Comment on "Critical point scaling of Ising spin glasses in a magnetic field"

T. Temesvári*

MTA-ELTE Theoretical Physics Research Group, Eötvös University, Pázmány Péter sétány 1/A, H-1117 Budapest, Hungary

(Received 14 April 2016; published 1 November 2016)

In a section of a recent paper [Phys. Rev. B **91**, 104432 (2015)], the authors discuss some of the arguments in the paper by Parisi and Temesvári [Nucl. Phys. B **858**, 293 (2012)]. In this Comment, it is shown how these arguments are misinterpreted and the existence of the Almeida-Thouless transition *in* the upper critical dimension six reasserted.

DOI: 10.1103/PhysRevB.94.176401

In a recent paper by Yeo and Moore [1] about the long debated existence of the Almeida-Thouless (AT) instability [2] in the short-ranged Ising spin glass below the upper critical dimension six, the authors criticize in Sec. III some of our statements and arguments in Ref. [3]. In that paper we have demonstrated: First the incorrect reasoning of Ref. [4] about the disappearance of the AT transition line when approaching the upper critical dimension from above; second we have computed the AT line staying exactly in six dimensions (and not by a limiting process); and third the ϵ expansion was used to compute the AT line below six dimensions, and the relatively smooth behavior of it while crossing $d = 6$ (with fixed bare parameters) was exhibited. In what follows, we want to comment on the discussion in Sec. III of Ref. [1].

AT AND ABOVE SIX DIMENSIONS

The first-order renormalization-group (RG) equations for the six-dimensional model are worked out and solved in Sec. 3 of Ref. [3], the AT line follows from that calculation (see Eq. (37) in Ref. [3])[1]:

$$h_{\mathrm{AT}}^2 = \frac{4}{(1 - w^2 \ln |r| + \frac{10}{3} w^2 \ln w)^4}\; w|r|^2 \approx \frac{4}{(1 - w^2 \ln |r|)^4}\; w|r|^2, \qquad d = 6, \tag{1}$$

where $w^2 \ll 1$ was used. (Note that a minus sign in the denominator of Eq. (13) has been left out in Ref. [1].) As it turns out from the discussion in Sec. 3 of Ref. [3], this approximation is valid if the scaling variable with zero scaling dimension (which is invariant under the RG in $d = 6$) is small, i.e.,

$$\frac{w^2}{1 + \frac{5}{3} w^2 \ln\, w^2 - w^2 \ln |r|} \ll 1, \tag{2}$$

and this condition is always satisfied whenever $|r| \ll 1$ and $w^2 \ll 1$; see also the middle part of Eq. (59) of that reference. Yeo and Moore [1] forget all about this derivation of the six-dimensional AT line; they deduce it from Eq. (11) of Ref. [1] by the limit $\epsilon \to 0$, and finally they argue that "Eq. (11) is not valid for this limit." We can absolutely agree with this last statement: The system at the upper critical dimension needs special care, physical quantities, such as the critical magnetic field where replica symmetry breaking sets in, cannot be obtained by a limiting process of $\epsilon \to 0$. The point is that ϵ in Eq. (11) may be small but fixed, whereas $|r| \ll 1$, and the $|r|^{\epsilon/2}$ term in the denominator must be ignored. Taking account of this, the AT line above dimension six, Eq. (11) of Ref. [1], must be written (consistently with the approximations used to derive it) as

$$h_{\mathrm{AT}}^2 \sim \frac{w|r|^{(d/2)-1}}{\left(\frac{2w^2}{\epsilon} + 1\right)^{(5d/6)-1}}, \quad d > 6. \tag{3}$$

This is just Eq. (28) of Ref. [3]. This equation for the AT line above six dimensions must be supplemented by the range of its applicability, otherwise false conclusions, such as Eq. (12) in Ref. [1] [which is obviously incompatible with (1)] could be deduced. For this reason, we briefly repeat the two steps needed for the derivation of (3):

(1) The RG equations for the three bare parameters, namely,

$$\dot{|r|} = \left(2 - \frac{10}{3} w^2\right)|r|, \tag{4}$$

$$\dot{w^2} = -\epsilon w^2 - 2w^4, \tag{5}$$

$$\dot{h^2} = \left(4 + \frac{\epsilon}{2} + \frac{1}{3} w^2\right) h^2 \tag{6}$$

are valid for $|r| \ll 1$ and $w^2 \ll 1$. One can introduce the nonlinear scaling fields [5] satisfying exactly, by definition, the linearized (around the fixed point) and diagonalized RG equations. For the system in (4) and for its Gaussian fixed point, one readily finds

$$\dot{g_{|r|}} = 2 g_{|r|}, \quad \dot{g_{w^2}} = -\epsilon g_{w^2} \quad \text{and} \quad \dot{g_{h^2}} = \left(4 + \frac{\epsilon}{2}\right) g_{h^2}.$$

*temtam@helios.elte.hu

[1] We use here the notations of Ref. [1]. In fact $|r|$ was called τ in Ref. [3], whereas r played the role of the nonlinear scaling field associated with τ. We also adapt here to the somewhat unconventional use of the symbol ϵ as $\epsilon = d - 6$.

The relations between bare parameters and nonlinear scaling fields were published in Ref. [3], for completeness we repeat them here

$$|r| = g_{|r|}\left(1-\frac{2}{\epsilon}g_{w^2}\right)^{-(5/3)}, \quad w^2 = g_{w^2}\left(1-\frac{2}{\epsilon}g_{w^2}\right)^{-1} \quad \text{and}$$
$$h^2 = g_{h^2}\left(1-\frac{2}{\epsilon}g_{w^2}\right)^{1/6}. \tag{7}$$

(2) The zeros of the scaling function of the replicon mass $\hat{\Gamma}_R$ are the locations of the AT instability. $\hat{\Gamma}_R$ depends on the bare parameters $|r|$, w^2, and h^2 through the RG invariants $x \equiv g_{w^2}g_{|r|}^{\epsilon/2}$ and $y \equiv g_{h^2}g_{|r|}^{-2-(\epsilon/4)}$. The AT instability line can then be written as $y = f(x)$ or

$$g_{h^2_{\rm AT}} = g_{|r|}^{2+(\epsilon/4)} f\left(g_{w^2}g_{|r|}^{(\epsilon/2)}\right) = \frac{g_{|r|}^2}{\sqrt{g_{w^2}}} g\left(g_{w^2}g_{|r|}^{\epsilon/2}\right), \quad \text{with}$$
$$g(x) \equiv \sqrt{x}\, f(x). \tag{8}$$

The following remarks are now in order:

(i) This form of the AT line is generic for the system where the zero-external-magnetic-field symmetry is broken only by the linear replica symmetric invariant in the Lagrangian whose bare coupling constant is h^2. (This model is used in Refs. [1,4] too.) Equation (7) cannot be used in this generic case to replace nonlinear scaling fields by bare couplings as they were derived from the one-loop RG equations in (4), (5), and (6).

(ii) Equation (14) of Ref. [1] formally agrees with (8), but the bare couplings are there instead of the g's. In this form it is not correct.

(iii) The function $g(x)$ of (8) can be calculated perturbatively, and the one-loop result was published in Ref. [3]: $g(x) = (-C')x$ where $-C'(\epsilon) > 0$ is analytic and positive around $\epsilon = 0$. Putting this into (8), one gets

$$g_{h^2_{\rm AT}} \sim g_{|r|}^{2+(\epsilon/2)}\sqrt{g_{w^2}},$$

and inserting the inverse relations of those in Eq. (7) one immediately arrives at (3).

As must be clear from the two-step process above, a mixture of renormalization *and* perturbation theory leads to Eq. (3). The leading linear contribution to $g(x)$ is free from a singularity at $d = 6$ as it comes from an ultraviolet convergent one-loop graph [3]. Triangular insertions in the next two-loop graphs, however, certainly produce singular terms, such as $g(x) \sim \frac{1}{\epsilon}x^2$, their neglect is acceptable only if $\frac{1}{\epsilon}x = \frac{1}{\epsilon}g_{w^2}g_{|r|}^{(\epsilon/2)} \ll 1$. Expressing this condition by the bare couplings, one can write the range of applicability of Eq. (3) as

$$|r| \ll 1, \quad w^2 \ll 1, \quad \text{and most importantly,}$$
$$\frac{1}{\epsilon}w^2|r|^{(\epsilon/2)}\left(1+\frac{2}{\epsilon}w^2\right)^{-1-(5/6)\epsilon} \ll 1. \tag{9}$$

The left-hand side of the third condition becomes of order unity (1/2) and thus breaks down when $\epsilon \to 0$ whereas $|r|$ and $w^2 \ll 1$ but otherwise fixed. This is just the limit leading to Eq. (12) of Ref. [1] (and to the conclusion of the disappearance of the AT line for $\epsilon \to 0$) and is the source of the basic fault in the original arguments in Ref. [4]. (See also Fig. 2(b) and the discussion around it in Ref. [3].) ϵ in (3) may be small but must be kept fixed. The simple first-order perturbational result is obtained for $w^2 \ll \epsilon$. The joint application of the perturbational method and RG (and not RG alone as Yeo and Moore [1] claim) provide (3) which is valid for $0 < \epsilon \ll w^2 \ll 1$ too. In this latter case the range of applicability of Eq. (3), according to (9), shrinks to zero as $-\ln|r| \gg \epsilon^{-1}$, together with the amplitude in (3). This phenomenon signals the appearance of the logarithmic correction in $d = 6$: $h^2_{\rm AT} \sim (\ln|r|)^{-4}|r|^2$, and it is not an indication of the disappearance of the AT line.

[1] J. Yeo and M. A. Moore, Phys. Rev. B **91**, 104432 (2015).
[2] J. R. L. de Almeida and D. J. Thouless, J. Phys. A **11**, 983 (1978).
[3] G. Parisi and T. Temesvári, Nucl. Phys. B **858**, 293 (2012).
[4] M. A. Moore and A. J. Bray, Phys. Rev. B **83**, 224408 (2011).
[5] F. J. Wegner, Phys. Rev. B **5**, 4529 (1972).