

Matrix model for non-Abelian quantum Hall states

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We propose a matrix quantum mechanics for a class of non-Abelian quantum Hall states. The model describes electrons which carry an internal $SU(p)$ spin. The ground states of the matrix model include spin-singlet generalizations of the Moore-Read and Read-Rezayi states and, in general, lie in a class previously introduced by Blok and Wen. The effective action for these states is a $U(p)$ Chern-Simons theory. We show how the matrix model can be derived from quantization of the vortices in this Chern-Simons theory and how the matrix model ground states can be reconstructed as correlation functions in the boundary WZW model.

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I. INTRODUCTION AND SUMMARY

Electrons in the lowest Landau level exhibit an astonishing array of compressible and incompressible states, the latter with both Abelian and non-Abelian topological order. For a number of these states, a description of the dynamics in terms of *matrix models* has proven useful. These typically involve a $U(N)$ gauge symmetry, with N the number of electrons. The gauge symmetry imposes constraints on the Hilbert space which project the dynamics onto the lowest Landau level.

The first such matrix model was developed by Pasquier and Haldane [1] (see also [2]) to describe the compressible state at half filling. Subsequently, Polychronakos introduced a matrix model for the Laughlin states [3], inspired by earlier work [4]. The Hilbert space of this matrix model not only lies in the lowest Landau level, but also captures the appropriate topological order. The purpose of this paper is to introduce a generalization of these matrix models describing a class of non-Abelian quantum Hall states.

A. A class of non-Abelian quantum Hall states

Before we describe the role played by the matrix model, we first summarize some properties of the non-Abelian Hall states that will emerge.

The original Moore-Read state [5], and its extension to the series of Read-Rezayi states [6], describe spin-polarized electrons. There are, however, a number of prominent non-Abelian Hall states in which the electrons carry an internal spin degree of freedom [7–9]. Typically, the quantum Hall ground states are singlets under the spin symmetry group. It is this kind of “non-Abelian spin-singlet state” which will be of interest in this paper.

In the context of quantum Hall physics, the “spin” degrees of freedom can be more general than the elementary spin of the electron. For example, in bilayer systems the layer index plays a similar role to the spin degree of freedom and is sometimes referred to as a “pseudospin.” In other systems, the electrons may carry more than two internal states. This

occurs, for example, in graphene where one should include both spin and valley degrees of freedom [10]. Here, we will consider systems in which each particle carries some number of internal states. This will include situations in which these states transform in a higher representation of $SU(2)$, but also situations in which the states transform under a general $SU(p)$ group. In all cases, we will refer to these internal states simply as the “spin” degrees of freedom of the particle.

When the symmetry group is $SU(2)$, one can construct non-Abelian spin-singlet states starting from the familiar Abelian (m, m, n) Halperin states [11]. It is well known that when particles carry spin $s = \frac{1}{2}$, only the Halperin states with $m = n + 1$ are spin singlets [12]. Apparently less well known is the statement that for particles carrying spin s , the (m, m, n) states, suitably interpreted, are spin singlets when $m = n + 2s$. Moreover, the presence of the spin degrees of freedom changes the universality class of these states and, for $s > \frac{1}{2}$, they have non-Abelian topological order. In particular, when the particles have spin $s = 1$, it is possible to rewrite these states in Pfaffian form and they lie in the same universality class as the Moore-Read states.

When the symmetry group is $SU(p)$, the obvious $(m, \dots, m, n, \dots, n)$ generalization of the Halperin states can again be used as the foundation to build non-Abelian states. When $m - n = 1$, these are spin singlets if each particle transforms in the fundamental representation of $SU(p)$. More generally, when $m - n = k$ one can build spin singlets if each particle transforms in the k th symmetric representation of $SU(p)$.

The states that arise in this way are not novel. They were first introduced many years ago by Blok and Wen [7], albeit using the rather different construction of conformal blocks in an $SU(p)_k$ WZW model. The states have filling fraction

$$\nu = \frac{p}{k + pn} \quad (1.1)$$

with p and k positive integers determined by the spin group and its representation, and n an arbitrary positive integer. For $p = 1$, these are simply the Laughlin states. For $p = k = 2$, these are spin-singlet generalizations of the Moore-Read states. For $p > 2$ and $k = 2$, these are spin-singlet generalizations of the Read-Rezayi states.

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B. Chern-Simons theories and matrix models

The effective description of the Blok-Wen states is a non-Abelian Chern-Simons theory. The gauge group and levels are given by

$$U(p)_{(k+pn)p,k} = \frac{U(1)_{(k+pn)p} \times SU(p)_k}{\mathbf{Z}_p}. \quad (1.2)$$

The allowed level of the $U(1)$ factor is strongly constrained by the fact that this is a $U(p)$ rather than $U(1) \times SU(p)$ theory [13].

Viewed in a certain slant of light, the Blok-Wen states are the most natural non-Abelian quantum Hall states. Let us take a quick aside to explain this. The long-distance physics of all non-Abelian quantum Hall states is described by some variant of non-Abelian Chern-Simons theories. This means, of course, that Wilson lines in this theory carry some representation under the non-Abelian group which, for us, is $SU(p)$. The corresponding “color” degrees of freedom are then interpreted as spin degrees of freedom of the underlying electron. This, in essence, is why non-Abelian quantum Hall states arise naturally from particles carrying internal spin.

In contrast, if one wants to describe the long-distance physics of spin-polarized non-Abelian Hall states, such as those of [5,6], one must work somewhat harder. This involves the introduction of yet further quotients of the 3d Chern-Simons theory [13] to eliminate the spin degrees of freedom. This is the sense in which the Blok-Wen states are particularly natural [14].

We are now in a position to explain how the matrix model arises. The electrons in the quantum Hall system correspond to vortices of the $U(p)$ Chern-Simons theory. The $U(N)$ matrix model is simply the description of the microscopic dynamics of N of these vortices. At present, we can construct this matrix model only for the choice $n = 1$ in Eqs. (1.1) and (1.2). It is to be expected that quantizing these vortices results in the quantum Hall ground state. The matrix model provides the technology to do this explicitly.

This relationship between Chern-Simons vortices, matrix models, and quantum Hall wave functions was explored for Abelian theories in Refs. [15,16]. The novelty in non-Abelian gauge theories is that the vortices are endowed with an internal orientation, or spin, degrees of freedom, as first explained in Refs. [17,18]. We will show that this results in the non-Abelian quantum Hall states described above. (An earlier, somewhat orthogonal attempt to describe a quantum Hall fluid of non-Abelian vortices was made in Ref. [19].)

C. Plan of the paper

The paper is written in a somewhat different order from the preceding introduction. In Sec. II, we introduce the matrix model but do not explain its Chern-Simons origins. Instead, we will take the matrix model as the starting point and show that it describes particles with spin moving in the lowest Landau level. We will see that, upon quantization, the ground state lies in the same universality class as the non-Abelian quantum Hall states previously introduced by Blok and Wen [7].

Section III can be read independently. We describe in some detail the Blok-Wen wave functions and their construction from spin generalizations of the Halperin-type states. We show,

in particular, how they describe spin-singlet generalizations of the Moore-Read [5] and Read-Rezayi [6] states. One rather cute fact is that the Read-Rezayi states arise in this picture from $SU(p)_2$ Chern-Simons theory; this is related by level-rank duality [20] to the more familiar $SU(2)_p$ coset constructions.

In Sec. IV, we return to the origin of the matrix model. We explain how it captures the dynamics of vortices in a Chern-Simons theory with gauge group (1.2).

Finally, in Sec. V, we complete the circle of ideas. We confirm that the matrix model wave functions, derived from the Chern-Simons theory, can be reconstructed as correlation functions in the boundary WZW model with algebra (1.2). In a follow-up paper [21], we will make the connection between the matrix model and the WZW model more direct: we will show that their partition functions agree.

II. THE QUANTUM HALL MATRIX MODEL

The purpose of this paper is to study a matrix model description of non-Abelian quantum Hall states. The model will describe N particles which we refer to as “electrons.”

The matrix model is a $U(N)$ gauged quantum mechanics, with a gauge field which we denote as α . This gauge field is coupled to an $N \times N$ complex matrix Z , together with a set of N -dimensional vectors φ_i which are labeled by an index $i = 1, \dots, p$. These transform under the gauge symmetry as

$$Z \rightarrow U Z U^\dagger \quad \text{and} \quad \varphi_i \rightarrow U \varphi_i \quad \text{for} \quad U \in U(N). \quad (2.1)$$

The dynamics is governed by the first-order action

$$S = \int dt \frac{iB}{2} \text{Tr}(Z^\dagger \mathcal{D}_t Z) + i \sum_{i=1}^p \varphi_i^\dagger \mathcal{D}_t \varphi_i - (k+p) \text{Tr} \alpha - \omega \text{Tr} Z^\dagger Z \quad (2.2)$$

with $\mathcal{D}_t Z = \partial_t Z - i[\alpha, Z]$ and $\mathcal{D}_t \varphi_i = \partial_t \varphi_i - i\alpha \varphi_i$.

The action depends on three parameters: B , ω , and k . We will see below that B is interpreted as the background magnetic field in which the electrons move, while ω is the strength of a harmonic trap which encourages the electrons to cluster close to the origin. Finally k , which appears in the combination $k+p$, is the coefficient of the quantum mechanical Chern-Simons term. Gauge invariance requires that k is an integer and we will further take it to be positive: $k \in \mathbf{Z}^+$.

In addition to the $U(N)$ gauge symmetry, our model also enjoys an $SU(p)$ global symmetry, under which the φ_i rotate. When $p = 1$, this action reduces to the model written by Polychronakos [3] to describe Laughlin states. The $p = 1$ matrix model was further studied in a number of papers, including [22–24] and we will review some of its properties below. The model with general p was previously discussed in Ref. [25], albeit with a different interpretation from that offered here.

Getting a feel for the matrix model

To gain some intuition for the physics underlying (2.2), let us first look at the example of a single particle, so $N = 1$ and our matrix model is an Abelian $U(1)$ gauge theory, with

dynamics

$$S_{N=1} = \int dt \frac{iB}{2} Z^\dagger \dot{Z} + \sum_{i=1}^p i \varphi_i^\dagger \mathcal{D}_t \varphi_i - (k+p)\alpha - \omega Z^\dagger Z.$$

In this case, the Z field decouples; the kinetic term, which is first order in time, describes the low-energy dynamics of an electron moving in a large external magnetic field B . When we come to the quantum theory, this will translate into the statement that the electron lies in the lowest Landau level. The term proportional to ω provides a harmonic trap for the electron.

Meanwhile, the φ_i variables describe the internal degrees of freedom of the electron. To see this, note that the equation of motion for α requires that $\sum_i |\varphi_i|^2 = k+p$ is constant. After dividing out by $U(1)$ gauge transformations, $\varphi_i \rightarrow e^{i\theta} \varphi_i$, we see that φ_i parametrize the space \mathbf{CP}^{p-1} . However, the action is first order in time derivatives, which means that \mathbf{CP}^{p-1} should be viewed as the *phase space* of the system, as opposed to the configuration space. This is important. Because the phase space has finite volume, the quantization of φ_i will result in a finite-dimensional internal Hilbert space for the electron. In other words, the electron carries “spin.”

Note that our usage of the word “spin” is somewhat more general than its standard meaning in condensed matter physics (or high-energy physics for that matter). Usually, one thinks of spin as referring to a representation of $SU(2)$; this corresponds to the choice $p=2$ in our model. More generally, our internal degree of freedom transforms in some representation of $SU(p)$. The choice of representation is determined by the parameter k . [We will show below that the electrons sit in the k th symmetric representation of $SU(p)$; in the case of $SU(2)$, this means that they carry spin $j=k/2$.]

We learn that the $U(1)$ matrix model describes a particle carrying spin, restricted to move in the lowest Landau level. The $U(N)$ matrix model simply describes N such particles. Roughly speaking, the N eigenvalues of the matrix Z correspond to the positions of the particles although, as we will see, there is some ambiguity in this when the particles are close. More precisely, we can again look at the equation of motion for the gauge field α . This results in the $u(N)$ -valued constraint

$$\frac{B}{2} [Z, Z^\dagger] + \sum_{i=1}^p \varphi_i \varphi_i^\dagger = (k+p) \mathbf{1}_N. \quad (2.3)$$

The phase space, \mathcal{M} , of the theory is now the space of solutions to (2.3), modulo the gauge action (2.1). This has real dimension $\dim \mathcal{M} = 2Np$. Our task is to quantize this phase space, with the harmonic potential $H = \omega \text{Tr } Z^\dagger Z$ providing the Hamiltonian.

A. Quantization

In this section, we study the quantization of our matrix model (2.2). The canonical commutation relations inherited from the action (2.2) are

$$\frac{B}{2} [Z_{ab}, Z_{cd}^\dagger] = \delta_{ad} \delta_{bc} \quad \text{and} \quad [\varphi_{ia}, \varphi_{jb}^\dagger] = \delta_{ab} \delta_{ij} \quad (2.4)$$

with $a, b = 1, \dots, N$ and $i, j = 1, \dots, p$. We choose a reference state $|0\rangle$ obeying

$$Z_{ab}|0\rangle = \varphi_i|0\rangle = 0.$$

The Hilbert space is then constructed in the usual manner by acting on $|0\rangle$ with Z^\dagger and φ_i^\dagger .

However, we still need to take into account the $U(N)$ gauge symmetry. This is implemented by requiring that all physical states obey the quantum version of the Gauss’ law constraint (2.3). Normal-ordering the terms in the matrix commutator, this reads

$$\frac{B}{2} : [Z, Z^\dagger] : + \sum_{i=1}^p \varphi_i \varphi_i^\dagger = (k+p) \mathbf{1}_N. \quad (2.5)$$

The traceless part of this equation is interpreted as the requirement that physical states are $SU(N)$ singlets. Meanwhile, the trace of this constraint requires all physical states to carry fixed charge under $U(1) \subset U(N)$. Here there is an ordering issue. Using the commutation relations (2.4), we find

$$\sum_{a=1}^N \sum_{i=1}^p \varphi_{ia} \varphi_{ia}^\dagger = (k+p)N \Rightarrow \sum_{a=1}^N \sum_{i=1}^p \varphi_{ia}^\dagger \varphi_{ia} = kN. \quad (2.6)$$

This tells us that all physical states carry charge kN under the $U(1)$. In other words, all states in the physical Hilbert space contain precisely kN copies of φ^\dagger acting on $|0\rangle$.

The spin of the particle revisited

We can now be more precise about the internal $SU(p)$ spin carried by each particle. Setting $N=1$, the spin states of a single particle take the form

$$|\Omega_{i_1 \dots i_k}\rangle = \varphi_{i_1}^\dagger \dots \varphi_{i_k}^\dagger |0\rangle.$$

Since each operator φ_i transforms in the fundamental of $SU(p)$, the spin states $|\Omega\rangle$ transform in the k th symmetric representation. In particular, for $k=1$ the electrons carry the fundamental representation of $SU(p)$.

Our main focus in this paper will be on quantum Hall states which are $SU(p)$ spin singlets. Some simple group theory tells us that for this to happen we must have the number of electrons N divisible by p . Indeed, we will see below that the ground states simplify in this case.

B. The ground states

The ground state of the matrix model with $p=1$ was constructed in Ref. [3]. We first review this example before explaining the straightforward generalization to $p>1$.

1. The $p=1$ ground state

When $p=1$, the electrons carry no internal spin. The constraint (2.6) tells us that all physical states have kN operators φ^\dagger acting on $|0\rangle$. Further, the Hamiltonian arising from (2.2) is

$$H = \omega \text{Tr } Z^\dagger Z, \quad (2.7)$$

which simply counts the number of Z^\dagger operators acting on $|0\rangle$. The route to constructing the ground state is then straightforward: we need to act with kN copies of φ^\dagger , keeping the number of Z^\dagger operators to a minimum. The subtleties arise from the requirement that the physical states are invariant under $SU(N)$ gauge transformations. Since we only have φ^\dagger operators to play with, the only way to achieve this is to construct a baryon operator of the form

$$\epsilon^{a_1 \dots a_N} (Z^{l_1} \varphi)_{a_1}^\dagger \dots (Z^{l_N} \varphi)_{a_N}^\dagger.$$

However, because φ is bosonic, the antisymmetrization inherent in $\epsilon^{a_1 \dots a_N}$ causes this operator to vanish unless all the exponents l_a are distinct. Because we pay an energy cost (2.7) for each insertion of Z^\dagger , it follows that the lowest energy operator is given by

$$\epsilon^{a_1 \dots a_N} (Z^0 \varphi)_{a_1}^\dagger (Z \varphi)_{a_2}^\dagger \dots (Z^{N-1} \varphi)_{a_N}^\dagger.$$

The trace constraint then tells us that the ground state is given by

$$|\text{ground}\rangle_k = [\epsilon^{a_1 \dots a_N} (Z^0 \varphi)_{a_1}^\dagger (Z \varphi)_{a_2}^\dagger \dots (Z^{N-1} \varphi)_{a_N}^\dagger]^k |0\rangle.$$

The interplay between the gauge symmetry and the Hamiltonian has resulted in the construction of a state with interesting correlations between the positions of particles, encoded in the operator Z . This will become increasingly apparent as we proceed; in particular, shortly we will write these states in the more familiar language of N -particle wave functions and see a close relationship to the Laughlin wave functions.

2. Ground states with $p \geq 2$

We now turn to the ground states when the electrons carry an internal spin. We anticipated above that the states will take a simpler form when N is divisible by p . And, indeed, this is the case.

(a) N divisible by p . When N is divisible by p , there is a unique ground state. This is an $SU(p)$ singlet. To describe the construction of this state, we first group p creation operators φ_i^\dagger together using the invariant antisymmetric tensor of $SU(p)$;

$$\mathcal{B}(r)_{a_1 \dots a_p}^\dagger = \epsilon^{i_1 \dots i_p} (Z^r \varphi)_{i_1 a_1}^\dagger \dots (Z^r \varphi)_{i_p a_p}^\dagger.$$

Following the terminology of the high-energy community, we refer to this state as a *baryon*. It is a singlet under the $SU(p)$ global symmetry, but transforms in the p th antisymmetric representation of the $U(N)$ gauge symmetry. To construct an $SU(N)$ singlet with the correct $U(1)$ charge (2.6), we make a ‘‘baryon of baryons.’’ The ground state is then

$$|\text{ground}\rangle_k = [\epsilon^{a_1 \dots a_N} \mathcal{B}(0)_{a_1 \dots a_p}^\dagger \mathcal{B}(1)_{a_{p+1} \dots a_{2p}}^\dagger \dots \mathcal{B}(N/p - 1)_{a_{N-p+1} \dots a_N}^\dagger]^k |0\rangle. \quad (2.8)$$

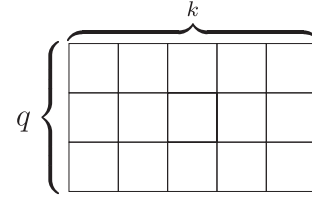
This state has energy $E = \frac{\omega k}{2B} \frac{N(N-p)}{p}$. This time the requirements of the $U(N)$ gauge invariance have resulted in interesting correlations between both position and spin degrees of freedom of the electrons. We will devote the rest of this section and the next to describing the structure of these states.

(b) $N = q \bmod p$. When N is not divisible by p , the ground state is no longer a singlet under the global $SU(p)$ symmetry. We write $N = mp + q$ with $m, q \in \mathbb{Z}^+$. One can check that

the ground states are

$$|\text{ground}\rangle_k = \prod_{l=1}^k [\epsilon^{a_1 \dots a_N} \mathcal{B}(0)_{a_1 \dots a_p}^\dagger \mathcal{B}(1)_{a_{p+1} \dots a_{2p}}^\dagger \dots \mathcal{B}(m-1)_{a_{N-p+1} \dots a_{N-q}}^\dagger (Z^m \varphi_{i(l,1)})_{a_{N-q+1}}^\dagger \dots (Z^m \varphi_{i(l,q)})_{a_N}^\dagger] |0\rangle,$$

where $i(l, \alpha)$, with $l = 1, \dots, k$ and $\alpha = 1, \dots, q$ are free indices labeling the degenerate ground states. These ground states transform in the k th-fold symmetrization of the q th antisymmetric representation of $SU(p)$. In terms of Young diagrams, this is the representation



We will see in Sec. V why these representations are special and might be expected to arise in quantum Hall states. In the meantime, we will primarily focus on the states (2.8) that arise when N is divisible by p .

C. The wave functions

The description of the ground states given above is in terms of a coherent state representation for matrices. To make connections with the more traditional form of the wave functions, we need to find a map between the creation operators Z^\dagger and the position space representation. For the $p = 1$ states, this was explained by Karabali and Sakita [23,24]. We first review their results and then provide the generalization to the $SU(p)$ matrix model.

1. $p = 1$ and the Laughlin wave functions

At the formal level, there is a clear similarity between the ground state for $p = 1$ theories,

$$|\text{ground}\rangle_k = [\epsilon^{a_1 \dots a_N} (Z^0 \varphi)_{a_1}^\dagger (Z \varphi)_{a_2}^\dagger \dots (Z^{N-1} \varphi)_{a_N}^\dagger]^k |0\rangle, \quad (2.9)$$

and the Laughlin wave functions at filling fraction $\nu = 1/m$,

$$\begin{aligned} \psi_m^{\text{Laughlin}}(z_a) &= \prod_{a < b} (z_a - z_b)^m e^{-B \sum |z_a|^2 / 4} \\ &= [\epsilon^{a_1 \dots a_N} z_{a_1}^0 z_{a_2}^1 \dots z_{a_N}^{N-1}]^m e^{-B \sum |z_a|^2 / 4}. \end{aligned}$$

However, this similarity can be misleading: the operators Z^\dagger and φ^\dagger are very different objects from the holomorphic position variables z_a . To make this connection precise, we need to be more careful about how to relate the two. In fact, there is no canonical map. There are, however, a number of natural ways to make the connection. Two of these, discussed in Ref. [23], are

(1) We work with a coherent state representation $\hat{Z}|Z, \varphi\rangle = Z|Z, \varphi\rangle$ and $\hat{\varphi}|Z, \varphi\rangle = \varphi|Z, \varphi\rangle$ where, for once, we have used hats to denote the difference between the quantum

operator \hat{Z} and the classical matrix Z . We then diagonalize $Z = VDV^{-1}$ with $D = \text{diag}(z_1, \dots, z_N)$ and express the resulting wave functions as $\psi(z_a) = \langle z_a | \Psi \rangle$.

(2) Alternatively, we could decompose the complex operator matrix $\hat{Z} = \hat{X} + i\hat{Y}$ and subsequently work in a coherent state representation $\hat{X}|X\rangle = X|X\rangle$. This picture has the advantage that the matrices \hat{X} and \hat{Y} are conjugate, giving us the representation

$$\hat{Z}_{ab}^\dagger = \frac{1}{\sqrt{2}} \left(X_{ab} - \frac{\partial}{\partial X_{ba}} \right).$$

Moreover, calculations in this approach are somewhat easier because the diagonalization $X = UXU^\dagger$ can be achieved by a unitary operator U . The resulting wave functions are written as $\psi(x_a) = \langle x_a | \Psi \rangle$. We then analytically continue $x_a \rightarrow z_a$ to provide holomorphic wave functions of the kind appropriate to describe the lowest Landau level.

Both of these approaches were described in Ref. [23]. The resulting wave functions differ in detail, but share their most important properties. First, for $k = 0$, the wave functions coincide with the Slater determinant for a fully filled Landau level

$$\langle z_a | \text{ground} \rangle_{k=0} = \prod_{a < b} (z_a - z_b) e^{-B \sum |z_a|^2 / 4}. \quad (2.10)$$

The exponential factor is the usual factor arising from the normalization of coherent states. The single factor of the Vandermonde determinant, which is not obvious in Eq. (2.9) when $k = 0$, is a Jacobian that arises in the transformation from matrix-valued objects to their eigenvalues.

Meanwhile, for $k \geq 1$, neither representation of the wave function coincides with the Laughlin state. Nonetheless, both have the property that

$$\langle z_a | \text{ground} \rangle_k \rightarrow \psi_{k+1}^{\text{Laughlin}}(z_a) \text{ as } |z_a - z_b| \rightarrow \infty.$$

In other words, the wave functions that arise from the matrix model coincide with the Laughlin wave functions only at large distances. This ensures that the matrix model ground state has filling fraction

$$\nu = \frac{1}{k+1}.$$

However, the matrix model state differs from the Laughlin wave function as particles approach to within a magnetic length. Indeed, one obtains the matrix model wave function from the Laughlin one by acting with exponentials of derivative operators $l_B(\partial/\partial z)$ on the polynomial part. In particular, the matrix model wave functions do not exhibit the familiar zero-of-order m that is characteristic of the Laughlin wave function.

The fact that matrix model and Laughlin states differ in detail is not a matter of concern. There is nothing privileged about the Laughlin wave function: it is merely a representative of a universality class of states, characterized by their topological order. The matrix model state lies in the same universality class. This can be shown explicitly; for certain calculations the coherent state representation (2.9) offers greater analytic control. In particular, it was shown in Ref. [16] that the fractional statistics of the quasihole can be computed in the coherent state representation following the

classic calculation of [26], but without resorting to the plasma analogy.

2. Wave functions for $p \geq 2$

For the case $p = 1$ described above, all physical states have the same dependence on φ^\dagger excitations; they differ only in their Z^\dagger excitations. This is the reason that no φ variables were needed when writing the wave functions. In contrast, when $p \geq 2$, different physical states can have a different structure of φ_i^\dagger excitations. These capture the way the state transforms under the $SU(p)$ symmetry.

We repeat the procedure described above, moving from coherent state representation to wave function. For $k = 0$ the wave function knows nothing about the spin degrees of freedom. This means that the $k = 0$ wave function is again given by (2.10), describing a fully filled Landau level with $\nu = 1$.

However, for $p \geq 2$ and $k > 1$, we have a new ingredient. Apart from the Vandermonde determinant (2.10), each time that a power of a particle coordinate z_a appears in the wave function, it is accompanied by a spin degree of freedom,

$$\sigma_a \in \{1, \dots, p\},$$

where $a = 1, \dots, N$ labels the particle.

For example, when $k = 1$, each particle has a single spin degree of freedom σ . This reflects the fact that, as we saw earlier, each particle transforms in the fundamental representation of $SU(p)$. More generally, the internal state of each particle is determined by k factors of the spin label σ . As we will explain in some detail in Sec. III, this is to be interpreted as specifying the k th symmetric representation under $SU(p)$.

The calculation to find the form of the wave function is now very similar to that presented in Ref. [23]. The result is simplest to state when N is divisible by p since, in this case, the resulting ground state wave function (2.8) is an $SU(p)$ spin singlet and have the property that

$$\langle z_a | \text{ground} \rangle_k \rightarrow \psi_{BW}(z_a) \text{ as } |z_a - z_b| \rightarrow \infty,$$

where $\psi_{BW}(z_a)$ are a class of non-Abelian wave functions constructed some time ago by Blok and Wen [7]. Like many non-Abelian quantum Hall states, the explicit description of the wave functions $\psi_{BW}(z_a)$ is straightforward, but somewhat fiddly. We will devote the next section to a more detailed description of these quantum Hall states and their properties. For now, we simply state that these states have filling fraction

$$\nu = \frac{p}{k+p}. \quad (2.11)$$

III. THE BLOK-WEN STATES

In this section, we describe the Blok-Wen wave functions in some detail. The original construction of [7] was in terms of conformal blocks of a conformal field theory (CFT) known as the $SU(p)$ Wess-Zumino-Witten (WZW) model and we will revisit this approach in Sec. V. Here we provide an alternative, more down-to-earth construction of the states. We start with wave functions carrying spin under $SU(2)$, moving on to the more general $SU(p)$ case in Sec. III B.

A. Particles with SU(2) spin

The simplest examples of wave functions describing particles with spin are due to Halperin [11]. We take N particles, with N even, and split them into two groups of $N/2$ particles, with positions z_a and w_a where each index now runs over $a = 1, \dots, N/2$. The (m, m, n) wave functions are

$$\psi(z, w) = \prod_{a < b}^{N/2} (z_a - z_b)^m \prod_{c < d}^{N/2} (w_c - w_d)^m \prod_{a, d} (z_a - w_d)^n, \quad (3.1)$$

where, throughout this section, we will omit the overall exponential factor common to all wave functions. Counting the angular momentum of particles shows that these states have filling fraction

$$\nu = \frac{2}{m+n}.$$

The (m, m, n) states (3.1) are really shorthand for wave functions with spin. As we review below, they should be dressed with explicit spin wave functions. This will result in the Blok-Wen states. These are actually a slightly more general

class of states than those that emerge from the matrix model. We will see that the matrix model gives states with $n = 1$ and $m = k + 1$.

Usually one thinks of the Halperin states as describing spin- $\frac{1}{2}$ particles, with z_a and w_a labeling the positions of those which are spin-up and spin-down, respectively. With this interpretation the $(n+1, n+1, n)$ states are spin singlets. However, we will show that we can also view (3.1) as describing particles with spin $s > \frac{1}{2}$. This is perhaps surprising as these particles have $2s+1$ spin states and it is not obvious how to decompose these into two groups. We will see that, with this interpretation, the (m, m, n) states are spin singlets when $s = (m-n)/2$. Matching to the matrix model parameters, this means $s = k/2$.

1. Spin- $\frac{1}{2}$

The standard interpretation of (3.1) is as a wave function for spin- $\frac{1}{2}$ particles. To dress the wave function with these spin states, it is useful to change notation slightly and label the positions of all N particles as z_a . Each particle carries a further internal spin degree of freedom σ_a which takes values $|\uparrow\rangle$ or $|\downarrow\rangle$. The (m, m, n) state for $m > n$ is then written as

$$\psi(z, \sigma) = \mathcal{A} \left[\prod_{a < b}^N (z_a - z_b)^n \prod_{a < b \text{ odd}} (z_a - z_b)^{m-n} \prod_{c < d \text{ even}} (z_c - z_d)^{m-n} |\uparrow\downarrow\uparrow\downarrow \dots \uparrow\downarrow| \right], \quad (3.2)$$

where \mathcal{A} stands for antisymmetrization over all particles, exchanging both positions and spins. This wave function describes fermions for m odd and bosons for m even.

It is well known that only the states with $m - n = 1$ are spin singlets [12]. In this case, the wave function factorizes as

$$\psi_{n+1, n+1, n}(z, \sigma) = \prod_{a < b}^N (z_a - z_b)^n \Phi(z, \sigma).$$

This describes fermions for n even and bosons for n odd. Here the first factor takes the familiar Laughlin-Jastrow form, while the second factor is the Slater determinant of two fully filled Landau levels, one for the up spins and one for the down spins. The resulting wave function can be written as

$$\Phi(z, \sigma) = \mathcal{A} \left[\prod_{a < b \text{ odd}} (z_a - z_b) \prod_{c < d \text{ even}} (z_c - z_d) |\uparrow_1\rangle |\downarrow_2\rangle |\uparrow_3\rangle \dots |\downarrow_{2N}\rangle \right]$$

or, equivalently, as

$$\Phi(z, \sigma) = \epsilon_{a_1 \dots a_{2N}} (z_{a_1} z_{a_2})^0 (z_{a_3} z_{a_4})^1 \dots (z_{a_{2N-1}} z_{a_{2N}})^{N-1} [|\uparrow_{a_1}\rangle |\downarrow_{a_2}\rangle |\uparrow_{a_3}\rangle |\downarrow_{a_4}\rangle \dots |\uparrow_{a_{2N-1}}\rangle |\downarrow_{a_{2N}}\rangle]. \quad (3.3)$$

In particular, this latter form makes it clear that the spins are paired in singlet states of the form $|\uparrow_{a_1}\rangle |\downarrow_{a_2}\rangle - |\downarrow_{a_1}\rangle |\uparrow_{a_2}\rangle$.

2. Spin-1

So far, we have just reproduced the usual story. Suppose now that $m = n + 2$. We claim that the following is a spin-singlet wave function for spin-1 particles,

$$\psi_{n+2, n+2, n}(z, \sigma) = \prod_{a < b} (z_a - z_b)^n \mathcal{P}[\Phi^2(z, \sigma)]. \quad (3.4)$$

This is a wave function for fermions when n is odd and bosons when n is even.

Our first task is to explain what this means. The factor Φ^2 includes two spin states for each particle. The tensor product of two spin- $\frac{1}{2}$ states gives $\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3}$. The operator \mathcal{P} projects onto the symmetric $\mathbf{3}$. (In the present case, this operation is not required as it is implemented automatically by the form of Φ^2 . However, we include it in our expression for clarity.) This means that we can interpret (3.4) as a quantum Hall state for spin-1 particles, with the map

$$|\uparrow\rangle |\uparrow\rangle = |1\rangle, \quad |\downarrow\rangle |\downarrow\rangle = |-1\rangle, \quad |\uparrow\rangle |\downarrow\rangle = |\downarrow\rangle |\uparrow\rangle = |0\rangle. \quad (3.5)$$

We further claim that (3.4) is a spin singlet. We will first motivate this by looking at the kinds of terms that arise. We

will subsequently provide a proof in the course of rewriting the wave function in a more familiar form.

Consider two particles, labeled 1 and 2, each of which carries spin- $\frac{1}{2}$. The spin singlet state is

$$|12\rangle_{\frac{1}{2}} = |\uparrow_1\rangle|\downarrow_2\rangle - |\downarrow_1\rangle|\uparrow_2\rangle,$$

where the subscript $\frac{1}{2}$ is there to remind us that this is the singlet built from two spin- $\frac{1}{2}$ particles. The simplest terms that occur in Eq. (3.4) are of the form $|12\rangle_{\frac{1}{2}}|12\rangle_{\frac{1}{2}}$. Using the map (3.5), we have

$$|12\rangle_{\frac{1}{2}}|12\rangle_{\frac{1}{2}} = |1_1\rangle|-1_2\rangle + |-1_1\rangle|1_2\rangle - 2|0_1\rangle|0_2\rangle,$$

which is indeed the singlet formed from two spin-1 states. To highlight this, we write the above equation as

$$|12\rangle_{\frac{1}{2}}|12\rangle_{\frac{1}{2}} = |12\rangle_1.$$

The next kind of term that arises in Eq. (3.4) involves four different particles. It is $|12\rangle_{\frac{1}{2}}|23\rangle_{\frac{1}{2}}|34\rangle_{\frac{1}{2}}|41\rangle_{\frac{1}{2}}$. We can similarly expand this in terms of spin-1 states and again find that only combinations of singlet states appear:

$$|12\rangle_{\frac{1}{2}}|23\rangle_{\frac{1}{2}}|34\rangle_{\frac{1}{2}}|41\rangle_{\frac{1}{2}} = |12\rangle_1|34\rangle_1 - |13\rangle_1|24\rangle_1 + |14\rangle_1|23\rangle_1.$$

The most general term in Eq. (3.4) has $2n$ particles. This too can be written as the linear combinations of n spin-1 singlet states. Rather than demonstrate this term by term, we will instead show that the wave function (3.4) has an alternative form written purely in terms of spin-1 singlets.

3. The spin-1 wave function as a Pfaffian

We will now show that the wave function (3.4) for spin-1 particles can be written as

$$\Phi^2(z, \sigma) = \text{Pf}\left(\frac{|ab\rangle_1}{z_a - z_b}\right) \prod_{a < b} (z_a - z_b) \quad (3.6)$$

with $\text{Pf}(M_{ab})$ the Pfaffian of the matrix M . This is a spin singlet version of the Moore-Read state [5]. It is sensible because the spin-1 singlet $|ab\rangle_1$ is symmetric in the two spins, in contrast to $|ab\rangle_{\frac{1}{2}}$ which is antisymmetric.

It was noticed long ago [7,27] that the (3,3,1) state is closely related to the Pfaffian state. In Ref. [7] the particles were spin-1 but projected onto the $m = 0$ spin component; in Ref. [27] the particles were taken to be spin- $\frac{1}{2}$ and the resulting state was not a spin singlet. Our result (3.6) is clearly closely related to these earlier results, both of which are proven using the Cauchy identity. However, the proof of (3.6) requires more sophisticated machinery which appears not to have been available at the time of [7,27].

Proof. The projective Hilbert space associated with the two spins is a Bloch sphere \mathbf{CP}^1 . We parameterize this by the inhomogeneous coordinate ζ . Formally, we then set $|\downarrow_a\rangle = 1$ and $|\uparrow_a\rangle = \zeta_a$ and write Φ as the polynomial

$$\Phi(z, \zeta) = \frac{1}{2^{N/2}} \epsilon_{a_1 \dots a_N} [(z_{a_1} z_{a_2})^0 \dots (z_{a_{N-1}} z_{a_N})^{N/2-1}] [(\zeta_{a_1} - \zeta_{a_2}) \dots (\zeta_{a_{N-1}} - \zeta_{a_N})].$$

This has the advantage that the right-hand side can be viewed as the determinant of a $N \times N$ matrix $\Delta[z; \zeta]$ with components

given by

$$\Delta[z; \zeta]_{a,b} = \begin{cases} z_a^{b-1}, & 1 \leq j \leq \frac{N}{2}, \\ \zeta_a z_a^{b-1}, & \frac{N}{2} + 1 \leq j \leq 2N. \end{cases} \quad (3.7)$$

To show the result (3.6), we then need to prove the polynomial identity

$$\det^2 \Delta[z; \zeta] \stackrel{?}{=} \text{Pf}\left(\frac{(\zeta_a - \zeta_b)^2}{z_a - z_b}\right) \prod_{a < b} (z_a - z_b).$$

In fact, this identity is a special case of a more general result proven in Refs. [28]. Theorem 2.4 of this paper shows (among other things) that two matrices $\Delta[z; \zeta]$ and $\Delta[z; \eta]$, each defined by (3.7), obey the relation

$$\det \Delta[z; \zeta] \det \Delta[z; \eta] = \text{Pf}\left(\frac{(\zeta_b - \zeta_a)(\eta_b - \eta_a)}{z_b - z_a}\right) \det(z_a^{b-1}).$$

Setting $\zeta_a = \eta_a$ yields the desired result. \blacksquare

4. Higher spin

The generalization to higher spins is now obvious. We construct the wave function

$$\psi_{n+2s, n+2s, n}(z, \sigma) = \prod_{a < b} (z_a - z_b)^n \mathcal{P}[\Phi^{2s}(z, \sigma)], \quad (3.8)$$

where \mathcal{P} is there to remind us that the spin states for each particle are projected onto the fully symmetrized product. This means that this is a wave function for particles with spin s . Once again, the final state is a spin singlet. This follows from some trivial group theory. The infinitesimal action of $\text{SU}(2)$ on the tensor product of $2s$ spin states is

$$T^\alpha = \sum_{a=1}^N t_a^\alpha \otimes \mathbf{1} \otimes \dots \otimes \mathbf{1} + \text{symmetric},$$

where t_a^α is the operator in the fundamental representation acting on the a th particle, and $\alpha = 1, 2, 3$ labels the three $\text{su}(2)$ generators. Because \mathcal{P} projects onto an irrep, we have

$$T^\alpha \mathcal{P}[\Phi^{2s}] = \mathcal{P}\left[\sum_a t_a^\alpha \Phi \otimes \Phi \otimes \dots \otimes \Phi\right] + \text{symmetric}.$$

But each of these terms vanishes because Φ is itself a spin singlet, which means that $\sum_a t_a^\alpha \Phi = 0$. This ensures that (3.8) is indeed a spin singlet.

Although (3.8) provides an explicit description of the state, it would be pleasing to find a simple expression purely in terms of the singlets $|ab\rangle_s$, analogous to the Pfaffian (3.6) for $s = 1$. We have not been able to do this; it may simply not be possible due to the entanglement structure between higher numbers of spins.

While the Halperin states (3.1) describe Abelian quantum Hall states, our spin-singlet states (3.6) and (3.8) with spin $s \geq 1$ are all non-Abelian quantum Hall states. Indeed, it has long been known that dressing a quantum Hall state with spin degrees of freedom can change the universality class of the state. We will see in Sec. V that these states are associated with $\text{SU}(2)_{2s}$ WZW models.

B. Particles with $SU(p)$ spin

We now generalize these ideas to particles that carry a “spin” under the group $SU(p)$. This means that each particle carries an internal Hilbert space which transforms under a particular representation of $SU(p)$.

The starting point is the p -component generalization of the Halperin states (3.1). We take N particles and split them into p groups, with positions w_{ia} , where $i = 1, \dots, p$ and $a = 1, \dots, N/p$. Then

$$\psi_{m,n}(z) = \left[\prod_{i=1}^p \prod_{a<b}^{N/p} (w_{ia} - w_{ib})^m \right] \left[\prod_{i<j}^p \prod_{c,d}^{N/p} (w_{ic} - w_{jd})^n \right]. \quad (3.9)$$

Multicomponent states of this form were first discussed in Ref. [29]. More recently they have been studied in Ref. [30] for $p = 4$ to describe both spin and valley indices of electrons in graphene, and more generally in Refs. [31,32]. The states (3.9) have filling fraction

$$\nu = \frac{p}{pn + (m - n)}. \quad (3.10)$$

It is natural to think of these wave functions as describing objects with p internal states. This corresponds to the situation where each particle sits in the fundamental representation, \mathbf{p} of $SU(p)$. However, as we will see, there is also a generalization of our previous construction in which each particle has more internal states, corresponding to the symmetric representations of $SU(p)$.

1. Fundamental representation

We start by describing the simplest situation where the particles sit in the fundamental representation, meaning that each carries an internal index, $\sigma_a \in \{1, 2, \dots, p\}$. In this case, the wave functions (3.9) are spin singlets when $m = n + 1$.

To see this, note that the smallest number of particles that can form a singlet state is p . To achieve this, the spin degrees of freedom are completely antisymmetrized in what high-energy physicists would call a “baryon,”

$$B_{a_1 \dots a_p} = \epsilon^{\sigma_{a_1} \dots \sigma_{a_p}} |\sigma_{a_1}\rangle \dots |\sigma_{a_p}\rangle.$$

We can then form a spin-singlet state of type $(n + 1, n)$ by writing

$$\psi_{n+1,n}(z, \sigma) = \prod_{a<b}^N (z_a - z_b)^n \Phi_{(p)}(z, \sigma),$$

where, in analogy with (3.3), $\Phi_{(p)}$ describes p fully filled Landau levels, one for each type of spin,

$$\begin{aligned} \Phi_{(p)}(z, \sigma) &= \epsilon_{a_1 \dots a_N} (z_{a_1} \dots z_{a_p})^0 (z_{a_{p+1}} \dots z_{a_{2p}})^1 \dots \\ &\quad (z_{a_{N-p+1}} \dots z_{a_N})^{N-1} B_{a_1 \dots a_p} B_{a_{p+1} \dots a_{2p}} \dots \\ &\quad B_{a_{N-p+1} \dots a_N}. \end{aligned} \quad (3.11)$$

In the language of [6,8], this state exhibits clustering at order p . This means that the factor $\Phi_{(p)}$ remains nonzero if the positions of up to p particles coincide. However, it vanishes if $p + 1$ or more particles coincide. We will see in Sec. V that this wave function actually describes an Abelian quantum Hall

state. To generate non-Abelian quantum Hall states, we need to look at higher representations of $SU(p)$.

2. Symmetric representations

For $m > n + 1$, we can still interpret (3.9) as a spin-singlet state, but now each particle must carry a spin in a higher representation. We define $k = m - n$ and write the wave function as

$$\psi_{n+k,n}(z, \sigma) = \prod_{a<b}^N (z_a - z_b)^n \mathcal{P}[\Phi_{(p)}^k(z, \sigma)], \quad (3.12)$$

where \mathcal{P} projects onto the symmetrized product of spin states, meaning that each particle transforms in the k th symmetric representation of $SU(p)$. These states are all spin singlets, by the same argument that we gave in Sec. III A 4.

These states, still with $n = 1$, exactly reproduce the long-distance behavior of the matrix model ground states described in the previous section, with the same values of k and p .

3. Relationship to Read-Rezayi states

When $k = 2$, our states describe particles transforming in the symmetric representation of $SU(p)$ with dimension $\frac{1}{2}p(p + 1)$. They are p -clustered states with filling fraction

$$\nu = \frac{p}{pn + 2}.$$

Both of these properties are shared by the Read-Rezayi states [6]. We will now show that our states are spin-singlet generalizations of the Read-Rezayi states.

A particularly simple form of the Read-Rezayi state was presented in Ref. [33],

$$\psi_n^{RR}(z) = \mathcal{S} \left[\prod_{i=1}^p \prod_{a<b}^{N/p} (w_{ia} - w_{ib})^2 \right] \prod_{c<d}^N (z_c - z_d)^n, \quad (3.13)$$

where \mathcal{S} means that we symmetrize over all possible divisions of the particles into the p groups, while the z_a factor simply means that we include all particle positions rather than restricting to those in a specific group.

We will show that, after projection onto a particular spin state, the Blok-Wen wave function (3.12) coincides with the Read-Rezayi wave function (3.13), i.e.,

$$\mathcal{P}[\Phi_{(p)}^k(z, \sigma)] \mapsto \mathcal{S} \left[\prod_{i=1}^p \prod_{a<b}^{N/p} (w_{ia} - w_{ib})^2 \right]. \quad (3.14)$$

Proof. Our first task is to explain what \mapsto means in the above expression. It is useful to first revisit the case of the Pfaffian (3.6). There, the spin singlet wave function included the factor

$$|ab\rangle_1 = |1_a\rangle|1_b\rangle + |-1_a\rangle|1_b\rangle - 2|0_a\rangle|0_b\rangle,$$

and we can project onto a spin-polarized state simply by replacing $|ab\rangle_1 \mapsto |0_a\rangle|0_b\rangle$. This point was also made in Ref. [7].

Unfortunately, there is no analogous procedure for $SU(p)$ spins. There is, however, a generalization of the projection onto the highest spin states. The states $|1_a\rangle = |\uparrow_a\rangle|\uparrow_a\rangle$ and $|-1_a\rangle = |\downarrow_a\rangle|\downarrow_a\rangle$ have the property that both fundamental spins lie in

the same direction. This is something which also makes sense for $SU(p)$ spins. We therefore define the projection \mapsto in Eq. (3.13) as an operator which correlates the fundamental $SU(p)$ spins associated to each individual particle

$$\prod_{a=1}^N |\sigma_a\rangle |\sigma'_a\rangle \mapsto \prod_{a=1}^N \delta_{\sigma_a \sigma'_a}.$$

In particular, when we project the state $\mathcal{P}[\Phi_{(p)}^2(z, \sigma)]$, we correlate the two antisymmetrizations of spins inside the two $\Phi_{(p)}$ factors. The projection picks out the states in which these two spins associated with a given particle are the same. Spins associated with different particles can be different.

Having defined the projection, we turn to the structure of the state (3.12). It is helpful to think about collecting terms with some particular allocation of spin to each particle. For definiteness, let us consider the term where the particle at position $z_{(a-1)p+i} \equiv w_{i a}$ is given spin i . (Here $a = 1, \dots, N/p$.) Now consider the polynomial in z which multiplies this spin state. It is the antisymmetrization over all ways of permuting particles at positions $w_{i a}$ and $w_{i a'}$ of

$$(w_{11} \dots w_{p1})^0 (w_{12} \dots w_{p2})^1 \dots (w_{1N/p} \dots w_{pN/p})^{N/p-1}.$$

But this is simply the product of p separate Laughlin-like factors $w_{a1}^0 w_{a2}^1 \dots w_{aN/p}^{N/p-1}$, and hence the coefficient of the spin state is proportional to

$$\prod_{a=1}^p \prod_{i < j}^{N/p} (w_{ai} - w_{aj}), \quad (3.15)$$

which is essentially half of the Read-Rezayi state. To complete the argument, notice that whatever sign a particular spin allocation comes with, it comes with the same sign in both copies of $\Phi_{(p)}$. Hence overall, we obtain the square of this expression, *symmetrized* over all spin allocations. The projection does indeed result in the Read-Rezayi state (3.14). ■

The Read-Rezayi states are associated with the parafermion CFT $SU(2)_k/U(1)_k$. Meanwhile, our states are associated with $SU(k)_2$. The two are related by level-rank duality [20]. This means that our states include the non-Abelian anyons of the Read-Rezayi state; for example, $SU(3)_2$ includes the Fibonacci anyons. We will see how these emerge in Sec. V when we review the connection to conformal field theory. However, the Blok-Wen spin-singlet states arise from a CFT with no quotient, and hence contain additional anyonic degrees of freedom that are not part of the Read-Rezayi sequence of states.

IV. THE VIEW FROM CHERN-SIMONS THEORY

Until now, we have focused only on the properties of the matrix model (2.2). In this section, we explain where it comes from. The main idea, first proposed in Ref. [15] and recently explored in some detail in Ref. [16], is that the matrix model describes the dynamics of vortices in a $d = 2 + 1$ dimensional Chern-Simons theory. In Ref. [16], this connection was explained for Abelian Chern-Simons theories; in this section we generalize this picture to the non-Abelian case.

Our starting point is a Chern-Simons theory with gauge group

$$U(p)_{k',k} = \frac{U(1)_{k'} \times SU(p)_k}{\mathbf{Z}_p}. \quad (4.1)$$

The \mathbf{Z}_p quotient places a strong restriction on the allowed values of k' which must obey [13]

$$k' - kp \in p^2 \mathbf{Z}. \quad (4.2)$$

We denote the $U(1)$ gauge field as \tilde{a} and the $SU(p)$ gauge field as a . Both are to be thought of as emergent gauge fields in the condensed matter system. Their dynamics is governed by the Chern-Simons action

$$S_{CS} = - \int d^3x \frac{k'}{4\pi} \epsilon^{\mu\nu\rho} \tilde{a}_\mu \partial_\nu \tilde{a}_\rho + \frac{k}{4\pi} \text{Tr} \epsilon^{\mu\nu\rho} \left(a_\mu \partial_\nu a_\rho - \frac{2i}{3} a_\mu a_\nu a_\rho \right).$$

To this we couple nonrelativistic matter. We consider N_f bosons ϕ_i , with $i = 1, \dots, N_f$, each transforming in the \mathbf{p} of $SU(p)$, with charge 1 under the $U(1)$. Their action is

$$S_{\text{matter}} = \int d^3x i \phi_i^\dagger \mathcal{D}_0 \phi_i - \frac{1}{2m} \mathcal{D}_n \phi_i^\dagger \mathcal{D}_n \phi_i - \frac{\pi}{mk'} (\phi_i^\dagger \phi_i)^2 - \frac{\pi}{mk} (\phi_i^\dagger t^\alpha \phi_i)^2.$$

Here the subscripts $\mu, \nu, \rho = 0, 1, 2$ are space-time indices while $n = 1, 2$ is a spatial index only. The $SU(p)$ generators t^α are in the fundamental representation.

The coefficients of the ϕ^4 terms—which each describe the strength of a delta-function interaction between particles—are seemingly fine-tuned to be proportional to the Chern-Simons levels $1/k'$ and $1/k$. However, these coefficient are known to run logarithmically under RG flow and we have simply set them to their fixed points [34,35]. As an alternative justification, one could add fermions to this theory and, with this choice of the coefficient, complete it in a manner consistent with supersymmetry [36].

The full action is then

$$S_{3d} = S_{CS} + S_{\text{matter}} - \int d^3x \mu \tilde{a}_0,$$

where we have introduced a background charge μ . This causes the scalars to condense in the vacuum, breaking the gauge symmetry. This symmetry breaking is complete whenever $N_f \geq p$. In what follows we will take $N_f = p$. There is a unique ground state of the theory given by

$$\phi_i^a = \sqrt{\frac{\mu}{p}} \delta_i^a$$

with $a = 1, \dots, p$ the gauge index and $i = 1, \dots, p$ the flavor index. In this vacuum, the gauge and flavor symmetries are broken to

$$U(1)_{\text{gauge}} \times SU(p)_{\text{gauge}} \times SU(p)_{\text{flavour}} \longrightarrow SU(p)_{\text{diag}}. \quad (4.3)$$

The low-energy physics of this broken phase is *not* that of a quantum Hall fluid. However, this can change in the presence of vortices.

A. Vortices

The symmetry-breaking pattern (4.3) allows for the existence of vortex excitations in which the phase of ϕ winds. These have a rather nice property in this theory. The fine-tuning of the potential term described above means that vortices lie at the “Bogomolnyi point”; they satisfy first-order differential equations, rather than second-order ones. The vortex equations are

$$\tilde{f}_{12} = \frac{2\pi}{k'}(|\phi_i|^2 - \mu), \quad f_{12}^\alpha = \frac{2\pi}{k}\phi_i^\dagger t^\alpha \phi_i, \quad \mathcal{D}_z \phi_i = 0, \quad (4.4)$$

where the first and second of these equations are Gauss’ law for the Abelian and non-Abelian gauge field, respectively. In the Abelian case, Bogomolnyi vortices also appeared in the context of quantum Hall physics in Ref. [37].

These equations coincide with the vortex equations that arise in certain non-Abelian relativistic theories [17,18]. Their properties have been studied in some detail over the years (see, for example, [38–40]), especially in the case $k' = kp$, which is consistent with (4.2), where $U(1)$ and $SU(p)$ gauge fields naturally combine into a $U(p)$ gauge fields with the same level, meaning that Gauss’ law reads

$$f_{12} = \frac{2\pi}{k} \left(\phi_i \phi_i^\dagger - \frac{\mu}{p} \right).$$

The most striking fact about these equations is that they do not have a unique solution. Instead, for N vortices the most general solution has $2pN$ parameters [17]. (This is shown by index theorem techniques, generalizing previous results for Abelian vortices [41,42].) These parameters can be thought of as labeling the positions and internal orientations of the N vortices. In particular, there are no forces between vortices. They can sit anywhere on the plane.

We can pick out a unique solution by adding an external harmonic trap. We choose a trap which, when evaluated on vortices, is proportional to their angular momentum

$$V_{\text{trap}} = -\omega \int d^2x \frac{\mu |z|^2}{2p} \tilde{f}_{12}.$$

This, of course, changes the equations of motion. The previous vortex solutions now precess around the origin. There is a unique, stationary, lowest-energy state which occurs when all vortices coalesce at the origin to form a rotationally invariant configuration.

For a large number of vortices N the solution looks like a disk of radius

$$R \approx \sqrt{\frac{k'N}{\pi\mu p}}.$$

Inside this disk, the scalar fields vanish, $\phi \approx 0$, and Gauss’ law is satisfied by the presence of a constant magnetic flux $\tilde{f}_{12} \approx -2\pi\mu/k'$. The end result is that we have manufactured a disk-shaped region, inside of which lives an unbroken Chern-Simons theory with $U(p)$ gauge group (4.1). We view this as a region of quantum Hall fluid.

B. Quantizing the vortices

We can start with a few simple observations. Our $d = 2 + 1$ dimensional theory has a background charge density μ . From

the perspective of the vortices, this looks like an effective external magnetic field [16],

$$B = \frac{2\pi\mu}{p}. \quad (4.5)$$

With this information, we can estimate the filling fraction of the quantum Hall fluid of vortices. In an area $A = \pi R^2$, the number of states in the lowest Landau level is $BA/2\pi = k'N/p^2$. Since we have filled this disk with N vortices, we expect a filling fraction

$$\nu = \frac{p^2}{k'}. \quad (4.6)$$

To understand the detailed properties of this Hall fluid, we must look in more detail at the microscopic dynamics of the vortices. As we mentioned above, in the absence of a trap, the most general solution to the vortex equations (4.4) has $2pN$ collective coordinates. We think of these as parametrizing a manifold $\mathcal{M}_{p,N}$ which is called the *moduli space*. Each point on the moduli space corresponds to a different solution. Because the action is first order in time derivatives, rather than second order, the moduli space $\mathcal{M}_{p,N}$ should be thought of as the *phase space* of vortices rather than the configuration space.

An expression for the first-order dynamics of these vortices was derived in Ref. [43]. (See also [16,44,45]; these papers all deal with vortices in the Abelian theory, but the generalization to non-Abelian vortices is straightforward.) Unfortunately, this result is somewhat abstract and, for a large number of closely packed vortices, not particularly useful.

Instead, we turn to a more versatile construction of the non-Abelian vortex moduli space $\mathcal{M}_{p,N}$ first derived in Ref. [17] using D-brane techniques. We introduce a complex $N \times N$ matrix Z and p complex N -vectors $\phi_i, i = 1, \dots, p$. Then the vortex moduli space is isomorphic to the space of solutions to

$$\frac{B}{2}[Z, Z^\dagger] + \sum_{i=1}^p \phi_i \phi_i^\dagger = k' \mathbf{1}_N$$

with solutions identified if they are related by $Z \rightarrow UZU^\dagger$ and $\phi_i \rightarrow U\phi_i$ where $U \in U(N)$. This, of course, is precisely the phase space of the matrix model (2.2), with the constraint above arising as Gauss’ law (2.3). Moreover, the dynamics of the matrix model coincides with the dynamics expected on the vortex moduli space.

The phase space of the matrix model and the vortex moduli space $\mathcal{M}_{p,N}$ are believed to coincide as complex manifolds, with the same Kähler class. However, the symplectic form on the phase space inherited from the quotient construction does not coincide with that associated with vortex dynamics. This means that the matrix model should be used with some caution in extracting detailed properties of the vortices. However, our interest lies in the universality class of the quantum Hall ground states and here the matrix model is expected to give the right answer. Indeed, we have seen that the ground state of the matrix model lies in the same universality class as the Blok-Wen states. In the next section, we will confirm that these Blok-Wen states are indeed the ground state wave functions associated with the Chern-Simons theory (4.1).

There is one final subtlety. Classically, the matrix model describes the dynamics of vortices when the $U(1)$ and $SU(p)$

levels are equal: $k' = kp$. However, at the quantum level, there is a shift of the level. In the 3d Chern-Simons theory, the $SU(p)$ level is renormalized at one loop to $k \rightarrow k + p$. The matrix model captures the quantum dynamics when these shifted levels coincide. This requires

$$k' = (k + p)p.$$

This too satisfies the requirement (4.2). This is the value that we have used in Eq. (2.2) and throughout this paper. In particular, we see that the filling fraction (4.6) becomes $\nu = p/(k + p)$ in agreement with the matrix model result (2.11).

V. THE VIEW FROM CONFORMAL FIELD THEORY

In the previous section, we used vortices to construct a disklike region of space in which the low-energy dynamics is described by an unbroken $U(p)$ Chern-Simons theory. The microscopic dynamics of these vortices are described by the matrix model (2.2) whose ground states lie in the same universality class as the Blok-Wen wave functions. In this section, we close the circle and describe these states from the perspective of the boundary.

Our vortex construction has presented us with a Chern-Simons theory on a manifold with boundary, where the boundary is now the edge of the large vortex. On general grounds, we expect this boundary to support a chiral $U_{(k+p)p,k}$ WZW model [46,47]. This should manifest itself in two ways. First, the excitations of the matrix model should coincide with the excitations of a (suitably discretized) WZW model. We will return to this in future work [21]. Second, the ground state wave function—which, as we have seen, is of the Blok-Wen type—should arise as the correlation function in the conformal field theory [5]. This, of course, was how Blok and Wen originally derived their wave functions [7]. Here we review this construction, including the effect of the Abelian factor in the gauge group.

Let us first review some simple properties of the WZW models. The irreducible representations of the $SU(p)$ Kac-Moody algebra at level k are labeled by the corresponding representation of the $SU(p)$ Lie algebra. The latter are well known to be described by Young tableaux with up to $p - 1$ rows. The representations of $SU(p)_k$ are those Young tableaux which have no more than k boxes in the first row.

Each irreducible representation of the Kac-Moody algebra gives rise to a primary operator in the corresponding WZW model. We call these operators \mathcal{O}_R where R denotes the representation. The usual candidates for quantum Hall wave functions are the correlation functions of strings of chiral operators

$$\langle \mathcal{O}_R(z_1) \dots \mathcal{O}_R(z_N) \rangle, \quad (5.1)$$

where \mathcal{O}_R is the “electron operator” in the CFT or, more generally, the operator associated with the particle which forms the quantum Hall state. (More precisely, these should be thought of as conformal blocks of the nonchiral WZW theory.)

There is, however, a problem in identifying (5.1) as a wave function: for most choices of \mathcal{O}_R , there is no unique answer due to monodromies in the correlation function as z_a are varied. Instead, the number of conformal blocks is the number of singlets that arises when the many copies of R are fused

together. Typically this number will increase exponentially with N . Of course, this growth of conformal blocks is precisely what is needed to describe non-Abelian quasiholes in a quantum Hall state, but this should only occur for correlation functions in which quasihole operators are inserted. For a sensible quantum Hall interpretation, we want to have a unique ground state, and this means that (5.1) should yield a unique answer when only electron operators are inserted.

There is, fortunately, a choice of R for which (5.1) has a unique answer. We take N to be a multiple of p and choose the representation R which is maximally symmetric. In terms of Young diagrams, it is a single row of k boxes

$$R = \overbrace{\begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline \end{array}}^k \quad (5.2)$$

For $SU(2)_k$, this corresponds to the spin $s = k/2$ representation; for $SU(p)_k$ it is the k th symmetric representation. This, of course, is precisely the representation carried by the particles described by our wave functions (3.12).

To see that there is indeed a unique singlet when we take (5.2), we need to look at the fusion rules. (See, for example [48], for a detailed description of how these are computed [49].) For $SU(2)_k$, it is straightforward to show that the fusion of two spin $s = k/2$ representations leaves only the singlet $s = 0$. Written in terms of the dimension $\mathbf{d} = 2s + 1$ of the representation, this reads

$$(\mathbf{k} + \mathbf{1}) \star (\mathbf{k} + \mathbf{1}) = \mathbf{1}.$$

For $SU(3)_k$, one finds that the k th symmetric representation, which we denote Sym_k , has fusion rules

$$\text{Sym}_k \star \text{Sym}_k = \overline{\text{Sym}}_k,$$

while

$$\text{Sym}_k \star \overline{\text{Sym}}_k = \mathbf{1},$$

which tells us that three copies of Sym_k can fuse only to the singlet. More generally, for $SU(p)_k$ we define the representations

$$Y_i = i \left\{ \overbrace{\begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array}}^k \right.$$

which consist of k columns (the maximum number allowed by the level) and i rows. These are precisely the representations that we saw in Sec. II B when discussing the ground states of the matrix model. In this notation, our maximally symmetric representation is $Y_1 = \text{Sym}_k$, while $Y_{N-1} = \overline{\text{Sym}}_k$. The Y_i have the nice property that they fuse only among themselves [50]

$$Y_i \star Y_j = Y_{i+j \bmod p}.$$

This is enough to ensure that

$$\overbrace{\text{Sym}_k \times \text{Sym}_k \times \dots \times \text{Sym}_k}^p = \mathbf{1},$$

which tells us that (5.1) has a unique answer when R is taken to be the maximally symmetric representation. Now our job is to compute it.

The wave function as a correlation function

The standard tool to compute correlation functions in WZW models is the Knizhnik-Zamolodchikov (KZ) equation [51]. Usually this is employed to compute 4-point functions but since we expect a unique solution to (5.1), we can hope to use it in the present case to compute higher-point functions.

The KZ equation reads

$$\left(\frac{\partial}{\partial z_a} - \frac{1}{k+p} \sum_{b \neq a}^N \frac{T_a^\alpha \otimes T_b^\alpha}{z_a - z_b} \right) \langle \mathcal{O}_R(z_1) \dots \mathcal{O}_R(z_N) \rangle = 0, \quad (5.3)$$

where T^α is the Hermitian generator for the k th symmetric representation. These obey the $SU(p)$ algebra

$$\begin{aligned} \mathcal{P}[\Phi_{(p)}^k(z, \sigma)] = & \left[\mathcal{A} \prod_{1 \leq a < b \leq N/p} (z_a - z_b) |\sigma_1\rangle \dots |\sigma_1\rangle \otimes \prod_{N/p < a < b \leq 2N/p} (z_a - z_b) |\sigma_2\rangle \dots |\sigma_2\rangle \otimes \dots \right. \\ & \left. \otimes \prod_{(p-1)N/p < a < b \leq N} (z_a - z_b) |\sigma_p\rangle \dots |\sigma_p\rangle \right]^k. \end{aligned} \quad (5.5)$$

Here we have placed the first N/p particles in the same spin state, the next N/p in a different spin state, and so on. The \mathcal{A} symbol means that we then antisymmetrize over all particles.

The generators T^α in Eq. (5.3) can be viewed as acting symmetrically on what were originally k distinct fundamental factors,

$$T^\alpha = \overbrace{t^\alpha \otimes \mathbf{1} \otimes \dots \otimes \mathbf{1}}^k + \text{symmetric permutations}$$

with t^α the generator in the fundamental representation. The normalization in the KZ equation ensures that we have $t_{ij}^\alpha t_{kl}^\alpha = \delta_{il} \delta_{jk} - \frac{1}{p} \delta_{ij} \delta_{kl}$, where the group indices $i, j, k, l = 1, \dots, p$. This means that if the operator $T_a^\alpha \otimes T_b^\alpha$ acts on a state where particles a and b have the same spin, this tensor product returns the same spin state multiplied by a factor $(1 - 1/p)$. By contrast, if the two particles have different spins, it returns a superposition of the same state with a factor $-1/p$, and a state with the particles swapped with no factor.

We will start by considering the action of the nonderivative part of the KZ operator on $\mathcal{P}[\Phi_{(p)}^k]$. Expanding out (5.5) will result in a slew of terms, each of them containing k factors. Let us look at one of these terms—call it X . We are going to figure out the coefficient in front of X after the action of the $T \otimes T$ term in the KZ operator. There are two contributions. One arises when $T \otimes T$ acts on X itself. The other arises from $T \otimes T$ hitting other terms in the expansion of $\mathcal{P}[\Phi_{(p)}^k]$ so that they are mapped into X . We deal with these in turn.

Let us first look at the action of $T \otimes T$ on X . Suppose that particle a has the same spin as particle b in d_b of the k factors.

$[T^\alpha, T^\beta] = i f^{\alpha\beta\gamma} T^\gamma$, with the normalization $f^{\alpha\gamma\delta} f^{\beta\gamma\delta} = 2p \delta^{\alpha\beta}$, where p appears in its role as the dual Coxeter number of $SU(p)$.

1. Solving the Knizhnik-Zamolodchikov equation

We will now show that the KZ equation (5.3) is solved by

$$\langle \mathcal{O}_R(z_1) \dots \mathcal{O}_R(z_N) \rangle = \prod_{a < b}^N (z_a - z_b)^{-k/p} \mathcal{P}[\Phi_{(p)}^k(z, \sigma)] \quad (5.4)$$

with $\Phi_{(p)}$ defined in Eq. (3.11).

It will be useful to first rewrite our ansatz in slightly more concrete form. As discussed up to (3.15), it is simple to check that, up to an unimportant normalization,

Then acting with $T \otimes T$ will, among other things, return X with a coefficient

$$\begin{aligned} & -\frac{1}{k+p} \sum_{b \neq a}^N \frac{k d_b (1 - 1/p) - k(k - d_b)/p}{z_a - z_b} \\ & = -\frac{1}{k+p} \sum_{b \neq a}^N \frac{k d_b - k^2/p}{z_a - z_b}. \end{aligned} \quad (5.6)$$

This is our first result.

Now let us see how we can get terms proportional to X by the action of $T \otimes T$ on some other term Y . This can occur only if Y differs from X by swapping the spins of i and j in just one factor. Suppose that, within Y , particle i has spin a and particle j has spin $b \neq a$. Then the $T \otimes T$ term in the KZ operator will map Y to X with coefficient

$$+\frac{1}{k+p} \frac{1}{z_a - z_b} \prod_{(d \neq a|i)} \frac{z_b - z_d}{z_a - z_d} \prod_{(d \neq b|j)} \frac{z_a - z_d}{z_b - z_d},$$

where the notation $(d \neq a|i)$ means that we take the product over all particles $d \neq a$ which carry spin i . Something nice now happens when this expression is summed over all particles j which carry spin b ; the terms combine to give

$$+\frac{1}{k+p} \left[- \sum_{(d \neq a|i)} \frac{1}{z_a - z_d} + \sum_{(d|i)} \frac{1}{z_a - z_d} \right]. \quad (5.7)$$

This is our second result.

The total coefficient multiplying the term X after the action of $T \otimes T$ is then given by the sum of (5.6) and (5.7). It is

$$\sum_{b \neq a} \frac{k/p - d_b}{z_a - z_b}.$$

The key point is that this coefficient is precisely canceled by the derivative term in the KZ equation, since $(z_a - z_b)$ appears with the power $-k/p + d_b$ in the correlation function (5.4). Note that the actual coefficient of a given term X typically includes, in addition to (5.8), a symmetry factor to account for the fact that X may appear many times in the original expansion of $\mathcal{P}[\Phi_{(p)}^k]$. This is not relevant for our final result; the same symmetry factor appears every time X arises. This concludes our proof that the correlation function (5.4) indeed solves the KZ equation (5.3).

2. The full wave function

The solution to the KZ equation ensures that our wave function (3.12) can be written as the product

$$\begin{aligned} \psi_{n+k,n}(z, \sigma) \\ = \prod_{a < b}^N (z_a - z_b)^{n+k/p} \langle \mathcal{O}_R(z_1) \dots \mathcal{O}_R(z_N) \rangle e^{-\sum_a |z_a|^2 / 4l_B^2} \end{aligned} \quad (5.8)$$

where, for once, we have restored the exponential factor common to all lowest Landau level wave functions. The first factor is, of course, a Laughlin wave function and can be expressed as a correlation function for a free compact boson. The fractional exponent is unusual, but has been seen before in constructing Halperin wave functions from a CFT [5,52], where the same factorization into “spin” degrees of freedom, captured by $\langle \mathcal{O}_R(z_1) \dots \mathcal{O}_R(z_N) \rangle$, and “charge” degrees of freedom captured by the Laughlin wave function also occurs.

The full wave function can be written as a correlation function in the WZW model with algebra

$$U(p)_{(k+np)p,k} = \frac{U(1)_{(k+np)p} \times SU(p)_k}{\mathbf{Z}_p}. \quad (5.9)$$

Note that our matrix model describes the Blok-Wen states with $n = 1$. Happily, in that case, the WZW model (5.9) indeed arises as the description of the boundary dynamics of the Chern-Simons theory with gauge group (4.1).

The $U(1)$ part is described by a compact chiral boson ϕ and the correlation function

$$\begin{aligned} \left\langle \prod_{a=1}^N e^{i\sqrt{(k+np)/p}\phi(z_a)} e^{-\int d^2z' \sqrt{(k+np)/p}\phi(z')/2\pi l_B^2} \right\rangle \\ = \prod_{a < b}^N (z_a - z_b)^{n+k/p} e^{-\sum_a |z_a|^2 / 4l_B^2} \end{aligned}$$

gives the Laughlin part of the wave function (5.8) in the usual manner [5].

3. Quasiholes as non-Abelian anyons

The presence of the $SU(p)_k$ factor ensures that our quantum Hall states have non-Abelian anyons for $k > 1$. These quasiholes are associated with primary operators in the WZW conformal field theory and their properties are expected to be determined by the fusion rules and braiding inherited from the CFT. Although this story is well known (see, for example, [53]), we pause here to point out a few of the more prominent examples.

An example: Ising anyons. Ising anyons are well known to appear in the Moore-Read state which is associated with the $SU(2)_2$ WZW model [54]. The primary operators carry spin $s = 0, \frac{1}{2}$, and 1. We denote these representations using their dimension $\mathbf{d} = 2s + 1$. As we saw above, we identify the electron with the spin 1, or $\mathbf{d} = 3$ representation. The $\mathbf{d} = 2$ primary is then identified with the quasihole, with the corresponding fusion rules given by

$$2 \star 2 = 1 \oplus 3, \quad 2 \star 3 = 2, \quad 3 \star 3 = 1.$$

These are the fusion rules for Ising anyons.

Another example: Fibonacci anyons. Fibonacci anyons are known to arise as the quasiholes in the \mathbf{Z}_3 parafermionic Read-Rezayi state. This is governed by the coset model $SU(2)_3/U(1)$. As we have seen above, these states arise in our construction as the $SU(3)_2$ WZW model. This is related to the parafermion CFT through level-rank duality and a quotient. The anyon is associated with the primary operator which transforms in the adjoint representation $\mathbf{8}$ of $SU(3)$. It is simple to compute the fusion rules in $SU(3)_2$ to find

$$8 \star 8 = 1 \oplus 8.$$

This is indeed the fusion rule for Fibonacci anyons. A nice review of these objects can be found in Ref. [55].

It remains an open problem to identify these anyonic states directly within the matrix model. For the Laughlin states, it was shown in Ref. [16] that the matrix model provides a construction of quasihole states which are analytically more tractable than the traditional approach. In particular, the Berry phase computation of [26] can be performed exactly, without resorting to the plasma analogy. It seems plausible that the matrix model may also prove useful in understanding the properties of non-Abelian anyons. We hope to return to this in the future.

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