PHYSICAL REVIEW B **94**, 155113 (2016)

Theory of (2+1)-dimensional fermionic topological orders and fermionic/bosonic topological orders with symmetries

Tian Lan,[1,2] Liang Kong,[3,4] and Xiao-Gang Wen[5,1]

[1]*Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada*
[2]*Department of Physics and Astronomy, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada*
[3]*Department of Mathematics & Statistics, University of New Hampshire, Durham, New Hampshire 03824, USA*
[4]*Center of Mathematical Sciences and Applications, Harvard University, Cambridge, Massachusetts 02138, USA*
[5]*Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*

(Received 28 August 2015; revised manuscript received 25 July 2016; published 10 October 2016)

We propose a systematic framework to classify (2+1)-dimensional (2+1D) fermionic topological orders without symmetry and 2+1D fermionic/bosonic topological orders with symmetry G. The key is to use the so-called symmetric fusion category $\mathcal{E}$ to describe the symmetry. Here, $\mathcal{E} = \mathrm{sRep}(\mathbb{Z}_2^f)$ describing particles in a fermionic product state without symmetry, or $\mathcal{E} = \mathrm{sRep}(G^f)$ [$\mathcal{E} = \mathrm{Rep}(G)$] describing particles in a fermionic (bosonic) product state with symmetry G. Then, topological orders with symmetry $\mathcal{E}$ are classified by nondegenerate unitary braided fusion categories over $\mathcal{E}$, plus their modular extensions and total chiral central charges. This allows us to obtain a list that contains all 2+1D fermionic topological orders without symmetry. For example, we find that, up to $p + \mathrm{i}\,p$ fermionic topological orders, there are only four fermionic topological orders with one nontrivial topological excitation: (1) the $K = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$ fractional quantum Hall state, (2) a Fibonacci bosonic topological order stacking with a fermionic product state, (3) the time-reversal conjugate of the previous one, and (4) a fermionic topological order with chiral central charge $c = \frac{1}{4}$, whose only topological excitation has non-Abelian statistics with spin $s = \frac{1}{4}$ and quantum dimension $d = 1 + \sqrt{2}$.

DOI: 10.1103/PhysRevB.94.155113

I. INTRODUCTION

A. Background

Topological order [1–3] is a new kind of order beyond Landau symmetry breaking theory. It cannot be characterized by the local order parameters associated with the symmetry breaking. However, topological order can be characterized/defined by (a) the topology-dependent ground state degeneracy [1,2] and (b) the non-Abelian geometric phases (S,T) of the degenerate ground states [3,4]. Those quantities are *robust against any local perturbations* [2]. Thus they are topological invariants that define new kinds of quantum phases—topologically ordered phases. Recently, it was found that, microscopically, topological order is related to long-range entanglement [5,6]. In fact, we can regard topological orders as patterns of long-range entanglement in many-body ground states [7], which is defined as the equivalence classes of gapped quantum liquid [8] states under local unitary transformations [9–11]. Chiral spin liquids [12,13], integer/fractional quantum Hall states [14–16], $\mathbb{Z}_2$ spin liquids [17–19], non-Abelian fractional quantum Hall states [20–23], etc., are examples of topologically ordered phases.

Topological order and long-range entanglement are truly new phenomena, which require new mathematical language to describe them. Tensor category theory [7,9,24–27] and simple current algebras [20,28–30] (or patterns of zeros [31–39]) may be parts of the new mathematical language. Using the new mathematical language, some systematic classification results for certain type of topological orders in low dimensions were achieved.

Using unitary fusion category (UFC) theory, we have developed a systematic and quantitative theory that classifies all the topological orders with gappable edge for 2+1D interacting bosonic systems [7,9]. A double Fibonacci bosonic topological order $2^B_{14/5} \boxtimes 2^B_{-14/5}$ was discovered [9]. We also developed a fermionic UFC theory, to classify topological orders with gappable edge for 2+1D interacting fermionic systems [25,27]. For 2+1D bosonic/fermionic topological orders (with gappable or ungappable edge) that have only Abelian statistics, we find that we can use integer K matrices to classify them [40] and use the following $U(1)$ Chern-Simons theory to describe them [4,40–45]:

$$\mathcal{L} = \frac{K_{IJ}}{4\pi} a_{I\mu} \partial_\nu a_{J\lambda} \epsilon^{\mu\nu\lambda}. \tag{1}$$

Such an effective theory can be realized by a multilayer fractional quantum Hall state:

$$\prod_{I;i<j} \left(z_i^I - z_j^I\right)^{K_{II}} \prod_{I<J;i,j} \left(z_i^I - z_j^J\right)^{K_{IJ}} \mathrm{e}^{-\frac{1}{4}\sum_{i,I} |z_i^I|^2}. \tag{2}$$

When diagonal K_{II}'s are all even, the K-matrices classify 2+1D bosonic Abelian topological orders [40]. When some diagonal K_{II}'s are odd, the K matrices classify 2+1D fermionic Abelian topological orders [40].

B. Invertible topological orders

We can stack two topologically ordered states together to form a new topologically ordered state. Such a stacking operation $\boxtimes$ makes the set of various topological orders into a commutative monoid [46]. (A monoid is almost a group except that elements may not have inverses.) A state has a trivial topological order if the stacking of such state with any other topological order give the same topological order back. It turns out that the states with a trivial topological order are always product states or short-range entangled states.

Although most topological orders do not have an inverse with respect to the stacking operation, some topological orders can have an inverse. Those topological orders are called invertible topological orders [46,47]. A topological order $\mathcal{C}$ is invertible if there exists another topological order $\mathcal{D}$, such that the stacking of $\mathcal{C}$ and $\mathcal{D}$ gives rise to a trivial topological order $\mathbf{1}$, i.e., $\mathcal{C} \boxtimes \mathcal{D} = \mathbf{1}$. In fact, such an inverse $\mathcal{D}$ can be obtained from $\mathcal{C}$ by a time-reversal transformation.

It turns out that a topological order is invertible if it has no nontrivial topological excitations [46,47]. In 2+1D, the set of all invertible bosonic topological orders form an Abelian group $\mathbb{Z}$, which is generated, via the stacking and time-reversal operations, by the E_8 bosonic quantum Hall state described by the following K matrix:

$$K_{E_8} = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}. \tag{3}$$

The E_8 bosonic quantum Hall state has no nontrivial topological excitations (since $\det(K) = 1$). However, the state has a nontrivial thermal Hall effect [48] and ungappable gapless chiral edge states [49,50] with a chiral central charge $c = 8$. Thus the E_8 state has a nontrivial invertible topological order.

C. Classify topological orders via (non-)Abelian statistics

If we overlook the invertible topological orders, i.e., consider only the quotient

$$\frac{\text{topological orders}}{\text{invertible topological orders}}, \tag{4}$$

then we can use non-Abelian statistics of topological excitations to describe and classify such a quotient. It is believed that non-Abelian statistics of topological excitations are fully described by unitary modular tensor categories (UMTC) [51–53], a notion of which is equivalent to that of a nondegenerate unitary braided fusion category [52,54], abbreviated as a nondegenerate UBFC (for an introduction to category and UMTC, see Appendix B).

Thus we can use the classification of nondegenerate UBFC's [55] to classify 2+1D bosonic topological orders (see Remark 1) up to invertible topological orders. In a recent paper [56], we have used such an approach to create a full list of simple 2+1D bosonic topological orders.

In this paper, we develop a theory for 2+1D fermionic topological orders without symmetry [25,27,57]—up to invertible topological orders, 2+1D fermionic topological orders without symmetry are classified by UMTC$_{/\mathcal{F}_0}$'s with modular extensions—where, by definition, a UMTC$_{/\mathcal{F}_0}$ is a nondegenerate UBFC over the symmetric fusion category (SFC) $\mathcal{F}_0$, and $\mathcal{F}_0 \equiv \text{sRep}(\mathbb{Z}_2^f)$ describes a fermionic product state without symmetry.

Several new concepts are used in the above statement. We will define SFC in Sec. I D and explain in Sec. II A why a SFC $\mathcal{E}$ describes a fermionic/bosonic product state with/without symmetry. The notion of "nondegenerate UBFC over SFC" will be explained in Sec. III, and we will explain the notion of modular extension in Secs. II C and VI.

Here we briefly discuss fermionic invertible topological orders. It is believed that all 2+1D fermionic invertible topological orders [57] form an Abelian group $\mathbb{Z}$ under the stacking operation $\boxtimes$. The Abelian group $\mathbb{Z}$ is generated by the $p + \mathrm{i}p$ superconductor of spinless fermions [58]. The $p + \mathrm{i}p$ superconductor has no nontrivial topological excitations. However, $p + \mathrm{i}p$ superconductor has a nontrivial thermal Hall effect and ungappable gapless chiral edge states with a chiral central charge $c = 1/2$, and thus has a nontrivial topological order. The most general 2+1D fermionic invertible topological orders can be obtained by stacking a finite number of layers of 2+1D $p \pm \mathrm{i}p$ superconductors.

To develop a simple theory for 2+1D fermionic topological orders, we assume that the non-Abelian statistics of topological excitations in 2+1D fermionic topological orders is fully described by the data (N_k^{ij}, s_i), where i,j,k label the types of topological excitations, s_i is the spin (mod 1) of the type-i topological excitation, and N_k^{ij} are the fusion coefficients of topological excitations. We find the conditions that the data (N_k^{ij}, s_i) must satisfy in order to describe a 2+1D fermionic topological order. By finding all the (N_k^{ij}, s_i)'s that satisfy the conditions, we obtain a classification of 2+1D fermionic topological orders (up to invertible topological orders). If we further include the chiral central charge c of the edge states, we believe that the data (N_k^{ij}, s_i, c) describe/classify all 2+1D fermionic topological orders (including the invertible ones).

We have numerically searched the (N_k^{ij}, s_i, c) that satisfy the conditions. This allows us to create complete lists of 2+1D fermionic topological orders (up to invertible topological orders) within certain bounds (see Tables I–VI). The invertible topological orders can be easily included by stacking with a number of layers of $p \pm \mathrm{i}p$ fermionic superconductors.

We would like to mention that the close relation between the bulk topological order and its boundary theory [46,59] suggests another way to understand/classify topological orders: one may use 1+1D boundary conformal field theories (CFT) to classify 2+1D bulk topological orders [59]. For the fermionic cases, one may use 1+1D boundary $\mathbb{Z}_2$-graded chiral algebra to classify 2+1D fermionic topological orders[60]. However, such an approach is not very fruitful, because the connection between 1+1D CFT (or $\mathbb{Z}_2$-graded chiral algebra) and 2+1D topological orders (or 2+1D fermionic topological orders) is not simple. In fact, 2+1D fermionic topological orders do not correspond to $\mathbb{Z}_2$-graded chiral algebra, rather they correspond to the gravitational anomalies (perturbative and/or global ones) in the $\mathbb{Z}_2$-graded chiral algebra [46,61–63]. So the relation between 1+1D $\mathbb{Z}_2$-graded chiral algebra and 2+1D fermionic topological orders is infinity-to-one: all the 1+1D $\mathbb{Z}_2$-graded chiral algebra with the same gravitational anomaly correspond to the same 2+1D fermionic topological order [46,61,62]. In contrast, the (N_k^{ij}, s_i, c) description used in this paper is a one-to-one description of 2+1D fermionic topological orders.

TABLE I. A list of simple fermionic topological orders (up to invertible ones) with N types of topological excitations (including the parent fermion) and chiral central charge c (mod 1/2). The excitations have quantum dimension d_i and spin s_i (mod 1). Θ_2 in the table is defined as $\Theta_2 \equiv D^{-1}\sum_i \mathrm{e}^{\mathrm{i}4\pi s_i} d_i^2$ where $D^2 = \sum_i d_i^2$. Also $\angle\Theta_2 \equiv \mathrm{Im}\ln(\Theta_2)$. The table contains all fermionic topological orders with $N = 2$, $N = 4$ (see Appendix A) and $D^2 \leqslant 600$, $N = 6$, and $D^2 \leqslant 400$. Here, $\zeta_n^m = \frac{\sin[\pi(m+1)/(n+2)]}{\sin[\pi/(n+2)]}$.

$N_c^F(\genfrac{}{}{0pt}{}{\lvert\Theta_2\rvert}{\angle\Theta_2/2\pi})$	D^2	$d_1,d_2,\ldots$	$s_1,s_2,\ldots$	Comments/K matrix
$2_0^F(\genfrac{}{}{0pt}{}{\zeta_2^1}{0})$	2	1,1	$0,\frac{1}{2}$	trivial $\mathcal{F}_0$
$4_0^F(\genfrac{}{}{0pt}{}{0}{0})$	4	1,1,1,1	$0,\frac{1}{2},\frac{1}{4},-\frac{1}{4}$	$\mathcal{F}_0 \boxtimes 2_1^B(\genfrac{}{}{0pt}{}{0}{0})$, $K = \begin{pmatrix}2&2\\2&1\end{pmatrix}$
$4_{1/5}^F(\genfrac{}{}{0pt}{}{\zeta_2^1\zeta_3^1}{3/20})$	7.2360	$1,1,\zeta_3^1,\zeta_3^1$	$0,\frac{1}{2},\frac{1}{10},-\frac{2}{5}$	$\mathcal{F}_0 \boxtimes 2_{-14/5}^B(\genfrac{}{}{0pt}{}{\zeta_3^1}{3/20})$
$4_{-1/5}^F(\genfrac{}{}{0pt}{}{\zeta_2^1\zeta_3^1}{-3/20})$	7.2360	$1,1,\zeta_3^1,\zeta_3^1$	$0,\frac{1}{2},-\frac{1}{10},\frac{2}{5}$	$\mathcal{F}_0 \boxtimes 2_{14/5}^B(\genfrac{}{}{0pt}{}{\zeta_3^1}{-3/20})$
$4_{1/4}^F(\genfrac{}{}{0pt}{}{\zeta_6^3}{1/2})$	13.656	$1,1,\zeta_6^2,\zeta_6^2 = 1+\sqrt{2}$	$0,\frac{1}{2},\frac{1}{4},-\frac{1}{4}$	$\mathcal{F}_{(A_1,6)}$
$6_0^F(\genfrac{}{}{0pt}{}{\zeta_2^1}{1/4})$	6	1,1,1,1,1,1	$0,\frac{1}{2},\frac{1}{6},-\frac{1}{3},\frac{1}{6},-\frac{1}{3}$	$\mathcal{F}_0 \boxtimes 3_{-2}^B(\genfrac{}{}{0pt}{}{1}{1/4})$, $K = (3)$, $\Psi_{1/3}(z_i)$
$6_0^F(\genfrac{}{}{0pt}{}{\zeta_2^1}{-1/4})$	6	1,1,1,1,1,1	$0,\frac{1}{2},-\frac{1}{6},\frac{1}{3},-\frac{1}{6},\frac{1}{3}$	$\mathcal{F}_0 \boxtimes 3_2^B(\genfrac{}{}{0pt}{}{1}{-1/4})$, $K = (-3)$, $\Psi_{1/3}^*(z_i)$
$6_0^F(\genfrac{}{}{0pt}{}{\zeta_6^3}{1/16})$	8	$1,1,1,1,\zeta_2^1,\zeta_2^1 = \sqrt{2}$	$0,\frac{1}{2},0,\frac{1}{2},\frac{1}{16},-\frac{7}{16}$	$\mathcal{F}_0 \boxtimes 3_{1/2}^B(\genfrac{}{}{0pt}{}{\zeta_6^1}{1/16})$, $\mathcal{F}_{U(1)_2/\mathbb{Z}_2}$
$6_0^F(\genfrac{}{}{0pt}{}{\zeta_6^3}{-1/16})$	8	$1,1,1,1,\zeta_2^1,\zeta_2^1$	$0,\frac{1}{2},0,\frac{1}{2},-\frac{1}{16},\frac{7}{16}$	$\mathcal{F}_0 \boxtimes 3_{-1/2}^B(\genfrac{}{}{0pt}{}{\zeta_6^1}{-1/16})$
$6_0^F(\genfrac{}{}{0pt}{}{1.0823}{3/16})$	8	$1,1,1,1,\zeta_2^1,\zeta_2^1$	$0,\frac{1}{2},0,\frac{1}{2},\frac{3}{16},-\frac{5}{16}$	$\mathcal{F}_0 \boxtimes 3_{3/2}^B(\genfrac{}{}{0pt}{}{0.7653}{3/16})$
$6_0^F(\genfrac{}{}{0pt}{}{1.0823}{-3/16})$	8	$1,1,1,1,\zeta_2^1,\zeta_2^1$	$0,\frac{1}{2},0,\frac{1}{2},-\frac{3}{16},\frac{5}{16}$	$\mathcal{F}_0 \boxtimes 3_{-3/2}^B(\genfrac{}{}{0pt}{}{0.7653}{-3/16})$
$6_{1/7}^F(\genfrac{}{}{0pt}{}{\zeta_2^1\zeta_5^2}{-5/14})$	18.591	$1,1,\zeta_5^1,\zeta_5^1,\zeta_5^2,\zeta_5^2$	$0,\frac{1}{2},\frac{5}{14},-\frac{1}{7},-\frac{3}{14},\frac{2}{7}$	$\mathcal{F}_0 \boxtimes 3_{8/7}^B(\genfrac{}{}{0pt}{}{\zeta_5^2}{-5/14})$
$6_{-1/7}^F(\genfrac{}{}{0pt}{}{\zeta_2^1\zeta_5^2}{5/14})$	18.591	$1,1,\zeta_5^1,\zeta_5^1,\zeta_5^2,\zeta_5^2$	$0,\frac{1}{2},-\frac{5}{14},\frac{1}{7},\frac{3}{14},-\frac{2}{7}$	$\mathcal{F}_0 \boxtimes 3_{-8/7}^B(\genfrac{}{}{0pt}{}{\zeta_5^2}{5/14})$
$6_0^F(\genfrac{}{}{0pt}{}{\zeta_{10}^5}{-1/12})$	44.784	$1,1,\zeta_{10}^2,\zeta_{10}^2,\zeta_{10}^4,\zeta_{10}^4$	$0,\frac{1}{2},\frac{1}{3},-\frac{1}{6},0,\frac{1}{2}$	$\mathcal{F}_{(A_1,-10)}$
$6_0^F(\genfrac{}{}{0pt}{}{\zeta_{10}^5}{1/12})$	44.784	$1,1,\zeta_{10}^2,\zeta_{10}^2,\zeta_{10}^4,\zeta_{10}^4$	$0,\frac{1}{2},-\frac{1}{3},\frac{1}{6},0,\frac{1}{2}$	$\mathcal{F}_{(A_1,10)}$

D. Symmetric fusion categories for bosonic/fermionic product states with symmetry

What is a SFC? A SFC describes a bosonic/fermionic product state with/without symmetry. It is characterized by the quasiparticle excitations. The SFC that describes a fermionic product state without symmetry (denoted by $\mathcal{F}_0$) contains only two types of quasiparticles (two simple objects): the trivial quasiparticle $\mathbf{1}$, and the parent fermion f that forms the fermionic system. The SFC that describes a bosonic product state with symmetry G (a finite group) is the category of G representations, denoted by Rep(G) (see Example 1). The quasiparticles (the simple objects) are all bosonic and correspond to irreducible G-representations. The SFC that describes a fermionic product state with full symmetry G^f, which contains, in particular, the fermion-number-parity symmetry $\mathbb{Z}_2^f$ [64], is the SFC of the super-representations of G^f (see Sec. II A and Example 2), denoted by sRep(G^f). The quasiparticles (the simple objects) correspond to the irreducible representations of G^f. They are fermionic if $\mathbb{Z}_2^f$ acts nontrivially on the corresponding representation, and bosonic if $\mathbb{Z}_2^f$ acts trivially.

E. Classify topological orders with symmetry

We propose a complete classification of 2+1D fermionic/bosonic topological orders with symmetry: 2+1D topological orders with the symmetry $\mathcal{E}$ are classified by $(\mathcal{C},\mathcal{M},c)$, where $\mathcal{C}$ is a UMTC$_{/\mathcal{E}}$, $\mathcal{M}$ is a modular extension of $\mathcal{C}$, and $c \in \mathbb{Q}$ is the total chiral central charge.

There are five main ingredients of above proposal. (1) By definition, UBFC describes topological excitations and their fusion-braiding properties (i.e., their non-Abelian statistics). It is clear that UBFC overlooks the edge states (i.e., cannot detect invertible topological orders).

(2) The SFC $\mathcal{E}$ is a special kind of UBFC that describes the excitations in boson/fermion product state with symmetry. In fact, the bosonic/fermionic symmetry is uniquely determined by $\mathcal{E}$. (See Sec. II for details) [65]. The SFC $\mathcal{E}$ is a categorical description of symmetry.

(3) The non-Abelian statistics of bulk topological excitations in a topological order with symmetry $\mathcal{E}$ is described by a UMTC$_{/\mathcal{E}}$ $\mathcal{C}$. The term "UMTC$_{/\mathcal{E}}$" in the above refers to a UBFC $\mathcal{C}$ such that (i) $\mathcal{C}$ contains $\mathcal{E}$. (In other words, $\mathcal{C}$ contains all the excitations of product state with the same symmetry); (ii) the excitations in $\mathcal{E}$ have trivial mutual statistics with all the excitations in $\mathcal{C}$; and (iii) the excitations that have trivial mutual statistics with all the excitations in $\mathcal{C}$ are *only* those in $\mathcal{E}$. (We note that when $\mathcal{E}$ is trivial UMTC$_{/\mathcal{E}}$ becomes UMTC.)

(4) Roughly speaking, a modular extension corresponds to gauging all the symmetry [66,67] (see Appendix D for a mathematical definition). Up to the E_8 states, the edge states of a UMTC$_{/\mathcal{E}}$ $\mathcal{C}$, are classified by the modular extensions of $\mathcal{C}$ (see Secs. II C and VI for detailed explanations). In particular, the modular extensions of $\mathcal{E}$ classify invertible gapped quantum liquid phases with symmetry $\mathcal{E}$ up to the E_8 states. (5) The remaining ambiguity, i.e., the number of layers of E_8 states, is fixed by the total chiral central charge c.

Invertible gapped quantum liquid phases are closely related to symmetry protected topological (SPT) phases [68–70].

TABLE II. A list of simple fermionic topological orders (up to invertible ones) with $N = 8$ types of topological excitations. The table contains all fermionic topological orders with $D^2 \leqslant 400$.

$N_c^F\binom{\lvert\Theta_2\rvert}{\angle\Theta_2/2\pi}$	D^2	$d_1, d_2, \ldots$	$s_1, s_2, \ldots$	Comments/K matrix
$8_0^F\binom{2\zeta_2^1}{0}$	8	$1,1,1,1,1,1,1,1$	$0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2}$	$\mathcal{F}_0 \boxtimes 4_0^B\binom{2}{0}$, $\begin{pmatrix}0&2\\2&0\end{pmatrix} \oplus \begin{pmatrix}0&1\\1&1\end{pmatrix}$
$8_0^F\binom{2}{1/8}$	8	$1,1,1,1,1,1,1,1$	$0,\frac{1}{2},0,\frac{1}{2},\frac{1}{8},-\frac{3}{8},\frac{1}{8},-\frac{3}{8}$	$\mathcal{F}_0 \boxtimes 4_1^B\binom{\zeta_2^1}{1/8}$, $-\begin{pmatrix}0&2\\2&1\end{pmatrix}$
$8_0^F\binom{2}{-1/8}$	8	$1,1,1,1,1,1,1,1$	$0,\frac{1}{2},0,\frac{1}{2},-\frac{1}{8},\frac{3}{8},-\frac{1}{8},\frac{3}{8}$	$\mathcal{F}_0 \boxtimes 4_{-1}^B\binom{\zeta_2^1}{-1/8}$, $\begin{pmatrix}0&2\\2&1\end{pmatrix}$
$8_0^F\binom{0}{0}$	8	$1,1,1,1,1,1,1,1$	$0,\frac{1}{2},0,\frac{1}{2},\frac{1}{4},-\frac{1}{4},\frac{1}{4},-\frac{1}{4}$	$\mathcal{F}_0 \boxtimes 4_0^B\binom{0}{0}$, $\begin{pmatrix}2&2\\2&1\end{pmatrix} \oplus \begin{pmatrix}2&2\\2&1\end{pmatrix}$
$8_{1/5}^F\binom{0}{0}$	14.472	$1,1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1$	$0,\frac{1}{2},\frac{1}{4},-\frac{1}{4},\frac{1}{10},-\frac{2}{5},-\frac{3}{20},\frac{7}{20}$	$\mathcal{F}_0 \boxtimes 4_{-9/5}^B\binom{0}{0}$
$8_{-1/5}^F\binom{0}{0}$	14.472	$1,1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1$	$0,\frac{1}{2},\frac{1}{4},-\frac{1}{4},-\frac{1}{10},\frac{2}{5},-\frac{7}{20},\frac{3}{20}$	$\mathcal{F}_0 \boxtimes 4_{9/5}^B\binom{0}{0}$
$8_0^F\binom{3.5915}{0.1699}$	24	$1,1,1,1,2,2,\sqrt{6},\sqrt{6}$	$0,\frac{1}{2},\frac{1}{2},0,\frac{1}{6},-\frac{1}{3},\frac{1}{16},-\frac{7}{16}$	$\mathcal{F}_{U(1)_6/\mathbb{Z}_2}$
$8_*^F\binom{1.7609}{-0.0288}$	24	$1,1,1,1,2,2,\sqrt{6},\sqrt{6}$	$0,\frac{1}{2},\frac{1}{2},0,\frac{1}{6},-\frac{1}{3},-\frac{1}{16},\frac{7}{16}$	primitive
$8_*^F\binom{3.5915}{0.3300}$	24	$1,1,1,1,2,2,\sqrt{6},\sqrt{6}$	$0,\frac{1}{2},\frac{1}{2},0,\frac{1}{6},-\frac{1}{3},\frac{3}{16},-\frac{5}{16}$	primitive
$8_*^F\binom{1.7609}{-0.4711}$	24	$1,1,1,1,2,2,\sqrt{6},\sqrt{6}$	$0,\frac{1}{2},\frac{1}{2},0,\frac{1}{6},-\frac{1}{3},-\frac{3}{16},\frac{5}{16}$	primitive
$8_*^F\binom{1.7609}{0.0288}$	24	$1,1,1,1,2,2,\sqrt{6},\sqrt{6}$	$0,\frac{1}{2},\frac{1}{2},0,-\frac{1}{6},\frac{1}{3},\frac{1}{16},-\frac{7}{16}$	primitive
$8_*^F\binom{3.5915}{-0.1699}$	24	$1,1,1,1,2,2,\sqrt{6},\sqrt{6}$	$0,\frac{1}{2},\frac{1}{2},0,-\frac{1}{6},\frac{1}{3},-\frac{1}{16},\frac{7}{16}$	primitive
$8_*^F\binom{1.7609}{0.4711}$	24	$1,1,1,1,2,2,\sqrt{6},\sqrt{6}$	$0,\frac{1}{2},\frac{1}{2},0,-\frac{1}{6},\frac{1}{3},\frac{3}{16},-\frac{5}{16}$	primitive
$8_*^F\binom{3.5915}{-0.3300}$	24	$1,1,1,1,2,2,\sqrt{6},\sqrt{6}$	$0,\frac{1}{2},\frac{1}{2},0,-\frac{1}{6},\frac{1}{3},-\frac{3}{16},\frac{5}{16}$	primitive
$8_{-1/10}^F\binom{\zeta_2^1\zeta_8^2}{3/10}$	26.180	$1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_8^2,\zeta_8^2$	$0,\frac{1}{2},\frac{1}{10},-\frac{2}{5},\frac{1}{10},-\frac{2}{5},-\frac{3}{10},\frac{1}{5}$	$\mathcal{F}_0 \boxtimes 4_{12/5}^B\binom{\zeta_8^2}{3/10}$
$8_0^F\binom{\zeta_2^1\zeta_8^2}{0}$	26.180	$1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_8^2,\zeta_8^2$	$0,\frac{1}{2},\frac{1}{10},-\frac{2}{5},-\frac{1}{10},\frac{2}{5},\frac{1}{2},0$	$\mathcal{F}_0 \boxtimes 4_0^B\binom{\zeta_8^2}{0}$
$8_{1/10}^F\binom{\zeta_2^1\zeta_8^2}{-3/10}$	26.180	$1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_8^2,\zeta_8^2$	$0,\frac{1}{2},-\frac{1}{10},\frac{2}{5},-\frac{1}{10},\frac{2}{5},\frac{3}{10},-\frac{1}{5}$	$\mathcal{F}_0 \boxtimes 4_{-12/5}^B\binom{\zeta_8^2}{-3/10}$
$8_{1/4}^F\binom{0}{0}$	27.313	$1,1,1,1,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^2$	$0,\frac{1}{2},\frac{1}{4},-\frac{1}{4},0,\frac{1}{2},\frac{1}{4},-\frac{1}{4}$	$4_{1/4}^F\binom{\zeta_6^3}{1/2} \boxtimes 2_1^B\binom{0}{0}$
$8_{1/6}^F\binom{\zeta_2^1\zeta_7^3}{-5/12}$	38.468	$1,1,\zeta_7^1,\zeta_7^1,\zeta_7^2,\zeta_7^2,\zeta_7^3,\zeta_7^3$	$0,\frac{1}{2},\frac{1}{6},-\frac{1}{3},\frac{5}{18},-\frac{2}{9},-\frac{1}{6},\frac{1}{3}$	$\mathcal{F}_0 \boxtimes 4_{-10/3}^B\binom{\zeta_7^3}{-5/12}$
$8_{-1/6}^F\binom{\zeta_2^1\zeta_7^3}{5/12}$	38.468	$1,1,\zeta_7^1,\zeta_7^1,\zeta_7^2,\zeta_7^2,\zeta_7^3,\zeta_7^3$	$0,\frac{1}{2},-\frac{1}{6},\frac{1}{3},-\frac{5}{18},\frac{2}{9},\frac{1}{6},-\frac{1}{3}$	$\mathcal{F}_0 \boxtimes 4_{10/3}^B\binom{\zeta_7^3}{5/12}$
$8_{-1/20}^F\binom{\zeta_6^3\zeta_3^1}{-7/20}$	49.410	$1,1,\zeta_3^1,\zeta_3^1,\zeta_6^2,\zeta_6^2,\zeta_3^1\zeta_6^2,\zeta_3^1\zeta_6^2$	$0,\frac{1}{2},\frac{1}{10},-\frac{2}{5},\frac{1}{4},-\frac{1}{4},-\frac{3}{20},\frac{7}{20}$	$4_{1/4}^F\binom{\zeta_6^3}{1/2} \boxtimes 2_{-14/5}^B\binom{\zeta_3^1}{3/20}$
$8_{1/20}^F\binom{\zeta_6^3\zeta_3^1}{7/20}$	49.410	$1,1,\zeta_3^1,\zeta_3^1,\zeta_6^2,\zeta_6^2,\zeta_3^1\zeta_6^2,\zeta_3^1\zeta_6^2$	$0,\frac{1}{2},-\frac{1}{10},\frac{2}{5},\frac{1}{4},-\frac{1}{4},-\frac{7}{20},\frac{3}{20}$	$4_{1/4}^F\binom{\zeta_6^3}{1/2} \boxtimes 2_{14/5}^B\binom{\zeta_3^1}{-3/20}$
$8_0^F\binom{2\zeta_6^2}{0}$	93.254	$1,1,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^2,(\zeta_6^2)^2,(\zeta_6^2)^2$	$0,\frac{1}{2},\frac{1}{4},-\frac{1}{4},\frac{1}{4},-\frac{1}{4},0,\frac{1}{2}$	$4_{1/4}^F\binom{\zeta_6^3}{1/2} \boxtimes_{\mathcal{F}_0} 4_{1/4}^F\binom{\zeta_6^3}{1/2}$
$8_{-1/8}^F\binom{\zeta_{14}^7}{7/16}$	105.09	$1,1,\zeta_{14}^2,\zeta_{14}^2,\zeta_{14}^4,\zeta_{14}^4,\zeta_{14}^6,\zeta_{14}^6$	$0,\frac{1}{2},\frac{3}{8},-\frac{1}{8},\frac{1}{8},-\frac{3}{8},-\frac{1}{4},\frac{1}{4}$	$\mathcal{F}_{(A_1,-14)}$
$8_{1/8}^F\binom{\zeta_{14}^7}{-7/16}$	105.09	$1,1,\zeta_{14}^2,\zeta_{14}^2,\zeta_{14}^4,\zeta_{14}^4,\zeta_{14}^6,\zeta_{14}^6$	$0,\frac{1}{2},-\frac{3}{8},\frac{1}{8},-\frac{1}{8},\frac{3}{8},\frac{1}{4},-\frac{1}{4}$	$\mathcal{F}_{(A_1,14)}$

Roughly speaking,

$$\{\text{invertible gapped quantum liquid phases}\}$$
$$= \{\text{invertible topological orders}\} \times \{\text{SPT phases}\}.$$

For bosonic cases, the only invertible topological orders are the E_8 states. Therefore bosonic SPT phases should be in one-to-one correspondence with the modular extensions of $\mathcal{E} = \text{Rep}(G)$, which are given by $\mathcal{H}^3[G,U(1)]$ [71]. This agrees with the classifications of 2+1D bosonic SPT phases with unitary finite symmetry group G [69].

Things are a bit complicated for fermionic cases. We are able to see both fermionic SPT phases and invertible fermionic topological orders in the modular extensions of $\mathcal{E} = \text{sRep}(G^f)$. More precisely, the modular extensions of $\text{sRep}(G^f)$ with central charge $c = 0$ correspond to fermionic SPT phases, while those with central charge $c \neq 0$ correspond to phases with both invertible topological orders and SPT orders. Two modular extensions with the same central charge have the same invertible fermionic topological orders (up to E_8 states), but may differ by some fermionic SPT phases. In other words, similar to the bosonic case, the central charge is the label for invertible fermionic topological orders. But unlike the bosonic case, where the minimal central charge is $c = 8$ for the E_8 state that is independent of the symmetry, the minimal central charge for invertible fermionic topological orders may depend on the symmetry G^f. For $G^f = \mathbb{Z}_2^f$, there are 16 modular extensions, forming a $\mathbb{Z}_{16}$ group (see Sec. VII B). The minimal invertible fermionic topological order (with no symmetry) has central charge $c = 8/16 = 1/2$. We expect that it is also the case if $G^f = G^b \times \mathbb{Z}_2^f$. (See also Ref. [72].) However, for generic G^f (not of the form $G^b \times Z_2^f$), the invertible fermionic topological orders can be different due to nontrivial interplays with other symmetries. We may have a different minimal central charge, which can be extracted from the modular extensions of $\text{sRep}(G^f)$.

TABLE III. A list of simple fermionic topological orders (up to invertible ones) with $N = 10$ types of topological excitations. The table contains all fermionic topological orders with $D^2 \leqslant 120$.

$N_c^F({}^{\lvert\Theta_2\rvert}_{\angle\Theta_2/2\pi})$	D^2	$d_1,d_2,\dots$	$s_1,s_2,\dots$	Comments/K matrix
$10_0^F({}^{\zeta_2^1}_{0})$	10	$1,1,1,1,1,1,1,1,1,1$	$0,\frac12,\frac1{10},-\frac25,\frac1{10},-\frac25,-\frac1{10},\frac25,-\frac1{10},\frac25$	$\mathcal{F}_0\boxtimes 5_4^B({}^{1}_{0})$, $\begin{pmatrix}4&3\\3&1\end{pmatrix}$
$10_0^F({}^{\zeta_2^1}_{1/2})$	10	$1,1,1,1,1,1,1,1,1,1$	$0,\frac12,\frac3{10},-\frac15,\frac3{10},-\frac15,-\frac3{10},\frac15,-\frac3{10},\frac15$	$\mathcal{F}_0\boxtimes 5_0^B({}^{1}_{1/2})$, $\begin{pmatrix}3&2\\2&3\end{pmatrix}$
$10_*^F({}^{2\zeta_2^1}_{-1/12})$	24	$1,1,1,1,\sqrt3,\sqrt3,\sqrt3,\sqrt3,2,2$	$0,\frac12,0,\frac12,0,\frac12,0,\frac12,\frac13,-\frac16$	primitive
$10_*^F({}^{2\zeta_2^1}_{1/12})$	24	$1,1,1,1,\sqrt3,\sqrt3,\sqrt3,\sqrt3,2,2$	$0,\frac12,0,\frac12,0,\frac12,0,\frac12,-\frac13,\frac16$	primitive
$10_*^F({}^{2\zeta_2^1}_{-5/12})$	24	$1,1,1,1,\sqrt3,\sqrt3,\sqrt3,\sqrt3,2,2$	$0,\frac12,0,\frac12,\frac14,-\frac14,\frac14,-\frac14,\frac13,-\frac16$	primitive
$10_*^F({}^{2\zeta_2^1}_{5/12})$	24	$1,1,1,1,\sqrt3,\sqrt3,\sqrt3,\sqrt3,2,2$	$0,\frac12,0,\frac12,\frac14,-\frac14,\frac14,-\frac14,-\frac13,\frac16$	primitive
$10_0^F({}^{\zeta_{10}^5}_{1/4})$	24	$1,1,1,1,\sqrt3,\sqrt3,\sqrt3,\sqrt3,2,2$	$0,\frac12,\frac12,0,\frac18,-\frac38,-\frac38,\frac18,\frac16,-\frac13$	$\mathcal{F}_0\boxtimes 5_{-2}^B({}^{\zeta_{10}^2}_{1/4})$
$10_0^F({}^{\sqrt6-\sqrt2}_{1/4})$	24	$1,1,1,1,\sqrt3,\sqrt3,\sqrt3,\sqrt3,2,2$	$0,\frac12,\frac12,0,\frac18,-\frac38,-\frac38,\frac18,-\frac16,\frac13$	$\mathcal{F}_0\boxtimes 5_{2}^B({}^{\sqrt3-1}_{1/4})$
$10_0^F({}^{\sqrt6-\sqrt2}_{-1/4})$	24	$1,1,1,1,\sqrt3,\sqrt3,\sqrt3,\sqrt3,2,2$	$0,\frac12,\frac12,0,-\frac18,\frac38,\frac38,-\frac18,\frac16,-\frac13$	$\mathcal{F}_0\boxtimes 5_{-2}^B({}^{\sqrt3-1}_{-1/4})$
$10_0^F({}^{\zeta_{10}^5}_{-1/4})$	24	$1,1,1,1,\sqrt3,\sqrt3,\sqrt3,\sqrt3,2,2$	$0,\frac12,\frac12,0,-\frac18,\frac38,\frac38,-\frac18,-\frac16,\frac13$	$\mathcal{F}_0\boxtimes 5_{2}^B({}^{\zeta_{10}^2}_{-1/4})$
$10_0^F({}^{4.2807}_{0.0874})$	40	$1,1,1,1,2,2,2,2,\sqrt{10},\sqrt{10}$	$0,\frac12,\frac12,0,\frac1{10},-\frac25,-\frac1{10},\frac25,\frac1{16},-\frac7{16}$	$\mathcal{F}_{U(1)_{10}/\mathbb{Z}_2}$
$10_*^F({}^{4.2807}_{-0.0874})$	40	$1,1,1,1,2,2,2,2,\sqrt{10},\sqrt{10}$	$0,\frac12,\frac12,0,\frac1{10},-\frac25,-\frac1{10},\frac25,-\frac1{16},\frac7{16}$	primitive
$10_*^F({}^{2.3823}_{0.3060})$	40	$1,1,1,1,2,2,2,2,\sqrt{10},\sqrt{10}$	$0,\frac12,\frac12,0,\frac1{10},-\frac25,-\frac1{10},\frac25,\frac3{16},-\frac5{16}$	primitive
$10_*^F({}^{2.3823}_{-0.3060})$	40	$1,1,1,1,2,2,2,2,\sqrt{10},\sqrt{10}$	$0,\frac12,\frac12,0,\frac1{10},-\frac25,-\frac1{10},\frac25,-\frac3{16},\frac5{16}$	primitive
$10_*^F({}^{2.3823}_{0.1939})$	40	$1,1,1,1,2,2,2,2,\sqrt{10},\sqrt{10}$	$0,\frac12,\frac12,0,\frac3{10},-\frac15,-\frac3{10},\frac15,\frac1{16},-\frac7{16}$	primitive
$10_*^F({}^{2.3823}_{-0.1939})$	40	$1,1,1,1,2,2,2,2,\sqrt{10},\sqrt{10}$	$0,\frac12,\frac12,0,\frac3{10},-\frac15,-\frac3{10},\frac15,-\frac1{16},\frac7{16}$	primitive
$10_*^F({}^{4.2807}_{0.4125})$	40	$1,1,1,1,2,2,2,2,\sqrt{10},\sqrt{10}$	$0,\frac12,\frac12,0,\frac3{10},-\frac15,-\frac3{10},\frac15,\frac3{16},-\frac5{16}$	primitive
$10_*^F({}^{4.2807}_{-0.4125})$	40	$1,1,1,1,2,2,2,2,\sqrt{10},\sqrt{10}$	$0,\frac12,\frac12,0,\frac3{10},-\frac15,-\frac3{10},\frac15,-\frac3{16},\frac5{16}$	primitive
$10_{-1/22}^F({}^{\zeta_2^1\zeta_9^4}_{2/11})$	69.292	$1,1,\zeta_9^1,\zeta_9^1,\zeta_9^2,\zeta_9^2,\zeta_9^3,\zeta_9^3,\zeta_9^4,\zeta_9^4$	$0,\frac12,\frac7{22},-\frac2{11},-\frac7{22},\frac2{11},-\frac9{22},\frac1{11},\frac1{22},-\frac5{11}$	$\mathcal{F}_0\boxtimes 5_{16/11}^B({}^{\zeta_9^4}_{2/11})$
$10_{1/22}^F({}^{\zeta_2^1\zeta_9^4}_{-2/11})$	69.292	$1,1,\zeta_9^1,\zeta_9^1,\zeta_9^2,\zeta_9^2,\zeta_9^3,\zeta_9^3,\zeta_9^4,\zeta_9^4$	$0,\frac12,-\frac7{22},\frac2{11},\frac7{22},-\frac2{11},\frac9{22},-\frac1{11},-\frac1{22},\frac5{11}$	$\mathcal{F}_0\boxtimes 5_{-16/11}^B({}^{\zeta_9^4}_{-2/11})$
$10_{1/14}^F({}^{\zeta_2^1\zeta_{12}^2}_{-5/28})$	70.684	$1,1,\zeta_5^2,\zeta_5^2,\zeta_5^2,\zeta_5^2,\zeta_{12}^2,\zeta_{12}^2,\zeta_{12}^4,\zeta_{12}^4$	$0,\frac12,\frac5{14},-\frac17,\frac5{14},-\frac17,-\frac5{14},\frac17,-\frac1{14},\frac37$	$\mathcal{F}_0\boxtimes 5_{18/7}^B({}^{\zeta_{12}^2}_{-5/28})$
$10_{-1/14}^F({}^{\zeta_2^1\zeta_{12}^2}_{5/28})$	70.684	$1,1,\zeta_5^2,\zeta_5^2,\zeta_5^2,\zeta_5^2,\zeta_{12}^2,\zeta_{12}^2,\zeta_{12}^4,\zeta_{12}^4$	$0,\frac12,-\frac5{14},\frac17,-\frac5{14},\frac17,\frac5{14},-\frac17,\frac1{14},-\frac37$	$\mathcal{F}_0\boxtimes 5_{-18/7}^B({}^{\zeta_{12}^2}_{5/28})$
$10_{-1/5}^F({}^{\zeta_{18}^9}_{-1/10})$	204.31	$1,1,\zeta_{18}^2,\zeta_{18}^2,\zeta_{18}^4,\zeta_{18}^4,\zeta_{18}^6,\zeta_{18}^6,\zeta_{18}^8,\zeta_{18}^8$	$0,\frac12,\frac25,-\frac1{10},\frac15,-\frac3{10},-\frac1{10},\frac25,\frac12,0$	$\mathcal{F}_{(A_1,-18)}$
$10_{1/5}^F({}^{\zeta_{18}^9}_{1/10})$	204.31	$1,1,\zeta_{18}^2,\zeta_{18}^2,\zeta_{18}^4,\zeta_{18}^4,\zeta_{18}^6,\zeta_{18}^6,\zeta_{18}^8,\zeta_{18}^8$	$0,\frac12,-\frac25,\frac1{10},-\frac15,\frac3{10},\frac1{10},-\frac25,\frac12,0$	$\mathcal{F}_{(A_1,18)}$

By combining with the theory of BF category developed in Ref. [46], the above proposal can be naturally generalized to higher dimensions. Up to invertible topological orders, $(d+1)$D fermionic/bosonic topological orders with/without symmetry are classified by nondegenerate unitary braided fusion $(d-1)$ categories over a symmetric fusion 1 category; the symmetric fusion 1-category, viewed as a unitary braided fusion $(d-1)$ category with only trivial k morphisms for $0 \leqslant k < d$, describes a $(d+1)$D fermionic/bosonic product state with/without symmetry. We also require that the unitary braided fusion $(d-1)$ category has a modular extension.

Fermionic/bosonic topological orders with symmetry will be thoroughly studied in an upcoming paper Ref. [73]. In this paper, we concentrate on 2+1D fermionic topological orders without symmetry, which are the simplest examples of nondegenerate UBFC's over a SFC.

F. Relation to G-crossed category

Note that our proposal in the bosonic cases—2+1D bosonic topological orders with symmetry G, up to invertible topological orders, are classified by $\text{UMTC}_{/\text{Rep}(G)}$ that have modular extensions—is different, but equivalent to another proposal in Ref. [74], using G-crossed UMTC's to classify 2+1D bosonic topological orders with symmetry G. Mathematically, a $\text{UMTC}_{/\text{Rep}(G)}$ $\mathcal{C}$ is related to a G-crossed UMTC $\mathcal{D} \cong \mathcal{M}_G$ via the de-equivariantization and equivariantization processes [54]. Let $\mathcal{D}_0$ be the neutral component (the full subcategory graded by the identity element of the group G) of $\mathcal{D}$. Note that $\mathcal{D}_0$ is a UMTC with a G action. We have

$$\begin{tikzcd}
\mathcal{C} \cong \mathcal{D}_0^G \arrow[r, shift left, "\text{de-equivariantization}"] \arrow[d, hook, "\mathcal{C}=\text{Rep}(G)_{\mathcal{M}}^{\text{cen}}"'] & \mathcal{C}_G \cong \mathcal{D}_0 \arrow[l, shift left, "\text{equivariantization}"] \arrow[d, hook, "\text{neutral component}"] \\
\mathcal{M} \cong \mathcal{D}^G \arrow[r, shift left, "\text{de-equivariantization}"] & \mathcal{M}_G \cong \mathcal{D} \arrow[l, shift left, "\text{equivariantization}"]
\end{tikzcd}$$

where de-equivariantization and equivariantization are inverse to each other. This is why we say that the two proposals are equivalent in the bosonic cases. We will further study their relation elsewhere. However, our proposal has the advantage

TABLE IV. A list of simple fermionic topological orders (up to invertible ones) with $N = 12$ types of topological excitations. The table contains all fermionic topological orders with $D^2 \leqslant 50$. Here, $\chi_n^m = m + \sqrt{n}$.

$N_c^F(\begin{smallmatrix}\lvert\Theta_2\rvert\\ \angle\Theta_2/2\pi\end{smallmatrix})$	D^2	$d_1,d_2,\ldots$	$s_1,s_2,\ldots$	Comments/K-matrix
$12^F_0(\begin{smallmatrix}0\\0\end{smallmatrix})$	12	$1,1,1,1,1,1,1,1,1,1,1,1$	$0,\frac{1}{2},\frac{1}{12},\frac{7}{12},\frac{1}{12},\frac{7}{12},\frac{1}{4},\frac{3}{4},\frac{1}{3},\frac{5}{6},\frac{1}{3},\frac{5}{6}$	$\mathcal{F}_0 \boxtimes 6^B_1(\begin{smallmatrix}0\\0\end{smallmatrix})$
$12^F_0(\begin{smallmatrix}0\\0\end{smallmatrix})$	12	$1,1,1,1,1,1,1,1,1,1,1,1$	$0,\frac{1}{2},\frac{1}{6},\frac{2}{3},\frac{1}{6},\frac{2}{3},\frac{1}{4},\frac{3}{4},\frac{5}{12},\frac{11}{12},\frac{5}{12},\frac{11}{12}$	$\mathcal{F}_0 \boxtimes 6^B_5(\begin{smallmatrix}0\\0\end{smallmatrix})$
$12^F_*(\begin{smallmatrix}2\\0\end{smallmatrix})$	16	$1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$	$0,\frac{1}{2},0,\frac{1}{2},\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4},0,\frac{1}{2},0,\frac{1}{2}$	primitive
$12^F_0(\begin{smallmatrix}0\\0\end{smallmatrix})$	16	$1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$	$0,\frac{1}{2},0,\frac{1}{2},\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4},\frac{1}{16},\frac{9}{16},\frac{5}{16},\frac{13}{16}$	$\mathcal{F}_0 \boxtimes 6^B_{3/2}(\begin{smallmatrix}0\\0\end{smallmatrix})$
$12^F_*(\begin{smallmatrix}2\\1/4\end{smallmatrix})$	16	$1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$	$0,\frac{1}{2},0,\frac{1}{2},\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4},\frac{1}{8},\frac{5}{8},\frac{1}{8},\frac{5}{8}$	primitive
$12^F_0(\begin{smallmatrix}0\\0\end{smallmatrix})$	16	$1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$	$0,\frac{1}{2},0,\frac{1}{2},\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4},\frac{3}{16},\frac{11}{16},\frac{7}{16},\frac{15}{16}$	$\mathcal{F}_0 \boxtimes 6^B_{1/2}(\begin{smallmatrix}0\\0\end{smallmatrix})$
$12^F_*(\begin{smallmatrix}2\\1/2\end{smallmatrix})$	16	$1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$	$0,\frac{1}{2},0,\frac{1}{2},\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4}$	primitive
$12^F_*(\begin{smallmatrix}2\\-1/4\end{smallmatrix})$	16	$1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$	$0,\frac{1}{2},0,\frac{1}{2},\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4},\frac{3}{8},\frac{7}{8},\frac{3}{8},\frac{7}{8}$	primitive
$12^F_{3/10}(\begin{smallmatrix}\zeta_2^1\zeta_3^1\\1/10\end{smallmatrix})$	21.708	$1,1,1,1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1$	$0,\frac{1}{2},\frac{1}{6},\frac{2}{3},\frac{1}{6},\frac{2}{3},\frac{1}{15},\frac{17}{30},\frac{1}{15},\frac{17}{30},\frac{2}{5},\frac{9}{10}$	$\mathcal{F}_0 \boxtimes 6^B_{4/5}(\begin{smallmatrix}\zeta_3^1\\1/10\end{smallmatrix})$
$12^F_{1/5}(\begin{smallmatrix}\zeta_2^1\zeta_3^1\\2/5\end{smallmatrix})$	21.708	$1,1,1,1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1$	$0,\frac{1}{2},\frac{1}{6},\frac{2}{3},\frac{1}{6},\frac{2}{3},\frac{1}{10},\frac{3}{5},\frac{4}{15},\frac{23}{30},\frac{4}{15},\frac{23}{30}$	$\mathcal{F}_0 \boxtimes 6^B_{16/5}(\begin{smallmatrix}\zeta_3^1\\2/5\end{smallmatrix})$
$12^F_{1/5}(\begin{smallmatrix}\zeta_2^1\zeta_3^1\\-1/10\end{smallmatrix})$	21.708	$1,1,1,1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1$	$0,\frac{1}{2},\frac{1}{3},\frac{5}{6},\frac{1}{3},\frac{5}{6},\frac{1}{10},\frac{3}{5},\frac{13}{30},\frac{14}{15},\frac{13}{30},\frac{14}{15}$	$\mathcal{F}_0 \boxtimes 6^B_{-4/5}(\begin{smallmatrix}\zeta_3^1\\-1/10\end{smallmatrix})$
$12^F_{3/10}(\begin{smallmatrix}\zeta_2^1\zeta_3^1\\-2/5\end{smallmatrix})$	21.708	$1,1,1,1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1$	$0,\frac{1}{2},\frac{1}{3},\frac{5}{6},\frac{1}{3},\frac{5}{6},\frac{7}{30},\frac{11}{15},\frac{7}{30},\frac{11}{15},\frac{2}{5},\frac{9}{10}$	$\mathcal{F}_0 \boxtimes 6^B_{-16/5}(\begin{smallmatrix}\zeta_3^1\\-2/5\end{smallmatrix})$
$12^F_{1/5}(\begin{smallmatrix}\zeta_3^1\zeta_6^3\\17/80\end{smallmatrix})$	28.944	$1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1,\zeta_2^1\zeta_3^1$	$0,\frac{1}{2},0,\frac{1}{2},\frac{1}{16},\frac{9}{16},\frac{1}{10},\frac{3}{5},\frac{1}{10},\frac{3}{5},\frac{13}{80},\frac{53}{80}$	$\mathcal{F}_0 \boxtimes 6^B_{17/10}(\begin{smallmatrix}\zeta_3^1\zeta_6^1\\17/80\end{smallmatrix})$
$12^F_{3/10}(\begin{smallmatrix}\zeta_3^1\zeta_6^3\\-7/80\end{smallmatrix})$	28.944	$1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1,\zeta_2^1\zeta_3^1$	$0,\frac{1}{2},0,\frac{1}{2},\frac{1}{16},\frac{9}{16},\frac{2}{5},\frac{9}{10},\frac{2}{5},\frac{9}{10},\frac{37}{80},\frac{77}{80}$	$\mathcal{F}_0 \boxtimes 6^B_{33/10}(\begin{smallmatrix}\zeta_3^1\zeta_6^1\\-7/80\end{smallmatrix})$
$12^F_{1/5}(\begin{smallmatrix}1.7513\\27/80\end{smallmatrix})$	28.944	$1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1,\zeta_2^1\zeta_3^1$	$0,\frac{1}{2},0,\frac{1}{2},\frac{3}{16},\frac{11}{16},\frac{1}{10},\frac{3}{5},\frac{1}{10},\frac{3}{5},\frac{23}{80},\frac{63}{80}$	$\mathcal{F}_0 \boxtimes 6^B_{27/10}(\begin{smallmatrix}1.2383\\27/80\end{smallmatrix})$
$12^F_{3/10}(\begin{smallmatrix}1.7513\\3/80\end{smallmatrix})$	28.944	$1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1,\zeta_2^1\zeta_3^1$	$0,\frac{1}{2},0,\frac{1}{2},\frac{3}{16},\frac{11}{16},\frac{2}{5},\frac{9}{10},\frac{2}{5},\frac{9}{10},\frac{7}{80},\frac{47}{80}$	$\mathcal{F}_0 \boxtimes 6^B_{3/10}(\begin{smallmatrix}1.2383\\3/80\end{smallmatrix})$
$12^F_{1/5}(\begin{smallmatrix}1.7513\\-3/80\end{smallmatrix})$	28.944	$1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1,\zeta_2^1\zeta_3^1$	$0,\frac{1}{2},0,\frac{1}{2},\frac{5}{16},\frac{13}{16},\frac{1}{10},\frac{3}{5},\frac{1}{10},\frac{3}{5},\frac{33}{80},\frac{73}{80}$	$\mathcal{F}_0 \boxtimes 6^B_{37/10}(\begin{smallmatrix}1.2383\\-3/80\end{smallmatrix})$
$12^F_{3/10}(\begin{smallmatrix}1.7513\\-27/80\end{smallmatrix})$	28.944	$1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1,\zeta_2^1\zeta_3^1$	$0,\frac{1}{2},0,\frac{1}{2},\frac{5}{16},\frac{13}{16},\frac{2}{5},\frac{9}{10},\frac{2}{5},\frac{9}{10},\frac{17}{80},\frac{57}{80}$	$\mathcal{F}_0 \boxtimes 6^B_{13/10}(\begin{smallmatrix}1.2383\\-27/80\end{smallmatrix})$
$12^F_{1/5}(\begin{smallmatrix}\zeta_3^1\zeta_6^3\\7/80\end{smallmatrix})$	28.944	$1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1,\zeta_2^1\zeta_3^1$	$0,\frac{1}{2},0,\frac{1}{2},\frac{7}{16},\frac{15}{16},\frac{1}{10},\frac{3}{5},\frac{1}{10},\frac{3}{5},\frac{3}{80},\frac{43}{80}$	$\mathcal{F}_0 \boxtimes 6^B_{7/10}(\begin{smallmatrix}\zeta_3^1\zeta_6^1\\7/80\end{smallmatrix})$
$12^F_{3/10}(\begin{smallmatrix}\zeta_3^1\zeta_6^3\\-17/80\end{smallmatrix})$	28.944	$1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_2^1\zeta_3^1,\zeta_2^1\zeta_3^1$	$0,\frac{1}{2},0,\frac{1}{2},\frac{7}{16},\frac{15}{16},\frac{2}{5},\frac{9}{10},\frac{2}{5},\frac{9}{10},\frac{27}{80},\frac{67}{80}$	$\mathcal{F}_0 \boxtimes 6^B_{23/10}(\begin{smallmatrix}\zeta_3^1\zeta_6^1\\-17/80\end{smallmatrix})$
$12^F_{1/7}(\begin{smallmatrix}0\\0\end{smallmatrix})$	37.183	$1,1,1,1,\zeta_5^1,\zeta_5^1,\zeta_5^1,\zeta_5^1,\zeta_5^2,\zeta_5^2,\zeta_5^2,\zeta_5^2$	$0,\frac{1}{2},\frac{1}{4},\frac{3}{4},\frac{3}{28},\frac{17}{28},\frac{5}{14},\frac{6}{7},\frac{1}{28},\frac{15}{28},\frac{2}{7},\frac{11}{14}$	$\mathcal{F}_0 \boxtimes 6^B_{1/7}(\begin{smallmatrix}0\\0\end{smallmatrix})$
$12^F_{5/14}(\begin{smallmatrix}0\\0\end{smallmatrix})$	37.183	$1,1,1,1,\zeta_5^1,\zeta_5^1,\zeta_5^1,\zeta_5^1,\zeta_5^2,\zeta_5^2,\zeta_5^2,\zeta_5^2$	$0,\frac{1}{2},\frac{1}{4},\frac{3}{4},\frac{1}{7},\frac{9}{14},\frac{11}{28},\frac{25}{28},\frac{3}{14},\frac{5}{7},\frac{13}{28},\frac{27}{28}$	$\mathcal{F}_0 \boxtimes 6^B_{41/7}(\begin{smallmatrix}0\\0\end{smallmatrix})$
$12^F_0(\begin{smallmatrix}\zeta_2^1\zeta_8^4\\0\end{smallmatrix})$	40	$1,1,1,1,2,2,2,2,\sqrt{5},\sqrt{5},\sqrt{5},\sqrt{5}$	$0,\frac{1}{2},0,\frac{1}{2},\frac{1}{10},\frac{3}{5},\frac{2}{5},\frac{9}{10},0,\frac{1}{2},0,\frac{1}{2}$	$\mathcal{F}_0 \boxtimes 6^B_4(\begin{smallmatrix}\zeta_8^4\\0\end{smallmatrix})$
$12^F_*(\begin{smallmatrix}\sqrt{12}\\0.1830\end{smallmatrix})$	40	$1,1,1,1,2,2,2,2,\sqrt{5},\sqrt{5},\sqrt{5},\sqrt{5}$	$0,\frac{1}{2},0,\frac{1}{2},\frac{1}{10},\frac{3}{5},\frac{2}{5},\frac{9}{10},\frac{1}{8},\frac{5}{8},\frac{1}{8},\frac{5}{8}$	primitive
$12^F_0(\begin{smallmatrix}1.7480\\1/2\end{smallmatrix})$	40	$1,1,1,1,2,2,2,2,\sqrt{5},\sqrt{5},\sqrt{5},\sqrt{5}$	$0,\frac{1}{2},0,\frac{1}{2},\frac{1}{10},\frac{3}{5},\frac{2}{5},\frac{9}{10},\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4}$	$\mathcal{F}_0 \boxtimes 6^B_4(\begin{smallmatrix}1.2360\\1/2\end{smallmatrix})$
$12^F_*(\begin{smallmatrix}\sqrt{12}\\-0.1830\end{smallmatrix})$	40	$1,1,1,1,2,2,2,2,\sqrt{5},\sqrt{5},\sqrt{5},\sqrt{5}$	$0,\frac{1}{2},0,\frac{1}{2},\frac{1}{10},\frac{3}{5},\frac{2}{5},\frac{9}{10},\frac{3}{8},\frac{7}{8},\frac{3}{8},\frac{7}{8}$	primitive
$12^F_0(\begin{smallmatrix}1.7480\\0\end{smallmatrix})$	40	$1,1,1,1,2,2,2,2,\sqrt{5},\sqrt{5},\sqrt{5},\sqrt{5}$	$0,\frac{1}{2},0,\frac{1}{2},\frac{1}{5},\frac{7}{10},\frac{3}{10},\frac{4}{5},0,\frac{1}{2},0,\frac{1}{2}$	$\mathcal{F}_0 \boxtimes 6^B_0(\begin{smallmatrix}1.2360\\0\end{smallmatrix})$
$12^F_*(\begin{smallmatrix}\sqrt{12}\\0.3169\end{smallmatrix})$	40	$1,1,1,1,2,2,2,2,\sqrt{5},\sqrt{5},\sqrt{5},\sqrt{5}$	$0,\frac{1}{2},0,\frac{1}{2},\frac{1}{5},\frac{7}{10},\frac{3}{10},\frac{4}{5},\frac{1}{8},\frac{5}{8},\frac{1}{8},\frac{5}{8}$	primitive
$12^F_0(\begin{smallmatrix}\zeta_2^1\zeta_8^4\\1/2\end{smallmatrix})$	40	$1,1,1,1,2,2,2,2,\sqrt{5},\sqrt{5},\sqrt{5},\sqrt{5}$	$0,\frac{1}{2},0,\frac{1}{2},\frac{1}{5},\frac{7}{10},\frac{3}{10},\frac{4}{5},\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4}$	$\mathcal{F}_0 \boxtimes 6^B_0(\begin{smallmatrix}\zeta_8^4\\1/2\end{smallmatrix})$
$12^F_*(\begin{smallmatrix}\sqrt{12}\\-0.3169\end{smallmatrix})$	40	$1,1,1,1,2,2,2,2,\sqrt{5},\sqrt{5},\sqrt{5},\sqrt{5}$	$0,\frac{1}{2},0,\frac{1}{2},\frac{1}{5},\frac{7}{10},\frac{3}{10},\frac{4}{5},\frac{3}{8},\frac{7}{8},\frac{3}{8},\frac{7}{8}$	primitive
$12^F_{1/4}(\begin{smallmatrix}\zeta_6^3\\-1/4\end{smallmatrix})$	40.970	$1,1,1,1,1,1,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^2$	$0,\frac{1}{2},\frac{1}{6},\frac{2}{3},\frac{1}{6},\frac{2}{3},\frac{1}{4},\frac{3}{4},\frac{5}{12},\frac{11}{12},\frac{5}{12},\frac{11}{12}$	$4^F_{1/4}(\begin{smallmatrix}\zeta_6^3\\1/2\end{smallmatrix}) \boxtimes 3^B_6(\begin{smallmatrix}1\\1/4\end{smallmatrix})$
$12^F_{1/4}(\begin{smallmatrix}\zeta_6^3\\1/4\end{smallmatrix})$	40.970	$1,1,1,1,1,1,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^2$	$0,\frac{1}{2},\frac{1}{3},\frac{5}{6},\frac{1}{3},\frac{5}{6},\frac{1}{12},\frac{7}{12},\frac{1}{12},\frac{7}{12},\frac{1}{4},\frac{3}{4}$	$4^F_{1/4}(\begin{smallmatrix}\zeta_6^3\\1/2\end{smallmatrix}) \boxtimes 3^B_2(\begin{smallmatrix}1\\-1/4\end{smallmatrix})$
$12^F_{1/4}(\begin{smallmatrix}\chi_8^2\\-7/16\end{smallmatrix})$	54.627	$1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^2,\chi_2^2,\chi_2^2$	$0,\frac{1}{2},0,\frac{1}{2},\frac{1}{16},\frac{9}{16},\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4},\frac{5}{16},\frac{13}{16}$	$4^F_{1/4}(\begin{smallmatrix}\zeta_6^3\\1/2\end{smallmatrix}) \boxtimes 3^B_{1/2}(\begin{smallmatrix}\zeta_6^1\\1/16\end{smallmatrix})$
$12^F_{1/4}(\begin{smallmatrix}2\\-5/16\end{smallmatrix})$	54.627	$1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^2,\chi_2^2,\chi_2^2$	$0,\frac{1}{2},0,\frac{1}{2},\frac{3}{16},\frac{11}{16},\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4},\frac{7}{16},\frac{15}{16}$	$4^F_{1/4}(\begin{smallmatrix}\zeta_6^3\\1/2\end{smallmatrix}) \boxtimes 3^B_{3/2}(\begin{smallmatrix}0.7653\\3/16\end{smallmatrix})$
$12^F_{1/4}(\begin{smallmatrix}2\\5/16\end{smallmatrix})$	54.627	$1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^2,\chi_2^2,\chi_2^2$	$0,\frac{1}{2},0,\frac{1}{2},\frac{5}{16},\frac{13}{16},\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4},\frac{1}{16},\frac{9}{16}$	$4^F_{1/4}(\begin{smallmatrix}\zeta_6^3\\1/2\end{smallmatrix}) \boxtimes 3^B_{5/2}(\begin{smallmatrix}0.7653\\-3/16\end{smallmatrix})$
$12^F_{1/4}(\begin{smallmatrix}\chi_8^2\\7/16\end{smallmatrix})$	54.627	$1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^2,\chi_2^2,\chi_2^2$	$0,\frac{1}{2},0,\frac{1}{2},\frac{7}{16},\frac{15}{16},\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4},\frac{3}{16},\frac{11}{16}$	$4^F_{1/4}(\begin{smallmatrix}\zeta_6^3\\1/2\end{smallmatrix}) \boxtimes 3^B_{7/2}(\begin{smallmatrix}\zeta_6^1\\-1/16\end{smallmatrix})$
$12^F_{2/35}(\begin{smallmatrix}\zeta_2^1\zeta_3^1\zeta_5^2\\-0.4928\end{smallmatrix})$	67.265	$1,1,\zeta_3^1,\zeta_3^1,\zeta_5^1,\zeta_5^1,\zeta_5^2,\zeta_5^2,\zeta_3^1\zeta_5^1,\zeta_3^1\zeta_5^1,\zeta_3^1\zeta_5^2,\zeta_3^1\zeta_5^2$	$0,\frac{1}{2},\frac{1}{10},\frac{3}{5},\frac{1}{7},\frac{9}{14},\frac{3}{14},\frac{5}{7},\frac{17}{70},\frac{26}{35},\frac{11}{35},\frac{57}{70}$	$\mathcal{F}_0 \boxtimes 6^B_{-138/35}(\begin{smallmatrix}\zeta_3^1\zeta_5^2\\-0.4928\end{smallmatrix})$
$12^F_{12/35}(\begin{smallmatrix}\zeta_2^1\zeta_3^1\zeta_5^2\\-0.2071\end{smallmatrix})$	67.265	$1,1,\zeta_3^1,\zeta_3^1,\zeta_5^1,\zeta_5^1,\zeta_5^2,\zeta_5^2,\zeta_3^1\zeta_5^1,\zeta_3^1\zeta_5^1,\zeta_3^1\zeta_5^2,\zeta_3^1\zeta_5^2$	$0,\frac{1}{2},\frac{1}{10},\frac{3}{5},\frac{5}{14},\frac{6}{7},\frac{2}{7},\frac{11}{14},\frac{16}{35},\frac{67}{70},\frac{27}{70},\frac{31}{35}$	$\mathcal{F}_0 \boxtimes 6^B_{-58/35}(\begin{smallmatrix}\zeta_3^1\zeta_5^2\\-0.2071\end{smallmatrix})$
$12^F_{11/70}(\begin{smallmatrix}\zeta_2^1\zeta_3^1\zeta_5^2\\0.2071\end{smallmatrix})$	67.265	$1,1,\zeta_3^1,\zeta_3^1,\zeta_5^1,\zeta_5^1,\zeta_5^2,\zeta_5^2,\zeta_3^1\zeta_5^1,\zeta_3^1\zeta_5^1,\zeta_3^1\zeta_5^2,\zeta_3^1\zeta_5^2$	$0,\frac{1}{2},\frac{2}{5},\frac{9}{10},\frac{1}{7},\frac{9}{14},\frac{3}{14},\frac{5}{7},\frac{3}{70},\frac{19}{35},\frac{4}{35},\frac{43}{70}$	$\mathcal{F}_0 \boxtimes 6^B_{58/35}(\begin{smallmatrix}\zeta_3^1\zeta_5^2\\0.2071\end{smallmatrix})$
$12^F_{31/70}(\begin{smallmatrix}\zeta_2^1\zeta_3^1\zeta_5^2\\0.4928\end{smallmatrix})$	67.265	$1,1,\zeta_3^1,\zeta_3^1,\zeta_5^1,\zeta_5^1,\zeta_5^2,\zeta_5^2,\zeta_3^1\zeta_5^1,\zeta_3^1\zeta_5^1,\zeta_3^1\zeta_5^2,\zeta_3^1\zeta_5^2$	$0,\frac{1}{2},\frac{2}{5},\frac{9}{10},\frac{5}{14},\frac{6}{7},\frac{2}{7},\frac{11}{14},\frac{9}{35},\frac{53}{70},\frac{13}{70},\frac{24}{35}$	$\mathcal{F}_0 \boxtimes 6^B_{138/35}(\begin{smallmatrix}\zeta_3^1\zeta_5^2\\0.4928\end{smallmatrix})$
$12^F_0(\begin{smallmatrix}0\\0\end{smallmatrix})$	89.569	$1,1,1,1,\zeta_{10}^2,\zeta_{10}^2,\zeta_{10}^2,\zeta_{10}^2,\zeta_{10}^4,\zeta_{10}^4,\zeta_{10}^4,\zeta_{10}^4$	$0,\frac{1}{2},\frac{1}{4},\frac{3}{4},\frac{1}{12},\frac{7}{12},\frac{1}{3},\frac{5}{6},0,\frac{1}{2},\frac{1}{4},\frac{3}{4}$	$6^F_0(\begin{smallmatrix}\zeta_{10}^5\\-1/12\end{smallmatrix}) \boxtimes 2^B_1(\begin{smallmatrix}0\\0\end{smallmatrix})$
$12^F_0(\begin{smallmatrix}0\\0\end{smallmatrix})$	89.569	$1,1,1,1,\zeta_{10}^2,\zeta_{10}^2,\zeta_{10}^2,\zeta_{10}^2,\zeta_{10}^4,\zeta_{10}^4,\zeta_{10}^4,\zeta_{10}^4$	$0,\frac{1}{2},\frac{1}{4},\frac{3}{4},\frac{1}{6},\frac{2}{3},\frac{5}{12},\frac{11}{12},0,\frac{1}{2},\frac{1}{4},\frac{3}{4}$	$6^F_0(\begin{smallmatrix}\zeta_{10}^5\\1/12\end{smallmatrix}) \boxtimes 2^B_1(\begin{smallmatrix}0\\0\end{smallmatrix})$
$12^F_{6/13}(\begin{smallmatrix}\zeta_2^1\zeta_{11}^5\\3/52\end{smallmatrix})$	113.49	$1,1,\zeta_{11}^1,\zeta_{11}^1,\zeta_{11}^2,\zeta_{11}^2,\zeta_{11}^3,\zeta_{11}^3,\zeta_{11}^4,\zeta_{11}^4,\zeta_{11}^5,\zeta_{11}^5$	$0,\frac{1}{2},\frac{5}{26},\frac{9}{13},\frac{9}{26},\frac{11}{13},\frac{6}{13},\frac{25}{26},\frac{1}{26},\frac{7}{13},\frac{1}{13},\frac{15}{26}$	$\mathcal{F}_0 \boxtimes 6^B_{-46/13}(\begin{smallmatrix}\zeta_{11}^5\\3/52\end{smallmatrix})$
$12^F_{1/26}(\begin{smallmatrix}\zeta_2^1\zeta_{11}^5\\-3/52\end{smallmatrix})$	113.49	$1,1,\zeta_{11}^1,\zeta_{11}^1,\zeta_{11}^2,\zeta_{11}^2,\zeta_{11}^3,\zeta_{11}^3,\zeta_{11}^4,\zeta_{11}^4,\zeta_{11}^5,\zeta_{11}^5$	$0,\frac{1}{2},\frac{4}{13},\frac{21}{26},\frac{2}{13},\frac{17}{26},\frac{1}{26},\frac{7}{13},\frac{6}{13},\frac{25}{26},\frac{11}{26},\frac{12}{13}$	$\mathcal{F}_0 \boxtimes 6^B_{46/13}(\begin{smallmatrix}\zeta_{11}^5\\-3/52\end{smallmatrix})$

TABLE V. A list of simple fermionic topological orders (up to invertible ones) with $N = 12$ types of topological excitations. The table contains all fermionic topological orders with $D^2 \leqslant 50$. Here $\chi_n^m = m + \sqrt{n}$.

$N_c^F(\substack{\lvert\Theta_2\rvert \\ \angle\Theta_2/2\pi})$	D^2	$d_1,d_2,\ldots$	$s_1,s_2,\ldots$	Comments/K-matrix
$12^F_{3/28}(\substack{\zeta_6^3\zeta_5^2 \\ -1/7})$	126.95	$1,1,\zeta_5^1,\zeta_5^1,\zeta_5^2,\zeta_5^2,\zeta_6^2,\zeta_6^2,\zeta_6^2\zeta_5^1,\zeta_6^2\zeta_5^1,\zeta_6^2\zeta_5^2,\zeta_6^2\zeta_5^2$	$0,\frac{1}{2},\frac{1}{7},\frac{9}{14},\frac{3}{14},\frac{5}{7},\frac{1}{4},\frac{3}{4},\frac{11}{28},\frac{25}{28},\frac{13}{28},\frac{27}{28}$	$3^B_{-8/7}\boxtimes 4^F_{1/4}(\substack{\zeta_6^3 \\ 1/2})$
$12^F_{-3/28}(\substack{\zeta_6^3\zeta_5^2 \\ 1/7})$	126.95	$1,1,\zeta_5^1,\zeta_5^1,\zeta_5^2,\zeta_5^2,\zeta_6^2,\zeta_6^2,\zeta_6^2\zeta_5^1,\zeta_6^2\zeta_5^1,\zeta_6^2\zeta_5^2,\zeta_6^2\zeta_5^2$	$0,\frac{1}{2},\frac{5}{14},\frac{6}{7},\frac{2}{7},\frac{11}{14},\frac{1}{4},\frac{3}{4},\frac{3}{28},\frac{17}{28},\frac{1}{28},\frac{15}{28}$	$3^B_{8/7}\boxtimes 4^F_{1/4}(\substack{\zeta_6^3 \\ 1/2})$
$12^F_{8/3}(\substack{6.2387 \\ 1/3})$	149.23	$1,1,\zeta_7^3,\zeta_7^3,\zeta_7^3,\zeta_7^3,\zeta_7^3,\zeta_7^3,\zeta_{16}^4,\zeta_{16}^4,\zeta_{16}^6,\zeta_{16}^6$	$0,\frac{1}{2},\frac{1}{9},\frac{11}{18},\frac{1}{9},\frac{11}{18},\frac{1}{9},\frac{11}{18},\frac{1}{3},\frac{5}{6},\frac{1}{6},\frac{2}{3}$	$6^B_{8/3}\boxtimes\mathcal{F}_0$
$12^F_{-8/3}(\substack{\zeta_2^1\zeta_{16}^4 \\ -1/3})$	149.23	$1,1,\zeta_7^3,\zeta_7^3,\zeta_7^3,\zeta_7^3,\zeta_7^3,\zeta_7^3,\zeta_{16}^4,\zeta_{16}^4,\zeta_{16}^6,\zeta_{16}^6$	$0,\frac{1}{2},\frac{7}{18},\frac{8}{9},\frac{7}{18},\frac{8}{9},\frac{7}{18},\frac{8}{9},\frac{1}{6},\frac{2}{3},\frac{1}{3},\frac{5}{6}$	$6^B_{-8/3}\boxtimes\mathcal{F}_0$
$12^F_{1/5}(\substack{\zeta_3^1\zeta_{10}^5 \\ 7/30})$	162.03	$1,1,\zeta_3^1,\zeta_3^1,\zeta_{10}^2,\zeta_{10}^2,\zeta_{10}^4,\zeta_{10}^4,\zeta_3^1\zeta_{10}^2,\zeta_3^1\zeta_{10}^2,\zeta_3^1\zeta_{10}^4,\zeta_3^1\zeta_{10}^4$	$0,\frac{1}{2},\frac{1}{10},\frac{3}{5},\frac{1}{6},\frac{2}{3},0,\frac{1}{2},\frac{4}{15},\frac{23}{30},\frac{1}{10},\frac{3}{5}$	$2^B_{-14/5}\boxtimes 6^F_0(\substack{\zeta_{10}^5 \\ 1/12})$
$12^F_{1/5}(\substack{\zeta_3^1\zeta_{10}^5 \\ 1/15})$	162.03	$1,1,\zeta_3^1,\zeta_3^1,\zeta_{10}^2,\zeta_{10}^2,\zeta_{10}^4,\zeta_{10}^4,\zeta_3^1\zeta_{10}^2,\zeta_3^1\zeta_{10}^2,\zeta_3^1\zeta_{10}^4,\zeta_3^1\zeta_{10}^4$	$0,\frac{1}{2},\frac{1}{10},\frac{3}{5},\frac{1}{3},\frac{5}{6},0,\frac{1}{2},\frac{13}{30},\frac{14}{15},\frac{1}{10},\frac{3}{5}$	$2^B_{-14/5}\boxtimes 6^F_0(\substack{\zeta_{10}^5 \\ -1/12})$
$12^F_{-1/5}(\substack{\zeta_3^1\zeta_{10}^5 \\ -1/15})$	162.03	$1,1,\zeta_3^1,\zeta_3^1,\zeta_{10}^2,\zeta_{10}^2,\zeta_{10}^4,\zeta_{10}^4,\zeta_3^1\zeta_{10}^2,\zeta_3^1\zeta_{10}^2,\zeta_3^1\zeta_{10}^4,\zeta_3^1\zeta_{10}^4$	$0,\frac{1}{2},\frac{2}{5},\frac{9}{10},\frac{1}{6},\frac{2}{3},0,\frac{1}{2},\frac{1}{15},\frac{17}{30},\frac{2}{5},\frac{9}{10}$	$2^B_{14/5}\boxtimes 6^F_0(\substack{\zeta_{10}^5 \\ 1/12})$
$12^F_{-1/5}(\substack{\zeta_3^1\zeta_{10}^5 \\ -7/30})$	162.03	$1,1,\zeta_3^1,\zeta_3^1,\zeta_{10}^2,\zeta_{10}^2,\zeta_{10}^4,\zeta_{10}^4,\zeta_3^1\zeta_{10}^2,\zeta_3^1\zeta_{10}^2,\zeta_3^1\zeta_{10}^4,\zeta_3^1\zeta_{10}^4$	$0,\frac{1}{2},\frac{2}{5},\frac{9}{10},\frac{1}{3},\frac{5}{6},0,\frac{1}{2},\frac{7}{30},\frac{11}{15},\frac{2}{5},\frac{9}{10}$	$2^B_{14/5}\boxtimes 6^F_0(\substack{\zeta_{10}^5 \\ -1/12})$
$12^F_{-2}(\substack{6.7759 \\ 1/4})$	201.23	$1,1,\frac{\chi_{21}^3}{2}\times 6,\frac{\chi_{21}^5}{2},\frac{\chi_{21}^5}{2},\frac{\chi_{21}^7}{2},\frac{\chi_{21}^7}{2}$	$0,\frac{1}{2},\frac{1}{14},\frac{4}{7},\frac{1}{7},\frac{9}{14},\frac{2}{7},\frac{11}{14},0,\frac{1}{2},\frac{1}{6},\frac{2}{3}$	$6^B_{-2}\boxtimes\mathcal{F}_0$
$12^F_{2}(\substack{6.7759 \\ -1/4})$	201.23	$1,1,\frac{\chi_{21}^3}{2}\times 6,\frac{\chi_{21}^5}{2},\frac{\chi_{21}^5}{2},\frac{\chi_{21}^7}{2},\frac{\chi_{21}^7}{2}$	$0,\frac{1}{2},\frac{3}{14},\frac{5}{7},\frac{5}{14},\frac{6}{7},\frac{3}{7},\frac{13}{14},0,\frac{1}{2},\frac{1}{3},\frac{5}{6}$	$6^B_{2}\boxtimes\mathcal{F}_0$
$12^F_{1/4}(\substack{\frac{11+\sqrt{1519}}{7} \\ -5/12})$	305.80	$1,1,\zeta_6^2,\zeta_6^2,\zeta_{10}^2,\zeta_{10}^2,\zeta_{10}^4,\zeta_{10}^4,\zeta_6^2\zeta_{10}^2,\zeta_6^2\zeta_{10}^2,\zeta_6^2\zeta_{10}^4,\zeta_6^2\zeta_{10}^4$	$0,\frac{1}{2},\frac{1}{4},\frac{3}{4},\frac{1}{6},\frac{2}{3},0,\frac{1}{2},\frac{5}{12},\frac{11}{12},\frac{1}{4},\frac{3}{4}$	$6^F_0(\substack{\zeta_{10}^5 \\ 1/12})\boxtimes 4^F_{1/4}(\substack{\zeta_6^3 \\ 1/2})$
$12^F_{1/4}(\substack{\frac{11+\sqrt{1519}}{7} \\ 5/12})$	305.80	$1,1,\zeta_6^2,\zeta_6^2,\zeta_{10}^2,\zeta_{10}^2,\zeta_{10}^4,\zeta_{10}^4,\zeta_6^2\zeta_{10}^2,\zeta_6^2\zeta_{10}^2,\zeta_6^2\zeta_{10}^4,\zeta_6^2\zeta_{10}^4$	$0,\frac{1}{2},\frac{1}{4},\frac{3}{4},\frac{1}{3},\frac{5}{6},0,\frac{1}{2},\frac{1}{12},\frac{7}{12},\frac{1}{4},\frac{3}{4}$	$6^F_0(\substack{\zeta_{10}^5 \\ -1/12})\boxtimes 4^F_{1/4}(\substack{\zeta_6^3 \\ 1/2})$
$12^F_{1/4}(\substack{\zeta_{22}^{11} \\ -5/12})$	352.17	$1,1,\zeta_{22}^2,\zeta_{22}^2,\zeta_{22}^4,\zeta_{22}^4,\zeta_{22}^6,\zeta_{22}^6,\zeta_{22}^8,\zeta_{22}^8,\zeta_{22}^{10},\zeta_{22}^{10}$	$0,\frac{1}{2},\frac{1}{12},\frac{7}{12},\frac{1}{4},\frac{3}{4},0,\frac{1}{2},\frac{1}{3},\frac{5}{6},\frac{1}{4},\frac{3}{4}$	$\mathcal{F}_{(A_1,22)}$
$12^F_{1/4}(\substack{\zeta_{22}^{11} \\ 5/12})$	352.17	$1,1,\zeta_{22}^2,\zeta_{22}^2,\zeta_{22}^4,\zeta_{22}^4,\zeta_{22}^6,\zeta_{22}^6,\zeta_{22}^8,\zeta_{22}^8,\zeta_{22}^{10},\zeta_{22}^{10}$	$0,\frac{1}{2},\frac{5}{12},\frac{11}{12},\frac{1}{4},\frac{3}{4},0,\frac{1}{2},\frac{1}{6},\frac{2}{3},\frac{1}{4},\frac{3}{4}$	$\mathcal{F}_{(A_1,-22)}$

that it easily generalizes to fermionic cases, by replacing Rep(G) with sRep(G^f).

Given a symmetry G, not all UBFC's are nondegenerate over Rep(G). Similarly, not all UMTC's admit a G action; there are group cohomological obstructions to define the G action on a UMTC [74]. They must vanish for a consistent G action on a UMTC. On the other hand, a UMTC$_{/\mathrm{Rep}(G)}$ $\mathcal{C}$ may not have modular extensions, and the corresponding UMTC $\mathcal{C}_G$ with G action may not have G-crossed extensions; there are also group cohomological obstructions for the extensions to exist [74]. Reference [67] showed that when the obstructions do not vanish, the anomalous symmetry action can still be realized on the surface of 3+1D systems. To study such anomalous cases, we need the higher dimensional analogs of our proposal.

G. Remarks

Remark 1. Without further announcement, all 2+1D topological orders considered in this work are anomaly-free (or closed) in the sense that they can be realized by a 2+1D lattice model with a local Hamiltonian [46].

Remark 2. In this paper, we use "nondegenerate UBFC over a SFC $\mathcal{E}$" and "UMTC$_{/\mathcal{E}}$" as synonyms. In the bosonic case with no symmetry, "over $\mathcal{E} = \mathcal{B}_0$," or the subscript "$_{/\mathcal{B}_0}$," will be omitted.

Remark 3. We restrict ourselves to *finite* symmetry groups in this work. The representations (or super-representations) of finite groups form symmetric *fusion* categories. For continuous groups, their representations (or super-representations) still form symmetric tensor categories, but not *fusion* categories (there are infinitely many nonisomorphic irreducible representations). It is not clear to what extent our results apply to cases of continuous groups.

Remark 4. Three types of tensor products are used in this work. We use $\boxtimes$ for the stacking product of two phases, $\otimes$ for the fusion product of particles, and $\otimes_{\mathbb{C}}$ for the usual tensor product of vector spaces over $\mathbb{C}$ and that of matrices with $\mathbb{C}$ entries.

II. CATEGORICAL DESCRIPTION OF TOPOLOGICAL ORDERS WITH SYMMETRY

In this section, we give a physically motivated discussion on how to find a categorical description of the particle statistics in a fermionic/bosonic topological order with symmetry. Readers who are not familiar with the categorical view of particle statistics are welcome to first read an elementary discussion of it in Appendix B.

A. Trivial topological orders with symmetry: Categorical view of symmetry

A 2+1D phase with trivial topological order (i.e., a product state) can have only *local particles*, which, by definition, are particles that can be created/annihilated by local operators. In a bosonic trivial phase without symmetry, there is only one type of (indecomposable) particle: the trivial particle **1**. When we localize the particle by a trap, the trapped trivial particle has no internal degrees of freedom (i.e., no degeneracy) and is described by a one-dimensional Hilbert space $\mathbb{C}$. For some very special traps, we may have accidental degeneracy described by a finite dimensional Hilbert space. Such a trapped particle with accidental degeneracy is called a composite particle and is a direct sum of the trivial particle. Therefore the bosonic product states without symmetry can be

TABLE VI. A list of simple fermionic topological orders (up to invertible ones) with $N = 14$ types of topological excitations. The table contains all fermionic topological orders with $D^2 \leqslant 40$. Here $\chi_n^m = m + \sqrt{n}$.

$N_c^F(\genfrac{}{}{0pt}{}{\lvert\Theta_2\rvert}{\angle\Theta_2/2\pi})$	D^2	$d_1,d_2,\ldots$	$s_1,s_2,\ldots$	Comments/K
$14_0^F(\genfrac{}{}{0pt}{}{\zeta_2^1}{1/4})$	14	1,1,1,1,1,1,1,1,1,1,1,1,1,1	$0,\frac{1}{2},\frac{1}{14},\frac{4}{7},\frac{1}{14},\frac{4}{7},\frac{1}{7},\frac{9}{14},\frac{1}{7},\frac{9}{14},\frac{2}{7},\frac{11}{14},\frac{2}{7},\frac{11}{14}$	$\mathcal{F}_0 \boxtimes 7_2^B(\genfrac{}{}{0pt}{}{1}{1/4})$
$14_0^F(\genfrac{}{}{0pt}{}{\zeta_2^1}{-1/4})$	14	1,1,1,1,1,1,1,1,1,1,1,1,1,1	$0,\frac{1}{2},\frac{3}{14},\frac{5}{7},\frac{3}{14},\frac{5}{7},\frac{5}{14},\frac{6}{7},\frac{5}{14},\frac{6}{7},\frac{3}{7},\frac{13}{14},\frac{3}{7},\frac{13}{14}$	$\mathcal{F}_0 \boxtimes 7_{-2}^B(\genfrac{}{}{0pt}{}{1}{-1/4})$
$14_*^F(\genfrac{}{}{0pt}{}{\sqrt{20}}{0.0512})$	32	1,1,1,1,1,1,1,1,2,2,2,2,2,2	$0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{8},\frac{5}{8}$	primitive
$14_*^F(\genfrac{}{}{0pt}{}{\sqrt{20}}{-0.0512})$	32	1,1,1,1,1,1,1,1,2,2,2,2,2,2	$0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{3}{8},\frac{7}{8}$	primitive
$14_*^F(\genfrac{}{}{0pt}{}{2}{1/8})$	32	1,1,1,1,1,1,1,1,2,2,2,2,2,2	$0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{8},\frac{5}{8},\frac{1}{4},\frac{3}{4}$	primitive
$14_*^F(\genfrac{}{}{0pt}{}{2}{-1/8})$	32	1,1,1,1,1,1,1,1,2,2,2,2,2,2	$0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{4},\frac{3}{4},\frac{3}{8},\frac{7}{8}$	primitive
$14_*^F(\genfrac{}{}{0pt}{}{\sqrt{20}}{0.1987})$	32	1,1,1,1,1,1,1,1,2,2,2,2,2,2	$0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{8},\frac{5}{8},\frac{1}{8},\frac{5}{8},\frac{1}{8},\frac{5}{8}$	primitive
$14_*^F(\genfrac{}{}{0pt}{}{2}{1/8})$	32	1,1,1,1,1,1,1,1,2,2,2,2,2,2	$0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{8},\frac{5}{8},\frac{1}{8},\frac{5}{8},\frac{3}{8},\frac{7}{8}$	primitive
$14_*^F(\genfrac{}{}{0pt}{}{2}{3/8})$	32	1,1,1,1,1,1,1,1,2,2,2,2,2,2	$0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{8},\frac{5}{8},\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4}$	primitive
$14_*^F(\genfrac{}{}{0pt}{}{2}{-1/8})$	32	1,1,1,1,1,1,1,1,2,2,2,2,2,2	$0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{8},\frac{5}{8},\frac{3}{8},\frac{7}{8},\frac{3}{8},\frac{7}{8}$	primitive
$14_*^F(\genfrac{}{}{0pt}{}{2}{-3/8})$	32	1,1,1,1,1,1,1,1,2,2,2,2,2,2	$0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4},\frac{3}{8},\frac{7}{8}$	primitive
$14_*^F(\genfrac{}{}{0pt}{}{\sqrt{20}}{-0.1987})$	32	1,1,1,1,1,1,1,1,2,2,2,2,2,2	$0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{3}{8},\frac{7}{8},\frac{3}{8},\frac{7}{8},\frac{3}{8},\frac{7}{8}$	primitive
$14_{1/4}^F(\genfrac{}{}{0pt}{}{\zeta_{14}^7}{-15/32})$	54.627	$1,1,1,1,\zeta_6^1,\zeta_6^1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^3,\zeta_6^3$	$0,\frac{1}{2},0,\frac{1}{2},\frac{11}{32},\frac{27}{32},\frac{11}{32},\frac{27}{32},\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4},\frac{7}{32},\frac{23}{32}$	$\mathcal{F}_0 \boxtimes 7_{1/4}^B(\genfrac{}{}{0pt}{}{\zeta_{14}^3}{-15/32})$
$14_{1/4}^F(\genfrac{}{}{0pt}{}{4.3454}{-13/32})$	54.627	$1,1,1,1,\zeta_6^1,\zeta_6^1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^3,\zeta_6^3$	$0,\frac{1}{2},0,\frac{1}{2},\frac{9}{32},\frac{25}{32},\frac{9}{32},\frac{25}{32},\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4},\frac{13}{32},\frac{29}{32}$	$\mathcal{F}_0 \boxtimes 7_{3/4}^B(\genfrac{}{}{0pt}{}{3.0727}{-13/32})$
$14_{1/4}^F(\genfrac{}{}{0pt}{}{2.9035}{-11/32})$	54.627	$1,1,1,1,\zeta_6^1,\zeta_6^1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^3,\zeta_6^3$	$0,\frac{1}{2},0,\frac{1}{2},\frac{15}{32},\frac{31}{32},\frac{15}{32},\frac{31}{32},\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4},\frac{11}{32},\frac{27}{32}$	$\mathcal{F}_0 \boxtimes 7_{5/4}^B(\genfrac{}{}{0pt}{}{2.0531}{-11/32})$
$14_{1/4}^F(\genfrac{}{}{0pt}{}{1.0195}{-9/32})$	54.627	$1,1,1,1,\zeta_6^1,\zeta_6^1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^3,\zeta_6^3$	$0,\frac{1}{2},0,\frac{1}{2},\frac{13}{32},\frac{29}{32},\frac{13}{32},\frac{29}{32},\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4},\frac{1}{32},\frac{17}{32}$	$\mathcal{F}_0 \boxtimes 7_{7/4}^B(\genfrac{}{}{0pt}{}{0.7209}{-9/32})$
$14_{1/4}^F(\genfrac{}{}{0pt}{}{1.0195}{9/32})$	54.627	$1,1,1,1,\zeta_6^1,\zeta_6^1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^3,\zeta_6^3$	$0,\frac{1}{2},0,\frac{1}{2},\frac{3}{32},\frac{19}{32},\frac{3}{32},\frac{19}{32},\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4},\frac{15}{32},\frac{31}{32}$	$\mathcal{F}_0 \boxtimes 7_{9/4}^B(\genfrac{}{}{0pt}{}{0.7209}{9/32})$
$14_{1/4}^F(\genfrac{}{}{0pt}{}{2.9035}{11/32})$	54.627	$1,1,1,1,\zeta_6^1,\zeta_6^1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^3,\zeta_6^3$	$0,\frac{1}{2},0,\frac{1}{2},\frac{1}{32},\frac{17}{32},\frac{1}{32},\frac{17}{32},\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4},\frac{5}{32},\frac{21}{32}$	$\mathcal{F}_0 \boxtimes 7_{11/4}^B(\genfrac{}{}{0pt}{}{2.0531}{11/32})$
$14_{1/4}^F(\genfrac{}{}{0pt}{}{4.3454}{13/32})$	54.627	$1,1,1,1,\zeta_6^1,\zeta_6^1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^3,\zeta_6^3$	$0,\frac{1}{2},0,\frac{1}{2},\frac{7}{32},\frac{23}{32},\frac{7}{32},\frac{23}{32},\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4},\frac{3}{32},\frac{19}{32}$	$\mathcal{F}_0 \boxtimes 7_{13/4}^B(\genfrac{}{}{0pt}{}{3.0727}{13/32})$
$14_{1/4}^F(\genfrac{}{}{0pt}{}{\zeta_{14}^7}{15/32})$	54.627	$1,1,1,1,\zeta_6^1,\zeta_6^1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^3,\zeta_6^3$	$0,\frac{1}{2},0,\frac{1}{2},\frac{5}{32},\frac{21}{32},\frac{5}{32},\frac{21}{32},\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4},\frac{9}{32},\frac{25}{32}$	$\mathcal{F}_0 \boxtimes 7_{15/4}^B(\genfrac{}{}{0pt}{}{\zeta_{14}^3}{15/32})$
$14_{2/5}^F(\genfrac{}{}{0pt}{}{\zeta_2^1\zeta_{13}^6}{3/10})$	173.50	$1,1,\zeta_{13}^1,\zeta_{13}^1,\zeta_{13}^2,\zeta_{13}^2,\zeta_{13}^3,\zeta_{13}^3,\zeta_{13}^4,\zeta_{13}^4,\zeta_{13}^5,\zeta_{13}^5,\zeta_{13}^6,\zeta_{13}^6$	$0,\frac{1}{2},\frac{1}{5},\frac{7}{10},\frac{11}{30},\frac{13}{15},0,\frac{1}{2},\frac{1}{10},\frac{3}{5},\frac{1}{6},\frac{2}{3},\frac{1}{5},\frac{7}{10}$	$\mathcal{F}_0 \boxtimes 7_{-8/5}^B(\genfrac{}{}{0pt}{}{\zeta_{13}^6}{3/10})$
$14_{1/10}^F(\genfrac{}{}{0pt}{}{\zeta_2^1\zeta_{13}^6}{-3/10})$	173.50	$1,1,\zeta_{13}^1,\zeta_{13}^1,\zeta_{13}^2,\zeta_{13}^2,\zeta_{13}^3,\zeta_{13}^3,\zeta_{13}^4,\zeta_{13}^4,\zeta_{13}^5,\zeta_{13}^5,\zeta_{13}^6,\zeta_{13}^6$	$0,\frac{1}{2},\frac{3}{10},\frac{4}{5},\frac{2}{15},\frac{19}{30},0,\frac{1}{2},\frac{2}{5},\frac{9}{10},\frac{1}{3},\frac{5}{6},\frac{3}{10},\frac{4}{5}$	$\mathcal{F}_0 \boxtimes 7_{8/5}^B(\genfrac{}{}{0pt}{}{\zeta_{13}^6}{-3/10})$
$14_0^F(\genfrac{}{}{0pt}{}{\chi_8^2}{1/8})$	186.50	$1,1,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^2,\chi_2^2,\chi_2^2,\chi_2^2,\chi_2^2,\chi_8^2,\chi_8^2,\chi_8^3,\chi_8^3$	$0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4},\frac{1}{8},\frac{5}{8},0,\frac{1}{2}$	$\mathcal{F}_0 \boxtimes 7_1^B(\genfrac{}{}{0pt}{}{\chi_2^2}{1/8})$
$14_0^F(\genfrac{}{}{0pt}{}{\chi_8^2}{-1/8})$	186.50	$1,1,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^2,\chi_2^2,\chi_2^2,\chi_2^2,\chi_2^2,\chi_8^2,\chi_8^2,\chi_8^3,\chi_8^3$	$0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4},\frac{3}{8},\frac{7}{8},0,\frac{1}{2}$	$\mathcal{F}_0 \boxtimes 7_{-1}^B(\genfrac{}{}{0pt}{}{\chi_2^2}{-1/8})$

described by the category of finite dimensional Hilbert spaces, denoted by $\mathcal{B}_0$, in which the one-dimensional Hilbert space $\mathbb{C}$ is the trivial particle.

For a 2+1D product state with symmetry (given by a finite group G), all the particles can be created/annihilated by local operators, and are local excitations. They can carry additional charges from the representations of the symmetry. As a consequence, (indecomposable) particles in a bosonic product state with symmetry are described by irreducible representations of G. Thus the trivial topological order with symmetry is described by the category of G representations, denoted by Rep(G) (see also Example 1).

For a fermionic product state with symmetry, we must include in G the fermion-number parity transformation z ($z \neq 1$), which is involutive, i.e., $z^2 = 1$, and commutes with other symmetries, i.e., $zg = gz$ for all $g \in G$. Therefore the fermonic symmetry is a pair $G^f = (G,z)$. The particles in the fermionic product state with symmetry G^f still have to be classified by irreducible representations of G. However, some particles are bosonic and some particles are fermionic: An irreducible representation is bosonic (or fermionic) if z acts as 1 (or -1) in the irreducible representation. These representations braid as bosons and fermions with trivial mutual statistics. Namely, by exchanging the positions of two fermions, we get an extra -1 sign (see Example 2 for a precise mathematical definition). Therefore the particles in a fermionic product state with symmetry G^f are described by the category sRep(G^f), which is the same category as Rep(G) but equipped with the braidings defined according to the fermion-number parity. For the fermonic trivial topological order without symmetry, there is no symmetry other than the fermion-number parity symmetry f, i.e., $G = \{1,z\} = \mathbb{Z}_2$ or $G^f = \mathbb{Z}_2^f = (\mathbb{Z}_2,z)$. In this case, we also denote sRep($\mathbb{Z}_2^f$) by $\mathcal{F}_0$ (see also Sec. I D).

The categories Rep(G) and sRep(G^f) are examples of symmetric fusion category (SFC), which is a UBFC with only trivial double braidings, i.e., trivial mutual statistics (see Sec. III B and Appendix D for precise definitions). It turns out that all SFC's are of these types [65]. More precisely, an SFC $\mathcal{E}$ is either Rep(G) for a unique group G or sRep(G^f) for a unique group G and a central involutive element $1 \neq z \in G$. In other words, SFC's are in one-to-one correspondence with (finite) bosonic/fermionic symmetry groups [G or $G^f = (G,z)$]. Therefore we can refer to a given bosonic/fermionic symmetry by a SFC $\mathcal{E}$, instead of the traditional way, by groups. This is the categorical way to describe symmetries.

In summary, we obtain the following result. All the excitations in a $2+1$D bosonic/fermionic product state with symmetry $\mathcal{E}$ are local, and are described by the SFC $\mathcal{E}$. Note that above statement also covers the cases without symmetry. In particular, when $\mathcal{E} = \mathcal{B}_0$, it describes a bosonic trivial topological order without symmetry; when $\mathcal{E} = \mathcal{F}_0$, it describes a fermonic trivial topological order without symmetry.

B. Nontrivial topological orders with symmetries

UBFC is the natural language to describe the particle statistics (braiding and fusion) in topological orders. The categorical description of symmetry, using the SFC $\mathcal{E}$ instead of the symmetry group, makes it more straightforward to consider nontrivial topological orders with symmetries. Roughly speaking, a UBFC $\mathcal{C}$ describing a nontrivial topological order with symmetry $\mathcal{E}$, must "contain" $\mathcal{E}$ in a certain way. More precisely, (1) $\mathcal{C}$ contains local excitations carrying all the irreducible representations of the symmetry group G. Mathematically, it means that $\mathcal{C}$ must contain $\mathcal{E}$ (either Rep(G) or sRep(G^f)) as a full subcategory (see Definition 3).

(2) Since local excitations, by definition, can be created/annihilated by local operators, they must have trivial mutual statistics with all particles (including themselves). Mathematically, it means that $\mathcal{E}$ lies in the centralizer $Z_2(\mathcal{C})$ of $\mathcal{C}$. The centralizer $Z_2(\mathcal{C})$ of $\mathcal{C}$ is defined as the full subcategory formed by objects that have trivial mutual braidings with all objects (including themselves). See Eq. (15) and Definition 4 for precise definitions.

(3) *Nondegeneracy condition*. In order for the phase to be anomaly-free (recall Remark 1), if a particle has trivial mutual statistics with all particles, it must be a local excitation. Mathematically, it just means that $Z_2(\mathcal{C}) = \mathcal{E}$.

A UBFC satisfying the above three properties is called a nondegenerate UBFC *over* $\mathcal{E}$ (UMTC$_{/\mathcal{E}}$ for short, see also Sec. III B and Definition 6) [54,75]. The precise requirements of the nondegeneracy condition on the S matrix is given in Sec. III. Note that the simplest UMTC$_{/\mathcal{E}}$ is just $\mathcal{E}$ itself, which is nothing but the trivial topological order with the symmetry $\mathcal{E}$.

In summary, we conclude that the bulk topological excitations in a bosonic/fermionic topological order with symmetry $\mathcal{E}$ is described by a UMTC$_{/\mathcal{E}}$. We describe the notion of a UMTC$_{/\mathcal{E}}$ by concrete computable data in Sec. III. For precise mathematical definition, see Appendix D or see Refs. [54,75]. In Appendix C, we provide yet another explanation of the above proposal from the point of view of local operator algebras that define the topological excitations in a topological phase with symmetry.

C. How to measure edge states categorically?

We have explained why a bosonic/fermionic topological order with a given symmetry $\mathcal{E}$ can be naturally described by a UMTC$_{/\mathcal{E}}$ $\mathcal{C}$. However, it also raises a few puzzles.

(1) The particles in $\mathcal{C}$ can be detected or distinguished via braiding only up to those local excitations $\mathcal{E}$. This ambiguity is protected by the symmetry. It raises a question: how to measure $\mathcal{C}$ and the symmetry $\mathcal{E}$ categorically?

(2) The category $\mathcal{C}$ only contains the information of the excitations in the bulk. It does not contain enough information of the edge states. It does not describe invertible topological orders. Unlike the no-symmetry cases, in which one can compute the central charge (mod 8) of a UMTC to get the information of the edge states, the notion of central charge is not defined for a UMTC$_{/\mathcal{E}}$, which is only a UBFC. It raises a question: how to measure the edge states of $\mathcal{C}$ (or invertible topological order) categorically?

Since the only categorical tool is the mutual braidings, the only thing we can do is to gauge the symmetry [66,67,74] by adding external particles to the system such that newly added particles can detect old particles in $\mathcal{E}$. Clearly, there are too many ways to add external particles. We impose the following two natural principles to the categorical detectors: (1) *the principle of efficiency*: a newly added particle should have nontrivial double braidings to at least one object in $\mathcal{E}$; and (2) *the principle of completeness*: the set of all new and old particles should be able to detect each other via double braidings. In other words, they must form a bosonic anomaly-free 2+1D topological order (without symmetry).

In other words, a categorical measurement must be "efficient" and "complete." These two principles lead us to the following precise definition of a categorical measurement, or a modular extension of $\mathcal{C}$.

A *categorical measurement* or a *modular extension* of a UMTC$_{/\mathcal{E}}$ $\mathcal{C}$ is a UMTC $\mathcal{M}$, together with a fully faithful embedding $\mathcal{C} \hookrightarrow \mathcal{M}$, such that the only particles in $\mathcal{M}$ that have trivial mutual braidings with all particles in $\mathcal{E}$ are those in $\mathcal{C}$. Mathematically, it means that the centralizer of $\mathcal{E}$ in $\mathcal{M}$ coincides with $\mathcal{C}$, i.e., $\mathcal{E}_{\mathcal{M}}^{\text{cen}} = \mathcal{C}$ (see Definitions 4 and 7). [Note that the centralizer of $\mathcal{C}$ in $\mathcal{C}$ is the centralizer of $\mathcal{C}$: $\mathcal{C}_{\mathcal{C}}^{\text{cen}} = Z_2(\mathcal{C})$.]

Physical realities lie in how $\mathcal{C}$ can be measured or detected by other nice categories, which, in this case, are nondegenerate UBFC's (or UMTC's). Therefore it is natural to require that a modular extension of a UMTC$_{/\mathcal{E}}$ always exists (see condition 8 in Sec. III A). In other words, it is always possible to gauge the symmetry $\mathcal{E}$ to obtain a modular extension of a UMTC$_{/\mathcal{E}}$. This is also necessary for the symmetry action to be anomaly free. For simplicity, in this paper, we will adopt a nonstandard definition of UBFC, by requiring UBFC to have a modular extension.

When $\mathcal{E} = \text{Rep}(G)$, the modular extensions of Rep(G) are given by the Drinfeld centers of fusion categories Vec_G^ω for $\omega \in \mathcal{H}^3[G,U(1)]$ [71], where Vec_G^ω is the category of G-graded vector spaces twisted by ω. In these cases, we see that the modular extensions of Rep(G) are consistent with the well-known classification of SPT phases by group cohomology [68–70]. We give more details of this case in Ref. [73,76]. In Sec. VII B, we further confirm this picture by explicitly identifying the modular extensions of $\mathcal{F}_0$ with the invertible fermionic topological orders generated by $p + \mathrm{i}p$ superconductors.

Given these evidences, we believe that the modular extension is the proper categorical way to measure the edge states and invertible topological orders that are missing from the categorical description of UMTC$_{/\mathcal{E}}$. Since UMTC's fix the central charge modulo 8, the only ambiguity left is that of E_8 states. This leads to our main proposal in Sec. I E.

III. NONDEGENERATE UBFC OVER SFC

In this section, we transform abstract data and axioms of a nondegenerate UBFC over a SFC to concrete data and equations. Due to the complexity of the axioms and the extra gauge degrees of freedom, expressing the data of a UBFC as concrete tensor entries is quite impractical. To avoid such complexity, we would like to work with the universal gauge-invariant data of a UBFC. Similar to the eigenvalues for matrices, the characters for group representations, for a UBFC, the gauge-invariant data are the fusion rules N^{ij}_k and the topological spins $\theta_i = \mathrm{e}^{2\pi\,\mathrm{i}\,s_i}$. Other gauge-invariant data, such as quantum dimensions and S and T matrices, can be expressed in terms of N^{ij}_k and s_i. These gauge-invariant data must satisfy finitely many algebraic equations according to the axioms of a UBFC.

For a nondegenerate UBFC over a SFC, we have some additional algebraic equations for the gauge-invariant data. This allows us to perform a finite search (for fixed rank within certain bounds) for topological orders with symmetry. In particular, when we choose the SFC to be $\mathcal{F}_0 = \mathrm{sRep}(Z_2^f)$, this leads to a classification and a table of simple 2+1D fermionic topological orders (see Tables I–VI).

A. A simple definition of unitary braided fusion category

A unitary braided fusion category (UBFC) (also called a unitary pre-modular category or a unitary ribbon fusion category) is a theory of the fusion-braiding properties of systems of anyons without the assumption of the nondegeneracy of the mutual braidings. Examples of such anyonic systems are those consisting of fermions, or bosons with some symmetries, as building blocks. The building blocks (the parent bosons/fermions) have trivial mutual braiding but can still be distinguished by fermion-number parity or other symmetry charges. This leads to degenerate mutual braidings.

In our simplified theory, a UBFC is described by a nonnegative integer tensor N^{ij}_k and a mod-1 rational vector s_i, where i,j,k run from 1 to N and N is called the rank of UBFC. We may simply denote a UBFC [the collection of data (N^{ij}_k,s_i)] by $\mathcal{C}$, a particle i in $\mathcal{C}$ by $i \in \mathcal{C}$. Sometimes it is more convenient to use abstract labels rather than 1 to N; we may also abuse $\mathcal{C}$ as the set of labels (particles).

Not all (N^{ij}_k,s_i) describe valid UBFC. In order to describe a UBFC $\mathcal{C}$, (N^{ij}_k,s_i) must satisfy the following conditions [51,53,55,59,77].

(1) *Fusion ring*. N^{ij}_k for the UBFC $\mathcal{C}$ are nonnegative integers that satisfy

$$N^{ij}_k = N^{ji}_k,\quad N^{1i}_j = \delta_{ij},\quad \sum_{k=1}^{N} N^{ik}_1 N^{kj}_1 = \delta_{ij},$$
$$\sum_m N^{ij}_m N^{mk}_l = \sum_n N^{in}_l N^{jk}_n \quad \text{or} \quad \sum_m N^{ij}_m N_m = N_i N_j, \tag{5}$$

where the matrix N_i is given by $(N_i)_{kj} = N^{ij}_k$, and the indices i,j,k run from 1 to N. In fact N^{ij}_1 defines a charge conjugation $i \to \bar{i}$:

$$N^{ij}_1 = \delta_{\bar{i}j}. \tag{6}$$

N^{ij}_k satisfying the above conditions define a fusion ring.

(2) *Rational condition*. N^{ij}_k and s_i for $\mathcal{C}$ satisfy [51,78–80]

$$\sum_r V^r_{ijkl} s_r = 0 \bmod 1, \tag{7}$$

where

$$V^r_{ijkl} = N^{ij}_r N^{kl}_{\bar{r}} + N^{il}_r N^{jk}_{\bar{r}} + N^{ik}_r N^{jl}_{\bar{r}} - (\delta_{ir} + \delta_{jr} + \delta_{kr} + \delta_{lr}) \sum_m N^{ij}_m N^{kl}_{\bar{m}}. \tag{8}$$

(3) *Verlinde fusion characters*. The topological S matrix is given by (see Eq. (223) in Ref. [52])

$$S_{ij} = \frac{1}{D} \sum_k N^{ij}_k \mathrm{e}^{2\pi\,\mathrm{i}\,(s_i+s_j-s_k)} d_k, \tag{9}$$

where d_i (called quantum dimension) is the largest eigenvalue of the matrix N_i and $D = \sqrt{\sum_i d_i^2}$ (called the total quantum dimension). Then [81]

$$\frac{S_{il}S_{jl}}{S_{1l}} = \sum_k N^{ij}_k S_{kl}. \tag{10}$$

(4) *Weak modularity*. The topological T matrix is given by

$$T_{ij} = \delta_{ij} \mathrm{e}^{2\pi\,\mathrm{i}\,s_i}. \tag{11}$$

Then (see Eq. (232) in Ref. [52])

$$S^\dagger T S = \Theta T^\dagger S^\dagger T^\dagger, \quad \Theta = D^{-1} \sum_i \mathrm{e}^{2\pi\,\mathrm{i}\,s_i} d_i^2. \tag{12}$$

(5) *Charge conjugation symmetry*.

$$S_{ij} = S^*_{i\bar{j}}, \quad s_i = s_{\bar{i}}, \quad \text{or} \quad S = S^\dagger C, \quad T = TC, \tag{13}$$

where the charge conjugation matrix C is given by $C_{ij} = N^{ij}_1 = \delta_{i\bar{j}}$.

(6) Let

$$\nu_i = \frac{1}{D^2} \sum_{jk} N^{jk}_i d_j d_k \mathrm{e}^{\mathrm{i}4\pi(s_j-s_k)}, \tag{14}$$

then $\nu_i \in \mathbb{Z}$ if $i = \bar{i}$ [82].

(7) The centralizer of $\mathcal{C}$, $Z_2(\mathcal{C})$, is the subset of the particle labels

$$Z_2(\mathcal{C}) = \left\{ i \mid S_{ij} = \frac{d_i d_j}{D}, \forall j \in \mathcal{C} \right\}. \tag{15}$$

Then, $Z_2(\mathcal{C})$ forms a fusion subring, and such a fusion subring is the fusion ring of a SFC. This leads to several conditions: (a) $d_i =$ integer, $\forall i \in Z_2(\mathcal{C})$, (b) if the fusion ring $Z_2(\mathcal{C})$ is simple (i.e., has no fusion subring), then the fusion ring $Z_2(\mathcal{C})$ is the fusion ring of the representaions of a simple finite group; and (c) every simple fusion subring of $Z_2(\mathcal{C})$ is the fusion ring of the representaions of a simple finite group.

(8) There exists a UMTC $\mathcal{M}$ containing $\mathcal{C}$ as a sub-UBFC, and the set

$$Z_2(\mathcal{C})^{\text{cen}}_{\mathcal{M}} = \left\{ i \in \mathcal{M} \mid S_{ij} = \frac{d_i d_j}{D}, \forall j \in Z_2(\mathcal{C}) \right\} \tag{16}$$

is the same as $\mathcal{C}$. For details, see Sec. VI, Definition 7. The above conditions are necessary and sufficient (due to condition 8) for (N^{ij}_k, s_i) to describe a UBFC.

According to tensor category theory, a UBFC is fully characterized by N^{ij}_k plus a F-tensor and a R tensor [51–53]. In our simplified theory, we use only the data (N^{ij}_k, s_i) to characterize a UBFC. In general, each (N^{ij}_k, s_i) may correspond to several UBFC's. However, for the examples found in this paper, each (N^{ij}_k, s_i) describes a single UBFC.

B. Nondegenerate UBFC over a SFC and classification of 2+1D bosonic/fermionic topological orders with/without symmetry

Two anyons i,j are said to be mutually local if and only if $S_{ij} = d_i d_j / D$. In other words, the mutual braiding (also called the double braiding) of i,j is trivial. In this sense, the centralizer of $\mathcal{C}$, $Z_2(\mathcal{C})$, defined in the last subsection is the subset of anyons that are mutually local to all anyons.

We have the following key definitions.

(1) A UBFC is *nondegenerate* (i.e., a UMTC) if $Z_2(\mathcal{C}) = \{1\}$. In this case, the data (N^{ij}_k, s_i) satisfy additional conditions: (a) S is a unitary matrix; (b) $\Theta = \exp(2\pi\,\mathrm{i}\,\frac{c}{8})$, where c is the chiral central charge; and (c) $\nu_i = 0$ if $i \neq \bar{i}$, and $\nu_i = \pm 1$ if $i = \bar{i}$ [53,55]. The above three conditions on (N^{ij}_k, s_i, c) plus those conditions in Sec. III A gives us a simplified theory of UMTC. Finding (N^{ij}_k, s_i, c) satisfying those conditions allows us to produce a list of simple 2+1D bosonic topological orders [56]. (2) A UBFC $\mathcal{E}$ is *symmetric* (i.e., a SFC) if $Z_2(\mathcal{E}) = \mathcal{E}$. (3) A UBFC $\mathcal{C}$ with an embedding $\mathcal{E} \hookrightarrow Z_2(\mathcal{C})$ is called a UBFC over the SFC $\mathcal{E}$. It is a nondegenerate UBFC over $\mathcal{E}$ (i.e., a UMTC$_{/\mathcal{E}}$) if $\mathcal{E} = Z_2(\mathcal{C})$. Put it simply, a UMTC$_{/\mathcal{E}}$ is a UBFC with $\mathcal{E}$ as its centralizer. One can also find more rigorous but abstract definitions of the above notions in Appendix D.

UMTC$_{/\mathcal{E}}$'s with modular extensions classify all 2+1D bosonic/fermionic topological orders with/without symmetry (up to invertible ones). (1) If we choose $\mathcal{E}$ to be trivial, i.e., $\mathcal{E} = \mathcal{B}_0$, then UMTC$_{/\mathcal{B}_0}$'s are just UMTC's, which classify all 2+1D bosonic topological orders without symmetry. (2) If we choose $\mathcal{E}$ to be the SFC for fermions, i.e., $\mathcal{E} = \mathcal{F}_0$, then UMTC$_{/\mathcal{F}_0}$'s classify all 2+1D fermionic topological orders without symmetry. (3) If we choose $\mathcal{E}$ to be the SFC of the representations of a group G, i.e., $\mathcal{E} = \text{Rep}(G)$, then UMTC$_{/\text{Rep}(G)}$'s classify all 2+1D bosonic topological orders with symmetry G. (4) If we choose $\mathcal{E}$ to be the SFC of the super-representations of fermionic symmetry G^f (recall Sec. II A), i.e., $\mathcal{E} = \text{sRep}(G^f)$, then UMTC$_{/\text{sRep}(G^f)}$'s classify all 2+1D fermionic topological orders with fermonic symmetry G^f.

The first case has been studied in Ref. [56]. In this paper, we concentrate on the second case. We leave the other two cases to Ref. [73].

C. Symmetric fusion category $\mathcal{F}_0$ for fermions

We have proposed that UMTC$_{/\mathcal{F}_0}$'s with modular extensions classify all 2+1D fermionic topological orders without symmetry, and the SFC $\mathcal{F}_0$ gives a fermionic system without topological order. But what is $\mathcal{F}_0$ in terms of gauge-invariant data? Let us list them: (1) The set of objects (particles) $\mathcal{F}_0 = \{1, f\}$. (2) The fusion coefficients N^{ij}_k: $N^{11}_1 = N^{ff}_1 = N^{1f}_f = N^{f1}_f = 1$. Other entries of N^{ij}_k are 0. In other words, the particle f only has a $\mathbb{Z}_2$ conservation: $f \otimes f = 1$ and $f \otimes 1 = f$. (3) $(\theta_1, \theta_f) = (1, -1)$ [i.e., $(s_1, s_f) = (0, \frac{1}{2})$ or $T_{\mathcal{F}_0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$]. In other words, the particle f has Fermi statistics. (4) $S_{\mathcal{F}_0} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. All the particles have trivial mutual statistics between them.

The above data, N^{ij}_k and $(s_1, s_f) = (0, \frac{1}{2})$, describes the SFC for fermions. There is only one such SFC for fermions.

IV. FERMIONIC TOPOLOGICAL ORDERS: UMTC$_{/\mathcal{F}_0}$

A. Conditions on (N^{ij}_k, s_i) for fermionic topological orders

Now we are ready to apply the general properties in Sec. III A for a UBFC, to obtain special properties of a UMTC$_{/\mathcal{F}_0}$. We find that a UMTC$_{/\mathcal{F}_0}$ (i.e., 2+1D fermionic topological orders) is described by (N^{ij}_k, s_i) that satisfy the conditions in Sec. III A plus the following conditions.

(1) Since f is Abelian, we know that for each i there is a unique j such that $N^{fi}_j = 1$, and for $j' \neq j$, $N^{fi}_{j'} = 0$. We denote such j by i^f. Thus fusion with f defines an involution, denoted by $i \mapsto i^f$. We have $(i^f)^f = i$, $N^{fi}_j = \delta_{ji^f}$. Also $d_{i^f} = d_i$.

(2) f is mutually local to all anyons:

$$S_{if} = \frac{1}{D} \frac{\theta_i \theta_f}{\theta_{i^f}} d_{i^f} = \frac{d_i}{D}. \tag{17}$$

Thus we have $\theta_{i^f} = -\theta_i$. This also means that $i^f \neq i$ and $i^f \neq \bar{i}$.

(3) N^{ij}_k and S_{ij} has some symmetries under $i \mapsto i^f$:

$$\begin{aligned} N^{ij}_k &= N^{i^f j^f}_k = N^{i^f j}_{k^f} = N^{ij^f}_{k^f}, \\ S_{ij} &= S_{ij^f}. \end{aligned} \tag{18}$$

This means that if we arrange the order of labels well, the S,T matrices have the form $S = \tilde{S} \otimes_{\mathbb{C}} S_{\mathcal{F}_0}$, $T = \tilde{T} \otimes_{\mathbb{C}} T_{\mathcal{F}_0}$. We may introduce the equivalence relation $i \sim i^f$. $\tilde{S}$ is indexed by the equivalent classes $[i] = [i^f]$. We shall call such equivalent classes $[i]$ *up-to-fermion* types.

(4) Using the fact that $Z_2(\mathcal{C}) = \{1, f\}$, one can show that $\tilde{S}$ must be unitary. Then for the fusion of equivalent classes we have the usual Verlinde formula

$$\begin{aligned} \tilde{N}^{[i][j]}_{[k]} &\overset{\text{def}}{=} N^{ij}_k + N^{ij}_{k^f} = \sum_{[l]} \frac{\tilde{S}_{[i][l]} \tilde{S}_{[j][l]} \tilde{S}^*_{[k][l]}}{\tilde{S}_{[1][l]}}, \\ &\tilde{S}_{[i][j]} \text{ is symmetric and unitary.} \end{aligned} \tag{19}$$

The above conditions plus those conditions in Sec. III A on (N^{ij}_k, s_i) give us a practical definition of UMTC$_{/\mathcal{F}_0}$, which classify 2+1D fermionic topological orders.

B. Numerical solutions for (N^{ij}_k, s_i)

To find (N^{ij}_k, s_i)'s that satisfy the above conditions plus those conditions in Sec. III A, we may start with $(\tilde N^{[i][j]}_{[k]}, \tilde S_{[i][j]})$ that satisfy

$$\begin{aligned} &\tilde N^{[i][j]}_{[k]} = N^{[j][i]}_{[k]}, \quad \tilde N^{[1][i]}_{[j]} = \delta_{[i][j]},\\ &\sum_{[k]} \tilde N^{[i][k]}_{[1]} \tilde N^{[k][j]}_{[1]} = \delta_{[i][j]}, \\ &\sum_{[m]} \tilde N^{[i][j]}_{[m]} \tilde N^{[m][k]}_{[l]} = \sum_{[n]} \tilde N^{[i][n]}_{[l]} \tilde N^{[j][k]}_{[n]}, \end{aligned} \tag{20}$$

and Eq. (19). We then split the value $\tilde N^{[i][j]}_{[k]}$ into two parts and construct N^{ij}_k via

$$\begin{aligned} \tilde N^{[i][j]}_{[k]} &= N^{ij}_k + N^{ij}_{k^f}, \\ N^{ij}_k &= N^{i^f j^f}_k = N^{i^f j}_{k^f} = N^{i j^f}_{k^f}, \\ N^{ij}_{k^f} &= N^{i^f j^f}_{k^f} = N^{i^f j}_{k} = N^{i j^f}_{k}. \end{aligned} \tag{21}$$

Such N^{ij}_k automatically satisfy Eq. (10) for a S that satisfies Eq. (18). So we only need to check if N^{ij}_k satisfies Eq. (5).

Using

$$\sum_k N^{ij}_k d_k = d_i d_j, \quad d_i = d_{i^f} = d_{[i]}, \tag{22}$$

we find that

$$\sum_{[k]} \tilde N^{[i][j]}_{[k]} d_{[k]} = d_{[i]} d_{[j]}. \tag{23}$$

Thus $d_{[i]}$ is also the largest eigenvalue of the matrix $\tilde N_{[i]}$ which is given by $(\tilde N_{[i]})_{[k][j]} = N^{[i][j]}_{[k]}$. The quantum dimensions d_i are already determined by $\tilde N^{[i][j]}_{[k]}$.

Following Ref. [56], we numerically searched (N^{ij}_k, s_i)'s that satisfy the above four conditions plus conditions 1–7 in Sec. III A. The results are summarized in Tables I–VI. We find that each entry corresponds to a valid fermionic topological order (up to invertible topological orders), even through we did not use the condition 8 when producing the tables. In the table, we used the notation $N^F_c(^{|\Theta_2|}_{\angle\Theta_2/2\pi})$ to denote fermionic topological orders with rank N, chiral central charge c (mod 1/2), and $\Theta_2 \equiv D^{-1}\sum_i e^{i4\pi s_i} d_i^2$ [$\angle\Theta_2 \equiv \mathrm{Im}\ln(\Theta_2)$]. The central charge c is given mod 1/2 since the minimal 2+1D invertible fermionic topological order has a central charge 1/2.

The topological excitations are labeled by $i = 1,\dots,N$. Note that $i = 1$ always label the trivial excitation, and $i = 2$ always label the excitation that corresponds to the parent fermion f. Also $2i$ and $2i-1$ always correspond to a pair of excitations that differs by f:

$$(2i)^f = 2i-1, \quad (2i-1)^f = 2i. \tag{24}$$

We like to remark that the rank N is the number of the types of topological excitations in the fermionic topological orders, which include the parent fermion as a nontrivial type. In literature, most people treat the parent fermion as a trivial type; so, the number of types of topological excitations usually referred in literature is, in our notion, the number of up-to-fermion types of topological excitations, $N/2$.

In the table, we also listed the quantum dimensions d_i and the spin s_i of the ith-type of topological excitations. We note that the quantum dimensions satisfy

$$d_i d_j = \sum_k N^{ij}_k d_k. \tag{25}$$

So in the table the quantum dimensions d_i partially represent the fusion coefficients N^{ij}_k.

The total quantum dimension

$$D^2 = \sum_{i=1}^N d_i^2 \tag{26}$$

is also listed. Note that in literature, people usually define $D_F^2 = \sum_{i=1}^{N/2} d_{2i}^2$ as the total quantum dimension. The topological entanglement entropy [5,6] is given by

$$S_{\text{top}} = \frac{1}{2}\log_2 D_F^2 = \frac{1}{2}\log_2 \frac{D^2}{2}. \tag{27}$$

From last column of the Tables I–VI, we see that most fermionic topological orders can be viewed as a stacking of a bosonic topological order (whose label was introduced in Ref. [56]) with the trivial fermionic topological order $\mathcal{F}_0$ (the fermionic product state). Some other fermionic topological orders can be viewed as a stacking of a bosonic topological order with a fermionic topological order, or as a stacking of two fermionic topological orders. There are also fermionic topological orders that are primitive, i.e., cannot be viewed as a stacking of two simpler nontrivial topological orders.

The simplest primitive fermionic topological order is the $4^F_{1/4}(^{\zeta_6^3}_{1/2})$ topological order. It is the first of a sequence of primitive fermionic topological orders with $4^F_{1/4}(^{\zeta_6^3}_{1/2})$, $(6^F_0(^{\zeta_{10}5}_{1/12})$, $8^F_{1/8}(^{\zeta_{14}^7}_{-7/16})$, $10^F_{1/5}(^{\zeta_{18}^9}_{1/10})$, etc. Another type of primitive fermionic topological orders are the 8^F_0 topological orders with $D^2 = 24$ (there are eight of them with different spins s_i). This is also the first of a sequence of primitive fermionic topological orders.

V. STACKING OPERATION FOR TOPOLOGICAL ORDERS

In this section, we discuss the stacking operation in details. In particular, we describe the stacking operation in terms of (N^{ij}_k, s_i, c).

A. Stacking fermionic/bosonic topological order with bosonic topological order

Suppose that we have two UBFC's, $\mathcal{C}$ and $\mathcal{D}$, with particles (simple objects) labeled by $i \in \mathcal{C}$, $a \in \mathcal{D}$. We can construct a new UBFC by simply stacking $\mathcal{C}$ and $\mathcal{D}$, denoted by $\mathcal{C} \boxtimes \mathcal{D}$. By definition, the combined system has no interaction between the two systems $\mathcal{C}$ and $\mathcal{D}$. Certainly, if we add a weak local interaction between the two systems, the combined system is still described by the same topological order $\mathcal{C} \boxtimes \mathcal{D}$, as long as the weak interaction does not drive a phase transition.

The anyon labels of $\mathcal{C}\boxtimes\mathcal{D}$ are pairs $(i,a), i\in\mathcal{C}, a\in\mathcal{D}$, and the topological data are given by (let $\mathcal{K}=\mathcal{C}\boxtimes\mathcal{D}$)

$$\begin{aligned}(N_{\mathcal{K}})^{(i,a)(j,b)}_{(k,c)} &= (N_{\mathcal{C}})^{ij}_{k}(N_{\mathcal{D}})^{ab}_{c},\\ s^{\mathcal{K}}_{(i,a)} &= s^{\mathcal{C}}_i + s^{\mathcal{D}}_a, \quad c_{\mathcal{K}} = c_{\mathcal{C}}+c_{\mathcal{D}},\\ T_{\mathcal{K}} &= T_{\mathcal{C}}\otimes_{\mathbb{C}} T_{\mathcal{D}},\\ S_{\mathcal{K}} &= S_{\mathcal{C}}\otimes_{\mathbb{C}} S_{\mathcal{D}}.\end{aligned} \tag{28}$$

This defines the stacking operation of fermionic/bosonic topological order with bosonic topological order in terms of the topological data (N^{ij}_k, s_i, c).

B. Abelian fermionic topological orders

It is proved in Ref. [54] that if a UMTC$_{/\mathcal{F}_0}$ $\mathcal{C}$ is Abelian, it must be the stacking of some UMTC $\mathcal{B}$ with $\mathcal{F}_0$, $\mathcal{C}=\mathcal{B}\boxtimes\mathcal{F}_0$. In other words, Abelian fermionic topological orders $\mathcal{C}$ can always be decomposed as bosonic topological orders $\mathcal{B}$ stacking with a layer of fermionic product state (with trivial fermionic topological order). However, this is not always true for non-Abelian cases, for example, the $4^F_{1/4}$ primitive fermionic topological order.

C. Stacking two fermionic topological orders

When we are considering two fermionic topological orders described by two UMTC$_{/\mathcal{F}_0}$'s, $\mathcal{C}$ and $\mathcal{D}$, we need a different notion of stacking, denoted by $\mathcal{C}\boxtimes_{\mathcal{F}_0}\mathcal{D}$. The physical idea is that $\mathcal{F}_0\subset\mathcal{C}$ and $\mathcal{F}_0\subset\mathcal{D}$ are the same fermion background; we would like to identify them. The stacking $\boxtimes$ operation defined above gives $\mathcal{C}\boxtimes\mathcal{D}$ which is a UMTC$_{/\mathcal{F}_0\boxtimes\mathcal{F}_0}$. However, the correct stacking $\boxtimes_{\mathcal{F}_0}$ operation should give us $\mathcal{C}\boxtimes_{\mathcal{F}_0}\mathcal{D}$ which is still a UMTC$_{/\mathcal{F}_0}$. To achieve this (i.e., to identify the two $\mathcal{F}_0$'s in $\mathcal{F}_0\boxtimes\mathcal{F}_0$ and reduce it to a single $\mathcal{F}_0$), we introduce the equivalent relation $(i,a)\sim(i^f,a^f)$, and the anyon labels of $\mathcal{C}\boxtimes_{\mathcal{F}_0}\mathcal{D}$ are the equivalent classes $[(i,a)]$. The topological data are given by (assume that $T_{\mathcal{C}}=\tilde{T}_{\mathcal{C}}\otimes_{\mathbb{C}}T_{\mathcal{F}_0}, S_{\mathcal{D}}=\tilde{S}_{\mathcal{D}}\otimes_{\mathbb{C}}S_{\mathcal{F}_0}$ and let $\mathcal{K}=\mathcal{C}\boxtimes_{\mathcal{F}_0}\mathcal{D}$)

$$\begin{aligned}(N_{\mathcal{K}})^{[(i,a)][(j,b)]}_{[(k,c)]} &= (N_{\mathcal{C}})^{ij}_{k}(N_{\mathcal{D}})^{ab}_{c} + (N_{\mathcal{C}})^{ij}_{kf}(N_{\mathcal{D}})^{ab}_{cf},\\ s^{\mathcal{K}}_{[(i,a)]} &= s^{\mathcal{C}}_i + s^{\mathcal{D}}_a = s^{\mathcal{C}}_{i^f} + s^{\mathcal{D}}_{a^f},\\ c_{\mathcal{K}} &= c_{\mathcal{C}}+c_{\mathcal{D}},\\ T_{\mathcal{K}} &= \tilde{T}_{\mathcal{C}}\otimes_{\mathbb{C}}\tilde{T}_{\mathcal{D}}\otimes_{\mathbb{C}}T_{\mathcal{F}_0},\\ S_{\mathcal{K}} &= \tilde{S}_{\mathcal{C}}\otimes_{\mathbb{C}}\tilde{S}_{\mathcal{D}}\otimes_{\mathbb{C}}S_{\mathcal{F}_0}.\end{aligned} \tag{29}$$

The above defines the stacking operation of two fermionic topological orders in terms of the topological data (N^{ij}_k, s_i, c). The stacking operation between fermionic topological orders also makes the set of fermionic topological orders into a monoid.

VI. MODULAR EXTENSIONS OF A FERMIONIC TOPOLOGICAL ORDER

In this section, we discuss how to calculate the modular extensions of a UMTC$_{/\mathcal{F}_0}$. First, note that if we have a UMTC $\mathcal{B}$ that contains fermions, $\mathcal{F}_0=\{1,f\}\subset\mathcal{B}$, it is possible to construct a UMTC$_{/\mathcal{F}_0}\mathcal{F}$ by taking the subset of anyons in $\mathcal{B}$ that are mutually local to (centralize) $\mathcal{F}_0$,

$$\mathcal{F}=(\mathcal{F}_0)^{\text{cen}}_{\mathcal{B}}=\{i \mid i\in\mathcal{B},\ S_{if}=d_i/D\}. \tag{30}$$

Such a UMTC$_{/\mathcal{F}_0}$ describes a fermionic topological order $\mathcal{F}$. By definition, $\mathcal{B}$ is the modular extension of the fermionic topological order $\mathcal{F}$. We consider it a physical requirement that fermionic topological orders must have modular extensions; in other words, the fermion-number-parity must be gaugable (see Secs. II C and VII B). This is nothing but condition 8 in Sec. III A.

Such modular extensions allow us to calculate the chiral central charge of the fermionic topological order $\mathcal{F}$. We conjecture that the chiral central charge c of all the modular extensions $\mathcal{B}$ of a given fermionic topological order $\mathcal{F}$ is the same modulo 1/2. Such a chiral central charge c mod 1/2 is the chiral central charge of the fermionic topological order.

How do we calculate the modular extension $\mathcal{B}_{\mathcal{F}}$ of a fermionic topological order $\mathcal{F}$ from the data of $\mathcal{F}$? We note that all the anyons in $\mathcal{F}$ are contained in $\mathcal{B}_{\mathcal{F}}$, and $\mathcal{B}_{\mathcal{F}}$ contains some additional anyons. Assume that the anyon labels of $\mathcal{B}_{\mathcal{F}}$ are $\{1,f,i,j,\dots,\underline{x},\underline{y},\dots\}$, where we use underline to indicate the additional anyons (not in $\mathcal{F}$). Let $\mathcal{N}^{ij}_k$, $\mathcal{S}_{ij}$ be the fusion coefficients and the S-matrix for $\mathcal{B}_{\mathcal{F}}$, and N^{ij}_k be the fusion coefficients for $\mathcal{F}$. Using Verlinde formula

$$\frac{\mathcal{S}_{f\underline{x}}}{\mathcal{S}_{1\underline{x}}}\frac{\mathcal{S}_{f\underline{x}}}{\mathcal{S}_{1\underline{x}}}=\frac{\mathcal{S}_{1\underline{x}}}{\mathcal{S}_{1\underline{x}}}=1, \tag{31}$$

we find that $\mathcal{S}_{f\underline{x}}=\pm\mathcal{S}_{1\underline{x}}=\pm d_{\underline{x}}/D_{\mathcal{B}_{\mathcal{F}}}$. However, by definition $\underline{x}\notin\mathcal{F}$, we must have $\mathcal{S}_{f\underline{x}}=-d_{\underline{x}}/D_{\mathcal{B}_{\mathcal{F}}}$. Since $\mathcal{S}$ is unitary, $0=\sum_a \mathcal{S}_{1a}\mathcal{S}_{fa}=\sum_{i\in\mathcal{F}} d_a^2/D^2_{\mathcal{B}_{\mathcal{F}}}-\sum_{\underline{x}\notin\mathcal{F}} d_{\underline{x}}^2/D^2_{\mathcal{B}_{\mathcal{F}}}$, therefore

$$\sum_{i\in\mathcal{F}} d_a^2=\sum_{\underline{x}\notin\mathcal{F}} d_{\underline{x}}^2. \tag{32}$$

Thus the total quantum dimension $D_{\mathcal{F}}$ of $\mathcal{F}$ and the total quantum dimension $D_{\mathcal{B}_{\mathcal{F}}}$ of its modular extension $\mathcal{B}_{\mathcal{F}}$ are directly related:

$$D^2_{\mathcal{F}}=\tfrac{1}{2}D^2_{\mathcal{B}_{\mathcal{F}}}. \tag{33}$$

The above also constraints the maximal number of additional anyons we can have.

Next we try to determine the fusion rules involving $\underline{x},\underline{y},\dots$. By Verlinde formula

$$\frac{\mathcal{S}_{i1}}{\mathcal{S}_{11}}\frac{\mathcal{S}_{\underline{x}1}}{\mathcal{S}_{11}}=\sum_{j\in\mathcal{F}}\mathcal{N}^{i\underline{x}}_{j}\frac{\mathcal{S}_{j1}}{\mathcal{S}_{11}}+\sum_{\underline{y}\notin\mathcal{F}}\mathcal{N}^{i\underline{x}}_{\underline{y}}\frac{\mathcal{S}_{\underline{y}1}}{\mathcal{S}_{11}}, \tag{34}$$

$$\frac{\mathcal{S}_{if}}{\mathcal{S}_{1f}}\frac{\mathcal{S}_{\underline{x}f}}{\mathcal{S}_{1f}}=\sum_{j\in\mathcal{F}}\mathcal{N}^{i\underline{x}}_{j}\frac{\mathcal{S}_{jf}}{\mathcal{S}_{1f}}+\sum_{\underline{y}\notin\mathcal{F}}\mathcal{N}^{i\underline{x}}_{\underline{y}}\frac{\mathcal{S}_{\underline{y}f}}{\mathcal{S}_{1f}}. \tag{35}$$

Adding the two we have $0=\sum_{j\in\mathcal{F}}\mathcal{N}^{i\underline{x}}_{j}d_j$, thus $\mathcal{N}^{i\underline{x}}_{j}=0$. Similarly, we can show $\mathcal{N}^{\underline{x}\underline{y}}_{\underline{z}}=0$. So the fusion coefficients of odd numbers of $\underline{x},\underline{y},\underline{z},\dots$ always vanish.

Therefore $\mathcal{N}_i$ for $i\in\mathcal{F}$ is block diagonal: $(\mathcal{N}_i)_{j\underline{x}}=(\mathcal{N}_i)_{\underline{x}j}=0$, where $i,j\in\mathcal{F}$ and $\underline{x}\notin\mathcal{F}$. In other words,

$$\mathcal{N}_i=N_i\oplus\check{N}_i, \tag{36}$$

where $(N_i)_{jk}=\mathcal{N}^{ij}_k=N^{ij}_k$ and $(\check{N}_i)_{\underline{x}\underline{y}}=\mathcal{N}^{i\underline{y}}_{\underline{x}}$, $i,j,k\in\mathcal{F}$, $\underline{x},\underline{y}\notin\mathcal{F}$.

If we pick a charge conjugation for the additional particles $\underline{x} \mapsto \bar{\underline{x}}$, the conditions for fusion rules reduce to

$$\mathcal{N}_{\underline{y}}^{i\underline{x}} = \mathcal{N}_{\underline{y}}^{\underline{x}i} = \mathcal{N}_{i}^{\bar{\underline{x}}\underline{y}} = \mathcal{N}_{\bar{\underline{x}}}^{i\bar{\underline{y}}}, \tag{37}$$

$$\sum_{k\in\mathcal{F}} N_k^{ij}\mathcal{N}_{\underline{y}}^{k\underline{x}} = \sum_{\underline{z}\notin\mathcal{F}} \mathcal{N}_{\underline{x}}^{i\underline{z}}\mathcal{N}_{\underline{z}}^{j\underline{y}}. \tag{38}$$

With a choice of charge conjugation, it is enough to construct (or search for) the matrices $\check{N}_i$ to determine all the extended fusion rules $\mathcal{N}_k^{ij}$. Then, it is straightforward to search for the spins s_i for the extend fusion rules $\mathcal{N}_k^{ij}$ to form some UMTC $\mathcal{B}$ and check if $\mathcal{B}$ contains $\mathcal{F}$.

Besides the general condition (38), there are also some simple constraints on $\check{N}_i$ that may speed up the numerical search. Firstly, observe that (38) is the same as

$$\check{N}_i\check{N}_j = \sum_{k\in\mathcal{F}} N_k^{ij}\check{N}_k, \tag{39}$$

where $i,j,k \in \mathcal{F}$. This means that $\check{N}_i$ satisfy the same fusion algebra as N_i, and $N_k^{ij} = \mathcal{N}_k^{ij}$ is the structure constant; therefore the eigenvalues of $\check{N}_i$ must be a subset of the eigenvalues of N_i.

Secondly, since $\sum_{\underline{y}\notin\mathcal{F}} \mathcal{N}_{\underline{y}}^{i\underline{x}} d_{\underline{y}} = d_i d_{\underline{x}}$, by Perron-Frobenius theorem, we know that d_i is the largest eigenvalue of $\check{N}_i$, with eigenvector $v, v_{\underline{x}} = d_{\underline{x}}$. ($d_i$ is also the largest absolute value of the eigenvalues of $\check{N}_i$.) Note that $\check{N}_{\bar{i}}\check{N}_i = \check{N}_i\check{N}_{\bar{i}}$, $\check{N}_{\bar{i}} = \check{N}_i^\dagger$. Thus d_i^2 is the largest eigenvalue of the positive semidefinite Hermitian matrix $\check{N}_i^\dagger\check{N}_i$. For any unit vector z, we have $z^\dagger\check{N}_i^\dagger\check{N}_i z \leqslant d_i^2$, in particular,

$$(\check{N}_i^\dagger\check{N}_i)_{\underline{x}\underline{x}} = \sum_{\underline{y}} \left(\mathcal{N}_{\underline{y}}^{i\underline{x}}\right)^2 \leqslant d_i^2. \tag{40}$$

The above result is very helpful to reduce the scope of numerical search.

Thirdly, since $\sum_{i\in\mathcal{F}} \mathcal{N}_{\underline{x}}^{i\underline{x}} d_i = d_{\underline{x}}^2$, combined with (32), we have

$$\sum_{i\in\mathcal{F}} d_i \mathrm{Tr}\, \check{N}_i = \sum_{\underline{x}\notin\mathcal{F}} d_{\underline{x}}^2 = \sum_{i\in\mathcal{F}} d_i^2. \tag{41}$$

This puts strong constraints on the traces of the matrices $\check{N}_i$, especially when d_i, d_i^2 are not all integers (but they are always algebraic numbers). For example, if d_i is of the form $k+\sqrt{l}, k,l \in \mathbb{Z}$ (41), essentially splits into two independent equations: the coefficients of $\sqrt{l}$ must be equal and the rest part must be equal. This is the case for the $4^F_{1/4}(^{\zeta_6^3}_{1/2})$ fermionic topological order. We can compute that $\mathrm{Tr}\,\check{N}_1 + \mathrm{Tr}\,\check{N}_f = 4$, thus $\mathrm{Tr}\,\check{N}_1 \leqslant 4$. Note that $\mathrm{Tr}\,\check{N}_1$ is exactly the number of additional particles. Therefore, combined with (40), we performed a finite search for modular extensions of $4^F_{1/4}(^{\zeta_6^3}_{1/2})$, as shown in Table VII.

VII. A CLASSIFICATION OF 2+1D INVERTIBLE FERMIONIC TOPOLOGICAL ORDERS

A. Quantization of chiral central charge c

Let us first review a standard argument for the quantization of chiral central charge c (see, for example, Refs. [46,57]). Consider a bosonic or a fermionic system with invertible topological order. After integrating out all the dynamical degrees of freedom, we obtain a partition function that may contain a gravitational Chern-Simons term:

$$Z[M^3] = \mathrm{e}^{\mathrm{i}\frac{2\pi c}{24}\int_{M^3}\omega_3}, \tag{42}$$

where $\mathrm{d}\omega_3 = p_1$ is the first Pontryagin class. When the tangent bundle of M^3 is nontrivial, the above expression $\int_{M^3}\omega_3$ is not well defined. In order to define the gravitational Chern-Simons term for arbitrary closed space-time manifold M^3, we note that the oriented cobordism group $\Omega_3^{SO} = 0$, i.e., any closed oriented 3-manifold M^3 is a boundary of a 4-manifold M^4: $M^3 = \partial M^4$. So, we can always define the gravitational Chern-Simons term as

$$\mathrm{e}^{\mathrm{i}\frac{2\pi c}{24}\int_{M^3=\partial M^4}\omega_3} = \mathrm{e}^{\mathrm{i}\frac{2\pi c}{24}\int_{M^4}p_1}. \tag{43}$$

However, the same oriented 3-manifold M^3 can be the boundary of two different 4-manifolds: $M^3 = \partial M^4 = \partial\tilde{M}^4$. In order for the above definition to be self-consistent, we require that

$$\mathrm{e}^{\mathrm{i}\frac{2\pi c}{24}\int_{M^4}p_1} = \mathrm{e}^{\mathrm{i}\frac{2\pi c}{24}\int_{\tilde{M}^4}p_1} \tag{44}$$

or

$$\mathrm{e}^{\mathrm{i}\frac{2\pi c}{24}\int_{M^4}p_1} = 1 \tag{45}$$

for any closed oriented 4-manifold $\partial M^4 = \emptyset$.

We note that

$$\int_{M^4} p_1 = 0 \bmod 3. \tag{46}$$

Therefore c must be quantized as

$$c = 0 \bmod 8 \tag{47}$$

to satisfy the condition Eq. (45). This implies that the central charge for bosonic invertible topological orders must be multiple of 8, where $c = 8$ is realized by the E_8 bosonic quantum Hall state.

But for fermionic invertible topological orders, the central charge is quantized differently. This is because M^4 must have a spin structure for fermion systems. In this case [83],

$$\int_{M^4_{\text{spin}}} p_1 = 0 \bmod 48. \tag{48}$$

Therefore c must be quantized as

$$c = 0 \bmod \tfrac{1}{2} \tag{49}$$

for 2+1D fermionic invertible topological orders. $c = \frac{1}{2}$ is realized by the $p + \mathrm{i}p$ fermionic superconducting state.

B. Classify 2+1D invertible fermionic topological orders via modular extentions

However, for each quantized c, do we have only one invertible fermionic topological order, or can we have several

TABLE VII. The 16 modular extensions of the $4^F_{1/4}(^{\zeta_6^3}_{1/2})$ fermionic topological order (in the first row).

$N_c^{F,B}$	S_{top}	D^2	$d_1,d_2,\dots$	$s_1,s_2,\dots$
$4^F_{1/4}(^{\zeta_6^3}_{1/2})$	1.3857	13.656	$1,1,\zeta_6^2,\zeta_6^2$	$0,\frac{1}{2},\frac{1}{4},-\frac{1}{4}$
$7^B_{9/4}$	2.3857	27.313	$1,1,\zeta_6^2,\zeta_6^2,\zeta_6^1,\zeta_6^3,\zeta_6^1$	$0,\frac{1}{2},\frac{1}{4},-\frac{1}{4},\frac{3}{32},\frac{15}{32},\frac{3}{32}$
$7^B_{-1/4}$	2.3857	27.313	$1,1,\zeta_6^2,\zeta_6^2,\zeta_6^1,\zeta_6^3,\zeta_6^1$	$0,\frac{1}{2},-\frac{1}{4},\frac{1}{4},\frac{5}{32},-\frac{7}{32},\frac{5}{32}$
$7^B_{-15/4}$	2.3857	27.313	$1,1,\zeta_6^2,\zeta_6^2,\zeta_6^1,\zeta_6^3,\zeta_6^1$	$0,\frac{1}{2},\frac{1}{4},-\frac{1}{4},\frac{11}{32},-\frac{9}{32},\frac{11}{32}$
$7^B_{7/4}$	2.3857	27.313	$1,1,\zeta_6^2,\zeta_6^2,\zeta_6^1,\zeta_6^3,\zeta_6^1$	$0,\frac{1}{2},-\frac{1}{4},\frac{1}{4},\frac{13}{32},\frac{1}{32},\frac{13}{32}$
$7^B_{-7/4}$	2.3857	27.313	$1,1,\zeta_6^2,\zeta_6^2,\zeta_6^1,\zeta_6^3,\zeta_6^1$	$0,\frac{1}{2},\frac{1}{4},-\frac{1}{4},-\frac{13}{32},-\frac{1}{32},-\frac{13}{32}$
$7^B_{15/4}$	2.3857	27.313	$1,1,\zeta_6^2,\zeta_6^2,\zeta_6^1,\zeta_6^3,\zeta_6^1$	$0,\frac{1}{2},-\frac{1}{4},\frac{1}{4},-\frac{11}{32},\frac{9}{32},-\frac{11}{32}$
$7^B_{1/4}$	2.3857	27.313	$1,1,\zeta_6^2,\zeta_6^2,\zeta_6^1,\zeta_6^3,\zeta_6^1$	$0,\frac{1}{2},\frac{1}{4},-\frac{1}{4},-\frac{5}{32},\frac{7}{32},-\frac{5}{32}$
$7^B_{-9/4}$	2.3857	27.313	$1,1,\zeta_6^2,\zeta_6^2,\zeta_6^1,\zeta_6^3,\zeta_6^1$	$0,\frac{1}{2},-\frac{1}{4},\frac{1}{4},-\frac{3}{32},-\frac{15}{32},-\frac{3}{32}$
$7^B_{-5/4}$	2.3857	27.313	$1,1,\zeta_6^2,\zeta_6^2,\zeta_6^3,\zeta_6^1,\zeta_6^1$	$0,\frac{1}{2},\frac{1}{4},-\frac{1}{4},-\frac{11}{32},\frac{1}{32},\frac{1}{32}$
$7^B_{13/4}$	2.3857	27.313	$1,1,\zeta_6^2,\zeta_6^2,\zeta_6^3,\zeta_6^1,\zeta_6^1$	$0,\frac{1}{2},-\frac{1}{4},\frac{1}{4},-\frac{13}{32},\frac{7}{32},\frac{7}{32}$
$7^B_{3/4}$	2.3857	27.313	$1,1,\zeta_6^2,\zeta_6^2,\zeta_6^3,\zeta_6^1,\zeta_6^1$	$0,\frac{1}{2},\frac{1}{4},-\frac{1}{4},-\frac{3}{32},\frac{9}{32},\frac{9}{32}$
$7^B_{-11/4}$	2.3857	27.313	$1,1,\zeta_6^2,\zeta_6^2,\zeta_6^3,\zeta_6^1,\zeta_6^1$	$0,\frac{1}{2},-\frac{1}{4},\frac{1}{4},-\frac{5}{32},\frac{15}{32},\frac{15}{32}$
$7^B_{11/4}$	2.3857	27.313	$1,1,\zeta_6^2,\zeta_6^2,\zeta_6^3,\zeta_6^1,\zeta_6^1$	$0,\frac{1}{2},\frac{1}{4},-\frac{1}{4},\frac{5}{32},-\frac{15}{32},-\frac{15}{32}$
$7^B_{-3/4}$	2.3857	27.313	$1,1,\zeta_6^2,\zeta_6^2,\zeta_6^3,\zeta_6^1,\zeta_6^1$	$0,\frac{1}{2},-\frac{1}{4},\frac{1}{4},\frac{3}{32},-\frac{9}{32},-\frac{9}{32}$
$7^B_{-13/4}$	2.3857	27.313	$1,1,\zeta_6^2,\zeta_6^2,\zeta_6^3,\zeta_6^1,\zeta_6^1$	$0,\frac{1}{2},\frac{1}{4},-\frac{1}{4},\frac{13}{32},-\frac{7}{32},-\frac{7}{32}$
$7^B_{5/4}$	2.3857	27.313	$1,1,\zeta_6^2,\zeta_6^2,\zeta_6^3,\zeta_6^1,\zeta_6^1$	$0,\frac{1}{2},-\frac{1}{4},\frac{1}{4},\frac{11}{32},-\frac{1}{32},-\frac{1}{32}$

distinct invertible fermionic topological orders? The above analysis of the quantization of the central charge c cannot answer this question. Here, we would like to propose the following conjecture to address this issue. Up to invertible bosonic topological orders, invertible fermionic topological orders are classified by the modular extensions of $\mathcal{F}_0$. More precisely, let $\mathrm{i}\mathcal{F}$ be an invertible fermionic topological order and define the equivalent relation $\sim$: $(\mathrm{i}\mathcal{F} \boxtimes E_8) \sim \mathrm{i}\mathcal{F}$. The quotient {invertible fermionic topological orders}/$\sim$ is classified by the modular extensions of $\mathcal{F}_0$.

The modular extensions of $\mathcal{F}_0$ are given by the bosonic topological orders that (a) contain a fermion f (with spin $1/2$ and quantum dimension 1) and (b) f has nontrivial mutual statistics with all nontrivial topological excitations. From Eq. (33), we see that a modular extension of $\mathcal{F}_0$ must have a total quantum dimension $D^2 = 4$. We find that the trivial fermionic topological order $\mathcal{F}_0$ has 16 modular extensions: eight Ising type UMTC 3^B_c with central charge $c = \pm 1/2, \pm 3/2, \pm 5/2, \pm 7/2$, and eight Abelian rank-4 UMTC 4^B_c with central charge $c = 0, \pm 1, \pm 2, \pm 3, 4$ (see Ref. [56]). This agrees with Kitaev's 16-fold way [52]. For a detailed exposition of the mathematical structures of these 16 UMTC's, see Refs. [52,54].

We conclude that, up to invertible bosonic topological orders, all invertible fermionic topological orders are classified by $\mathbb{Z}_{16}$ generated by the $p + \mathrm{i}p$ fermionic superconducting state. This is a generally believed result, which is one of the reasons that motivates the above conjecture.

For nontrivial fermionic topological orders, we further conjecture. The fermionic topological orders with a given set of bulk topological excitations $\mathcal{F}$ are classified by the modular extensions of $\mathcal{F}$ up to invertible bosonic topological orders. They have the same set of bulk topological excitations $\mathcal{F}$, but different edge states. This a special case of our general proposal mentioned in Sec. I E.

For the fermionic topological order of the form $\mathcal{F} = \mathcal{F}_0 \boxtimes \mathcal{B}$ (i.e., a stacking of trivial fermionic topological order $\mathcal{F}_0$ and a bosonic topological order $\mathcal{B}$), it has the modular extensions (up to invertible bosonic topological orders) given by $\mathcal{B}_{\mathcal{F}} = \mathcal{B}_{\mathcal{F}_0} \boxtimes \mathcal{B}$, where $\mathcal{B}_{\mathcal{F}_0}$ is one of the 16 modular extensions of $\mathcal{F}_0$. They correspond to fermionic topological orders that have the same set of bulk excitations, but different edge states. Also, the 16 modular extensions of the $4^F_{1/4}(^{\zeta_6^3}_{1/2})$ primitive fermionic topological order is listed in Table VII. Again, they correspond to fermionic topological orders that have the same set of bulk excitations, but different edge states. Physically, those 16 fermionic topological orders for 16 modular extensions correspond the condensing the fermion into $0, 1, \dots, 15$ layers of $p + \mathrm{i}p$ superconducting states.

In Table II, there are seven entries with 8^F_* label, where * mean that the central charge is undetermined. However, the 8^F_* entries and one 8^F_0 entry all belong to the same $D^2 = 24$ block. Those eight fermionic topological orders all contain a topological nontrivial fermion. We believe that they are related by condensing such fermion into integer quantum Hall states. Thus their central charge should differ only by integers. In other words, all eight entries have central charge $c = 0$ mod $1/2$. Similar phenomena also happen in other tables.

Before we end this section, we briefly remark on the relation between the modular extensions of $\mathcal{F}_0$ and the Witt groups [84]. The 16 modular extensions of $\mathcal{F}_0$ does not form a group under the stacking product $\boxtimes$ because they are not invertible. However, they do form a $\mathbb{Z}_{16}$ group if we carefully define the stacking $\boxtimes_{\mathcal{F}_0}$ for modular extensions [73,76]. Moreover, the Witt classes of these 16 modular extensions of $\mathcal{F}_0$ also form a $\mathbb{Z}_{16}$ subgroup of the bosonic Witt group $\mathcal{W}$ [84].

This subgroup is precisely the kernel of the canonical group homomorphism $\mathcal{W} \to \mathcal{W}_{/\mathcal{F}_0}$ [54,75], where $\mathcal{W}_{/\mathcal{F}_0}$ is the Witt group for $\text{UMTC}_{/\mathcal{F}_0}$'s. This is not an accident, it turns out that, by taking the Witt class, the set of all modular extensions of a generic SFC $\mathcal{E}$ maps onto the kernel of the canonical group homomorphism $\mathcal{W} \to \mathcal{W}_{/\mathcal{E}}$ [76], where $\mathcal{W}_{/\mathcal{E}}$ is the Witt group for $\text{UMTC}_{/\mathcal{E}}$'s. Details will be given in Ref. [76].

VIII. EXAMPLES AND REALIZATIONS OF FERMIONIC TOPOLOGICAL ORDERS

A. Fermionic Abelian topological orders

The fermionic Abelian topological orders with $d_i = 1$ in the Tables I, II, IV, V, and VI are described by K matrices (which are included in the last column of the tables). Their many-body wave functions are given by Eq. (2). Note that $K = (m)$ corresponds to the filling fraction $\nu = 1/m$ Laughlin state $\Psi_{1/m}(z_i)$.

We also note that most fermionic topological orders are stacking of a bosonic topological order and a fermion product state. The wave functions for the bosonic topological orders (described by the K matrix and/or simple current algebra) are given in Refs. [55,85] and Appendix E.

B. Fermionic topological orders from the $\frac{U(1)_M}{\mathbb{Z}_2}$-orbifold simple-current algebra

We can also use the conformal field theory (CFT) (or more precisely, a simple-current algebra) to construct 2+1D topological orders. In fact, we regard the correlation function of N_p simple-current operators as an N_p electron wave function $\Psi(z_1,\dots,z_{N_p})$ [20,28–33]. Such an N_p electron wave function describes a purely chiral fermionic topological order, if the simple current operators have half-integer conformal weights. (If all simple current operators have integer conformal weights, the correlation function of simple-current operators describes a bosonic many-body state.)

The adjoint representation generated by the simple-current operators corresponds to the trivial up-to-fermion type of topological excitations (see Sec. IV A for an explanation of up-to-fermion type of topological excitations). While other irreducible representations of the simple-current algebra correspond to nontrivial up-to-fermion type of topological excitations. The number of the up-to-fermion types of topological excitations is given by the number of the irreducible representations of the simple-current algebra.

For example, a bosonic topological state can be constructed through $\frac{U(1)_M}{\mathbb{Z}_2}$-orbifold CFT. The $\frac{U(1)_M}{\mathbb{Z}_2}$-orbifold CFT is a simple-current algebra generated by the spin-M simple current $\psi = \cos(\sqrt{2M}\phi)$ (for details, see Ref. [86]). Since the conformal dimension (the spin) of the simple-current ψ is an integer M, ψ is an bosonic operator. The correlation of ψ's gives rise to a many-boson wave function with a bosonic topological order (for details, see Ref. [85]).

The topological excitations in such a topologically ordered state correspond to the irreducible representations of the $\frac{U(1)_M}{\mathbb{Z}_2}$-orbifold simple-current algebra, which is listed in Table VIII. The spins s_i and quantum dimensions d_i of those topological excitations are given by the conformal dimensions h_i, $s_i = h_i$ mod 1, and the quantum dimensions d_i of those irreducible representations. The S matrix (i.e., the mutual statistics) of those topological excitations is given in Tables IX and X. We denote such bosonic topological order and the corresponding UMTC as $\mathcal{B}_{U(1)_M/\mathbb{Z}_2} = \{1, j, \phi^\alpha_M, \phi_\gamma, \tau^\alpha, \sigma^\alpha\}$, where $\alpha = 1,2$ and $\gamma = 1,\dots,M-1$.

TABLE VIII. The irreducible representations $\mathcal{V}_i^{\frac{U(1)_M}{\mathbb{Z}_2}}$ of $\frac{U(1)_M}{\mathbb{Z}_2}$-orbifold simple current algebra. The second column is the conformal dimensions h_i of the corresponding primary fields. The third column is the quantum dimensions d_i of the representations.

Label i	h_i	d_i	
1	0	1	
j	1	1	
ϕ^α_M	$M/4$	1	$\alpha = 1,2$
σ^α	1/16	$\sqrt{M}$	$\alpha = 1,2$
τ^α	9/16	$\sqrt{M}$	$\alpha = 1,2$
ϕ_γ	$\gamma^2/4M$	2	$\gamma = 1,\dots,M-1$

In fact, the above UMTC $\mathcal{B}_{U(1)_M/\mathbb{Z}_2}$ with $M = 6$ is a modular extension of the $8^F_0(^{3.5915}_{0.1699})$ fermionic topological order with $s_i = (0,\frac12,\frac12,0,\frac16,-\frac13,-\frac{7}{16},\frac{1}{16})$. From the S matrix in Table IX, we see that the objects/particles in $\mathcal{F}_0 = \{1,\phi^1_6\}$, a subset of $\mathcal{B}_{U(1)_M/\mathbb{Z}_2}$, are mutually local with respect to each other. Thus the spin-6/4 operator ϕ^1_6 corresponds to the parent fermion f. From the S matrix in Table IX, we also see that the topological excitations in $\mathcal{F} = \{1,\phi^1_6,\phi^2_6,j,\phi_2,\phi_4,\tau^1,\sigma^1\}$, another subset of $\mathcal{B}_{U(1)_M/\mathbb{Z}_2}$, are local with respect to $\mathcal{F}_0$. Thus $\mathcal{F}$ is a UBFC over $\mathcal{F}_0$. In fact it is a $\text{UMTC}_{/\mathcal{F}_0}$.

The conformal dimensions and the quantum dimensions of the topological excitations in $\mathcal{F}$ are given by $h_i = (0,\frac32,\frac32,1,\frac16,\frac23,\frac{9}{16},\frac{1}{16})$ and $d_i = (1,1,1,1,2,2,\sqrt6,\sqrt6)$. Thus $\mathcal{F}$ is the $\text{UMTC}_{/\mathcal{F}_0}$ that describes the $8^F_0(^{3.5915}_{0.1699})$ fermionic topological order (see Table II). The fusion of such a $8^F_0(^{3.5915}_{0.1699})$ fermionic topological order is given in Table XI.

The above results help us to obtain the many-body wave function that realize the $8^F_0(^{3.5915}_{0.1699})$ fermionic topological order. In fact, naively, the correlation of the spin-3/2 fermionic simple-current operator ϕ^1_6's

$$\Psi(\{z_i\}) \propto \lim_{z_\infty\to\infty}\left\langle \hat{V}(z_\infty)\prod \phi^1_6(z_i)\right\rangle \tag{50}$$

gives rise to a quantum-Hall many-fermion wave function $\Psi(\{z_i\})\mathrm{e}^{-\frac14\sum|z_i|^2}$ with the above $8^F_0(^{3.5915}_{0.1699})$ fermionic topological order. The edge excitations of such a quantum Hall state are described by the $\frac{U(1)_M}{\mathbb{Z}_2}$-orbifold CFT [30,50,87,88].

However, the above construction has a problem: the correlation of ϕ^1_6 (i.e., $\Psi(\{z_i\})$) has poles as $z_i \to z_j$. But this is only a technical problem that can be fixed as pointed out in Ref. [85]. We may put the wave function on a lattice or adding additional factors $\prod |z_i - z_j|^3$ to make the wave function finite. This is a realization of the $8^F_0(^{3.5915}_{0.1699})$ topological order.

We may also introduce three complex chiral fermions ψ_1, ψ_2, and ψ_3. This allows us to construct a four-layer quantum-Hall wave function as the following correlation in

TABLE IX. The S matrix for the $\frac{U(1)_M}{\mathbb{Z}_2}$-orbifold simple current algebra with $M =$ even. Here, $\gamma,\lambda = 1,\dots,M-1$, $\alpha,\beta = 1,2$, and $\sigma_{\alpha\beta} = 2\delta_{\alpha\beta} - 1$.

S_{ij}	1	j	ϕ_M^α	σ^α	τ^α	ϕ_γ
1	1	1	1	$\sqrt{M}$	$\sqrt{M}$	2
j	1	1	1	$-\sqrt{M}$	$-\sqrt{M}$	2
ϕ_M^β	1	1	1	$\sigma_{\alpha\beta}\sqrt{M}$	$\sigma_{\alpha\beta}\sqrt{M}$	$2(-)^\gamma$
σ^β	$\sqrt{M}$	$-\sqrt{M}$	$\sigma_{\alpha\beta}\sqrt{M}$	$\delta_{\alpha\beta}\sqrt{2M}$	$-\delta_{\alpha\beta}\sqrt{2M}$	0
τ^β	$\sqrt{M}$	$-\sqrt{M}$	$\sigma_{\alpha\beta}\sqrt{M}$	$-\delta_{\alpha\beta}\sqrt{2M}$	$\delta_{\alpha\beta}\sqrt{2M}$	0
ϕ_λ	2	2	$2(-)^\lambda$	0	0	$4\cos(\pi\frac{\gamma\lambda}{M})$

a CFT:

$$\Psi(\{z_i,w_i,u_i,v_i\}) \propto \left\langle \hat{V}(z_\infty)\prod c_1(z_i)c_2(w_i)c_3(u_i)c_4(v_i)\right\rangle,$$
$$c_i = \psi_i,\ i = 1,2,3, \quad c_4 = \psi_1\psi_2\psi_3\phi_6^1. \tag{51}$$

In such a four-layer quantum-Hall state, the particles in the first three layers are fermions and the particles in the fourth layer are bosons. Such a wave function is finite, and its edge excitations are described by the $\frac{U(1)_M}{\mathbb{Z}_2} \times U^3(1)$ CFT [30,50,87,88], where $U^3(1)$ CFT describes the edge excitation of $\nu = 3$ integer quantum Hall states (generated by ψ_i, $i = 1,2,3$). Therefore the $8_0^F(^{3.5915}_{0.1699})$ fermionic topological order described by the wave function $\Psi(\{z_i,w_i,u_i,v_i\})\mathrm{e}^{-\frac14\sum|z_i|^2+|w_i|^2+|u_i|^2+|v_i|^2}$ only differs from the $8_0^F(^{3.5915}_{0.1699})$ fermionic topological order described by $\Psi(\{z_i\})\mathrm{e}^{-\frac14\sum|z_i|^2}$ by an invertible fermionic topological order of the $\nu = 3$ integer quantum Hall state.

The above discussion also apply to $\frac{U(1)_M}{\mathbb{Z}_2}$-orbifold CFT with $M = 2+4n$. When $M = 2$ (i.e., $n = 0$), the corresponding fermionic topological order is the $6_0^F(^{\zeta_6^3}_{1/16})$ topological order. The case $M = 6$ (i.e., $n = 1$) was discussed above. The case $M = 10$ (i.e., $n = 2$) gives rise to the $10_0^F(^{4.2807}_{0.0874})$ topological order in Table III. The larger n gives a sequence of fermionic topological orders. We denote those fermionic topological orders by $\mathcal{F}_{U(1)_M/\mathbb{Z}_2}$. One of its modular extensions is $\mathcal{B}_{U(1)_M/\mathbb{Z}_2}$.

We note that fermionic topological orders $\mathcal{F}_{U(1)_M/\mathbb{Z}_2}$, $M = 2+4n$, always contain a fermionic topological excitation, apart from the parent fermion. When those fermionic topological excitations condense into invertible integer quantum Hall states, it changes the $\mathcal{F}_{U(1)_M/\mathbb{Z}_2}$ topological order to some other topological order with the same quantum dimensions d_i but different spins s_i. We can see those related fermionic topological orders in Tables I–III.

C. Fermionic topological orders from the (A_1,k) Kac-Moody algebra

The (A_1,k) Kac-Moody algebra (i.e., the $SU(2)$ level k Kac-Moody algebra), for $k \in \mathbb{Z}$, also gives rise to a sequence of UMTC's. The gauge-invariant data of (A_1,k) are as follows.

(1) The objects (particles) are labeled by $i \in \{0,1,2,\dots,k\}$. They carry the $SU(2)$ isospin $S = i/2$. The corresponding primary fields are denoted by V_i^m, $m = -\frac{k}{2},-\frac{k}{2}+1,\dots,\frac{k}{2}$.

(2) Fusion rules: $i\otimes j = |i-j| \oplus (|i-j|+2) \oplus (|i-j|+4)\oplus\cdots\oplus \min(i+j,2k-i-j)$.

(3) Conformal dimensions $h_i = \dfrac{i(i+2)}{4(k+2)}$. (Spins $s_i = h_i$ mod 1.)

(4) Quantum dimensions $d_i = \zeta_k^i \equiv \frac{\sin[\pi(i+1)/(k+2)]}{\sin[\pi/(k+2)]}$.

(5) Chiral central charge $c = \dfrac{3k}{k+2}$.

The above data (fusion rules and spins) describe a bosonic topological order denoted by $\mathcal{B}_{(A_1,k)}$, whose S matrix can be calculated from Eq. (9).

Observe that for $k = 4l+2$, $l \in \mathbb{Z}$, the last particle $i = 4l+2$ in $\mathcal{B}_{(A_1,4l+2)}$ is a fermion. The corresponding conformal field is a simple current operator. We identity $\mathcal{F}_0 = \{0, f = 4l+2\} \subset \mathcal{B}_{(A_1,4l+2)}$. Then, we have a sequence of fermionic topological orders

$$\begin{aligned}\mathcal{F}_{(A_1,4l+2)} &= \{i \in \mathcal{B}_{(A_1,4l+2)}|S_{i,4l+2} = d_i/D\}\\ &= \{0,2,4,\dots,4l+2\} \subset \mathcal{B}_{(A_1,4l+2)},\end{aligned} \tag{52}$$

such that $\mathcal{B}_{(A_1,4l+2)}$ is a modular extension of $\mathcal{F}_{(A_1,4l+2)}$. For $l = 0$, $\mathcal{F}_{(A_1,2)} \cong \mathcal{F}_0$ is the trivial fermionic topological

TABLE X. The S matrix for the $\frac{U(1)_M}{\mathbb{Z}_2}$-orbifold simple current algebra with $M =$ odd. Here, $\gamma,\lambda = 1,\dots,M-1$, $\alpha,\beta = 1,2$, and $\sigma_{\alpha\beta} = 2\delta_{\alpha\beta} - 1$.

S_{ij}	1	j	ϕ_M^α	σ^α	τ^α	ϕ_γ
1	1	1	1	$\sqrt{M}$	$\sqrt{M}$	2
j	1	1	1	$-\sqrt{M}$	$-\sqrt{M}$	2
ϕ_M^β	1	1	-1	$\mathrm{i}\sigma_{\alpha\beta}\sqrt{M}$	$\mathrm{i}\sigma_{\alpha\beta}\sqrt{M}$	$2(-)^\gamma$
σ^β	$\sqrt{M}$	$-\sqrt{M}$	$\mathrm{i}\sigma_{\alpha\beta}\sqrt{M}$	$\mathrm{e}^{\pi\,\mathrm{i}\sigma_{\alpha\beta}/4}\sqrt{2M}$	$-\mathrm{e}^{\pi\,\mathrm{i}\sigma_{\alpha\beta}/4}\sqrt{2M}$	0
τ^β	$\sqrt{M}$	$-\sqrt{M}$	$\mathrm{i}\sigma_{\alpha\beta}\sqrt{M}$	$-\mathrm{e}^{\pi\,\mathrm{i}\sigma_{\alpha\beta}/4}\sqrt{2M}$	$\mathrm{e}^{\pi\,\mathrm{i}\sigma_{\alpha\beta}/4}\sqrt{2M}$	0
ϕ_λ	2	2	$2(-)^\lambda$	0	0	$4\cos(\pi\frac{\gamma\lambda}{M})$

TABLE XI. Fusion rule $j \otimes i$ for the $8_0^F(^{3.5915}_{0.1699})$ fermionic topological order with $d_i = (1,1,1,1,2,2,\sqrt{6},\sqrt{6})$.

d_i	1	1	1	1	2	2	$\sqrt{6}$	$\sqrt{6}$
$j\backslash i$	$\mathbf{1}$	f	a	a^f	α	α^f	β	β^f
$\mathbf{1}$	$\mathbf{1}$	f	a	a^f	α	α^f	β	β^f
f	f	$\mathbf{1}$	a^f	a	α^f	α	β^f	β
a	a	a^f	$\mathbf{1}$	f	α^f	α	β	β^f
a^f	a^f	a	f	$\mathbf{1}$	α	α^f	β^f	β
α	α	α^f	α^f	α	$\mathbf{1} \oplus a^f \oplus \alpha^f$	$f \oplus a \oplus \alpha$	$\beta \oplus \beta^f$	$\beta \oplus \beta^f$
α^f	α^f	α	α	α^f	$f \oplus a \oplus \alpha$	$\mathbf{1} \oplus a^f \oplus \alpha^f$	$\beta \oplus \beta^f$	$\beta \oplus \beta^f$
β	β	β^f	β	β^f	$\beta \oplus \beta^f$	$\beta \oplus \beta^f$	$\mathbf{1} \oplus a \oplus \alpha \oplus \alpha^f$	$f \oplus a^f \oplus \alpha \oplus \alpha^f$
β^f	β^f	β	β^f	β	$\beta \oplus \beta^f$	$\beta \oplus \beta^f$	$f \oplus a^f \oplus \alpha \oplus \alpha^f$	$\mathbf{1} \oplus a \oplus \alpha \oplus \alpha^f$

order. The $l = 1$ case has been studied in Ref. [89], whose fusion rule is listed in Table XII. This sequence appears in our numerical calculations $[4^F_{1/4}(^{\zeta_6^3}_{1/2}), 6^F_0(^{2\zeta_{10}^1}_{1/12}), 8^F_{1/8}(^{\zeta_{14}^7}_{-7/16}), 10^F_{1/5}(^{\zeta_{18}^9}_{1/10})$ in Tables I–III]. In fact, all fermionic topological orders in this sequence are primitive.

For $l = 1$, the simple current operator carries isospin-3 and is given by $V_3^m, m = -3,-2,\dots,3$, with conformal dimension $h_6 = \frac{3}{2}$. To obtain a many-body wave function that gives rise to the $4^F_{1/4}(^{\zeta_6^3}_{1/2})$ fermionic topological order, we may again introduce three complex chiral fermions ψ_1, ψ_2, and ψ_3. This allows us to construct a four-layer quantum-Hall wave function as the following correlation in a $SU(2)_6 \times U^3(1)$ Kac-Moody algebra [28]:

$$\Psi(\{z_i,w_i,u_i,v_i,m_i\}) \propto \left\langle \hat{V}(z_\infty) \prod c_1(z_i) c_2(w_i) c_3(u_i) c_4^{m_i}(v_i) \right\rangle,$$
$$c_i = \psi_i,\ i = 1,2,3, \quad c_4^m = \psi_1\psi_2\psi_3 V_6^m. \tag{53}$$

In such a four-layer quantum-Hall state, the particles in the first three layers are fermions and the particles in the fourth layer are isospin-3 bosons. Such a wave function is finite, and its edge excitations are described by the $(A_1,6) \times U^3(1)$ CFT [30,50,87,88], where $U^3(1)$ CFT describes the edge excitation of $\nu = 3$ integer quantum Hall states (generated by ψ_i, $i = 1,2,3$). The wave function $\Psi(\{z_i,w_i,u_i,v_i,m_i\})\mathrm{e}^{-\frac{1}{4}\sum |z_i|^2+|w_i|^2+|u_i|^2+|v_i|^2}$ gives rise to the $4^F_{1/4}(^{\zeta_6^3}_{1/2})$ or $\mathcal{F}_{(A_1,6)}$ fermionic topological order.

TABLE XII. Fusion rule $j \otimes i$ for $4^F_{1/4}(^{\zeta_6^3}_{1/2})$ fermionic topological order. $\zeta_6^2 = 1 + \sqrt{2}$.

d_i	1	1	ζ_6^2	ζ_6^2
$j\backslash i$	$\mathbf{1}$	f	α	α^f
$\mathbf{1}$	$\mathbf{1}$	f	α	α^f
f	f	$\mathbf{1}$	α^f	α
α	α	α^f	$\mathbf{1} \oplus \alpha \oplus \alpha^f$	$f \oplus \alpha \oplus \alpha^f$
α^f	α^f	α	$f \oplus \alpha \oplus \alpha^f$	$\mathbf{1} \oplus \alpha \oplus \alpha^f$

The $\mathcal{B}_{(A_1,6)}$ is one of the modular extensions of the $4^F_{1/4}(^{\zeta_6^3}_{1/2})$ fermionic topological order. Such a modular extension is the $N_c^B = 7^B_{9/4}$ bosonic topological order in Table VII.

IX. SUMMARY

In this paper, we proposed that 2+1D bosonic/fermionic topological orders with symmetry G are classified, up to invertible topological orders, by UMTC$_{/\mathcal{E}}$'s with modular extensions. $\mathcal{E}$ is the category Rep(G) of G-representations in bosonic cases and the category sRep(G^f) for fermionic cases, where $G^f = (G,z)$ and $z \in G$ is the fermion-number parity symmetry. The case of $G = \{1\}$ (or $G^f = (\mathbb{Z}_2,z) = \mathbb{Z}_2^f$) corresponds to the bosonic (or fermionic) case without symmetry.

We developed a simplified theory for nondegenerate UBFC over the SFC $\mathcal{F}_0 = \mathrm{sRep}(\mathbb{Z}_2^f)$, which allows us to obtain a list of simple fermionic topological orders with no symmetry. We find two sequences of primitive fermionic topological orders $\mathcal{F}_{(A_1,4l+2)}$ and $\mathcal{F}_{U(1)_M/\mathbb{Z}_2}$.

ACKNOWLEDGMENTS

We would like to thank Zheng-Cheng Gu for many very helpful discussions. This research is supported by NSF Grant No. DMR-1005541, and NSFC 11274192. It is also supported by the John Templeton Foundation No. 39901. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research. Liang Kong is supported by the Center of Mathematical Sciences and Applications at Harvard University.

APPENDIX A: PROOF THAT THERE ARE ONLY FOUR $N = 4$ UMTC$_{/\mathcal{F}_0}$

In this section, we prove that there are only four different $N = 4$UMTC$_{/\mathcal{F}_0}$ using the conditions listed in the main text. Let us label the four particles by $\{1,f,a,a^f\}$. Since $a^f = a \otimes f$, all the fusion rules can actually be generated by

$$a \otimes a = 1 \oplus zf \oplus xa \oplus ya^f,$$

where x,y,z are nonnegative integers.

Firstly,

$$a \otimes a^f = a \otimes a \otimes f = f \oplus z1 \oplus xa^f \oplus ya.$$

Since $a = \bar{a} \neq a^f$, we must have $z = 0$. The quantum dimension $d_a = d_{a^f}$ is then given by

$$d_a^2 = 1 + (x+y)d_a,\, d_a = \frac{x + y + \sqrt{(x+y)^2 + 4}}{2}.$$

Secondly, if either of x,y is zero, the fusion rule splits, and the $N = 4\mathrm{UMTC}_{/\mathcal{F}_0}$ is not primitive. It is the stacking of a $N = 2$ UMTC with $\mathcal{F}_0$. The classification of $N = 2$ UMTC is clear: semion UMTC, Fibonacci UMTC and their time-reversal conjugates (the first four in Table XIII). After stacking with $\mathcal{F}_0$, semion UMTC and its time-reversal produce the same $\mathrm{UMTC}_{/\mathcal{F}_0}$. This way we obtain in total three different $\mathrm{UMTC}_{/\mathcal{F}_0}$.

It remains to show that there is only one primitive $N = 4$ $\mathrm{UMTC}_{/\mathcal{F}_0}$. Firstly, we know that the S matrix has the form $S = \tilde{S} \otimes_{\mathbb{C}} S_{\mathcal{F}_0}$ and the rank 2 matrix $\tilde{S}$ is unitary. So S can only be

$$\frac{1}{D}\begin{pmatrix} 1 & d_a \\ d_a & -1 \end{pmatrix} \otimes_{\mathbb{C}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

In particular,

$$-1 = DS_{aa} = \sum_k N_k^{aa}\, \mathrm{e}^{2\pi\, \mathrm{i}(s_a + s_a - s_k)} d_k.$$

We also know that $s_1 = 0,\ s_f = 1/2,\ s_{a^f} = s_a + 1/2$. The above reduces to

$$-1 = \mathrm{e}^{4\pi\, \mathrm{i} s_a} + (x - y) d_a\, \mathrm{e}^{2\pi\, \mathrm{i} s_a}$$

or

$$-2\cos(2\pi s_a) = (x - y)d_a. \tag{A1}$$

But note that for a primitive $\mathrm{UMTC}_{/\mathcal{F}_0}$, $x \geqslant 1$, $y \geqslant 1$, thus $d_a \geqslant \frac{1+1+\sqrt{(1+1)^2+4}}{2} = 1 + \sqrt{2}$. If $x \neq y$ we must have $|(x-y)d_a| > 2$, which is contradictory to Eq. (A1). So it is only possible that $x = y$ and $s_a = 1/4$ or $s_a = 3/4$.

Secondly, we check the condition that

$$\nu_a = \frac{1}{D^2} \sum_{jk} N_a^{jk} d_j d_k\, \mathrm{e}^{\mathrm{i}4\pi(s_j - s_k)}$$

is an integer. Substituting the results we obtained so far,

$$\nu_a = \frac{-4d_a + 4xd_a^2}{2 + 2d_a^2}, \quad d_a = x + \sqrt{x^2 + 1}.$$

Thus

$$2x^3 + (2x^2 - 1)\sqrt{x^2+1} = (x^2+1)\nu_a + x\nu_a\sqrt{x^2+1}.$$

Since ν_a is an integer ($\sqrt{x^2+1}$ is never an integer for $x = 1,2,\ldots$), we have

$$\begin{cases} 2x^3 = (x^2+1)\nu_a, \\ 2x^2 - 1 = x\nu_a. \end{cases}$$

Substituting the second into the first, we get

$$2x^3 = x(2x^2 - 1) + \nu_a,$$

thus, $x = \nu_a, x^2 = 1$. The only non-negative integer solution is $x = \nu_a = 1$. This solution does give us a valid $\mathrm{UMTC}_{/\mathcal{F}_0}$. The two choices of spins, $s_a = 1/4, s_{a^f} = 3/4$, or $s_a = 3/4, s_{a^f} = 1/4$, turn out to be isomorphic. This completes our proof.

APPENDIX B: CATEGORICAL VIEW OF PARTICLE STATISTICS

In 3+1D, particles can have two different kinds of statistics, bosonic or fermionic. Besides, if the system has certain physical symmetry, particles also carry group representations. The Bose/Fermi statistics and representations of symmetry groups can be unified by a single mathematical framework *symmetric categories*.

Before giving a rigorous mathematical definition, here we try to give a physical picture of "categories." Physically, tensor category theory can be viewed as a theory that describe quasiparticle excitations in a gapped state. The particles (pointlike excitations) correspond to objects in category theory, and the operators or operations acting on the particles correspond to morphisms in category theory. Two particles that can be connected by local operators are regarded as equivalent and correspond to two isomorphic objects in category theory.

Under such an equivalence relation, the local operators are regarded as trivial (or null) operations, that correspond to trivial morphisms. Other operations, such as moving one particle around another, braiding two particles, etc., correspond to nontrivial morphisms. Those operations are described by the product of hopping operators, i.e., the string operators (or Wilson loop operators). In other words, local operators are trivial morphisms, while string operators can be nontrivial morphisms.

String operators also have an equivalence relation: Two string operators are considered equivalent if they (1) have the same matrix elements among the low energy states or (2) equivalently, differ by only local operators. (Those string operators are also called logic operators in topological quantum computing.) It is the equivalent classes of string operators that correspond to morphisms in category theory.

Besides, if there is some physical symmetry, we require the operators to preserve the symmetry, i.e., they intertwine (commute with) the symmetry actions. For example, two particles, carrying different irreducible representations of the $SO(3)$ symmetry group, cannot have morphism between them, i.e., there is no symmetry preserving operations that can change one particle into the other. On the other hand, if one particle carry a reducible representation of spin-1 and spin-2, and the other particle carry a reducible representation of spin-2 and spin-3, then there is a morphism between the two particles (objects), (i.e., symmetry preserving operations may turn the first particle into the second particle with a nonzero amplitude). We denote the first particle as spin-1$\oplus$ spin-2 and the second particle as spin-2$\oplus$ spin-3, and the morphism as an arrow between the two particles:

$$(\text{spin-1} \oplus \text{spin-2}) \to (\text{spin-2} \oplus \text{spin-3}). \tag{B1}$$

In category theory, the irreducible representations, such as spin-1, correspond to simple objects, and the reducible representations, spin-1$\oplus$ spin-2, correspond to composite objects. The composite objects are direct sums $\oplus$ of simple objects.

TABLE XIII. A list of simple bosonic topological orders (up to invertible ones) with N types of topological excitations and chiral central charge c (mod 8). The excitations have quantum dimension d_i and spin s_i (mod 1). The table contains all topological orders with $N \leqslant 4$ as well as $N = 5$ and $D^2 \leqslant 120$. Here, $\zeta_n^m = \frac{\sin[\pi(m+1)/(n+2)]}{\sin[\pi/(n+2)]}$.

$N_c^B\binom{\lvert\Theta_2\rvert}{\angle\Theta_2/2\pi}$	D^2	$d_1,d_2,\ldots$	$s_1,s_2,\ldots$	Comments, SCA, K matrix, wave function
$2^B_1\binom{0}{0}$	2	1,1	$0,\frac{1}{4}$	$K = (2)$, $\Psi_{1/2}(z_i)$
$2^B_{-1}\binom{0}{0}$	2	1,1	$0,-\frac{1}{4}$	$K = (-2)$, $\Psi^*_{1/2}(z_i)$
$2^B_{14/5}\binom{\zeta_3^1}{-3/20}$	3.6180	$1,\zeta_3^1$	$0,\frac{2}{5}$	$(A_1,3)_{1/2}$, $(G_2,1)$
$2^B_{-14/5}\binom{\zeta_3^1}{3/20}$	3.6180	$1,\zeta_3^1$	$0,-\frac{2}{5}$	$(A_1,-3)_{1/2}$, $(G_2,-1)$
$3^B_2\binom{1}{-1/4}$	3	1,1,1	$0,\frac{1}{3},\frac{1}{3}$	$K = (2\ 2; 1)$
$3^B_{-2}\binom{1}{1/4}$	3	1,1,1	$0,-\frac{1}{3},-\frac{1}{3}$	$K = (-2\ -2; -1)$
$3^B_{3/2}\binom{0.7653}{3/16}$	4	$1,1,\zeta_2^1$	$0,\frac{1}{2},\frac{3}{16}$	$(A_1,2)$, $(B_9,1)$, $\mathcal{A}(\frac{1}{z_1-z_2}\frac{1}{z_3-z_4}\cdots)\Psi_1(z_i)$ [20]
$3^B_{5/2}\binom{0.7653}{-3/16}$	4	$1,1,\zeta_2^1$	$0,\frac{1}{2},\frac{5}{16}$	$(B_2,1)$, $\Psi_2(z_i)\Psi_2(z_i)$ [21,28]
$3^B_{7/2}\binom{\zeta_6^1}{-1/16}$	4	$1,1,\zeta_2^1$	$0,\frac{1}{2},\frac{7}{16}$	$(B_3,1)$
$3^B_{-7/2}\binom{\zeta_6^1}{1/16}$	4	$1,1,\zeta_2^1$	$0,\frac{1}{2},-\frac{7}{16}$	$(B_4,1)$
$3^B_{-5/2}\binom{0.7653}{3/16}$	4	$1,1,\zeta_2^1$	$0,\frac{1}{2},-\frac{5}{16}$	$(B_5,1)$, $\Psi_2^*(z_i)\Psi_2^*(z_i)$ [21,28]
$3^B_{-3/2}\binom{0.7653}{-3/16}$	4	$1,1,\zeta_2^1$	$0,\frac{1}{2},-\frac{3}{16}$	$(B_6,1)$
$3^B_{-1/2}\binom{\zeta_6^1}{-1/16}$	4	$1,1,\zeta_2^1$	$0,\frac{1}{2},-\frac{1}{16}$	$(B_7,1)$
$3^B_{1/2}\binom{\zeta_6^1}{1/16}$	4	$1,1,\zeta_2^1$	$0,\frac{1}{2},\frac{1}{16}$	$(B_8,1)$
$3^B_{8/7}\binom{\zeta_5^2}{-5/14}$	9.2958	$1,\zeta_5^1,\zeta_5^2$	$0,-\frac{1}{7},\frac{2}{7}$	$(A_1,5)_{1/2}$
$3^B_{-8/7}\binom{\zeta_5^2}{5/14}$	9.2958	$1,\zeta_5^1,\zeta_5^2$	$0,\frac{1}{7},-\frac{2}{7}$	$(A_1,-5)_{1/2}$
$4^B_1\binom{\zeta_2^1}{1/8}$	4	1,1,1,1	$0,\frac{1}{8},\frac{1}{8},\frac{1}{2}$	$K = (4)$, $\Psi_{1/4}(z_i)$
$4^B_2\binom{0}{0}$	4	1,1,1,1	$0,\frac{1}{4},\frac{1}{4},\frac{1}{2}$	$2^B_1\binom{0}{0} \boxtimes 2^B_1\binom{0}{0}$
$4^B_3\binom{\zeta_2^1}{-1/8}$	4	1,1,1,1	$0,\frac{3}{8},\frac{3}{8},\frac{1}{2}$	$K = (2\ 2\ 2; 1\ 1; 1)$
$4^B_4\binom{2}{0}$	4	1,1,1,1	$0,\frac{1}{2},\frac{1}{2},\frac{1}{2}$	$K = (2\ 2\ 2\ 2; 1\ 0\ 0; 1\ 0; 1)$
$4^B_{-3}\binom{\zeta_2^1}{1/8}$	4	1,1,1,1	$0,-\frac{3}{8},-\frac{3}{8},\frac{1}{2}$	$K = -(2\ 2\ 2; 1\ 1; 1)$
$4^B_{-2}\binom{0}{0}$	4	1,1,1,1	$0,-\frac{1}{4},-\frac{1}{4},\frac{1}{2}$	$2^B_{-1}\binom{0}{0} \boxtimes 2^B_{-1}\binom{0}{0}$
$4^B_{-1}\binom{\zeta_2^1}{-1/8}$	4	1,1,1,1	$0,-\frac{1}{8},-\frac{1}{8},\frac{1}{2}$	$K = (-4)$, $\Psi^*_{1/4}(z_i)$
$4^B_0\binom{2}{0}$	4	1,1,1,1	$0,0,0,\frac{1}{2}$	$K = (0\ 0; 2)$
$4^B_0\binom{0}{0}$	4	1,1,1,1	$0,0,\frac{1}{4},-\frac{1}{4}$	$2^B_1\binom{0}{0} \boxtimes 2^B_{-1}\binom{0}{0}$
$4^B_{9/5}\binom{0}{0}$	7.2360	$1,1,\zeta_3^1,\zeta_3^1$	$0,-\frac{1}{4},\frac{3}{20},\frac{2}{5}$	$2^B_{-1}\binom{0}{0} \boxtimes 2^B_{14/5}\binom{\zeta_3^1}{-3/20}$
$4^B_{-9/5}\binom{0}{0}$	7.2360	$1,1,\zeta_3^1,\zeta_3^1$	$0,\frac{1}{4},-\frac{3}{20},-\frac{2}{5}$	$2^B_1\binom{0}{0} \boxtimes 2^B_{-14/5}\binom{\zeta_3^1}{3/20}$
$4^B_{19/5}\binom{0}{0}$	7.2360	$1,1,\zeta_3^1,\zeta_3^1$	$0,\frac{1}{4},-\frac{7}{20},\frac{2}{5}$	$2^B_1\binom{0}{0} \boxtimes 2^B_{14/5}\binom{\zeta_3^1}{-3/20}$
$4^B_{-19/5}\binom{0}{0}$	7.2360	$1,1,\zeta_3^1,\zeta_3^1$	$0,-\frac{1}{4},\frac{7}{20},-\frac{2}{5}$	$2^B_{-1}\binom{0}{0} \boxtimes 2^B_{-14/5}\binom{\zeta_3^1}{3/20}$
$4^B_0\binom{\zeta_8^2}{0}$	13.090	$1,\zeta_3^1,\zeta_3^1,\zeta_8^2$	$0,\frac{2}{5},-\frac{2}{5},0$	$2^B_{14/5}\binom{\zeta_3^1}{-3/20} \boxtimes 2^B_{-14/5}\binom{\zeta_3^1}{3/20}$
$4^B_{12/5}\binom{\zeta_8^2}{3/10}$	13.090	$1,\zeta_3^1,\zeta_3^1,\zeta_8^2$	$0,-\frac{2}{5},-\frac{2}{5},\frac{1}{5}$	$2^B_{-14/5}\binom{\zeta_3^1}{3/20} \boxtimes 2^B_{-14/5}\binom{\zeta_3^1}{3/20}$
$4^B_{-12/5}\binom{\zeta_8^2}{-3/10}$	13.090	$1,\zeta_3^1,\zeta_3^1,\zeta_8^2$	$0,\frac{2}{5},\frac{2}{5},-\frac{1}{5}$	$2^B_{14/5}\binom{\zeta_3^1}{-3/20} \boxtimes 2^B_{14/5}\binom{\zeta_3^1}{-3/20}$
$4^B_{10/3}\binom{\zeta_7^3}{5/12}$	19.234	$1,\zeta_7^1,\zeta_7^2,\zeta_7^3$	$0,\frac{1}{3},\frac{2}{9},-\frac{1}{3}$	$(A_1,7)_{1/2}$
$4^B_{-10/3}\binom{\zeta_7^3}{-5/12}$	19.234	$1,\zeta_7^1,\zeta_7^2,\zeta_7^3$	$0,-\frac{1}{3},-\frac{2}{9},\frac{1}{3}$	$(A_1,-7)_{1/2}$, $(G_2,2)$
$5^B_0\binom{1}{1/2}$	5	1,1,1,1,1	$0,\frac{1}{5},\frac{1}{5},-\frac{1}{5},-\frac{1}{5}$	$K = (2\ 2; 3)$
$5^B_4\binom{1}{0}$	5	1,1,1,1,1	$0,\frac{2}{5},\frac{2}{5},-\frac{2}{5},-\frac{2}{5}$	$K = (2\ 2\ 2\ 2; 1\ 1\ 0; 1\ 0; 1)$
$5^B_2\binom{\sqrt{3}-1}{1/4}$	12	$1,1,\zeta_4^1,\zeta_4^1,2$	$0,0,\frac{1}{8},-\frac{3}{8},\frac{1}{3}$	$(A_1,4)$
$5^B_2\binom{\zeta_{10}^2}{-1/4}$	12	$1,1,\zeta_4^1,\zeta_4^1,2$	$0,0,-\frac{1}{8},\frac{3}{8},\frac{1}{3}$	$[(A_1,4) \boxtimes 2^B_1 \boxtimes 2^B_1]_{1/4}$
$5^B_{-2}\binom{\zeta_{10}^2}{1/4}$	12	$1,1,\zeta_4^1,\zeta_4^1,2$	$0,0,\frac{1}{8},-\frac{3}{8},-\frac{1}{3}$	$[(A_1,-4) \boxtimes 2^B_1 \boxtimes 2^B_1]_{1/4}$
$5^B_{-2}\binom{\sqrt{3}-1}{-1/4}$	12	$1,1,\zeta_4^1,\zeta_4^1,2$	$0,0,-\frac{1}{8},\frac{3}{8},-\frac{1}{3}$	$(A_1,-4)$
$5^B_{16/11}\binom{\zeta_9^4}{2/11}$	34.646	$1,\zeta_9^1,\zeta_9^2,\zeta_9^3,\zeta_9^4$	$0,-\frac{2}{11},\frac{2}{11},\frac{1}{11},-\frac{5}{11}$	$(A_1,9)_{1/2}$, $(F_4,2)$
$5^B_{-16/11}\binom{\zeta_9^4}{-2/11}$	34.646	$1,\zeta_9^1,\zeta_9^2,\zeta_9^3,\zeta_9^4$	$0,\frac{2}{11},-\frac{2}{11},-\frac{1}{11},\frac{5}{11}$	$(A_1,-9)_{1/2}$, $(E_8,3)$
$5^B_{18/7}\binom{\zeta_{12}^2}{-5/28}$	35.342	$1,\zeta_5^2,\zeta_5^2,\zeta_{12}^2,\zeta_{12}^4$	$0,-\frac{1}{7},-\frac{1}{7},\frac{1}{7},\frac{3}{7}$	$(A_1,12)_{1/4}$, $(A_2,4)_{1/3}$
$5^B_{-18/7}\binom{\zeta_{12}^2}{5/28}$	35.342	$1,\zeta_5^2,\zeta_5^2,\zeta_{12}^2,\zeta_{12}^4$	$0,\frac{1}{7},\frac{1}{7},-\frac{1}{7},-\frac{3}{7}$	$(A_3,3)_{1/4}$

If we view two particles (objects) i and j from far away, the two particles can be regarded as a single particle k. This defines a fusion operation $\otimes$:

$$i \otimes j = k. \tag{B2}$$

If we include such a fusion operation between objects of a category, we get a tensor category, where $\otimes$ is also called the tensor product.

To summarize, the equivalence classes of particles form a set of objects. If we add arrows (morphisms) between objects, we turn the set into a category. If we further add the fusion operation, we turn the category into a tensor category.

It is the philosophy of category theory, also the physical idea of second quantization, that we can focus on only the operators (morphisms) while treat particles (objects) as black boxes, but still have all the information of the system. In other words, the particles (objects) are defined by all their relations (morphisms) to other particles (objects).

Usually, when we try to understand an object, we like to divide the object into smaller pieces (or more basic components). If we can do that, we gain a better understanding of the object. This is the reductionist approach. But there is another approach. We do not think about the internal structure of the object, and pretend the internal structure is not there (i.e., treating the object as a black box). (Maybe the internal structure really does not exist.) We try to understand an object through its relations (i.e., morphisms) to all objects. In fact, we use all those relations to define the object. In other words, there are no objects, just relations. An object is uniquely determined by its relation to all objects (called Yoneda Lemma in category theory). In other words, the very existence of an object is in the form of the relations (morphisms). This is the philosophy of category theory. We see that category theory is essentially a theory of relations.

On the other hand, this categorical point of view is also the point of view taken by most physicists who pretend (or perhaps just get used to claim that) they are reductionists. Indeed, from a physical point of view, there is no more fundamental reality than the relations, or interactions, between particles, because what can be measured in physics are not particles but only their interactions. Perhaps, a physical object only arise as an illusion of an observer after a sophisticated process of computation based on the data from relations or interactions.

Now we try to introduce the operators (morphisms) in the category of particle statistics. One of the most important examples of nontrivial operators are those string operators (the product of local hopping operators) that generate *braidings*. Such a string operator, exchanging the positions of two particles a,b along a given path γ, corresponds to an isomorphism $c_{a,b} : a \otimes b \to b \otimes a$. Since local operators are quotiented out, the braiding operator depends only on the isotopy class of the path γ. In 2+1D, there are two isotopy classes of paths with winding numbers ± 1, clockwise and counter-clockwise. They are inverse to each other. However, in 3+1D, clockwise and counter-clockwise paths fall into the same isotopy class; the braiding must be the inverse of itself. Such braidings are call symmetric. (This is what the term "symmetric" in "symmetric category" means; it refers to "symmetric braiding" rather than some physical symmetry.) Therefore, in 3+1D, the braidings of identical particles can only be either $+1$ or -1, corresponding to bosonic or fermionic statistics. A system of such particles is described by a symmetric category. In contrast, in 2+1D, the braidings are allowed to be more complicated, known as anyonic or even non-Abelian statistics. Those particles are described by a braided fusion category, which is explained later.

Other examples of topological operators are the fusion and splitting operators. In 3+1D, they become important if we take into account the physical symmetry. Consider two particles, carrying two irreducible representations U,V of the symmetry group. We bring them together to form a composite particle, carrying the tensor product representation $U \otimes V$. Usually, $U \otimes V$ is not irreducible, and can fuse into another particle carrying an irreducible representation W via symmetry preserving operations. Such a process $f : U \otimes V \to W$ is a fusion operator, corresponding to a morphism in a category; its Hermitian conjugate $f^\dagger : W \to U \otimes V$ is a splitting operator (another morphism), corresponding to the process of splitting one particle into two. We need more data to describe these fusion and splitting operators, for example, the Clebsch-Gordan coefficients for spins. Furthermore, if more than three particles are fused, the $6j$ symbols kicks in. They measure the difference between fusing particles in different orders. In 3+1D, this seems just a different way to study group representations, by focusing on how representations fuse/split rather than how the group acts. However, the fusion and splitting operators become very rich in 2+1D. Because anyons do not necessarily carry group representations, the fusion and splitting operators are much more than merely the interwiners between group representations. This leads to rich non-Abelian statistics in 2+1D.

In summary, particle statistics in 3+1D and physical symmetry are described by symmetric categories. In 2+1D, there are new kinds of particle statistics beyond symmetric categories (i.e., Bose/Fermi statistics). But those 2+1D statistics is still not arbitrary.

First, there is a series of self-consistent conditions among the braiding, fusion and splitting operators. These lead to the mathematical structure of a *unitary braided fusion category* (UBFC).

Secondly, we would also assume the theory to be "complete." By "complete" we mean that "everything can be physically measured." Recall in quantum mechanics, we assume that states have inner products. Theoretically, physical measurements are made by taking inner products. Here "inner products" are "nondegenerate" bilinear forms. Nondegeneracy means that if two states produce the same inner products with all states (the same measurement outputs), they must be the same state. Thus the nondegeneracy means the theory is "complete." Now, the particle statistics are measured by the mutual braiding, so we expect similar "braiding nondegeneracy." More precisely, the braiding measurement is performed as follows. Assume that a particle a is waiting to be measured. We first create a pair of test particles i and its antiparticle $\bar{i}$, then move i around a, i.e., a double braiding, and finally annihilate $i\bar{i}$. The amplitude of such process is proportional to the (a,i) entry of the topological S matrix, S_{ai}. So the S matrix is the output of the braiding measurement, and, we should impose the nondegeneracy condition to the S matrix.

For 2+1D bosonic topological orders with no symmetry, the only (topological) measurement is the mutual braiding.

Thus different particles should be fully distinguished by their distinct mutual braiding statistics with other particles. If two particles have the same mutual braiding statistics with all particles, then the two particle must be equivalent (i.e., connected by local operators). [This is an application of the philosophy of category theory: an object is defined by its relations (morphisms) with all objects.] Indeed, a complete set of the equivalent classes of particles (i.e., the topological excitations) are described by a UBFC such that its S-matrix is nondegenerate. Such a UBFC is called *nondegenerate*. It is equivalent to the notion of a *unitary modular tensor category* (UMTC) [52,54]. This is why we say that the topological excitations (and their non-Abelian statistics) of a 2+1D bosonic topological order are fully described by a UMTC, or equivalently, a nondegenerate UBFC.

APPENDIX C: TOPOLOGICAL ORDERS WITH SYMMETRY FROM THE POINT OF VIEW OF LOCAL OPERATOR ALGEBRAS

In this section, we try to explain how to obtain a tensor-categorical description of topological bulk excitations in a 2+1D topological order with symmetry from the perspective of a local operator algebra that defines these topological excitations. Let us first recall what is known in the no-symmetry cases. Consider a 2+1D bosonic topological order without symmetry that can be realized by a lattice model. We have a local operator algebra A acting on the total Hilbert space $\mathcal{H}_R$ associated to a disklike region R. A topological (particlelike) excitation localized within a disklike region R in the lattice can be defined as a subspace of $\mathcal{H}_R$. Such a topological excitation can not be created/annihilated by any local operators. As a consequence, a topological excitation must be a module over the local operator algebra A, or an A module. This fact was fully established in Levin-Wen models that can realize all topological orders with gappable boundaries [26,62,90]. This fact must also hold for all topological orders because any topological order $\mathcal{C}$ can be viewed as a subsystem of a boundary-gappable topological order $\mathcal{C}\boxtimes\mathcal{D}$, where $\mathcal{D}$ can be chosen to be the time-reversal conjugate of $\mathcal{C}$. Then the topological excitations in $\mathcal{C}$ can all be realized as modules over a local operator algebra in a Levin-Wen model that realizes the phase $\mathcal{C}\boxtimes\mathcal{D}$. The choice of A is almost never unique even for a given lattice model. It usually depends on the choice of the region R. However, its dependence on R is not essential as it was proved in Levin-Wen models that different local operator algebras are all Morita equivalent [90]. In other words, the Morita class of A, or equivalently, the category of A-modules, denoted by A-Mod, is unique. We remark that this uniqueness should hold not only for a given Levin-Wen Hamiltonian but also for a class of Hamiltonians connected by local perturbations.

In general, A is naturally equipped with a structure (somewhat equivalent to that of an E_2 algebra [91]) such that the category A-Mod is a braided monoidal category. Moreover, due to the requirement of unitarity in physics, we expect more structures on A, such as certain $*$ structure and semisimpleness, such that the category A-Mod is a nondegenerate UBFC (or a UMTC). Macroscopically, the local operator algebra A is not observable and not a topological invariant either. Instead, only its Morita class (or equivalently, the category A-Mod) is a macroscopic observable and a topological invariant.

For a bosonic topological order with a symmetry group G, let us consider a lattice model realizing it. In this lattice model, we still have a local operator algebra A (not respecting the symmetry) acting on the total Hilbert space $\mathcal{H}_R$ associated to a disklike region R, in which there is a topological excitation. As in the cases without symmetry, we do not worry about the dependence of A on the region R. We assume that the existence of the lattice model realizing the bosonic topological order with symmetry G is equivalent to the existence of a local operator algebra A equipped with a G action, i.e., a group homomorphism $f: G\to \mathrm{Aut}(A)$. Actually, if G is on-site, G should also act on $\mathcal{H}_R$ as local operators in A. As a result, there is a natural group homomorphism $G\to\mathrm{Aut}(A)$, defined by $g\mapsto(a\mapsto gag^{-1})$ for $a\in A$ and $g\in G$. Also note that $a\mapsto gag^{-1}$ is an algebraic isomorphism. Therefore the microscopic data (A,f) completely determines the topological order with symmetry. What we would like to do is to use the pair (A,f) as the initial data to derive a natural macroscopic description of this topological order with symmetry G.

Note that the final macroscopic observables should respect the symmetry G in some sense. In the microscopic world, the local operators that respect the symmetry G are those living in the fix-point algebra $A^G := \{a\in A|\, ga = ag, \forall g\in G\}$, which is a subalgebra of A. Naively, it seems that the category of A^G modules, denoted by A^G-Mod, should be a natural choice for the categorical description of the topological excitations in this topological order with symmetry. However, this naive choice is not good for many reasons. The main reason is that we lose a lot of information in the process of replacing "A with a G action" by A^G. What we would like to do is to find the correct replacement of the category "A^G-Mod". We do that in two steps. In the first step, we carefully throw away the right amount microscopic data in "A with a G action" so that all the macroscopic data remain intact; in the second step, we try to find a fix-point construction which lose no more information.

Similar to the nosymmetry cases, the macroscopic data of "A with a G action" is encoded in its "Morita class." Therefore the first step amounts to find a proper notion of the category of modules over "A with a G action." It turns out that a G action on A naturally determines a G action on the category A-Mod as functors. More precisely, assuming that G is on-site for convenience, an A-module M, i.e., a pair $(M,\rho: A\otimes_{\mathbb{C}} M\to M)$, can be twisted by an element $g\in G$ to give a new A module (M,ρ^g) with the action ρ^g defined by $\rho^g(a\otimes_{\mathbb{C}} m)=\rho(gag^{-1}\otimes_{\mathbb{C}} m)$. For each $g\in G$, there is a functor $T_g: A\text{-Mod}\to A\text{-Mod}$ which maps (M,ρ) to (M,ρ^g) and maps an A-module map $M\to N$ to the same linear map (which automatically intertwines the actions ρ^g). We expect that T_g also respects the monoidal and braiding structures on A-Mod. Namely, it is a braided monoidal equivalence. These functors $T_g,\forall g\in G$, give arise to a G action $T:\hat{G}\to \mathrm{Aut}^{br}(A\text{-Mod})$ on A-Mod, where $\hat{G}$ is the monoidal category with objects given by elements in G and morphisms given by identity morphisms. Recall that only the Morita class of "A with a G action" is macroscopically meaningful. Moreover, if we equip the category A-Mod with the forgetful functor to Vec, each g action on A for $g\in G$ can be recovered from the functor T_g by the unique natural isomorphism from the identity functor to T_g. Therefore this "A-Mod with G action" can be regarded as the Morita class of "A with a G action." This already suggests that a proper categorical description of a topological order with symmetry is given by a nondegenerate UBFC $\mathcal{C}$ equipped with a G action T, i.e., a pair $(\mathcal{C},T)$. In particular, the trivial phase with symmetry G is given by the trivial nondegenerate UBFC $\mathcal{B}_0$ with a G action.

TABLE XIV. The table contains all bosonic topological orders with $N=6$ and $D^2 \leqslant 111$ (continued on the next page).

$N_c^B\binom{\lvert\Theta_2\rvert}{\angle\Theta_2/2\pi}$	D^2	$d_1,d_2,\dots$	$s_1,s_2,\dots$	Comments, SCA, K matrix
$6^B_{1}\binom{0}{0}$	6	1,1,1,1,1,1	$0,\frac{1}{12},\frac{1}{12},-\frac{1}{4},\frac{1}{3},\frac{1}{3}$	$2^B_{-1}\binom{0}{0}\boxtimes 3^B_{2}\binom{1}{-1/4}$
$6^B_{-1}\binom{0}{0}$	6	1,1,1,1,1,1	$0,-\frac{1}{12},-\frac{1}{12},\frac{1}{4},-\frac{1}{3},-\frac{1}{3}$	$2^B_{1}\binom{0}{0}\boxtimes 3^B_{-2}\binom{1}{1/4}$
$6^B_{3}\binom{0}{0}$	6	1,1,1,1,1,1	$0,\frac{1}{4},\frac{1}{3},\frac{1}{3},-\frac{5}{12},-\frac{5}{12}$	$2^B_{1}\binom{0}{0}\boxtimes 3^B_{2}\binom{1}{-1/4}$
$6^B_{-3}\binom{0}{0}$	6	1,1,1,1,1,1	$0,-\frac{1}{4},-\frac{1}{3},-\frac{1}{3},\frac{5}{12},\frac{5}{12}$	$2^B_{-1}\binom{0}{0}\boxtimes 3^B_{-2}\binom{1}{1/4}$
$6^B_{1/2}\binom{0}{0}$	8	$1,1,1,1,\zeta_2^1,\zeta_2^1$	$0,\frac{1}{4},-\frac{1}{4},\frac{1}{2},-\frac{1}{16},\frac{3}{16}$	$2^B_{1}\binom{0}{0}\boxtimes 3^B_{-1/2}\binom{\zeta_6^1}{-1/16}$
$6^B_{-1/2}\binom{0}{0}$	8	$1,1,1,1,\zeta_2^1,\zeta_2^1$	$0,\frac{1}{4},-\frac{1}{4},\frac{1}{2},\frac{1}{16},-\frac{3}{16}$	$2^B_{-1}\binom{0}{0}\boxtimes 3^B_{1/2}\binom{\zeta_6^1}{1/16}$
$6^B_{3/2}\binom{0}{0}$	8	$1,1,1,1,\zeta_2^1,\zeta_2^1$	$0,\frac{1}{4},-\frac{1}{4},\frac{1}{2},\frac{1}{16},\frac{5}{16}$	$2^B_{1}\binom{0}{0}\boxtimes 3^B_{1/2}\binom{\zeta_6^1}{1/16}$
$6^B_{-3/2}\binom{0}{0}$	8	$1,1,1,1,\zeta_2^1,\zeta_2^1$	$0,\frac{1}{4},-\frac{1}{4},\frac{1}{2},-\frac{1}{16},-\frac{5}{16}$	$2^B_{-1}\binom{0}{0}\boxtimes 3^B_{-1/2}\binom{\zeta_6^1}{-1/16}$
$6^B_{5/2}\binom{0}{0}$	8	$1,1,1,1,\zeta_2^1,\zeta_2^1$	$0,\frac{1}{4},-\frac{1}{4},\frac{1}{2},\frac{3}{16},\frac{7}{16}$	$2^B_{1}\binom{0}{0}\boxtimes 3^B_{3/2}\binom{0.7653}{3/16}$
$6^B_{-5/2}\binom{0}{0}$	8	$1,1,1,1,\zeta_2^1,\zeta_2^1$	$0,\frac{1}{4},-\frac{1}{4},\frac{1}{2},-\frac{3}{16},-\frac{7}{16}$	$2^B_{-1}\binom{0}{0}\boxtimes 3^B_{-3/2}\binom{0.7653}{-3/16}$
$6^B_{7/2}\binom{0}{0}$	8	$1,1,1,1,\zeta_2^1,\zeta_2^1$	$0,\frac{1}{4},-\frac{1}{4},\frac{1}{2},\frac{5}{16},-\frac{7}{16}$	$2^B_{1}\binom{0}{0}\boxtimes 3^B_{5/2}\binom{0.7653}{-3/16}$
$6^B_{-7/2}\binom{0}{0}$	8	$1,1,1,1,\zeta_2^1,\zeta_2^1$	$0,\frac{1}{4},-\frac{1}{4},\frac{1}{2},-\frac{5}{16},\frac{7}{16}$	$2^B_{-1}\binom{0}{0}\boxtimes 3^B_{-5/2}\binom{0.7653}{3/16}$
$6^B_{4/5}\binom{\zeta_3^1}{1/10}$	10.854	$1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1$	$0,-\frac{1}{3},-\frac{1}{3},\frac{1}{15},\frac{1}{15},\frac{2}{5}$	$2^B_{14/5}\binom{\zeta_3^1}{-3/20}\boxtimes 3^B_{-2}\binom{1}{1/4}$
$6^B_{-4/5}\binom{\zeta_3^1}{-1/10}$	10.854	$1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1$	$0,\frac{1}{3},\frac{1}{3},-\frac{1}{15},-\frac{1}{15},-\frac{2}{5}$	$2^B_{-14/5}\binom{\zeta_3^1}{3/20}\boxtimes 3^B_{2}\binom{1}{-1/4}$
$6^B_{16/5}\binom{\zeta_3^1}{2/5}$	10.854	$1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1$	$0,-\frac{1}{3},-\frac{1}{3},\frac{4}{15},\frac{4}{15},-\frac{2}{5}$	$2^B_{-14/5}\binom{\zeta_3^1}{3/20}\boxtimes 3^B_{-2}\binom{1}{1/4}$
$6^B_{-16/5}\binom{\zeta_3^1}{-2/5}$	10.854	$1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1$	$0,\frac{1}{3},\frac{1}{3},-\frac{4}{15},-\frac{4}{15},\frac{2}{5}$	$2^B_{14/5}\binom{\zeta_3^1}{-3/20}\boxtimes 3^B_{2}\binom{1}{-1/4}$
$6^B_{3/10}\binom{1.2383}{3/80}$	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_3^1\zeta_2^1$	$0,\frac{1}{2},-\frac{5}{16},-\frac{1}{10},\frac{2}{5},\frac{7}{80}$	$2^B_{14/5}\binom{\zeta_3^1}{-3/20}\boxtimes 3^B_{-5/2}\binom{0.7653}{3/16}$
$6^B_{-3/10}\binom{1.2383}{-3/80}$	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_3^1\zeta_2^1$	$0,\frac{1}{2},\frac{5}{16},\frac{1}{10},-\frac{2}{5},-\frac{7}{80}$	$2^B_{-14/5}\binom{\zeta_3^1}{3/20}\boxtimes 3^B_{5/2}\binom{0.7653}{-3/16}$
$6^B_{7/10}\binom{\zeta_3^1\zeta_6^1}{7/80}$	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_3^1\zeta_2^1$	$0,\frac{1}{2},\frac{7}{16},\frac{1}{10},-\frac{2}{5},\frac{3}{80}$	$2^B_{-14/5}\binom{\zeta_3^1}{3/20}\boxtimes 3^B_{7/2}\binom{\zeta_6^1}{-1/16}$
$6^B_{-7/10}\binom{\zeta_3^1\zeta_6^1}{-7/80}$	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_3^1\zeta_2^1$	$0,\frac{1}{2},-\frac{7}{16},-\frac{1}{10},\frac{2}{5},-\frac{3}{80}$	$2^B_{14/5}\binom{\zeta_3^1}{-3/20}\boxtimes 3^B_{-7/2}\binom{\zeta_6^1}{1/16}$
$6^B_{13/10}\binom{1.2383}{-27/80}$	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_3^1\zeta_2^1$	$0,\frac{1}{2},-\frac{3}{16},-\frac{1}{10},\frac{2}{5},\frac{17}{80}$	$2^B_{14/5}\binom{\zeta_3^1}{-3/20}\boxtimes 3^B_{-3/2}\binom{0.7653}{-3/16}$
$6^B_{-13/10}\binom{1.2383}{27/80}$	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_3^1\zeta_2^1$	$0,\frac{1}{2},\frac{3}{16},\frac{1}{10},-\frac{2}{5},-\frac{17}{80}$	$2^B_{-14/5}\binom{\zeta_3^1}{3/20}\boxtimes 3^B_{3/2}\binom{0.7653}{3/16}$
$6^B_{17/10}\binom{\zeta_3^1\zeta_6^1}{17/80}$	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_3^1\zeta_2^1$	$0,\frac{1}{2},-\frac{7}{16},\frac{1}{10},-\frac{2}{5},\frac{13}{80}$	$2^B_{-14/5}\binom{\zeta_3^1}{3/20}\boxtimes 3^B_{-7/2}\binom{\zeta_6^1}{1/16}$
$6^B_{-17/10}\binom{\zeta_3^1\zeta_6^1}{-17/80}$	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_3^1\zeta_2^1$	$0,\frac{1}{2},\frac{7}{16},-\frac{1}{10},\frac{2}{5},-\frac{13}{80}$	$2^B_{14/5}\binom{\zeta_3^1}{-3/20}\boxtimes 3^B_{7/2}\binom{\zeta_6^1}{-1/16}$
$6^B_{23/10}\binom{\zeta_3^1\zeta_6^1}{-17/80}$	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_3^1\zeta_2^1$	$0,\frac{1}{2},-\frac{1}{16},-\frac{1}{10},\frac{2}{5},\frac{27}{80}$	$2^B_{14/5}\binom{\zeta_3^1}{-3/20}\boxtimes 3^B_{-1/2}\binom{\zeta_6^1}{-1/16}$
$6^B_{-23/10}\binom{\zeta_3^1\zeta_6^1}{17/80}$	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_3^1\zeta_2^1$	$0,\frac{1}{2},\frac{1}{16},\frac{1}{10},-\frac{2}{5},-\frac{27}{80}$	$2^B_{-14/5}\binom{\zeta_3^1}{3/20}\boxtimes 3^B_{1/2}\binom{\zeta_6^1}{1/16}$
$6^B_{27/10}\binom{1.2383}{27/80}$	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_3^1\zeta_2^1$	$0,\frac{1}{2},-\frac{5}{16},\frac{1}{10},-\frac{2}{5},\frac{23}{80}$	$2^B_{-14/5}\binom{\zeta_3^1}{3/20}\boxtimes 3^B_{-5/2}\binom{0.7653}{3/16}$
$6^B_{-27/10}\binom{1.2383}{-27/80}$	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_3^1\zeta_2^1$	$0,\frac{1}{2},\frac{5}{16},-\frac{1}{10},\frac{2}{5},-\frac{23}{80}$	$2^B_{14/5}\binom{\zeta_3^1}{-3/20}\boxtimes 3^B_{5/2}\binom{0.7653}{-3/16}$
$6^B_{33/10}\binom{\zeta_3^1\zeta_6^1}{-7/80}$	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_3^1\zeta_2^1$	$0,\frac{1}{2},\frac{1}{16},-\frac{1}{10},\frac{2}{5},\frac{37}{80}$	$2^B_{14/5}\binom{\zeta_3^1}{-3/20}\boxtimes 3^B_{1/2}\binom{\zeta_6^1}{1/16}$
$6^B_{-33/10}\binom{\zeta_3^1\zeta_6^1}{7/80}$	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_3^1\zeta_2^1$	$0,\frac{1}{2},-\frac{1}{16},\frac{1}{10},-\frac{2}{5},-\frac{37}{80}$	$2^B_{-14/5}\binom{\zeta_3^1}{3/20}\boxtimes 3^B_{-1/2}\binom{\zeta_6^1}{-1/16}$
$6^B_{37/10}\binom{1.2383}{-3/80}$	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_3^1\zeta_2^1$	$0,\frac{1}{2},-\frac{3}{16},\frac{1}{10},-\frac{2}{5},\frac{33}{80}$	$2^B_{-14/5}\binom{\zeta_3^1}{3/20}\boxtimes 3^B_{-3/2}\binom{0.7653}{-3/16}$
$6^B_{-37/10}\binom{1.2383}{3/80}$	14.472	$1,1,\zeta_2^1,\zeta_3^1,\zeta_3^1,\zeta_3^1\zeta_2^1$	$0,\frac{1}{2},\frac{3}{16},-\frac{1}{10},\frac{2}{5},-\frac{33}{80}$	$2^B_{14/5}\binom{\zeta_3^1}{-3/20}\boxtimes 3^B_{3/2}\binom{0.7653}{3/16}$
$6^B_{1/7}\binom{0}{0}$	18.591	$1,1,\zeta_5^1,\zeta_5^1,\zeta_5^2,\zeta_5^2$	$0,-\frac{1}{4},-\frac{1}{7},-\frac{11}{28},\frac{1}{28},\frac{2}{7}$	$2^B_{-1}\binom{0}{0}\boxtimes 3^B_{8/7}\binom{\zeta_5^2}{-5/14}$
$6^B_{-1/7}\binom{0}{0}$	18.591	$1,1,\zeta_5^1,\zeta_5^1,\zeta_5^2,\zeta_5^2$	$0,\frac{1}{4},\frac{1}{7},\frac{11}{28},-\frac{1}{28},-\frac{2}{7}$	$2^B_{1}\binom{0}{0}\boxtimes 3^B_{-8/7}\binom{\zeta_5^2}{5/14}$
$6^B_{15/7}\binom{0}{0}$	18.591	$1,1,\zeta_5^1,\zeta_5^1,\zeta_5^2,\zeta_5^2$	$0,\frac{1}{4},\frac{3}{28},-\frac{1}{7},\frac{2}{7},-\frac{13}{28}$	$2^B_{1}\binom{0}{0}\boxtimes 3^B_{8/7}\binom{\zeta_5^2}{-5/14}$
$6^B_{-15/7}\binom{0}{0}$	18.591	$1,1,\zeta_5^1,\zeta_5^1,\zeta_5^2,\zeta_5^2$	$0,-\frac{1}{4},-\frac{3}{28},\frac{1}{7},-\frac{2}{7},\frac{13}{28}$	$2^B_{-1}\binom{0}{0}\boxtimes 3^B_{-8/7}\binom{\zeta_5^2}{5/14}$

TABLE XV. The table contains all bosonic topological orders with $N = 6$ and $D^2 \leqslant 111$. Here, $\chi_n^m \equiv m + \sqrt{n}$.

$N_c^B(^{\lvert\Theta_2\rvert}_{\angle\Theta_2/2\pi})$	D^2	$d_1, d_2, \ldots$	$s_1, s_2, \ldots$	Comments, SCA, K matrix
$6_0^B(^{1.2360}_{0})$	20	$1,1,2,2,\sqrt{5},\sqrt{5}$	$0,0,\frac{1}{5},-\frac{1}{5},0,\frac{1}{2}$	$(D_5,2)_{1/4}$, $(U(1)_5/\mathbb{Z}_2)_{1/2}$
$6_0^B(^{\zeta_8^4}_{1/2})$	20	$1,1,2,2,\sqrt{5},\sqrt{5}$	$0,0,\frac{1}{5},-\frac{1}{5},\frac{1}{4},-\frac{1}{4}$	$(6_0^{B,a} \boxtimes 2_1^B \boxtimes 2_{-1}^B)_{1/4}$
$6_4^B(^{\zeta_8^4}_{0})$	20	$1,1,2,2,\sqrt{5},\sqrt{5}$	$0,0,\frac{2}{5},-\frac{2}{5},0,\frac{1}{2}$	$(6_4^{B,a} \boxtimes 2_1^B \boxtimes 2_{-1}^B)_{1/4}$
$6_4^B(^{1.2360}_{1/2})$	20	$1,1,2,2,\sqrt{5},\sqrt{5}$	$0,0,\frac{2}{5},-\frac{2}{5},\frac{1}{4},-\frac{1}{4}$	$(B_2,2)$
$6_{58/35}^B(^{\zeta_3^1\zeta_5^2}_{0.2071})$	33.632	$1,\zeta_3^1,\zeta_5^1,\zeta_5^2,\zeta_3^1\zeta_5^1,\zeta_3^1\zeta_5^2$	$0,\frac{2}{5},\frac{1}{7},-\frac{2}{7},-\frac{16}{35},\frac{4}{35}$	$2_{14/5}^B(^{\zeta_3^1}_{-3/20}) \boxtimes 3_{-8/7}^B(^{\zeta_5^2}_{5/14})$
$6_{-58/35}^B(^{\zeta_3^1\zeta_5^2}_{-0.2071})$	33.632	$1,\zeta_3^1,\zeta_5^1,\zeta_5^2,\zeta_3^1\zeta_5^1,\zeta_3^1\zeta_5^2$	$0,-\frac{2}{5},-\frac{1}{7},\frac{2}{7},\frac{16}{35},-\frac{4}{35}$	$2_{-14/5}^B(^{\zeta_3^1}_{3/20}) \boxtimes 3_{8/7}^B(^{\zeta_5^2}_{-5/14})$
$6_{138/35}^B(^{\zeta_3^1\zeta_5^2}_{0.4928})$	33.632	$1,\zeta_3^1,\zeta_5^1,\zeta_5^2,\zeta_3^1\zeta_5^1,\zeta_3^1\zeta_5^2$	$0,\frac{2}{5},-\frac{1}{7},\frac{2}{7},\frac{9}{35},-\frac{11}{35}$	$2_{14/5}^B(^{\zeta_3^1}_{-3/20}) \boxtimes 3_{8/7}^B(^{\zeta_5^2}_{-5/14})$
$6_{-138/35}^B(^{\zeta_3^1\zeta_5^2}_{-0.4928})$	33.632	$1,\zeta_3^1,\zeta_5^1,\zeta_5^2,\zeta_3^1\zeta_5^1,\zeta_3^1\zeta_5^2$	$0,-\frac{2}{5},\frac{1}{7},-\frac{2}{7},-\frac{9}{35},\frac{11}{35}$	$2_{-14/5}^B(^{\zeta_3^1}_{3/20}) \boxtimes 3_{-8/7}^B(^{\zeta_5^2}_{5/14})$
$6_{46/13}^B(^{\zeta_{11}^5}_{-3/52})$	56.746	$1,\zeta_{11}^1,\zeta_{11}^2,\zeta_{11}^3,\zeta_{11}^4,\zeta_{11}^5$	$0,\frac{4}{13},\frac{2}{13},-\frac{6}{13},\frac{6}{13},-\frac{1}{13}$	$(A_1,11)_{1/2}$
$6_{-46/13}^B(^{\zeta_{11}^5}_{3/52})$	56.746	$1,\zeta_{11}^1,\zeta_{11}^2,\zeta_{11}^3,\zeta_{11}^4,\zeta_{11}^5$	$0,-\frac{4}{13},-\frac{2}{13},\frac{6}{13},-\frac{6}{13},\frac{1}{13}$	$(A_1,-11)_{1/2}$
$6_{8/3}^B(^{\zeta_{16}^4}_{1/3})$	74.617	$1,\zeta_7^3,\zeta_7^3,\zeta_7^3,\zeta_{16}^4,\zeta_{16}^6$	$0,\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{3},-\frac{1}{3}$	$(A_1,16)_{1/4}$
$6_{-8/3}^B(^{\zeta_{16}^4}_{-1/3})$	74.617	$1,\zeta_7^3,\zeta_7^3,\zeta_7^3,\zeta_{16}^4,\zeta_{16}^6$	$0,-\frac{1}{9},-\frac{1}{9},-\frac{1}{9},-\frac{1}{3},\frac{1}{3}$	$(A_1,-16)_{1/4}$
$6_2^B(^{\chi_{21}^5/2}_{-1/4})$	100.61	$1,\frac{1}{2}\chi_{21}^3,\frac{1}{2}\chi_{21}^3,\frac{1}{2}\chi_{21}^3,\frac{1}{2}\chi_{21}^5,\frac{1}{2}\chi_{21}^7$	$0,-\frac{1}{7},-\frac{2}{7},\frac{3}{7},0,\frac{1}{3}$	$(G_2,-3)$
$6_{-2}^B(^{\chi_{21}^5/2}_{1/4})$	100.61	$1,\frac{1}{2}\chi_{21}^3,\frac{1}{2}\chi_{21}^3,\frac{1}{2}\chi_{21}^3,\frac{1}{2}\chi_{21}^5,\frac{1}{2}\chi_{21}^7$	$0,\frac{1}{7},\frac{2}{7},-\frac{3}{7},0,-\frac{1}{3}$	$(G_2,3)$

Note that the pair $(\mathcal{C},T)$ is not a G-invariant description since the G action is explicit. So we take the second step to find a G-invariant description. This can be achieved by simply replacing the category $\mathcal{C}$ with a G action by the fix-point category $\mathcal{C}^G$, which consists of those objects in $\mathcal{C}$ that is invariant under the G action, i.e., those objects $X \in \mathcal{C}$ such that $T_g(X) \simeq X, \forall g \in G$. The category $\mathcal{C}^G$ is also called the equivariantization of $(\mathcal{C},T)$ (see Ref. [54] for a precise definition). It turns out to be a nondegenerate UBFC over Rep(G) [54].

For example, for the trivial phase $(\mathcal{B}_0,T)$ with symmetry G, the category $\mathcal{B}_0^G$ is nothing but Rep(G). Different from the replacement of "A with a G action" by A^G, which loses information, that of "$\mathcal{C}$ with a G action" by $\mathcal{C}^G$ loses no information at all. Indeed, one can recover the former structure from the later one by a condensation process [54]. Mathematically, the 2-category of nondegenerate UBFC's equipped with a G action is canonically equivalent to that of nondegenerate UBFC's over Rep(G) [54]. Therefore this notion of a nondegenerate UBFC over Rep(G) is the correct replacement to the category "A^G-Mod" that we are looking for.

Working with a nondegenerate UBFC over Rep(G) has some advantages over working with a nondegenerate UBFC with a G action. For example, it can be generalized easily to fermonic topological orders with/without symmetry by replacing Rep(G) by a SFC $\mathcal{E}$, which determines bosonic/fermionic symmetry uniquely. An object in $\mathcal{E}$ is a local excitation (a trivial A^G module). It can be created/annihilated by local operators, and has trivial mutual statistics with all excitations (the local operators break the symmetry if the excitation carries a nontrivial representation). In other words, $\mathcal{E}$ describes the local excitations in a topological order with symmetry. In particular, the SFC $\mathcal{E}$ should be viewed as the categorical description of the trivial phase with symmetry.

In summary, our analysis leads us to the proposal in Sec. II B that *the bulk excitations in a* 2+1D *topological order with symmetry $\mathcal{E}$ are described by a nondegenerate* UBFC *over $\mathcal{E}$*.

APPENDIX D: MATHEMATICAL DEFINITIONS

For the reader's convenience, we collect the some relevant mathematical definitions in this section. We would assume a basic knowledge on tensor category theory. Readers can consult with Refs. [54,75] for more details.

Definition 1. A *fusion category* is a rigid semisimple $\mathbb{C}$-linear tensor category, which has only finitely many isomorphic classes of simple objects, finite dimensional hom spaces, and simple unit object. A *braided fusion category* is a fusion category endowed with a braiding satisfying the hexagon equations. (For a detailed definition, see, e.g., Refs. [51,52].)

For physical reasons, we would assume that all the categories are *unitary*, i.e., one can take the Hermitian conjugate of the morphisms (physically they are operators between Hilbert spaces), and such Hermitian conjugate is compatible with the fusion and braiding structures. A unitary fusion category has a canonical spherical structure [52]. As a result, a unitary braided fusion category (UBFC) is automatically a ribbon category, or a premodular category.

Definition 2. The pair of objects X,Y in a UBFC $\mathcal{C}$ are said to *centralize* (mutually local to) each other if $c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X\otimes Y}$, where $c_{X,Y} : X \otimes Y \xrightarrow{\simeq} Y \otimes X$ is the braiding in $\mathcal{C}$. If X,Y are simple, this is equivalent to $S_{XY} = d_X d_Y / D$, where S is the S matrix.

TABLE XVI. The table contains all bosonic topological orders with $N = 7$ and $D^2 \leqslant 40$. Here, $\chi_n^m \equiv m + \sqrt{n}$.

$N_c^B({}^{\lvert\Theta_2\rvert}_{\angle\Theta_2/2\pi})$	D^2	$d_1,d_2,\ldots$	$s_1,s_2,\ldots$	Comments, SCA, K matrix
$7^B_2({}^{1}_{1/4})$	7	$1,1,1,1,1,1,1$	$0,\frac{1}{7},\frac{1}{7},\frac{2}{7},\frac{2}{7},\frac{4}{7},\frac{4}{7}$	$(4\,4;3)$
$7^B_6({}^{1}_{-1/4})$	7	$1,1,1,1,1,1,1$	$0,\frac{3}{7},\frac{3}{7},\frac{5}{7},\frac{5}{7},\frac{6}{7},\frac{6}{7}$	$-(4\,4;3)$, $(A_6,1)$
$7^B_{9/4}({}^{0.7209}_{9/32})$	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},\frac{3}{32},\frac{3}{32},\frac{1}{4},\frac{3}{4},\frac{15}{32}$	$(A_1,6)$
$7^B_{13/4}({}^{3.0727}_{13/32})$	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},\frac{7}{32},\frac{7}{32},\frac{1}{4},\frac{3}{4},\frac{19}{32}$	$(7^B_{9/4}\boxtimes 4^B_1)_{1/4}$
$7^B_{-15/4}({}^{\zeta_{14}^3}_{-15/32})$	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},\frac{11}{32},\frac{11}{32},\frac{1}{4},\frac{3}{4},\frac{23}{32}$	$(7^B_{13/4}\boxtimes 4^B_1)_{1/4}$
$7^B_{-11/4}({}^{2.0531}_{-11/32})$	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},\frac{15}{32},\frac{15}{32},\frac{1}{4},\frac{3}{4},\frac{27}{32}$	$(7^B_{-15/4}\boxtimes 4^B_1)_{1/4}$
$7^B_{-7/4}({}^{0.7209}_{9/32})$	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},\frac{19}{32},\frac{19}{32},\frac{1}{4},\frac{3}{4},\frac{31}{32}$	$(7^B_{-11/4}\boxtimes 4^B_1)_{1/4}$
$7^B_{-3/4}({}^{3.0727}_{13/32})$	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},\frac{23}{32},\frac{23}{32},\frac{1}{4},\frac{3}{4},\frac{3}{32}$	$(7^B_{-7/4}\boxtimes 4^B_1)_{1/4}$
$7^B_{1/4}({}^{\zeta_{14}^3}_{-15/32})$	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},\frac{27}{32},\frac{27}{32},\frac{1}{4},\frac{3}{4},\frac{7}{32}$	$(7^B_{-3/4}\boxtimes 4^B_1)_{1/4}$
$7^B_{5/4}({}^{2.0531}_{-11/32})$	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},\frac{31}{32},\frac{31}{32},\frac{1}{4},\frac{3}{4},\frac{11}{32}$	$(7^B_{1/4}\boxtimes 4^B_1)_{1/4}$
$7^B_{7/4}({}^{0.7209}_{-9/32})$	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},\frac{13}{32},\frac{13}{32},\frac{1}{4},\frac{3}{4},\frac{1}{32}$	$(C_6,1)$
$7^B_{11/4}({}^{2.0531}_{11/32})$	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},\frac{17}{32},\frac{17}{32},\frac{1}{4},\frac{3}{4},\frac{5}{32}$	$(7^B_{7/4}\boxtimes 4^B_1)_{1/4}$
$7^B_{15/4}({}^{\zeta_{14}^3}_{15/32})$	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},\frac{21}{32},\frac{21}{32},\frac{1}{4},\frac{3}{4},\frac{9}{32}$	$(7^B_{11/4}\boxtimes 4^B_1)_{1/4}$
$7^B_{-13/4}({}^{3.0727}_{-13/32})$	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},\frac{25}{32},\frac{25}{32},\frac{1}{4},\frac{3}{4},\frac{13}{32}$	$(7^B_{15/4}\boxtimes 4^B_1)_{1/4}$
$7^B_{-9/4}({}^{0.7209}_{-9/32})$	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},\frac{29}{32},\frac{29}{32},\frac{1}{4},\frac{3}{4},\frac{17}{32}$	$(7^B_{-13/4}\boxtimes 4^B_1)_{1/4}$
$7^B_{-5/4}({}^{2.0531}_{11/32})$	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},\frac{1}{32},\frac{1}{32},\frac{1}{4},\frac{3}{4},\frac{21}{32}$	$(7^B_{-9/4}\boxtimes 4^B_1)_{1/4}$
$7^B_{-1/4}({}^{\zeta_{14}^3}_{15/32})$	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},\frac{5}{32},\frac{5}{32},\frac{1}{4},\frac{3}{4},\frac{25}{32}$	$(7^B_{-5/4}\boxtimes 4^B_1)_{1/4}$
$7^B_{3/4}({}^{3.0727}_{-13/32})$	27.313	$1,1,\zeta_6^1,\zeta_6^1,\zeta_6^2,\zeta_6^2,\zeta_6^3$	$0,\frac{1}{2},\frac{9}{32},\frac{9}{32},\frac{1}{4},\frac{3}{4},\frac{29}{32}$	$(7^B_{-1/4}\boxtimes 4^B_1)_{1/4}$
$7^B_2({}^{\chi_7^1}_{1/4})$	28	$1,1,2,2,2,\sqrt{7},\sqrt{7}$	$0,0,\frac{1}{7},\frac{2}{7},\frac{4}{7},\frac{1}{8},\frac{5}{8}$	$(U(1)_7/\mathbb{Z}_2)_{1/2}$
$7^B_2({}^{1.6457}_{-1/4})$	28	$1,1,2,2,2,\sqrt{7},\sqrt{7}$	$0,0,\frac{1}{7},\frac{2}{7},\frac{4}{7},\frac{3}{8},\frac{7}{8}$	$(7^B_2({}^{\chi_7^1}_{1/4})\boxtimes 2^B_1\boxtimes 2^B_{-1})_{1/4}$
$7^B_{-2}({}^{1.6457}_{1/4})$	28	$1,1,2,2,2,\sqrt{7},\sqrt{7}$	$0,0,\frac{3}{7},\frac{5}{7},\frac{6}{7},\frac{1}{8},\frac{5}{8}$	$(7^B_{-2}({}^{\chi_7^1}_{-1/4})\boxtimes 2^B_1\boxtimes 2^B_{-1})_{1/4}$
$7^B_{-2}({}^{\chi_7^1}_{-1/4})$	28	$1,1,2,2,2,\sqrt{7},\sqrt{7}$	$0,0,\frac{3}{7},\frac{5}{7},\frac{6}{7},\frac{3}{8},\frac{7}{8}$	$(B_3,2)$, $(D_7,2)_{1/2}$
$7^B_{8/5}({}^{\zeta_{13}^6}_{-3/10})$	86.750	$1,\zeta_{13}^1,\zeta_{13}^2,\zeta_{13}^3,\zeta_{13}^4,\zeta_{13}^5,\zeta_{13}^6$	$0,\frac{4}{5},\frac{2}{15},0,\frac{2}{5},\frac{1}{3},\frac{4}{5}$	$(A_1,13)_{1/2}$
$7^B_{32/5}({}^{\zeta_{13}^6}_{3/10})$	86.750	$1,\zeta_{13}^1,\zeta_{13}^2,\zeta_{13}^3,\zeta_{13}^4,\zeta_{13}^5,\zeta_{13}^6$	$0,\frac{1}{5},\frac{13}{15},0,\frac{3}{5},\frac{2}{3},\frac{1}{5}$	$(A_1,-13)_{1/2}$
$7^B_1({}^{\chi_2^2}_{1/8})$	93.254	$1,\zeta_6^2,\zeta_6^2,\chi_2^2,\chi_2^2,\chi_8^2,\chi_8^3$	$0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{5}{8},0$	$(A_2,5)_{1/3}$
$7^B_7({}^{\chi_2^2}_{-1/8})$	93.254	$1,\zeta_6^2,\zeta_6^2,\chi_2^2,\chi_2^2,\chi_8^2,\chi_8^3$	$0,\frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{3}{4},\frac{3}{8},0$	$(A_2,-5)_{1/3}$

Physically, two particles "centralize" or "mutually local to" each other means that the two particles have trivial mutual statistics with each other.

Definition 3. A *full subcategory* $\mathcal{D}$ of the category $\mathcal{C}$ is a subcategory of $\mathcal{C}$ such that every morphism in $\mathcal{C}$ between two objects in $\mathcal{D}$ is also a morphism in $\mathcal{D}$, i.e., $\mathrm{Hom}_{\mathcal{D}}(x,y) = \mathrm{Hom}_{\mathcal{C}}(x,y)$ for all $x,y \in \mathcal{D}$.

Definition 4. Given a full subcategory $\mathcal{D}$ of a braided fusion category $\mathcal{C}$, the *centralizer* of $\mathcal{D}$ in $\mathcal{C}$, denoted by $\mathcal{D}_{\mathcal{C}}^{\mathrm{cen}}$, is the full subcategory of objects in $\mathcal{C}$ that centralize all the objects in $\mathcal{D}$. In particular, $Z_2(\mathcal{C}) \equiv \mathcal{C}_{\mathcal{C}}^{\mathrm{cen}}$ is called the *centralizer* of $\mathcal{C}$.

Definition 5. A *symmetric fusion category* (SFC) $\mathcal{E}$ is a UBFC $\mathcal{E}$ such that $Z_2(\mathcal{E}) = \mathcal{E}$. In other words, $\mathcal{E}$ is symmetric if $c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X\otimes Y}$ for objects $X,Y \in \mathcal{E}$.

This means that all particles in a SFC have trivial mutual statistics with respect to each other. SFC's are closely related to physical symmetries (groups).

Example 1. For a finite group G, the category of G representations, denoted by $\mathrm{Rep}(G)$, is an example of SFC. The category $\mathrm{Rep}(G)$ is equipped with the tensor product $\otimes$ given by the usual vector space tensor product $\otimes_{\mathbb{C}}$ and the standard symmetric braiding:

$$c_{X,Y}(x \otimes_{\mathbb{C}} y) = y \otimes_{\mathbb{C}} x, \quad \forall x \in X, y \in Y. \tag{D1}$$

In particular, the category $\mathcal{B}_0 = \mathrm{Rep}(\{1\})$ corresponds to bosonic systems without symmetry.

Example 2. Let G^f be a pair (G,z), where G is a finite group and z is involutive central nontrivial element in G, i.e., $z^2 = 1$, $zg = gz$ for all $g \in G$, and $z \neq 1$. Such element z acts on G-representations as the fermion-number parity, i.e., $zx = x$ if x is even and $zx = -x$ if x is odd. In other words, the pair $\mathbb{Z}_2^f = (\{1,z\},z)$ is the fermion-number-parity subgroup of G. We define the category $\mathrm{sRep}(G^f)$ as the same fusion category as $\mathrm{Rep}(G)$ but equipped with a modified braiding:

$$c_{X,Y}(x \otimes_{\mathbb{C}} y) = \begin{cases} -y \otimes_{\mathbb{C}} x, & x,y \text{ both odd}, \\ y \otimes_{\mathbb{C}} x, & \text{otherwise}. \end{cases} \tag{D2}$$

sRep(G^f) is also an example of SFC. It is a "super" or "fermionic" version of Rep(G) that describes fermionic symmetries. In particular, sRep($\mathbb{Z}_2^f$) $= \mathcal{F}_0$ corresponds to fermionic systems without symmetry.

By Deligne's theorem [65], a SFC is equivalent to either Rep(G) or sRep(G^f).

Definition 6. A UBFC $\mathcal{C}$ *over* a SFC $\mathcal{E}$ is a UBFC $\mathcal{C}$ with a fully faithful braided tensor embedding $\mathcal{E} \hookrightarrow Z_2(\mathcal{C})$. A UBFC $\mathcal{C}$ over a SFC $\mathcal{E}$ is *nondegenerate*, i.e., a UMTC$_{/\mathcal{E}}$, if $\mathcal{E}$ coincides with its centralizer, $\mathcal{E} = Z_2(\mathcal{C})$.

If we take $\mathcal{E} = \mathcal{B}_0$, we recover the usual definition of nondegenerate UBFC, or unitary modular tensor category (UMTC).

Definition 7. A *modular extension* of a UBFC $\mathcal{C}$ is a pair $(\mathcal{M},\eta)$, where $\mathcal{M}$ is a UMTC and $\eta : \mathcal{C} \hookrightarrow \mathcal{M}$ is a fully faithful braided tensor embedding, such that $Z_2(\mathcal{C})^{\text{cen}}_{\mathcal{M}} = \mathcal{C}$ (identify the image of η with $\mathcal{C}$).

Two modular extensions $(\mathcal{M}_1,\eta_1),(\mathcal{M}_2,\eta_2)$ of $\mathcal{C}$ are equivalent if there is a braided tensor equivalence $F : \mathcal{M}_1 \to \mathcal{M}_2$, such that $F \circ \eta_1 = \eta_2$.

Mathematically, the notion of a UBFC $\mathcal{C}$, or a UBFC $\mathcal{C}$ over $\mathcal{E}$, is self-contained. All the definitions and conditions can be checked within $\mathcal{C}$. There is no need to require $\mathcal{C}$ to be embedded into a larger UMTC. We include having a modular extension (condition 8 in Sec. III A) in the definition of UBFC in the main text, because it is a physical anomaly-free condition (see discussions in Sec. II C), and also ensures that UBFC's in terms of gauge-invariant data (N^{ij}_k,s_i) can be concretely realized by subcategories of certain UMTC's.

APPENDIX E: A LIST OF SIMPLE BOSONIC TOPOLOGICAL ORDERS

Many fermionic topological orders can be viewed as bosonic topological orders stacked with a fermionic product state or some other simpler fermionic topological orders. For completeness, here we list simple bosonic topological orders obtained in Ref. [56] (see Table XIII, XIV, XV, and XVI). The Abelian topological orders with $d_i = 1$ are described by K-matrices, denoted by the notation $(K_{11}K_{22}\dots;K_{12}K_{23}\dots;K_{13}K_{24}\dots;\dots)$. In Ref. [85], we show that all the non-Abelian topological orders in the table can be generated by simple current algebra (SCA) [28–30]. (See also https://www.math.ksu.edu/~gerald/voas/) Their many-body wave functions are given by the correlation of the simple currents in the SCA [20,28–30]. The SCA's are denoted by $(R,\pm k)_\alpha$ (see Refs. [55,85]), where $R = A_n,B_n,C_n,D_n$, etc., and $(R,-k)_\alpha$ is the time-reversal conjugate of $(R,+k)_\alpha$. The last column of Table XIII indicates how the corresponding topological order is realized by the K-matrix state, the SCA state, or the stacking of simpler topological orders. For some cases, we also indicate the many-body wave function that realize the corresponding topological order, where we use $\Psi_{1/m}(z_i)$ to indicate the filling fraction $\nu = 1/m$ Laughline wave function and $\Psi_n(z_i)$ to indicate the fermionic wave function with n-filled Landau level.

We also like to point out that the 16 red entries in Table XIII are all the 16 modular extentions of $\mathcal{F}_0$, which describe the 2+1D invertible fermionic topological orders up to E_8 states.

[1] X.-G. Wen, Phys. Rev. B **40**, 7387 (1989).
[2] X.-G. Wen and Q. Niu, Phys. Rev. B **41**, 9377 (1990).
[3] X.-G. Wen, Int. J. Mod. Phys. B **04**, 239 (1990).
[4] E. Keski-Vakkuri and X.-G. Wen, Int. J. Mod. Phys. B **07**, 4227 (1993).
[5] M. Levin and X.-G. Wen, Phys. Rev. Lett. **96**, 110405 (2006).
[6] A. Kitaev and J. Preskill, Phys. Rev. Lett. **96**, 110404 (2006).
[7] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B **82**, 155138 (2010).
[8] B. Zeng and X.-G. Wen, Phys. Rev. B **91**, 125121 (2015).
[9] M. A. Levin and X.-G. Wen, Phys. Rev. B **71**, 045110 (2005).
[10] F. Verstraete, J. I. Cirac, J. I. Latorre, E. Rico, and M. M. Wolf, Phys. Rev. Lett. **94**, 140601 (2005).
[11] G. Vidal, Phys. Rev. Lett. **99**, 220405 (2007).
[12] V. Kalmeyer and R. B. Laughlin, Phys. Rev. Lett. **59**, 2095 (1987).
[13] X.-G. Wen, F. Wilczek, and A. Zee, Phys. Rev. B **39**, 11413 (1989).
[14] K. von Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980).
[15] D. C. Tsui, H. L. Stormer, and A. C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982).
[16] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[17] N. Read and S. Sachdev, Phys. Rev. Lett. **66**, 1773 (1991).
[18] X.-G. Wen, Phys. Rev. B **44**, 2664 (1991).
[19] R. Moessner and S. L. Sondhi, Phys. Rev. Lett. **86**, 1881 (2001).
[20] G. Moore and N. Read, Nucl. Phys. B **360**, 362 (1991).
[21] X.-G. Wen, Phys. Rev. Lett. **66**, 802 (1991).
[22] R. Willett, J. P. Eisenstein, H. L. Störmer, D. C. Tsui, A. C. Gossard, and J. H. English, Phys. Rev. Lett. **59**, 1776 (1987).
[23] I. P. Radu, J. B. Miller, C. M. Marcus, M. A. Kastner, L. N. Pfeiffer, and K. W. West, Science **320**, 899 (2008).
[24] M. Freedman, C. Nayak, K. Shtengel, K. Walker, and Z. Wang, Ann. Phys. (NY) **310**, 428 (2004).
[25] Z.-C. Gu, Z. Wang, and X.-G. Wen, Phys. Rev. B **91**, 125149 (2015).
[26] A. Kitaev and L. Kong, Commun. Math. Phys. **313**, 351 (2012).
[27] Z.-C. Gu, Z. Wang, and X.-G. Wen, Phys. Rev. B **90**, 085140 (2014).
[28] B. Blok and X.-G. Wen, Nucl. Phys. B **374**, 615 (1992).
[29] X.-G. Wen and Y.-S. Wu, Nucl. Phys. B **419**, 455 (1994).
[30] Y.-M. Lu, X.-G. Wen, Z. Wang, and Z. Wang, Phys. Rev. B **81**, 115124 (2010).
[31] X.-G. Wen and Z. Wang, Phys. Rev. B **77**, 235108 (2008).
[32] X.-G. Wen and Z. Wang, Phys. Rev. B **78**, 155109 (2008).
[33] M. Barkeshli and X.-G. Wen, Phys. Rev. B **79**, 195132 (2009).
[34] A. Seidel and D.-H. Lee, Phys. Rev. Lett. **97**, 056804 (2006).
[35] E. J. Bergholtz, J. Kailasvuori, E. Wikberg, T. H. Hansson, and A. Karlhede, Phys. Rev. B **74**, 081308 (2006).
[36] A. Seidel and K. Yang, Phys. Rev. Lett. **101**, 036804 (2008).

[37] B. A. Bernevig and F. D. M. Haldane, Phys. Rev. Lett. **100**, 246802 (2008).
[38] B. A. Bernevig and F. D. M. Haldane, Phys. Rev. B **77**, 184502 (2008).
[39] B. A. Bernevig and F. D. M. Haldane, Phys. Rev. Lett. **101**, 246806 (2008).
[40] X.-G. Wen and A. Zee, Phys. Rev. B **46**, 2290 (1992).
[41] B. Blok and X.-G. Wen, Phys. Rev. B **42**, 8145 (1990).
[42] N. Read, Phys. Rev. Lett. **65**, 1502 (1990).
[43] J. Fröhlich and T. Kerler, Nucl. Phys. B **354**, 369 (1991).
[44] D. Belov and G. W. Moore, arXiv:hep-th/0505235.
[45] A. Kapustin and N. Saulina, Nucl. Phys. B **845**, 393 (2011).
[46] L. Kong and X.-G. Wen, arXiv:1405.5858.
[47] D. S. Freed, arXiv:1406.7278.
[48] C. L. Kane and M. P. A. Fisher, Phys. Rev. B **55**, 15832 (1997).
[49] X.-G. Wen, Phys. Rev. B **43**, 11025 (1991).
[50] X.-G. Wen, Adv. Phys. **44**, 405 (1995).
[51] B. Bakalov and A. Kirillov, *Lectures on Tensor Categories and Modular Functors* (American Mathematical Society, Providence, RI, 2001).
[52] A. Kitaev, Ann. Phys. **321**, 2 (2006).
[53] Z. Wang, *Topological Quantum Computation*, CBMS Regional Conference Series in Mathematics (American Mathematical Society, Providence, RI, 2010).
[54] V. Drinfeld, S. Gelaki, D. Nikshych, and V. Ostrik, arXiv:0906.0620.
[55] E. Rowell, R. Stong, and Z. Wang, Commun. Math. Phys. **292**, 343 (2009).
[56] X.-G. Wen, Natl. Sci. Rev. **3**, 68 (2016).
[57] A. Kapustin, R. Thorngren, A. Turzillo, and Z. Wang, J. High Energy Phys. 12 (2015) 052.
[58] N. Read and D. Green, Phys. Rev. B **61**, 10267 (2000).
[59] E. Witten, Commun. Math. Phys. **121**, 351 (1989).
[60] R. Dijkgraaf and E. Witten, Commun. Math. Phys. **129**, 393 (1990).
[61] X.-G. Wen, Phys. Rev. D **88**, 045013 (2013)
[62] T. Lan and X.-G. Wen, Phys. Rev. B **90**, 115119 (2014).
[63] A. Kapustin, arXiv:1403.1467.
[64] Z.-C. Gu and X.-G. Wen, Phys. Rev. B **90**, 115141 (2014).
[65] P. Deligne, Mosc. Math. J. **2**, 227 (2002).
[66] M. Levin and Z.-C. Gu, Phys. Rev. B **86**, 115109 (2012).
[67] X. Chen, F. J. Burnell, A. Vishwanath, and L. Fidkowski, Phys. Rev. X **5**, 041013 (2015).
[68] X. Chen, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B **84**, 235141 (2011).
[69] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B **87**, 155114 (2013).
[70] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Science **338**, 1604 (2012).
[71] V. Drinfeld, S. Gelaki, D. Nikshych, and V. Ostrik, arXiv:0704.0195.
[72] M. Cheng, Z. Bi, Y.-Z. You, and Z.-C. Gu, arXiv:1501.01313.
[73] T. Lan, L. Kong, and X.-G. Wen, arXiv:1602.05946.
[74] M. Barkeshli, P. Bonderson, M. Cheng, and Z. Wang, arXiv:1410.4540.
[75] A. Davydov, D. Nikshych, and V. Ostrik, Selecta Math. **19**, 237 (2013).
[76] Tian Lan, Liang Kong, and Xiao-Gang Wen, Commun. Math. Phys. 1 (2016).
[77] D. Gepner and A. Kapustin, Phys. Lett. B **349**, 71 (1995).
[78] C. Vafa, Phys. Lett. B **206**, 421 (1988).
[79] G. Andersen and G. Moore, Commun. Math. Phys. **117**, 441 (1988).
[80] P. Etingof, Math. Res. Lett. **9**, 651 (2002).
[81] E. Verlinde, Nucl. Phys. B **300**, 360 (1988).
[82] P. Bruillard, arXiv:1204.4836.
[83] J. F. Ebert, arXiv:math/0611612.
[84] A. Davydov, M. Müger, D. Nikshych, and V. Ostrik, J. Reine Angew. Math. **677**, 135 (2013).
[85] K. Schoutens and X.-G. Wen, Phys. Rev. B **93**, 045109 (2016).
[86] R. Dijkgraaf, C. Vafa, E. Verlinde, and H. Verlinde, Commun. Math. Phys. **123**, 485 (1989).
[87] X.-G. Wen, Y.-S. Wu, and Y. Hatsugai, Nucl. Phys. B **422**, 476 (1994).
[88] X.-G. Wen, Phys. Rev. B **60**, 8827 (1999).
[89] L. Fidkowski, X. Chen, and A. Vishwanath, Phys. Rev. X **3**, 041016 (2013).
[90] L. Kong, *Proceedings of XVIITH International Congress of Mathematical Physics* (World Scientific, 2012), pp. 444–455.
[91] J. Lurie, *Higher Algebra*, a book available at http://www.math.harvard.edu/~lurie/ (2014).