Theory of driven nonequilibrium critical phenomena

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A system driven in the vicinity of its critical point by varying a relevant field in an arbitrary function of time is a generic system that possesses a long relaxation time compared with the driving time scale and thus represents a large class of nonequilibrium systems. For such a manifestly nonlinear nonequilibrium strongly fluctuating system, we show that there exists universal nonequilibrium critical behavior that is incredibly well described by its equilibrium critical properties. A dynamic renormalization-group theory is developed to account for the behavior. The weak driving may give rise to several time scales depending on its form and thus rich nonequilibrium phenomena of various regimes and their crossovers, negative susceptibilities, as well as a violation of fluctuation-dissipation theorem and hysteresis. An initial condition that can be in either equilibrium or nonequilibrium but has longer correlations than the driving scales also results in a unique regime and complicates the situation. The implication of the results on measurement is also discussed. The theory may shed light on the study of other nonequilibrium systems and even nonlinear science.

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I. INTRODUCTION

Although equilibrium statistical physics has achieved great success, equilibrium systems are an exception rather than the rule: nonequilibrium phenomena are far more abundant and thus attract considerable attention [1–12]. Even though a systematic framework similar to the equilibrium statistical mechanics is still elusive, unifying principles for some nonequilibrium systems have emerged. For small systems, for instance, the key role of fluctuations and various fluctuation theorems for them have given birth to the stochastic thermodynamics [13] and the thermodynamics of information [14]. However, how about macroscopic systems?

Nonequilibrium systems are disparate and a systematic classification is still absent. One way can be to classify them according to the drives that bring them into nonequilibrium states and are thus not necessarily periodic [15]. One category is then to change some controlling parameters of a system instantaneously to their new values. The system then enters a nonequilibrium relaxation process. It can result in either a new equilibrium state or a nonequilibrium steady state [1–6,10–12,16] depending on whether a finite current flows through the system. Another one is to change the parameters infinitely slowly [8]. This is an adiabatical way that is usually invoked in theoretical studies such as linear response to study small deviations from equilibrium. The third category, on which we focus here, is to change the parameters within a finite time. Jarzynski's work theorem for small systems was derived for such processes [17]. For a macroscopic system, however, such a driving does not necessarily take it into nonequilibrium states.

Whether a driven system is in equilibrium or not depends on its relaxation time and the time scale of the driving. If the former is shorter than the latter, the system can follow the variation of the external driving adiabatically and hence stays in quasiequilibrium or adiabatic states. Only in the reverse case can a system fall genuinely out of equilibrium. The larger the difference between the two time scales, the more strongly the system deviates from its equilibrium states. A system in a glassy state has a long relaxation time. A system close to its critical point also possesses a divergent correlation time. Moreover, the equilibrium properties of the latter system have a well-established theoretical framework of the renormalization-group (RG) theory [18–26]. Accordingly, driving a system in the vicinity of its critical point within a finite time is a prototype of genuine nonequilibrium systems and is well fitted for studying whether universal nonequilibrium behavior exists or not. For comparison, relaxing a critical system in the first category has led to a critical initial slip [27] and the corresponding method of short-time critical dynamics has been applied extensively to estimate critical properties [28–30].

Indeed, some aspects of such driven systems have already been studied. On the one hand, the Kibble-Zurek (KZ) mechanism [31–34], first proposed in cosmology and then applied to condensed matter physics, provides a mechanism for nonequilibrium topological defect formation after a system is cooled through a continuous phase transition to a symmetry-broken ordered phase. Upon combining the equilibrium scaling near the critical point with the adiabatic-impulse-adiabatic approximation, a universal KZ scaling for the defect density has been proposed [33–36]. It has then been tested intensively in many systems, ranging from classical [37–53] to quantum [35,36,54–66]. A recent experiment on the Bose-Einstein condensation found agreement with the KZ scaling [52], though another one about the Mott insulator to superfluid transition on optical lattices concluded further theories were needed [66]. This is in line with the fact that most experimental results require additional assumptions for an interpretation of their consistency with the KZ scaling [67]. As defect counting is not easy [68] and whether phase ordering plays a role or not is yet to be clarified [69], it was proposed recently to detect the scaling of other observables [68]. The two recent experiments, for example, measured the domain size [52] and the correlation length [66] instead of the defect density.

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On the other hand, finite-time scaling (FTS) [70,71] offers a different perspective on the problem. From the analogy between the space domain of a diverging correlation length that may get longer than a system's size and the time domain of a diverging correlation time t_{eq} that may be longer than its allowable relaxation time, FTS was proposed as a temporal counterpart of the well-known finite-size scaling (FSS). A linear driving with a rate R_1 was found to specify a readily tunable driving time scale t_{R_1} that is asymptotically proportional to R_1^{-z/r_1}, where r_1 is the RG eigenvalue of R_1 and z the dynamic critical exponent. Similar to FSS, in the FTS regime, $t_{R_1} \ll t_{eq}$; the system falls out of equilibrium and just lies in the impulse regime of the KZ mechanism. This means that t_{R_1} divides the adiabatic and impulse regimes and governs the evolution of the latter, thus improving its understanding [72]. FTS has been successfully applied to classical [70–79] and quantum systems [80–82] to determine their critical properties. In particular, a positive specific-heat critical exponent α and thus violation of the bound for the correlation-length critical exponent ν [83] was found for a randomness-rounded first-order phase transition [75], corroborated by subsequent studies [84,85], and the critical behaviors of heating and cooling were observed to be qualitatively different [72]. In addition, FTS has been combined with the critical initial slip, extending the KZ mechanism to beyond adiabaticity [86].

So far, most work for the KZ mechanism and the FTS considers primarily a linear driving across the critical point. For a driving that is not exactly linear in time t, it is linearized near the critical point [35,58,87]. For a nonlinear driving, a monomial form t^n is usually considered with a nonunity integer n [35,58,70,76,87]. An advantage of these forms is that essentially only one parameter is involved and the driving may appear simple. However, it is not easy to confirm a driving to be linear in experiments. Questions rising naturally are then how about a general driving of an arbitrary form within a finite time. How can one generalize the understanding gained in FTS to such a general case? Does such a case possess universal behavior, and if yes, how to describe it?

Note that driving in a general form within a finite time near a critical point is highly nontrivial. Within a finite time, the system inevitably falls out of equilibrium due to critical slowing down. Also, it is characterized by a set of usually nonunity critical exponents and thus behaves strongly nonlinearly there [88]. The theory of FTS [70,71,88], which deals with both the nonequilibrium behavior in the FTS regime and the equilibrium behavior in the adiabatic regime, is mathematically a stochastic nonlinear time-dependent Landau-Ginzburg equation [23–26]. Upon a general driving, one of the controlling parameters becomes an arbitrary function of time. So, whether or not such a nonlinear partial differential equation in nonequilibrium situations shows universal behavior is surely not obvious even though the magnitude of the driving is small.

In this paper, we study the behavior of a system that is driven weakly close to its critical point within a finite time in a form that does not generate resonances but otherwise is arbitrary. We shall show that the system exhibits universal nonequilibrium critical behavior as may be expected. What is unexpected is that, incredibly, this driven critical behavior, far off equilibrium as the fluctuation-dissipation theorem is violated and the susceptibility can take on negative values due to hysteresis, is well described by only the equilibrium static and dynamic critical exponents, though the scaling functions can still involve singularities that need the exponent of the driving. A dynamic RG theory will be developed for the system to account for its universal nonequilibrium critical behavior. It shows that there exist various time scales determined by the driving parameters themselves and their combinations. As a result, the system can lie in different nonequilibrium regimes controlled by different time scales, with crossovers between them depending on the parameters. This generalizes the theory of FTS in which a single time scale, arising from a linear driving, governs the evolution of the system in the nonequilibrium regime. An initial condition that has longer correlations than the driving scales also gives rise to a unique regime and complicates the situation. This is opposite to the critical initial slip in which a nonequilibrium initial state has shorter correlations than those of the equilibrium state at the values of the initial driving parameters. Our theory furnishes a corrected understanding of experimental measurements in which an external driving is applied to a system with long relaxation times. As the system studied is a generic nonequilibrium one, the theory may shed light on the study of other nonequilibrium systems. It may also be instructive to nonlinear science as the driving may help to probe scaling behavior there.

We note that the driving form can be arbitrary except that sometimes the driving itself may generate some kinds of resonance depending on the systems considered. At present, we can only detect this from the results *a posteriori*. In case they do not fit the theory, some resonance may be in effect. There exist a class of driven dissipative systems in which a strong driving leads to new nonequilibrium phase transitions [89–93], and a new critical exponent associated with the nonequilibrium driving has been found [91,92]. These may be regarded as a kind of resonances in which the driving acts to create and maintain the new transitions, whereas in the present case the driving serves as a probe of the transition; it does not change radically the existing transition. We believe, however, that the present approach can also apply to that case to study the critical properties.

In the following, we shall first develop a dynamic RG theory in Sec. II and study the effect of initial conditions in Sec. III. We then apply the theory to several specific forms of driving and discuss its implication to measurements in Secs. IV and V, respectively. In order to test our result, we perform Monte Carlo (MC) simulations using the model and method in Sec. VI with the results being detailed in Sec. VII. Conclusions are given in Sec. VIII.

II. DYNAMIC RG THEORY

In this section, a dynamic RG theory is first developed to analyze the universal behavior of a system driven by a general temporal form near its critical point. Then different timescales are identified and crossovers are briefly discussed from the scaling forms obtained. We only consider the cases in which the system starts with an equilibrium initial condition far away from the critical point. The effect of initial conditions is left to Sec. III.

A. Theory

The dynamic RG theory for a system with a driving was initiated in a theory of first-order phase transitions [94]. It was then applied back to critical phenomena [71,88]. Here we shall generalize the theory to a driving of a general form and identify the restriction on the driving with which different behavior may emerge.

Any relevant parameter such as the temperature T or an externally applied field can serve as a driving field. Without loss of generality, we use the terminology of magnetism and choose the external magnetic field H as the driving throughout. For clarity, we shall often set the reduced temperature $\tau = T - T_c = 0$ and ignore the effect of finite system sizes L, where T_c is the critical temperature. They can be taken into account straightforwardly, though finite-time finite-size scaling may emerge in cooling when L is considered [72].

Consider a ϕ^4 free energy functional [19–22]

$$\mathcal{F}[\phi] = \int d\mathbf{r}\left[\frac{1}{2}\bar{\tau}\phi^2 + \frac{1}{4!}g\phi^4 + \frac{1}{2}(\nabla\phi)^2 - H\phi\right], \qquad (1)$$

where ϕ is a coarse-grained field variable, g a coupling constant, and $\bar{\tau}$ the distance to the mean-field T_c at $H = 0$. The dynamics is governed by the Langevin equation

$$\frac{\partial\phi}{\partial t} = -\lambda\frac{\delta\mathcal{F}}{\delta\phi} + \zeta, \qquad (2)$$

where λ is a kinetic coefficient and ζ is a Gaussian white noise satisfying $\langle\zeta(\mathbf{r},t)\rangle = 0$ and $\langle\zeta(\mathbf{r},t)\zeta(\mathbf{r}',t')\rangle = 2\lambda T\delta(\mathbf{r} - \mathbf{r}')\delta(t - t')$. The dynamic model, Eqs. (1) and (2), constitutes the simplest equilibrium critical dynamics of Model A for the nonconserved order parameter [23]. It is a nonlinear stochastic partial differential equation that cannot be solved exactly generally. Moreover, perturbation expansions near the critical point are plagued with infrared divergences [19].

However, universal long-wavelength long-time properties can be found by the RG theory without solving the equation. This can be done systematically using field-theoretic techniques [20,21]. It has been shown that the model of Eqs. (1) and (2) is equivalent to a dynamical field theory described by the dynamical functional [25,26,95]

$$I[\phi,\tilde{\phi}] = \int d\mathbf{r}dt\left\{\tilde{\phi}\left[\dot{\phi} + \lambda(\bar{\tau} - \nabla^2)\phi + \frac{1}{3!}\lambda g\phi^3 - \lambda H\right] - \lambda T\tilde{\phi}^2\right\}, \qquad (3)$$

where $\tilde{\phi}$ is a response field [96]. In the field-theoretic framework, the universal critical behavior is determined by the renormalization factors that remove the divergences arising at long times and when the underlying lattice constant of the original theory is sent to vanishing.

For a constant external field H and an equilibrium initial condition, because of the supersymmetry of $I[\phi,\tilde{\phi}]$ [20], it is well known that only the following four independent renormalization factors Z defined as

$$\begin{aligned}
&\phi \to \phi_0 = Z_\phi^{1/2}\phi, \quad \tilde{\varphi} \to \tilde{\phi}_0 = Z_{\tilde{\phi}}^{1/2}\tilde{\phi},\\
&g \to g_0 = N_d\mu^\epsilon Z_\phi^{-2}Z_u u, \quad \lambda \to \lambda_0 = Z_\phi^{1/2}Z_{\tilde{\phi}}^{-1/2}\lambda, \qquad (4)\\
&\bar{\tau} \to \bar{\tau}_0 = Z_\phi^{-1}Z_\tau\tau + \bar{\tau}_c, \quad H \to H_0 = Z_\phi^{-1/2}H,
\end{aligned}$$

are needed, where the subscripts 0 denote bare parameters, μ is an arbitrary momentum scale, $\bar{\tau}_c$ the shifted critical point, $\epsilon = 4 - d$, and $N_d = 2/[(4\pi)^{d/2}\Gamma(d/2)]$ with d being the space dimensionality and Γ the Euler Gamma function. We have directly renormalized the field H in Eq. (4). It results from the expansion of response functions with H [20,21]. The four Z determine the fixed point and three independent critical exponents including the dynamic one. However, when the initial state is out of equilibrium with a short correlation in the vicinity of the critical point, it was found that another new Z factor is required to cancel the new divergence due to the initial time. This leads to an independent critical initial-slip exponent [27,95].

Now, for a driving with a time-dependent H, upon ignoring the effects from the nonequilibrium initial conditions, which have been studied [86], whether new exponents are needed hinges on whether new intrinsic divergences are generated. Since we only focus on a spatially homogeneous driving, possible new divergences can only stem from the time domain. When H blows up with t as in the linearly driving case, a divergence at $t \to \infty$ is always present. However, it is extrinsic as it arises from the driving itself. A nontrivial divergence must originate from a resonancelike interaction of the driving with the system considered. This must then result in new exponents [91,92]. Interesting as it is, this is not the case on which we focus here as our aim here is to bring a system out of equilibrium with weak drives. In this case, we can again expand the response function with H at each instant as in the time-independent case. Therefore the four Z suffice to remove all the divergences and no new exponents are needed! We shall meet a new singularity in some monomial driving, but that is generated completely by the form of the field itself and no critical exponents are needed there.

With the renormalization factors, the universal behavior can be determined by the RG equation. It can be derived formally from $\langle\phi\rangle \equiv M(\mu,\lambda,u,\tau,H)$ by the fact that the bare quantities are independent of μ as [20,21,71,88]

$$\left[\mu\partial_\mu + \varsigma\lambda\partial_\lambda + \beta\partial_u + \gamma_\tau\tau\partial_\tau + \tfrac{1}{2}\gamma_\phi(H\partial_H + 1)\right]M = 0, \qquad (5)$$

where the Wilson functions are defined as

$$\begin{aligned}
\varsigma &= \mu\partial_\mu \ln\lambda, \quad \gamma_\phi = \mu\partial_\mu \ln Z_\phi,\\
\gamma_\tau &= \mu\partial_\mu \ln\tau, \quad \beta(u) = \mu\partial_\mu u \qquad (6)
\end{aligned}$$

at constant bare parameters. At the fixed point $u = u^*$ at which $\beta(u^*) = 0$, combining the solution of Eq. (5) with the result of dimension analysis, one arrives at

$$M(t,H,\tau) = b^{-\beta/\nu}M(tb^{-z},Hb^{\beta\delta/\nu},\tau b^{1/\nu}), \qquad (7)$$

where b is a length rescaling factor and the critical exponents are given as usual by

$$\begin{aligned}
&\eta = \gamma_\phi^*, \quad \nu^{-1} = 2 - \gamma_\tau^*, \quad z = 2 + \varsigma^*,\\
&\beta/\nu = (d - 2 + \eta)/2, \;\; \delta = (d + 2 - \eta)/(d - 2 + \eta) \qquad (8)
\end{aligned}$$

with the stars marking the values at the fixed point. Equation (7) gives the scale transform of M and applies to both a constant and a time-dependent H. It can give rise to various scaling forms. For example, choosing a scale such that $\tau b^{1/\nu}$ is a constant leads to

$$M(t,H,\tau) = \tau^{\beta} f_{\tau}(t\tau^{\nu z}, H\tau^{-\beta\delta}), \tag{9}$$

where f_τ is a universal scaling function.

We now turn to the parameters that specify the time dependence of the driving. Let $H = H(t,B_1,\ldots,B_p)$, where $B_i, i = 1,\ldots,B_p$ are p independent parameters. Using B_i and t instead of H and t as variables because they are not independent, we can write formally the RG equation as [71]

$$\left[\mu\partial_\mu + \varsigma t\partial_t + \beta\partial_u + \sum_{i=1}^{p}(\gamma_{B_i} B_i \partial_{B_i}) + \frac{1}{2}\gamma_\phi\right] M = 0, \tag{10}$$

where $\gamma_{B_i} = \mu\partial_\mu \ln B_i$ are the Wilson function of B_i and we have replaced λ with t directly and suppressed τ by considering the critical theory only. As a result, a similar method then gives rise to the solution

$$M(t,B_1,\ldots,B_p) = b^{-\beta/\nu} M(tb^{-z}, B_1 b^{r_{B_1}},\ldots,B_p b^{r_{B_p}}), \tag{11}$$

where the RG eigenvalue r_{B_i} of B_i is given by

$$r_{B_i} \equiv d_{B_i} - \gamma^*_{B_i} \tag{12}$$

with d_{B_i} being the naïve dimension of B_i [71,88].

To determine r_{B_i}, note that the t dependence in Eq. (10) results both explicitly from M itself and implicitly from H for the driving. So, one has formally [71] $\partial_t = \partial_t + (\partial_t H)\partial_H$ and $\partial_{B_i} = (\partial_{B_i} H)\partial_H$. Substituting them into Eq. (10) and comparing the outcome with Eq. (5), we find, at the fixed point,

$$\varsigma^*(\partial_{\ln t} \ln H) + \sum_i \gamma^*_{B_i}(\partial_{\ln B_i} \ln H) = \frac{1}{2}\gamma^*_\phi. \tag{13}$$

This is a single equation of all $\gamma^*_{B_i}$ for a general H. Yet, it must be valid at each instant. This solves all $\gamma^*_{B_i}$ and hence $r^*_{B_i}$ via Eq. (12) by comparing similar terms as can be seen from the examples below. Note that all $\gamma^*_{B_i}$ are determined by γ^*_ϕ and ς^* or η and z from Eq. (8). So are all $r^*_{B_i}$. In other words, the usual static and dynamic critical exponents are sufficient for the driven nonequilibrium critical phenomena as has been pointed out. It will be seen later on that $r^*_{B_i}$ can also be found from direct scale transforms among H, t, and B_i, similar to the linear case [88] without explicitly solving Eq. (13).

B. Time scales and crossovers

We now discuss briefly the meaning associated with the parameters of the driving.

From the scaling form (9), one can identify the equilibrium correlation time $t_{\rm eq}$ and a time scale pertinent to the field t_H as

$$t_{\rm eq} \sim |\tau|^{-\nu z}, \qquad t_H \sim |H|^{-\nu z/\beta\delta}, \tag{14}$$

respectively. Both time scales diverge as expected at the exact critical point $\tau = 0$ and $H = 0$ but are tamed by a finite τ or H.

Similarly, for each parameter, one finds from Eq. (11) that there is an associated time scale t_{B_i} asymptotically proportional to $B_i^{-z/r_{B_i}}$, viz.,

$$t_{B_i} \sim B_i^{-z/r_{B_i}}. \tag{15}$$

This is a universal form near the critical point for all the parameters once their correct r_{B_i} is used, even for τ and H, in which case it returns to Eq. (14). We shall frequently use it both explicitly and inexplicitly in the following. In the scaling regime, the shortest long time scale controls the evolution of the system. If t_{B_i} is just such a time scale, the scaling form is then

$$M = B_i^{\beta/\nu r_{B_i}} f_{B_i}\left(t B_i^{z/r_{B_i}}, \tau B_i^{-1/r_{B_i}\nu}, B_j B_i^{-r_{B_j}/r_{B_i}},\ldots\right) \tag{16}$$

from Eq. (11), where $j \neq i$, f_{B_i} is the associated scaling function, and the ellipsis stands for the parameters except B_i and B_j. We have put back τ in Eq. (16). Equation (16) implies that physical observables can be rescaled by B_i in the critical regime. It can also be written as

$$M = t_{B_i}^{-\beta/\nu z} f_{t_{B_i}}(t/t_{B_i}, t_{B_i}/t_{\rm eq}, t_{B_i}/t_{B_j},\ldots) \tag{17}$$

using the time scales. In Eqs. (16) and (17), each argument in the scaling functions must be vanishingly small to ensure that f_{B_i} and $f_{t_{B_i}}$ are analytic there. This means that $t_{B_i} \ll t_{\rm eq}$ and $t_{B_i} \ll t_{B_j}$ for all $j \neq i$ consistently. The former indicates that the driving time scale t_{B_i} is shorter than the equilibrium correlation time $t_{\rm eq}$ and thus the system falls out of equilibrium. So, Eqs. (16) and (17) are the generalizations of the FTS and we shall also refer them as FTS forms.

If conditions change such that another time scale, t_{B_j} say, becomes the shortest, or $t_{B_i}/t_{B_j} \gg 1$. In this case, it is now the dominant time scale and governs the leading singularity. Accordingly, f_{B_i} is singular near the critical point and behaves as $(B_j B_i^{-r_{B_j}/r_{B_i}})^{\beta/\nu r_{B_j}}$ in order to cancel the original singularity. This is a crossover from the regime governed by t_{B_i} to that by t_{B_j} with the crossover condition $t_{B_i} \sim t_{B_j}$. For a general driving with several parameters, such phenomena can be rich.

Moreover, we shall see in the following that there exist time scales that are determined by more than one independent parameter. One case is the first expansion coefficient in t for a general driving. It can be a dominant time scale. By contrast, some time scales may only be transient and never dominate. Although finding the dominant time scale is sometimes not easy, we shall see that the present theory still describes the driven nonequilibrium critical phenomena well.

In addition, there exist crossovers to regimes that are specified by other parameters than B_i [70,71]. For example, when τ is large or $t_{\rm eq}$ dominates, there is a crossover to the adiabatic or (quasi-)equilibrium regime that is governed by it and is described by the scaling form (9) with all B_i present. Similar results can be obtained by other relevant parameters such as L. We shall not pursue them further in the following. We note in passing that near the critical point, because the correlation length $\xi \sim t_{\rm eq}^{1/z}$, each time scale considered then relates to a corresponding length scale by a similar relation.

III. INITIAL CONDITIONS

In this section, we focus on the effect of initial conditions on driving. Recall that the dynamic equation (2) is a first-order stochastic differential equation. So, mathematically initial conditions are necessary. An initial condition comprises of two parts: a starting field $H_{\rm in}$, which characterizes the distance to the critical point, and an initial state that is specified by a distribution $\mathcal{P}_{\rm in}$ of the order parameter ϕ, or equivalently, all orders of the moments of $\mathcal{P}_{\rm in}$ if exist. If a given initial state is identical with the equilibrium state at $H_{\rm in}$, the given state is referred to as an equilibrium initial state. If it is not, it is in general a nonequilibrium initial state. Such a nonequilibrium state can well be the equilibrium state at another field value. It can also be a genuine nonequilibrium state that is in a far-off equilibrium condition and thus unlikely to be an equilibrium state of the system at any of its particular field values. Such a state can be created by driving the system considered to near its critical point within the FTS regime of the driving. We shall see that all the correlations of a system are then specified remarkably by the driving time scale or driving length scale rather than by the field value itself when all other variables are fixed.

In the previous section, $H_{\rm in}$ is chosen far away from the critical point and thus the initial state plays no role no matter whether it is in equilibrium or not as the system can equilibrate quickly there. By contrast, near the critical point, a nonequilibrium initial state with correlations *shorter* than the equilibrium ones at $H_{\rm in}$ results in the critical initial slip [27] even when H is varied linearly [86]. Here, we consider the effect of initial states that have *longer* correlations than the driving ones and that can be in either equilibrium or nonequilibrium relative to the equilibrium state at $H_{\rm in}$.

In general, upon suppressing other scales, the FTS form including the initial condition is

$$M = t_D^{-\beta/\nu z} f_D\big(H t_D^{\beta\delta/\nu z}, H_{\rm in} t_D^{\beta\delta/\nu z}, V(t_D,\mathcal{P}_{\rm in})\big), \qquad (18)$$

where t_D is the dominant time scale of the driving, $H_{\rm in} t_D^{\beta\delta/\nu z}$ characterizes the rescaled initial distance to the critical point, and $V(t_D,\mathcal{P}_{\rm in})$ is a universal characteristic function describing the rescaled initial distribution. For a general length rescaling factor b, $V(b,\mathcal{P}_{\rm in})$ is a generalization of the critical characteristic function, $U(b,M_{\rm in})$, for an initial state with an arbitrary $M_{\rm in}$ and vanishing correlations [82,97], for which V returns to U. For sufficiently small $M_{\rm in}$, $U(b,M_{\rm in}) = M_{\rm in} b^{x_0}$ with x_0 being related to the critical initial slip exponent [82,97]. From Eq. (18), the FTS regime controlled by t_D satisfies $|H| t_D^{\beta\delta/\nu z} \ll 1$ and falls within $|H| \ll \hat{H}$, where $\hat{H} \sim t_D^{-\beta\delta/\nu z}$ represents the boundary of the FTS regime.

If $|H_{\rm in}| t_D^{\beta\delta/\nu z} \gg 1$, $|H_{\rm in}| \gg \hat{H}$ and then locates in the adiabatic regime. Accordingly, an initial state at $H_{\rm in}$ either in equilibrium or in nonequilibrium decays exponentially to the equilibrium one quickly with the driving and the initial condition is irrelevant. One can then simply start a driving just beyond the FTS regime with an equilibrium distribution $\mathcal{P}_{\rm eq}(H)$ at $|H| \gtrsim \hat{H}$.

If $|H_{\rm in}| t_D^{\beta\delta/\nu z} \ll 1$ to the contrary, $H_{\rm in}$ then lies in the FTS regime. As a result, the information of the initial state distribution cannot be ignored. In this case, how the initial condition affects the evolution depends on $\mathcal{P}_{\rm in}$. In the following two sections, we shall consider two simple cases that will be needed in later sections.

A. Equilibrium initial conditions

When the initial state is an equilibrium state at $H_{\rm in}$, the critical initial slip does not matter. In this case, $\mathcal{P}_{\rm in} = \mathcal{P}_{\rm eq}(H_{\rm in})$ and is determined solely by $H_{\rm in}$ because $\tau = 0$, e.g., $M_{\rm in} \sim |H_{\rm in}|^{1/\delta}$ for small $|H_{\rm in}|$. So, V can be expressed by $H_{\rm in} t_D^{\beta\delta/\nu z}$. That $H_{\rm in}$ lies in the FTS regime implies $t_D < |H_{\rm in}|^{-\nu z/\beta\delta} \sim t_{H_{\rm in}}$, i.e., the correlation time of the equilibrium initial condition is longer than the driving time t_D. So is the correlation length of the initial condition. In this case, Eq. (18) becomes

$$M = |H_{\rm in}|^{1/\delta} f_{H_{\rm in}}(H/H_{\rm in}, t_D |H_{\rm in}|^{\nu z/\beta\delta}), \qquad (19)$$

and the initial condition dominates within a time $t_{H_{\rm in}}$ even though $t_D < t_{H_{\rm in}}$. The reason why the longer scale instead of the shorter scale dominates is that here the former originating from the initial condition already exist there and thus dominate the short ones that are still setting up from the background of the longer correlations. Once the short scale is done, the driving t_D takes over.

B. Nonequilibrium initial conditions: continuous piecewise driving

We next study a specific genuine nonequilibrium initial condition that has longer correlations than the driving scales, though an equilibrium state with such correlations at other field values than the initial one and thus a nonequilibrium state at the initial field can also work. To this end, consider a process with the following two steps. First, start from the adiabatic regime $|H_{\rm in}| > |\hat{H}|$ with a certain form of driving $H_{{\rm st}_1}$, whose dominant time scale is t_{D_1}, and stop at H_1 inside the FTS regime of t_{D_1}, i.e., $|H_1| t_{D_1}^{\beta\delta/\nu z} \ll 1$. Because H_1 is inside the FTS regime, the distribution $\mathcal{P}_1$ is not the equilibrium distribution at H_1 but a nonequilibrium one. Its dominating shortest scale is completely determined by t_{D_1} and all other longer scales do not equilibrate according to the theory. Second, just at H_1, change the form of driving to $H_{{\rm st}_2}$. Therefore $\{H_1,\mathcal{P}_1\}$ serves as the nonequilibrium initial condition of the second step.

If the dominant time scale of the second step $t_{D_2} \ll t_{D_1}$, then $|H_1| t_{D_2}^{\beta\delta/\nu z} \ll |H_1| t_{D_1}^{\beta\delta/\nu z} \ll 1$, so that H_1 falls also inside the FTS regime of t_{D_2}. The condition of a longer initial scale t_{D_1} than t_{D_2}, the scale to emerge, albeit both nonequilibrium, eliminates the initial slip of the increase of M too. The scaling form of the second step is thus

$$M = t_{D_1}^{-\beta/\nu z} f_{{\rm st}_2}\big(H t_{D_1}^{\beta\delta/\nu z}, H_1 t_{D_1}^{\beta\delta/\nu z}, t_{D_2}/t_{D_1}\big), \qquad (20)$$

which is dominated by the initial state characterized by t_{D_1} rather than by t_{D_2} from the driving at work, where we have dropped all other possible scales.

IV. SPECIFIC FORMS OF DRIVING

We now apply the results from the last two sections to some specific examples of driving. First, we shall consider a monomial driving in Sec. IV A, which is simple and heuristic. Then, we turn to polynomial cases in Sec. IV B, whose results are useful to analyze driving with more complicated forms. Right after that, we shall use the polynomial results as a method to discuss a sinusoidal driving in Sec. IV C. Finally, the Gaussian approximation of the above examples will be given in Sec. IV D as an appreciation of the various timescales associated with a driving. We shall denote the parameters of the driving for an nth order monomial by R_n and the amplitude and the frequency of a sinusoidal form by A and Ω to differentiate them from the general parameters B_i.

A. Monomial driving

Consider a driving in a monomial form [58,70,76,87]

$$H(t,R_n) = R_n t^n. \tag{21}$$

Without loss of generality, we assume $R_n > 0$ in order to simplify the following expressions. Note that the critical point lies exactly at $t = 0$ and $H = 0$.

Substituting Eq. (21) into Eq. (13), we have

$$\gamma^*_{R_n} = -n\varsigma^* + \gamma^*_\phi/2. \tag{22}$$

Using Eq. (8) and the naïve dimension of R_n [71], $d_{R_n} = (d+2)/2 + 2n$, which is the difference between the dimensions of H and t^n, one finds

$$r_n = d_{R_n} - \gamma^*_{R_n} = \beta\delta/\nu + nz \tag{23}$$

and thus $t_{R_n} \sim R_n^{-z/r_n}$ from Eq. (15).

We can also reach Eq. (23) from the scale transforms of H and t similar to the FTS for linearly varying field [70,71]. After a scale transform, $H' = Hb^{\beta\delta/\nu}$ and $t' = tb^{-z}$ from Eq. (7). The definition of r_n in Eq. (11), viz., $R'_n = R_n b^{r_n}$, and Eq. (21) then result directly in Eq. (23), since Eq. (21) is also valid when coarse grained, which can also be regarded as a definition of R'_n.

From Eq. (23), for R_n to be relevant, $r_n > 0$, i.e., $n > -\beta\delta/\nu z$. When $n = 1$, the FTS for a linearly varying field is recovered [70,71].

Since R_n is the only parameter of H, there is only one driving time scale and its FTS form reads

$$M = R_n^{\beta/\nu r_n} f_n^t\left(t R_n^{z/r_n}\right) \tag{24}$$

directly from Eq. (16), or,

$$M = R_n^{\beta/\nu r_n} f_n^H\left(H R_n^{-\beta\delta/\nu r_n}\right), \tag{25}$$

where we have simplified the subscripts.

There exists a unique singularity stemming from the peculiar property of the scaling functions for a nonlinear driving. Although $t R_n^{z/r_n} = (H R_n^{-\beta\delta/\nu r_n})^{1/n}$, one usually does not care for the exponent and freely applies either Eq. (24) or Eq. (25) to describe the process, believing that no new singularity will occur except for the confluent ones [98]. This is not true for $n > 1$, however. To see this, note that in general, a scaling function can be Taylor expanded near a critical point. We find, however, that only the expansion of f_n^t can describe M near the critical point, whereas that of f_n^H cannot. This is a manifest of nonequilibrium. It may arise from the fact that the evolution is with the time but not with the field and thus the RG equation for t is better than that for H in this case. Yet, substituting $t = (H/R_n)^{1/n}$ into the expansion of the former works well for the latter. This indicates that f_n^H is singular at $H = 0$ for $n > 1$.

Moreover, the singularity of f_n^H leads to a new leading singularity for the susceptibility χ at $H = 0$, which is

$$\chi = R_n^{-\frac{\gamma}{\nu r_n}}\left[\frac{a_1^{\gtrless}}{n}\left(H R_n^{-\frac{\beta\delta}{\nu r_n}}\right)^{1/n-1} + \dots\right], \tag{26}$$

where $\gamma = \beta(\delta - 1)$, a_1 is an expansion coefficient of f_n^t independent on both R_n and H, and the superscripts $\gtrless$ represent the expansions for $t > 0$ and $t < 0$, respectively. When $n > 1$, χ diverges at the critical point $H = 0$ and $R_n = 0$ as $H^{(n-1)/n}$, even in the FTS regime, though it collapses well for different R_n in the plane of $\chi R_n^{\gamma/\nu r_n}$ versus $H R_n^{-\beta\delta/\nu r_n}$ as Eq. (26) indicates. Note that in equilibrium, $\chi \sim H^{-\gamma/\beta\delta}$ near $H = 0$ but changes to the present nonequilibrium one once R_n is finite, in which case the transition occurs near $H R_n^{-\beta\delta/\nu r_n} \sim 1$ rather than at $H = 0$, showing hysteresis. So, the singularity is all due to the driving as only n is involved. In addition, the leading singularity of χ also turns into $R_n^{-\beta(\delta-n)/n\nu r_n}$, whose exponent changes sign for $n > \delta$.

We now briefly discuss the relation [72,80] with the KZ theory [31–36]. The theory considers only cooling in classical transitions and the parameters that play the role of temperature in quantum transitions. It comprises of the KZ mechanism and the KZ scaling that are different but interwoven. The former is a mechanism to produce topological defects. Its essence is a finite frozen correlation length $\hat{\xi}$, with which two regions of about $2\hat{\xi}$ apart are causally independent and their boundary can then be a topological defect. To estimate $\hat{\xi}$, one divides the cooling process into two adiabatic regimes and a nonadiabatic impulse regime in between that contains the critical point. By assuming evolutionless in the impulse regime, its boundary with the adiabatic regime then gives rise to the maximally possible correlation length and thus $\hat{\xi}$. When "the remaining time until the transition" [33] $\hat{t} = \hat{\tau}/R = t_{\rm eq}$ for a linear cooling with a rate R, one thus finds $\hat{t} \sim R^{-\nu z/(1+\nu z)}$ and $\hat{\xi} \sim |\hat{\tau}|^{-\nu z} \sim R^{-\nu/(1+\nu z)}$ and thus follows the KZ scaling for the density of the topological defects. Within the present theory, one considers a specific variable, the correlation length ξ, whose FTS form in the absence of H is [80,86]

$$\xi = R^{-1/r} f_\xi(\tau R^{-1/r\nu}) = \hat{\xi} f_\xi(\tau R^{-1/r\nu}), \tag{27}$$

similar to Eq. (25) with a similar $r = z + 1/\nu$ [70,71,88] and a numerical factor has been ignored between the two scaling functions f_ξ. One sees that ξ is $\hat{\xi}$, the driving length corresponding to the time scale $\hat{t}$, only (except the critical point $\tau = 0$) at the boundary (up to a constant multiplier) corresponding to $\hat{t}$ and $\hat{\tau}$ at which the FTS regime crosses over to the adiabatic regime in which

$$\xi = |\tau|^{-\nu} f'_\xi(R|\tau|^{-r\nu}), \tag{28}$$

similar to Eq. (9). One sees therefore that the defect density is only one special, but possibly difficult to be quantified [68], observable and the boundary is only one special locus within the present theory, which describes all observables in the whole cooling process including the impulse regime.

Moreover, when a system is driven from near—rather than far off—its critical point with a nonequilibrium initial condition, the adiabaticity to reckon the KZ scaling in the KZ theory breaks from the beginning as no initial adiabatic stage exists at all. We have shown, however, that the KZ mechanism and the KZ scaling still work when FTS is combined with the critical initial slip [86].

If we include other parameters such τ and L in Eq. (24), we can have crossovers to other regimes. However, as pointed out in Sec. II B, we shall not consider them further.

The monomial driving with $n \neq 1$ can improve the adiabaticity of a transition [99], but requires a better experimental control since it has to be nonlinear exactly at the critical point [35]. In the following, we shall see that polynomial forms can be a better approximation.

B. Polynomial driving

Suppose a quadratic driving has small deviations δt and δh from the zero point, which usually can not be avoided in experiment. Then

$$H = R_2(t + \delta t)^2 + \delta h. \tag{29}$$

For such a driving, we can expand it about the critical point at $H = 0$, which lead to a polynomial driving

$$H = R_2 t'^2 + R_1 t', \tag{30}$$

where $t' = t - t_o$, $H(t_o) = 0$, and R_1 is the coefficient of the linear term. Moreover, generally such a driving crosses or approaches $H = 0$ several times. In the following, we shall study a driving that crosses $H = 0$ only once and multiple times separately.

1. Single trans-critical driving

We first discuss the case in which a driving crosses $H = 0$ only once. The dominant time scale in such a polynomial driving turns out to be quite simple; it is just among all the t_{R_i}. So, all we need is to compare and find the smallest of them. To be concrete, consider

$$H(R_1, R_3, t) = R_1 t + R_3 t^3 \tag{31}$$

with positive R_1 and R_3 without loss of generality to ensure $t = 0$ is the only real solution at the critical point $H = 0$. Equation (13) gives

$$R_1 t\left(\varsigma^* + \lambda^*_{R_1} - \frac{\gamma^*}{2}\right) + R_3 t^3\left(3\varsigma^* + \lambda^*_{R_3} - \frac{\gamma^*}{2}\right) = 0. \tag{32}$$

Consequently, Eqs. (22) and (23) for $n = 1$ and 3 are the solutions to $\lambda^*_{R_1}$ and $\lambda^*_{R_3}$ and hence r_1 and r_3, respectively. These results are natural from the direct method of scale transforms as both monomial terms scale with the magnetic field.

As there are two parameters, one has two time scales and two different FTS regimes. Their scaling forms are

$$M = R_1^{\beta/\nu r_1} f_{R_1}\left(t R_1^{z/r_1}, R_3 R_1^{-r_3/r_1}\right) \tag{33}$$

or

$$M = R_3^{\beta/\nu r_3} f_{R_3}\left(t R_3^{z/r_3}, R_1 R_3^{-r_1/r_3}\right) \tag{34}$$

from Eq. (16). The first form describes the regime when $R_3 R_1^{-r_3/r_1} \ll 1$, or, $t_{R_1} \ll t_{R_3}$, i.e., t_{R_1} dominates in the critical region. By contrast, when $R_3 R_1^{-r_3/r_1} \gg 1$, or, $t_{R_1} \gg t_{R_3}$, the scaling function $f_{R_1} \sim (R_3 R_1^{-r_3/r_1})^{\beta/\nu r_3}$ and crosses over to Eq. (34) dominated by t_{R_3}. In general, the power n_i of each term in the polynomial driving need not be an integer, as in the monomial case, but it must satisfy $n_i > -\beta\delta/\nu z$ to keep R_{n_i} relevant.

2. Multi-trans-critical driving

Here we focus on the case in which the process crosses the critical point $H = 0$ several times. Consider

$$H = R_3 t^3 - R_1 t \tag{35}$$

with $R_3 > 0$ and $R_1 > 0$ for simplicity. The driving changes direction twice at $t_{v_\pm} = \pm\sqrt{R_1/3R_3}$ and crosses the critical point at three instants: $t_0 = 0$ and $t_\pm = \pm\sqrt{R_1/R_3}$, see Fig. 1(a). Accordingly, we can divide the process into three parts, 1, $t = (-\infty, t_{v_-}]$, 2, $t = (t_{v_-}, t_{v_+}]$, and 3, $t = (t_{v_+}, +\infty)$, such that H crosses $H = 0$ only once in each part. Figure 1(b) shows the three divergent peaks and the two valleys of t_H, Eq. (14). If the dominant driving time scale $t_D \gg t_{H_{v_\pm}}$, the time scale at the valleys, the system can equilibrate near $t_{v_\pm}$ and thus the three parts can be treated separately. Conversely, when $t_D \ll t_{H_{v_\pm}}$, it stays in the FTS regime even at the valleys and thus the initial conditions are important.

To be specific, we now expand the driving field about each $H = 0$. Let us start from $t < 0$ outside the FTS regime. The initial condition then plays no role for the first part. Near t_-, let $t' = t - t_-$,

$$H = R_3 t'^3 - 3\sqrt{R_1 R_3} t'^2 + 2R_1 t'. \tag{36}$$

A quadratic term $R_2 \equiv 3\sqrt{R_1 R_3}$ emerges. As $t_{R_2}/t_{R_1} \sim (R_1 R_3^{-r_1/r_3})^{r_3/2r_2}$ and $t_{R_3}/t_{R_2} \sim (R_1 R_3^{-r_1/r_3})^{1/2}$, if $R_1 R_3^{-r_1/r_3} \gg 1$, $t_{R_1} \ll t_{R_2} \ll t_{R_3}$ and R_1 dominates; if $R_1 R_3^{-r_1/r_3} \ll 1$ to the contrary, the relation among the three time scales reverses and

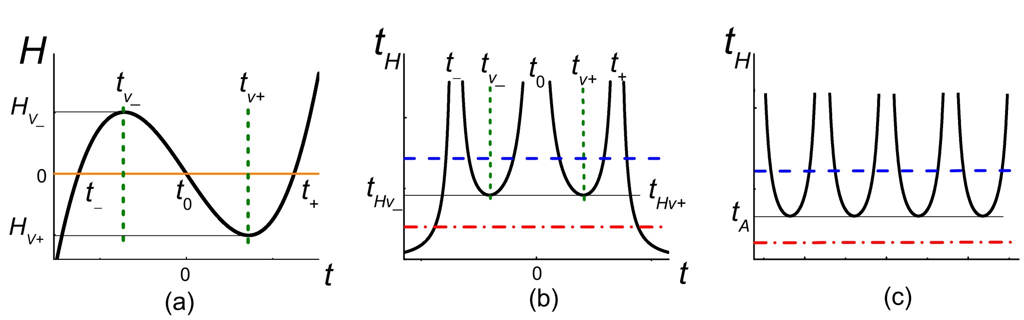

FIG. 1. (a) Generic $H = R_3 t^3 - R_1 t$ curve. (b) Schematic picture of its t_H and other time scales vs t. (c) Schematic picture of t_H and other time scales vs t for the sinusoidal driving $H = A \sin \Omega t$. The red dash-dot lines correspond to the case in which the initial state is dominant, while the blue dash lines to the case in which equilibrium can be achieved. See the text for details.

R_3 rules the game. Therefore the new generated term does not overwhelm the existing ones.

For $t_0 = 0$, no expansion is necessary but the initial condition can be important. When $R_1 R_3^{-r_1/r_3} \gg 1$, R_1 again dominates. So, $t_{R_1}/t_{H_{v_\pm}} \sim (R_1 R_3^{-r_1/r_3})^{\nu z r_3/2\beta\delta r_1} \gg 1$ and thus $t_{R_1} \gg t_{H_{v_\pm}}$. Therefore t_{v_-} locates outside the FTS regime and part 2 can be treated separately. The FTS form in this case is given by Eq. (33). When $R_1 R_3^{-r_1/r_3} \ll 1$ and R_3 dominates, in contrast, the initial state is important. From the discussion of t_-, the initial state is also dominated by t_{R_3}. So the FTS form in this case should be described by a form similar to Eq. (20). However, as t_{v_-} is also determined by R_1 and R_3, $t_{v_-} R_3^{z/r_3}$ can be reduced to $R_1 R_3^{-r_1/r_3}$. Accordingly, the scaling form now resembles Eq. (34), with the R_3 factor stemming from the initial state from t_-. Similar analysis can be applied to the third part t_+.

We can also understand roughly the effects of the initial condition from t_0, $t_\pm$, and $t_{v_\pm}$ themselves. For $R_1 R_3^{-r_1/r_3} \gg 1$, they are far apart and so there is sufficient time for the system to equilibrate. As a result, the initial condition is irrelevant. While for $R_1 R_3^{-r_1/r_3} \ll 1$, they are close to each other and the initial condition has to be taken into account.

C. Sinusoidal driving

Sinusoidal driving is widely used in experimental and theoretical studies [15]. The periodicity of the drive changes the Hamiltonian of the system from the Floquet formalism [100,101], which is the time domain analog of the Bloch theorem. For strong driving at high frequencies, this can result in coherent destruction of tunneling for driven quantum tunneling [15] and similar phenomena in driven quantum phase transitions [102–112]. The changed Hamiltonian can even lead to Floquet topological insulators [113]. Sinusoidal driving has also been applied to classical continuous [114] and first-order phase transitions [115–120]. A dynamic phase transition was reported in the kinetic Ising model under a time-dependant oscillating field [115]. This may be another example of the resonance interaction. In addition, sinusoidal functions are fundamental in Fourier analysis, so it is instructive to study sinusoidal driving. Here we shall study the low-energy universal critical properties of classical continuous phase transitions under a sinusoidal driving by the theory developed in Sec. II without considering the Floquet formalism. The results are expected to apply to quantum phase transitions as well for low frequencies and crossing the critical point only once, since the KZ scaling has been found to be applicable in this case [105,106,109]. Whether or not the interference found in the quantum case for multiple crossing has any effect in the classical case is left for future study.

Consider

$$H(A,\Omega,t) = A \sin \Omega t, \qquad (37)$$

where A is the amplitude and Ω is the angular frequency of the driving. The RG result, Eq. (13), reads

$$(\varsigma^* + \lambda_\Omega^*) A\Omega t \cos \Omega t + \left(\lambda_A^* - \frac{\gamma^*}{2}\right) A \sin \Omega t = 0, \qquad (38)$$

which indicates that $\lambda_\Omega^* = -\varsigma^*$ and $\lambda_A^* = \gamma^*/2$ and thus r_Ω and r_A are simply z and $\beta\delta/\nu$, respectively. This is obvious because A must transform as H and Ω as t^{-1} from Eq. (37). Accordingly,

$$t_A \sim A^{-\nu z/\beta\delta}, \qquad t_\Omega \sim \Omega^{-1}, \qquad (39)$$

from Eq. (15).

The scaling forms for Ω and A dominating are, respectively,

$$M(t,A,\Omega) = \Omega^{\beta/\nu z} f_\Omega(\Omega t, A\Omega^{-\beta\delta/\nu z}), \qquad (40)$$

$$M(H,A,\Omega) = A^{1/\delta} f_A(HA^{-1}, \Omega A^{-\nu z/\beta\delta}), \qquad (41)$$

since $t_\Omega \ll t_A$ or $A\Omega^{-\beta\delta/\nu z} \ll 1$ for the former and vice versa for the latter. We can of course choose H and t as their respective parameters.

However, there exists yet another way to study the driving. We can expand the driving near the critical point at $t = 0$ and $H = 0$ and utilize the theory for the polynomial in Sec. IV B. When the first-order term dominates, we find the scaling form

$$M = (A\Omega)^{\beta/\nu r_1} f_{A\Omega}(H(A\Omega)^{-\beta\delta/\nu r_1}, \Omega A^{-\nu z/\beta\delta}), \qquad (42)$$

because all the rescaled higher-order terms can simply be reduced to $A^{-1}\Omega^{\beta\delta/\nu z}$. To be consistent, $t_{A\Omega} \sim (A\Omega)^{-z/r_1} \ll t_H$ and $\Omega A^{-\nu z/\beta\delta} \ll 1$. One sees therefore that there is a new dominating timescale $t_{A\Omega}$ determined by two parameters, whose combination cannot be identified directly from the field itself!

In order to reveal the relationship among Eqs. (40) to (42), we notice that, using Eq. (39),

$$\frac{t_{A\Omega}}{t_A} \sim (A\Omega^{-\frac{\beta\delta}{\nu z}})^{\frac{\nu z^2}{\beta\delta r_1}}, \quad \frac{t_{A\Omega}}{t_\Omega} \sim (\Omega A^{-\frac{\nu z}{\beta\delta}})^{\frac{\beta\delta}{\nu z}}. \qquad (43)$$

Therefore, for $\Omega A^{-\nu z/\beta\delta} \ll 1$, $t_A \ll t_{A\Omega} \ll t_\Omega$, while for $\Omega A^{-\nu z/\beta\delta} \gg 1$, $t_\Omega \ll t_{A\Omega} \ll t_A$. This appears to indicate that t_A would dominate the former regime and t_Ω the latter. In other words, no regime dominated by $t_{A\Omega}$ would emerge. In fact, it is $t_{A\Omega}$ that dominates the former regime and the initial state that governs the latter. t_A is only transient and never dominant.

To see this, note that t_A is just the minima of t_H from Fig. 1(c). Accordingly, it is only a transient time scale in the sense that it only appears at the instant when H assumes its maxima or minima and thus cannot be a constantly existing dominating time scale. Therefore, for $\Omega A^{-\nu z/\beta\delta} \ll 1$, although $t_{A\Omega} \gg t_A$ from Eq. (43), corresponding to the blue dashed line in Fig. 1(c), $t_{A\Omega}$ is the dominant scale. In addition, the system can equilibrate at the valleys and the hysteresis loops are saturate. If $\Omega A^{-\nu z/\beta\delta} \gg 1$ to the contrary, although t_Ω is the shortest, the system always stays in the FTS regime and the initial condition dominates; only after the initial state decays away can t_Ω take over and dominate. Moreover, the higher order terms of the field expansion are larger than the linear term, or the time scales corresponding to the higher order terms are shorter than that of the linear term, and the expansion method is invalid.

As the initial state dominates for $\Omega A^{-\nu z/\beta\delta} \gg 1$, the hysteresis loops become unsaturate. If we start the process in equilibrium at $H_{\rm in} = -A$ and $\mathcal{P}_{\rm in} = \mathcal{P}_{\rm eq}(-A)$, the scaling in this regime is described by

$$M = A^{1/\delta} f_{H_{\rm in}=-A}(HA^{-1}, \Omega^{-1} A^{\nu z/\beta\delta}), \qquad (44)$$

where the subscript differentiates it from Eq. (41), as Eq. (44) is a special case of Eq. (19) for a specific initial condition used. It is valid for a small A and a large Ω, in opposite to Eq. (41), which never dominates.

D. Gaussian approximation

As an appreciation of the various timescales associated with a driving and a verification of the above results, in this section, we consider the Gaussian approximation of the model (1) and (2).

In this approximation, the model simplifies to

$$\frac{\partial\langle\phi(k,t)\rangle}{\partial t} = -\lambda[(\tau + k^2)\langle\phi(k,t)\rangle - H(t)\delta(\mathbf{k})] \tag{45}$$

in the wave number $\mathbf{k}$ space, where δ here is the Dirac delta function. The solution is

$$M(t) = M_{\rm in}e^{-\tau\lambda(t-t_{\rm in})} - g(t_{\rm in})e^{-\tau\lambda(t-t_{\rm in})} + g(t), \tag{46}$$

where $M_{\rm in}$ is the initial uniform magnetization. The first two terms in Eq. (46) are the contributions of initial state. They decay exponentially. The third term is the effect of driving and depends on its detail. If $t - t_{\rm in} \simeq 0$, the last two terms nearly cancel out and the first term dominates. This means the time is too short for the effect of driving to be significant and the result mainly reflects the property of $M_{\rm in}$. By contrast, when $t - t_{\rm in} \gg 0$, only the driving term survives.

For the polynomial form of driving (31) but with all t replaced by λt in order to yield a correct time unit, one finds

$$g(\lambda t) = R_3(\lambda t)^3/\tau - 3R_3(\lambda t)^2/\tau^2 + (R_1/\tau + 6R_3/\tau^3)\lambda t - (R_1/\tau^2 + 6R_3/\tau^4). \tag{47}$$

The critical exponents of the Gaussian model [19–21]

$$\nu = \frac{1}{2},\ z = 2,\ \beta = \frac{d-2}{4},\ \text{and}\ \delta = \frac{d+2}{d-2} \tag{48}$$

give rise to

$$r_1 = (d+6)/2, \qquad r_3 = (d+14)/2, \tag{49}$$

from Eq. (23). As a result, for the parameters chosen,

$$t_{\rm eq} = (\lambda\tau)^{-1}, \qquad \lambda t_{R_1} = R^{-\frac{4}{d+6}}, \qquad \lambda t_{R_3} = R^{-\frac{4}{d+14}}. \tag{50}$$

These then turn Eq. (46) into

$$M = t_{R_3}^{-(d-2)/4} f_{G3}\left(t_{\rm eq}/t_{R_3}, t/t_{R_3}, t_{R_1}/t_{R_3}\right), \tag{51}$$

where $f_{G3}(X,Y,Z) = XY^3 - 3X^2Y^2 + (Z^{-2}X + 6X^3)Y - (Z^{-2}X^2 + 6X^4)$. Equation (51) has the same form as Eq. (17) and verifies it. We can of course rescale all time scales by t_{R_1}.

In the case of sinusoidal driving,

$$g(\lambda t) = \frac{A\sin(\Omega\lambda t - \theta)}{\sqrt{\Omega^2+\tau^2}} = \frac{At_{\rm eq}\sin(t/t_\Omega - \theta)}{\sqrt{1 + t_{\rm eq}^2/t_\Omega^2}}, \tag{52}$$

where $\theta = \arctan(\Omega/\tau)$. When $\Omega/\tau = t_{\rm eq}/t_\Omega \ll 1$, the period is significantly longer than the correlation time. So, t_Ω is not relevant and M will saturate at some time in the process. In the region $t \simeq 0$, Eq. (52) can be approximated to be $M \simeq (A\Omega)^{\beta/\nu r_1}(t_{\rm eq}/t_{A\Omega})(t/t_{A\Omega})$, consistent with Eq. (42) in which $t_{A\Omega}$ dominates using Eqs. (48) and (49).

When $\Omega/\tau \gg 1$, the situation reverses and the system can not equilibrate during the whole process, which corresponds to the unsaturated case. In this case, $\theta \simeq \pi/2$, Eq. (52) approximates to $M \simeq A/\Omega = A^{1/\delta}(\Omega^{-1}A^{\nu z/\beta\delta})$ in agreement with Eq. (44) near $t \simeq 0$ using Eq. (48). One sees that no regime is controlled by t_A, consistent with the theory.

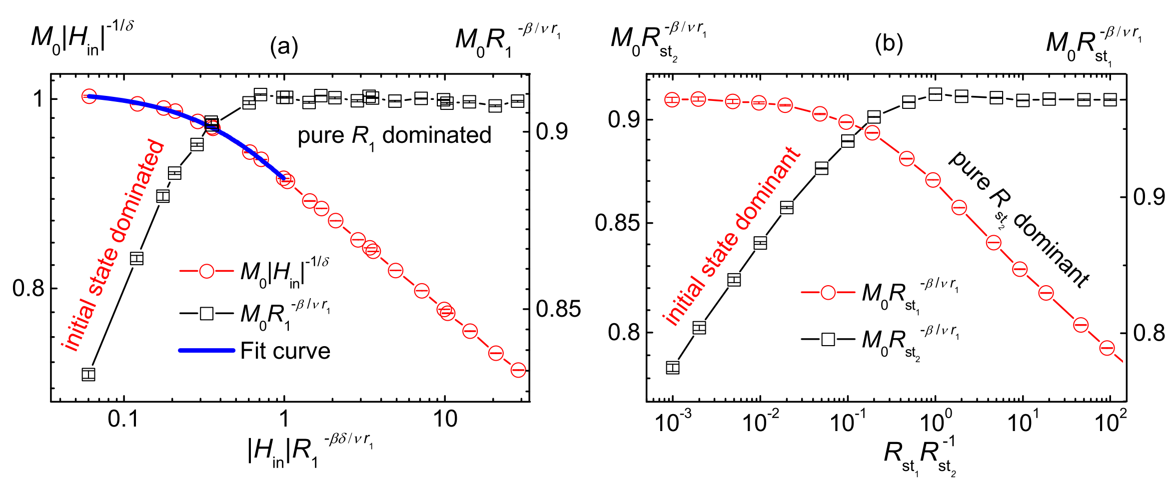

FIG. 2. Effects of (a) equilibrium initial conditions and (b) nonequilibrium initial conditions of the stepwise linear driving with $t_1R_{\rm st_1}^{z/r_1} = 0.5$. On the right-hand side of each panel, the black curves are flat showing the irrelevance of the initial state; while the red curves have slopes of $-0.0674(3)$ in (a) and $-0.0294(3)$ in (b), consistent with the theoretic value $-1/\delta = -0.0666$ and $-\beta/\nu r_1 = -0.0309$, respectively. On the left-hand side, the black curves depend on the initial conditions and the red curves tend to be flat, showing clearly the existence of the initial state dominated regime. Correspondingly, the slopes of the black curves become 0.0538(1) for the leftmost three data in (a) and 0.0311(2) in (b), close to the theoretical values $1/\delta$ and $\beta/\nu r_1$, respectively. The thick blue curve in (a) is a fit to the expansion of the scaling function in Eq. (19) to order 2. Note the different scales of the right and the left vertical axes. Thin lines connecting symbols are only a guide to the eye.

V. IMPLICATION TO MEASUREMENT

We discuss a possible implication of our results to experiments here. Experimentally, one often applies a weak driving field to study the property of a system [114,121]. For example, in Ref. [121], the authors measured the linear conductivity between 3 Hz to 3 MHz, and found a dynamic scaling near the vortex-glass transition. In [114], the correlations of order parameters are related to light scattering intensities. In both cases, the amplitude of the driving is ignored in the scaling function by assuming that the amplitude is small. In theories, the linear response [8], for example, one also applies a weak external field to compute the response of a system. However, the field is sent to zero after the computation. In experiments, however, the field is always there no matter how small it is. This usually incurs only a small perturbation. However, near the critical point where correlations are long ranged, it may be problematic. Recall that in Eqs. (40) to (42), there exists an additional term containing both A and Ω. Upon omitting this term, the method to obtain critical properties from data collapses, as was done in [121], is thus presumably flawed. A naïve way to overcome this from the theory is to vary A with Ω in a such way that $\Omega A^{-\nu z/\beta\delta}$ is fixed.

The problem can also be seen more fundamentally from the fluctuation-dissipation theorem (FDT) [8], which is

$$\chi \equiv (\partial M/\partial H)_T = C/T \equiv L^d \langle(\phi - \langle\phi\rangle)^2\rangle/T \qquad (53)$$

here. It enables one to measure the equilibrium correlation C via the response of a system to its external probes, the susceptibility χ. We shall see in the following unambiguously nonequilibrium behavior and violation of FDT for a weak driving near a critical point. Therefore one cannot obtain accurate correlation functions by measuring the responses and vice versa near the critical point even for a vanishingly small A! Nonequilibrium still, we shall find that the scaling law holds between the critical exponents of χ and C. Accordingly, one can still employ C or χ to estimate the critical exponents as in equilibrium with due attention to the effect of the amplitude.

VI. MODEL AND METHOD

To verify our results, we study the classical $d = 2$ Ising model with nearest-neighbor interaction

$$\mathcal{H} = -J \sum_{\langle i,j\rangle} s_i s_j - H \sum_i s_i, \qquad (54)$$

where $J > 0$ is a coupling constant and $s_i = \pm 1$ is the spin at site i. Periodic boundary conditions are applied throughout. The critical temperature and critical exponents are known exactly: $T_c = 2J/\ln(1+\sqrt{2})$, $\beta = 1/8$, $\delta = 15$, and $\nu = 1$, while the dynamic exponent is chosen as $z = 2.1667$ [122]. We shall mainly study the order parameter defined as $M = \langle\sum_i s_i\rangle/N$ for the N spins as usual, since topological defects, domain walls between regions of up and down spins, are difficult to count especially at finite temperatures at which thermal fluctuations are violent. Moreover, as pointed out in Sec. IV A, they must follow similar scaling.

We use MC with a single site METROPOLIS algorithm [123]. To minimize the finite size effect, the minimum lattice size chosen is 512×512. The sample sizes are between 500 to 3000, resulting in small relative errors to be seen in the error bars displayed. To reduce variables in the scaling functions, all simulations are performed at $T = T_c$, and $M_0 \equiv |M(H=0)|$ is frequently used.

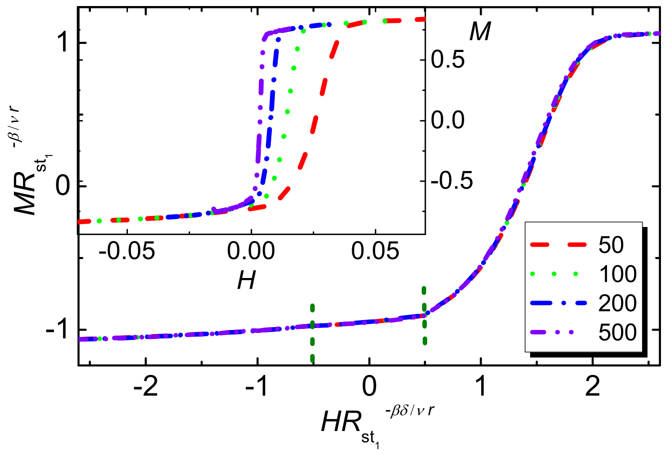

FIG. 3. Magnetization of the three-step piecewise linear driving. $t_1 R_{st_1}^{z/r_1} = 0.5$ and $R_{st_1}/R_{st_2} = 0.2$. These choices yield $H_1 R_{st_2}^{-\beta\delta/\nu r_1} = 0.2370 < 1$ and so H_1 lies inside the FTS regime of R_{st_2}. The numbers list the values of t_1. The two vertical dashed lines demarcate the three steps. Inset: Original curves before rescaled.

VII. NUMERICAL RESULTS

A. Initial condition

Figure 2(a) shows M_0 of a linear driving starting from an equilibrium initial condition at different $|H_{\rm in}|R_1^{-\beta\delta/\nu z}$ for $H_{\rm in} < 0$. One sees clearly a crossover from an R_1 dominated regime described by Eq. (18) in the absence of V to an initial-state dominated regime described by Eq. (19) with $t_D = t_{R_1}$. Although there is a 19% relative error in the slope of the initial-state dominated regime due to the few data, the good

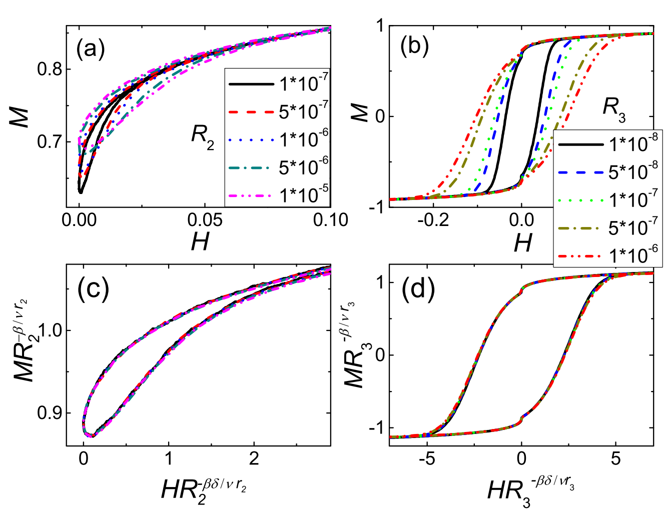

FIG. 4. Hysteresis loops for $H = R_2 t^2$ (left column) and $H = R_3 t^3$ (right column) for various listed R_2 and R_3, respectively. The original loops (top) collapse well after rescaled (bottom), verifying Eq. (25).

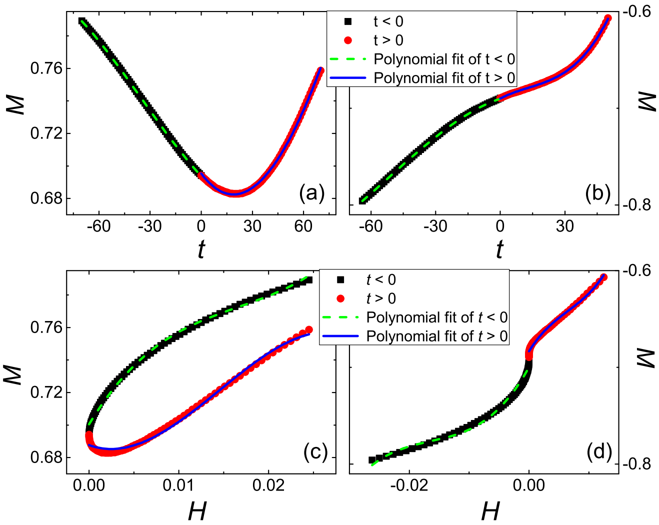

FIG. 5. M vs t (top) and M vs H (bottom) for $H = R_n t^n$ with $n = 2$ (left) and $n = 3$ (right), respectively. The data are chosen to satisfy $|t|R_n^{z/r_n} < 1$ near the critical point and are fitted to polynomials up to order 3 for $t > 0$ and $t < 0$, respectively. The goodnesses of the fits in (a) and (b) confirms the analyticity of f_n^t in Eq. (24).

fit to Eq. (19) both confirms the scaling and demonstrates the analyticity of the scaling function. These results show convincingly the effects of the initial condition and the validity of the theory.

Figure 3 shows the results of the piecewise driving for the nonequilibrium initial condition. A three-step linear driving is simulated. It starts at $H_{\rm in} < -\hat{H}$ of the first driving. For simplicity, we choose $R_{\rm st_3} = R_{\rm st_1}$ and stop the second step at $H_2 = -H_1 > 0$. We also set $H_1 = -R_{\rm st_1} t_1$. As a result, the four free parameters of the initial conditions, t_1, t_2, $R_{\rm st_1}/R_{\rm st_2}$, and $R_{\rm st_2}/R_{\rm st_3}$, are reduced to t_1 and $R_{\rm st_1}/R_{\rm st_2}$. In Fig. 3, H_1 is chosen to fall inside the FTS regime of the second driving, but then H_2 lies outside. Also $t_{R_{\rm st_1}} > t_{R_{\rm st_2}}$. Accordingly, the second stage is described by Eq. (19), while the other two by Eq. (25), though all three stages are rescaled by $R_{\rm st_1}$. The good collapses show the good applicability of FTS to this case.

The nonequilibrium initial conditions dominated regime and its crossover for the driving are displayed in Fig. 2(b). Here, we vary the value of $R_{\rm st_1}/R_{\rm st_2}$, which also changes $H_1 R_{\rm st_2}^{-\beta\delta/\nu r_1}$. It is clear that the initial state is dominant for $R_{\rm st_1}/R_{\rm st_2} \ll 1$, but irrelevant to the opposite. A crossover appears near $R_{\rm st_1}/R_{\rm st_2} \sim 1$. The results show remarkably that, first, the single driving scale does determine all correlations of the system in the FTS regime, and second, in the nonequilibrium initial state dominated regime, all correlations of the initial state still evolve in a concerted way as if the previous driving were still in effect.

B. Monomial

In Fig. 4, we verify Eq. (25) for $n = 2$ and 3. Although the forms of the driving are qualitatively different as $n = 2$ is even but $n = 3$ is odd, they both obey the theory.

In Fig. 5, we investigate the behavior of f_n in Eqs. (24) and (25) near $H = 0$ and $t = 0$ for $n = 2$ and 3. One sees that the polynomial fits are very good in (a) and (b) but deviate significantly from the data in (c) and (d). In the latter case, we have found that fits to polynomials up to order 9 and far

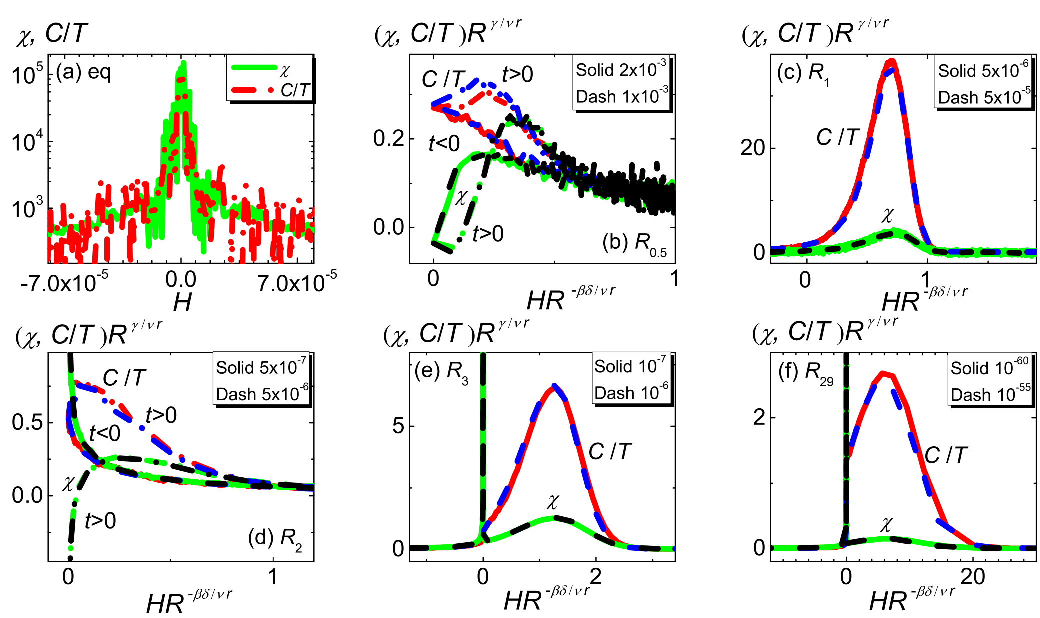

FIG. 6. (a) χ and C/T vs H at equilibrium and (b) to (f) their rescaled for the driving $H = R_n t^n$ with (b) $n = 0.5$, (c) 1, (d) 2, (e) 3, and (f) 29, respectively. Note that subscripts are absent in R and r of the axis titles. In (b) to (f), the red and blue curves, marked with C/T, are the rescaled curves $CR^{-\gamma/\nu r}/T$, while the green and black curves, marked with χ, are the rescaled curves $\chi R^{-\gamma/\nu r}$. The solid and dashed curves represent the different rates listed. In (b) and (d), the solid and dashed curves correspond to $t < 0$, while the dash-dotted curves to $t > 0$ as indicated. Note that χ is negative for t close to 0^+. As predicted by Eq. (26), χ vanishes, is finite, and diverges at $H = 0$ for $n < 1$ (b), $n = 1$ (c), and $n > 1$ (d) to (f), respectively. The collapses of C/T are not as good as χ due to large fluctuations. The two curves in (a) are averaged over 10 000 000 MC steps.

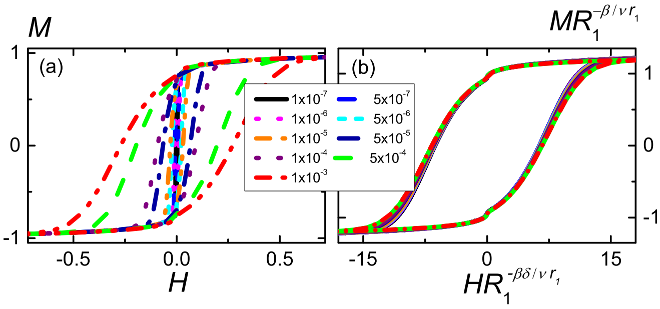

FIG. 7. Hysteresis loops for $H = R_1 t + R_3 t^3$ with $R_1 R_3^{-r_1/r_3} = 0.1$. The legend lists R_1 used. (a) is original data and (b) is rescaled by R_1. The collapses get somehow poor when the absolute values of horizontal axis are large, possibly due to corrections to scaling.

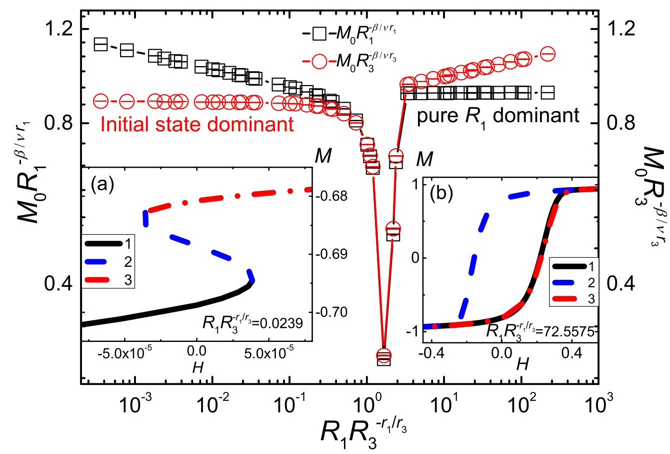

FIG. 9. Different regimes and their crossover for $H = R_3 t^3 - R_1 t$ near t_0. Black and red curves are the same data rescaled by R_1 and R_3, respectively. When $R_1 R_3^{-r_1/r_3} \gg 1$, the process can be treated as a pure R_1 driving. As a result, the black curve is flat, whereas the slope of the red curve is 0.0310(5), in agreement with the theoretical value of $\beta/\nu r_1$. When $R_1 R_3^{-r_1/r_3} \ll 1$, the red curve becomes flat, while the black curve has a slope $-0.0320(2)$, consistent again with $-\beta/\nu r_1$ from Eqs. (33) and (34). Lines connecting symbols are only a guide to the eye. Insets: Generic M vs H curves in the two corresponding regimes. Black solid, blue dashed, and red dash-dotted curves represent the 1, 2, and 3 parts of the process, respectively. In (a), the blue curve begins at $H > 0$ and $M < 0$, which is evidently nonequilibrium. In (b), the curves of the first (black) and the third (red) parts nearly coincide.

smaller ranges are also poor and thus invalid the analyticity of f_n^H in Eq. (25). This indicates that f_n behaves analytically with respect to t but not to H near the critical point.

Figure 6 shows the rescaled curves of χ and C/T for two different rates. The extra singularity of χ at $H = 0$ as exemplified in Eq. (26) is seen as spikes in (e) and (f). However, C exhibits no such peaks. Note that χ is negative in (b) and (d) for t close to 0^+. In equilibrium, however, χ is related via the FDT (53) to C, which is non-negative. So the negative χ implies that the system cannot follow the driving and is definitely out of equilibrium. It can also be seen that χ and C/T separate significantly near the peaks for all forms of driving shown. This violation of the FDT is again manifestly a nonequilibrium effect, which maximizes near the peaks where the transition takes place. Yet, the good collapses of both χ and C indicate that the scaling law between the critical exponents holds even there. One might regard the scaling with the driving field in the absence of a new leading exponent as a kind of adiabaticity in which one replaces the field directly by its time dependent form and the system would just evolve according to it. The present results demonstrate that this is not all.

C. Polynomial

Figure 7 depicts the hysteresis loops of the polynomial driving Eq. (31). According to Eq. (33), for fixed $R_1 R_3^{-r_1/r_3}$, all rescaled curves collapsed well onto each other as shown. The loops can also be rescaled according to Eq. (34) as $R_1 R_3^{-r_1/r_3}$ is fixed. Therefore both R_1 and R_3 can describe the scaling well.

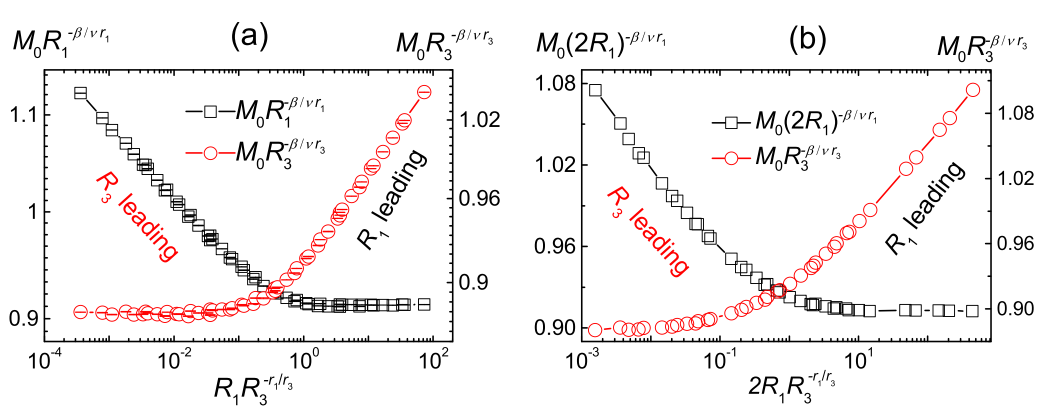

FIG. 8. Different FTS regimes and their crossover for (a) $H = R_1 t + R_3 t^3$ and (b) $H = R_3 t^3 - R_1 t$ near t_-. Black and red curves are the same data rescaled by R_1 [$2R_1$ in (b)] and R_3, respectively. Each rescaled curve consists of a leading R_1 section (where $R_1 R_3^{-r_1/r_3} \gg 1$), a leading R_3 section (where $R_1 R_3^{-r_1/r_3} \ll 1$) with different slopes and a crossover between them (where $R_1 R_3^{-r_1/r_3} \sim 1$). The slope of the black (red) curve in the R_3 (R_1) dominated regime is $-0.0305(1)$ [0.0309(1)] in (a) and $-0.0309(1)$ [0.0299(4)] in (b), consistent with theoretical absolute value of $\beta/\nu r_1$. No error bars appear in (b) as M_0 is obtained by interpolating the averaged magnetization curves at the first $H = 0$. However, they cannot be appreciably larger than those displayed in Fig. 9 below. Lines connecting symbols are only a guide to the eye.

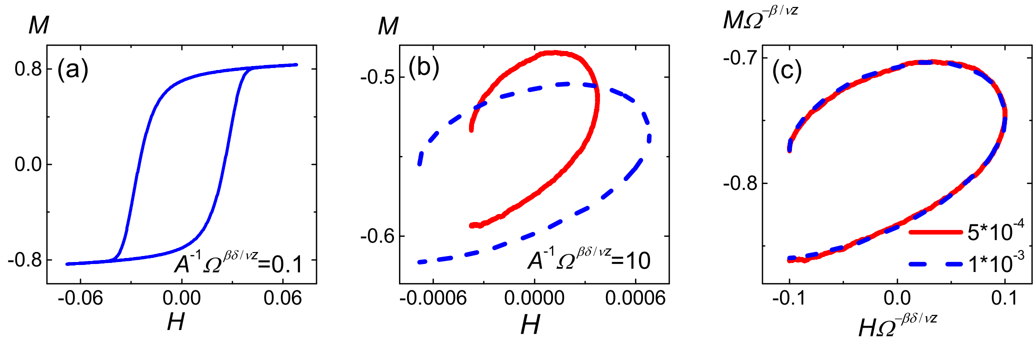

FIG. 10. Hysteresis loops of the sinusoidal driving $H = A\sin\Omega t$. (a) For $A^{-1}\Omega^{\beta\delta/\nu z} = 0.1 < 1$, M saturates at large H. (b) For $A^{-1}\Omega^{\beta\delta/\nu z} = 10 > 1$, M does not saturate. (c) The hysteresis loops in (b) rescaled by Ω. The legend in (c) lists Ω used in (b) and (c).

To investigate the different regimes and their crossover, we again pick the data at $H = 0$, so that the scaling function in Eqs. (33) and (34) are only affected by $R_1R_3^{-r_1/r_3}$. The results of the rescaling are shown in Fig. 8(a). On the one hand, each curve becomes flat when rescaled by the right rescaling parameter at its corresponding regime, implying that the effect of the other parameter can be ignored in that regime. On the other hand, it gets incline in the regime where the other parameter dominates with a slope determined by the exponent to ensure the leading behavior of that parameter. The good agreement with the theory shown confirms the latter. A similar result appears for the driving (35) near t_-, Fig. 8(b). This indicates that the newly emerged second-order term never dominates, though it gives rise to a larger region of crossover by comparing Figs. 8(a) and 8(b).

Near $t_0 = 0$, the graph now appears somehow different as shown in Fig. 9, whose insets demonstrate manifestly the dramatic effect of nonequilibrium initial conditions. For $R_1R_3^{-r_1/r_3} \ll 1$, the system does not have enough time to relax to equilibrium near t_{v_-}. Consequently, the early part affects the later one in contrast to the opposite regime in which all three curves start and end in equilibrium and can thus be treated separately. M_0 is thus still negative as seen in inset (a) and results in the dip as absolute values are used. This indicates that the initial state controls the evolution in this regime. Because the dominant time scale of the first part is t_{R_3}, the initial state is again dominated by R_3. This is why the two original parameters can describe the scaling well using Eqs. (33) and (34). Nonetheless, separating the process and considering the initial conditions reveal far rich phenomena and physics.

D. Sinusoidal

Here, all simulations are performed with the initial condition $H_{\text{in}} = -A$ and $\mathcal{P}_{\text{in}} = \mathcal{P}_{\text{eq}}(-A)$. We first show the hysteresis loops for the saturated and unsaturated cases in Fig. 10. Note that in the unsaturated case, the loops are not close and the range of M is relatively small. This is because equilibrium can not be achieved during the whole process and the initial condition is important. One sees that even though they are not dominated by Ω, the hysteresis loops can be rescaled well by it according to Eq. (40). Similarly, as $A^{-1}\Omega^{\beta\delta/\nu z}$ is fixed, the loops can also be rescaled by A as well according to Eq. (41).

Figure 11 is the verification of the scaling forms (42), (40), and (44). One sees that there exists no regime in which t_Ω dominates, even in the regime $\Omega^{-1}A^{\nu z/\beta\delta} \ll 1$ in which it is

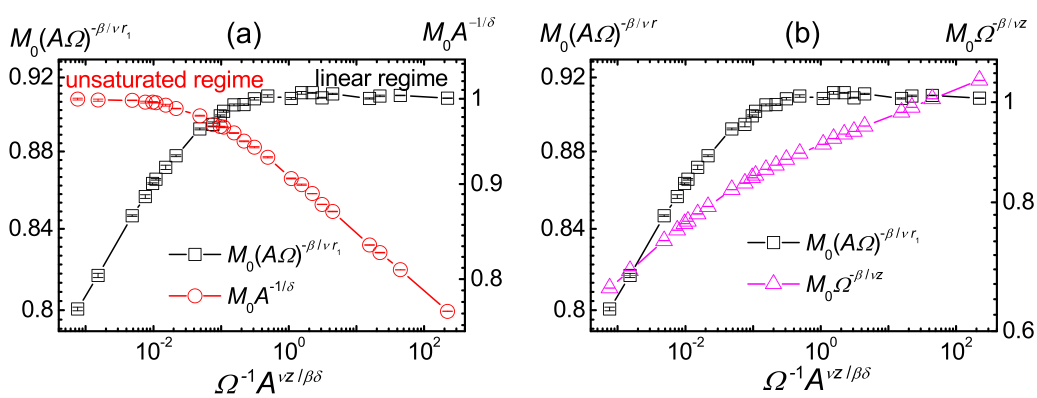

FIG. 11. Different regimes and their crossover for $H = A\sin\Omega t$. Rescaled of the same data (a) by $A\Omega$ (black squares) and A (red circles) and (b) by $A\Omega$ (black squares) and Ω (magenta triangles). The black curves are identical in (a) and (b). The magenta curve is not flat in any regime, implying that Ω does not dominate. The black curves are flat at $\Omega^{-1}A^{\nu z/\beta\delta} \gg 1$, indicating the dominance of $t_{A\Omega}$ in this regime. Correspondingly, the slopes of red and magenta curves are $-0.0312(2)$ and $0.0559(4)$, in agreement with the theoretic values of $-\beta/\nu r_1$ and $\beta/\nu z = 0.0577$ according to Eqs. (42) and (44) and Eqs. (40) and (44), respectively. For $\Omega^{-1}A^{\nu z/\beta\delta} \ll 1$, the red curve becomes flat and the slopes of the black and magenta curves are 0.0289(5) and 0.0269(2), consistent with the theoretic values of $\beta/\nu r_1$ and $(1/z - 1/r_1)\beta/\nu = 0.0268$, respectively, showing the importance of the initial condition. Lines connecting symbols are only a guide to the eye.

the shortest time scale. In this regime, the initial condition is dominant and the hysteresis loops are unsaturate as Fig. 10(b) shows. Accordingly, the leading scaling with A here stems from the initial condition rather than from t_A. After the initial condition decays away, t_Ω may take over. In the opposite regime in which t_A is shortest from Eq. (43), it is $t_{A\Omega}$ instead of t_A that dominates, as seen in Fig. 11. In other words, t_A never dominates, because it is a transient scale. All these confirm well our theory of the sinusoidal driving.

VIII. CONCLUSION

We have studied a generic class of nonequilibrium systems represented by a critical system that possesses a long relaxation time and is weakly driven within a finite time in a form that does not cause resonances but otherwise is arbitrary. An RG theory has been developed to account for such driven nonequilibrium critical phenomena. From the theory, the driving generates finite time scales that can well be shorter than the equilibrium correlation time and thus driving the system far off equilibrium. This can create topological defects of the KZ mechanism. Moreover, the finite time scales can control different regimes and thus crossovers among them can take place when conditions change. These nonequilibrium phenomena are well described in the theory with just the usual static and dynamic critical exponents. Yet, this does not mean that a kind of adiabaticity in which the field is replaced directly with its time dependent form is all the story, because nonequilibrium behaviors such as violation of the fluctuation-dissipation theorem, hysteresis, and even negative values of the susceptibility appear. Still, the scaling law among the critical exponents holds between the response and the correlation, either of which can thus be employed to estimate the critical exponents as in equilibrium.

We have identified a unique type of initial conditions that dominates the evolution under the driving. Opposite to the nonequilibrium initial conditions that lead to the critical initial slip, this type of initial conditions has longer correlations than the driving ones and thus the critical initial slip does not emerge and can be in either equilibrium or nonequilibrium. An example of the latter is one arising from a driving that changes continuously to a subsequent one with a shorter dominant time scale. Under the latter driving, all correlations of the system are still governed remarkably by the time scales of the former driving in the initial-state-controlled regime.

Applications of the theory to some specific forms of driving have discovered results that have not been found before. Besides the negative susceptibility, a monomial driving with $n > 1$ involves a singularity that originates from the difference in the scaling functions between their different forms of rescaled arguments and that is characterized completely by n. A polynomial driving can exhibit initial state dominated regimes when it crosses or approaches the critical point several times in addition to the different regimes and their crossovers determined by the driving parameters. A sinusoidal driving and other forms of general drivings are dominated by their first expansion coefficient in time, often the linear one, when their overall amplitudes are sufficiently large. Some time scales determined by their parameters may only be transient or may be dominated by the initial state and thus do not control any regime before the initial state decays away. The presence of the amplitude-involving rescaled arguments, no matter whether as a small variable in the scaling functions or more seriously as a parameter that gives rise to crossovers, cautions experimental measurements in which an external driving such as the sinusoidal one is applied to a system with long relaxation times.

We remark that the result of no new critical exponents except the equilibrium ones is built on their origin in the field-theoretic RG theory, viz., no new divergences are generated from the driving. It is thus unlikely to be false at least to the precision of the present numerical results. For the critical initial slip of an initial increase of the order parameter and its associated exponent, they ought to play no role as we have purposely chosen the initial states that have longer correlations than the driving one. However, they may well enter for a general initial condition. In this case, they may not be as prominent as that exhibits in the conventional correlationless nonequilibrium initial state and may thus complicate the analysis. Although we have only studied the standard ϕ^4 theory for the Ising universality class, the theory should apply to other universality classes as well. As the system studied is a generic nonlinear nonequilibrium one, the theory may shed light on the study of other nonequilibrium systems. It may also be instructive to nonlinear science as the driving may help to probe scaling behavior there.

ACKNOWLEDGMENT

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