gaumé, S. Della Pietra, and G. Moore, Anomalies and odd dimensions, Ann. Phys. **163**, 288 (1985).

[4] C. Wang and T. Senthil, Interacting fermionic topological insulators/superconductors in three dimensions, Phys. Rev. B **89**, 195124 (2014).

[5] M. A. Metlitski, L. Fidkowski, X. Chen, and A. Vishwanath, Interaction effects on 3d topological superconductors: Surface topological order from vortex condensation, the 16-fold way, and fermionic Kramers doublets, arXiv:1406.3032.

[6] C.-T. Hsieh, G. C. Cho, and S. Ryu, Global anomalies on the surface of symmetry-protected topological phases in $(3 + 1)$ dimensions, Phys. Rev. B **93**, 075135 (2016).

[7] E. Witten, Fermion path integrals and topological phases, Rev. Mod. Phys. **88**, 035001 (2016).

[8] A. Kapustin, R. Thorngren, A. Turzillo, and Z. Wang, Fermionic symmetry protected topological phases and cobordisms, J. High Energy Phys. **52**, 1512 (2015).

[9] M. Metlitski, S-duality of $\mathcal{U}(1)$ gauge theory with $\theta = \pi$ on non-orientable manifolds: Applications to topological insulators and superconductors, arXiv:1510.05663.

[10] L. Fidkowski, X. Chen, and A. Vishwanath, Non-Abelian Topological Order on the Surface of a 3d Topological Superconductor From an Exactly Solved Model, Phys. Rev. X **3**, 041016 (2013).

[11] A. Kitaev, Homotopy-theoretic approach to SPT phases in action: $\mathbb{Z}_{16}$ classification of three-dimensional superconductors, lecture notes available at http://www.ipam.ucla.edu/abstract/?tid=12389.

[12] Y. Tachikawa and K. Yonekura, Gauge interactions and topological phases of matter, arXiv:1604.06184.

[13] M. A. Metlitski, C. L. Kane, and M. P. A. Fisher, A symmetry-respecting topologically-ordered surface phase of 3D electron topological insulators, Phys. Rev. B **92**, 125111 (2015).

[14] P. Bonderson, C. Nayak, and X. L. Qi, A time-reversal invariant topological phase at the surface of a 3D topological insulator, J. Stat. Mech. (2013) P09016.

[15] X. Chen, L. Fidkowski, and A. Vishwanath, Symmetry enforced non-Abelian topological order at the surface of a topological insulator, Phys. Rev. B **89**, 165132 (2014).

[16] C. Wang, A. C. Potter, and T. Senthil, Gapped symmetry preserving surface state for the electron topological insulator, Phys. Rev. B **88**, 115137 (2013).

[17] C. Wang, A. C. Potter, and T. Senthil, Classification of interacting electronic topological insulators in three dimensions, Science **343**, 629 (2014).

[18] M. A. Metlitski and A. Vishwanath, Particle-vortex duality of 2D Dirac fermion from electric-magnetic duality of 3d topological insulators, Phys. Rev. B **93**, 245151 (2016).

[19] D. F. Mross, A. Essin, and J. Alicea, Composite Dirac Liquids: Parent States for Symmetric Surface Topological Order, Phys. Rev. X **5**, 011011 (2015).

[20] A. Vishwanath and T. Senthil, Physics of Three Dimensional Bosonic Topological Insulators: Surface Deconfined Criticality and Quantized Magnetoelectric Effect, Phys. Rev. X **3**, 011016 (2013).

[21] M. A. Metlitski, L. Fidkowski, X. Chen, and A. Vishwanath, Interaction effects on 3D topological superconductors: Surface topological order from vortex condensation, the 16-fold way, and fermionic Kramers doublets, arXiv:1406.3032.

[22] N. Seiberg and E. Witten, Gapped boundary phases of topological insulators via weak coupling, Prog. Theor. Exp. Phys. (2016) 12C101.

[23] A. M. Polyakov, Compact gauge fields and the infrared catastrophe, Phys. Lett. B **59**, 82 (1975).

[24] E. Witten, On flux quantization in m theory and the effective action, J. Geom. Phys. **22**, 1 (1997).

[25] C. G. Callan, Jr., and J. A. Harvey, Anomalies and fermion zero-modes on strings and domain walls, Nucl. Phys. B **250**, 427 (1985).

[26] J. Distler, D. Freed, and G. W. Moore, Spin structures and superstrings, arXiv:1007.4581.

[27] J. Distler, D. Freed, and G. W. Moore, Orientifold precis, in *Mathematical Foundations of Quantum Field Theory and Perturbative String Theory*, edited by H. Sati, Proceedings of the Symposium on Pure Mathematics Vol. 83 (American Mathematical Society, Providence, RI, 2011).

[28] X. Dai and D. S. Freed, η-invariants and determinant lines, J. Math. Phys. **35**, 5155 (1994).

[29] K. J. Yonekura, J. High Energ. Phys. 09 (2016) 022.

[30] M. F. Atiyah, I. M. Singer, and V. Patodi, Spectral asymmetry in Riemannian geometry, Math. Proc. Cambridge Philos. Soc. **77**, 43 (1975).

[31] M. Z. Hasan and C. L. Kane, Topological insulators, Rev. Mod. Phys. **82**, 3045 (2010).

[32] X.-L. Qi and S.-C. Zhang, Topological insulators and superconductors, Rev. Mod. Phys. **83**, 1057 (2011).

[33] B. A. Bernewig with T. L. Hughes, *Topological Insulators and Topological Superconductors* (Princeton University Press, Princeton, NJ, 2013).

[34] D. S. Freed and M. J. Hopkins, Reflection positivity and invertible topological phases, arXiv:1604.06527.

[35] V. Borokhov, A. Kapustin, and X. K. Wu, Topological disorder operators in three-dimensional conformal field theory, J. High Energy Phys. 11 (2002) 049.

[36] R. Jackiw and C. Rebbi, Solitons with fermion number 1/2, Phys. Rev. D **13**, 3398 (1976).

[37] N. Read and D. Green, Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries, and the fractional quantum Hall effect, Phys. Rev. B **61**, 10267 (2000).

[38] G. W. Moore and N. Read, Nonabelions in the fractional quantum Hall effect, Nucl. Phys. B **360**, 362 (1991).

[39] R. Dijkgraaf and E. Witten, Topological gauge theories and group cohomology, Commun. Math. Phys. **129**, 393 (1990).

[40] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Symmetry protected topological orders and the group cohomology of their symmetry group, Phys. Rev. B **87**, 155114 (2013).

[41] D. S. Freed and G. W. Moore, Setting the quantum integrand of M-theory, Commun. Math. Phys. **369**, 89 (2006).

[42] X.-G. Wen, Quantum orders and symmetric spin liquids, Phys. Rev. B **65**, 165113 (2002).

[43] A. M. Essin and M. Hermele, Classifying fractionalization: Symmetry classification of gapped $\mathbb{Z}_2$ spin liquids in two dimensions, Phys. Rev. B **87**, 104406 (2013).

[44] A. Mesaros and Y. Ran, A classification of symmetry enriched topological phases with exactly solvable models, Phys. Rev. B **87**, 155115 (2013).

[45] L.-Y. Hung and X.-G. Wen, Quantized topological terms in weak-coupling gauge theories with a global symmetry and their connection to symmetry-enriched topological phases, Phys. Rev. B **87**, 165107 (2013).

[46] X. Chen, F. Burnell, A. Vishwanath, and L. Fidkowski, Anomalous Symmetry Fractionalization and Surface Topological Order, Phys. Rev. X **5**, 041013 (2015).

[47] E. Diaconescu, G. Moore, and E. Witten, E_8 gauge theory, and a derivation of K-theory from M-theory, Adv. Math. Theor. Phys. **6**, 1031 (2003).