Quantum variance: A measure of quantum coherence and quantum correlations for many-body systems

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Quantum coherence is a fundamental common trait of quantum phenomena, from the interference of matter waves to quantum degeneracy of identical particles. Despite its importance, estimating and measuring quantum coherence in generic, mixed many-body quantum states remains a formidable challenge, with fundamental implications in areas as broad as quantum condensed matter, quantum information, quantum metrology, and quantum biology. Here, we provide a quantitative definition of the variance of quantum coherent fluctuations (the *quantum variance*) of any observable on generic quantum states. The quantum variance generalizes the concept of thermal de Broglie wavelength (for the position of a free quantum particle) to the space of eigenvalues of any observable, quantifying the degree of coherent delocalization in that space. The quantum variance is generically measurable and computable as the difference between the static fluctuations and the static susceptibility of the observable; despite its simplicity, it is found to provide a tight lower bound to most widely accepted estimators of "quantumness" of observables (both as a feature as well as a resource), such as the Wigner-Yanase skew information and the quantum Fisher information. When considering bipartite fluctuations in an extended quantum system, the quantum variance expresses genuine quantum correlations among the two parts. In the case of many-body systems, it is found to obey an area law at finite temperature, extending therefore area laws of entanglement and quantum fluctuations of pure states to the mixed-state context. Hence the quantum variance paves the way to the measurement of macroscopic quantum coherence and quantum correlations in most complex quantum systems.

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I. INTRODUCTION

Quantum mechanics introduces two fundamentally new traits in a physical system: (1) an intrinsic uncertainty on the knowledge of observables (Heisenberg's uncertainty or coherent quantum fluctuations) and (2) a superior form of correlation among degrees of freedom, stemming from correlated quantum uncertainties (or entanglement) [1–3]. Quantum uncertainty of observables persists even at zero temperature in the form of so-called zero-point fluctuations, responsible for macroscopic quantum phenomena such as the inability of liquid Helium to solidify at ambient pressure [4]. On the other hand, two quantum systems (hereafter called A and B) can exhibit correlations far exceeding any classical counterpart, which for pure quantum states are embodied by entanglement [5].

The supremacy of both fluctuations and correlations in quantum systems, as compared to classical ones, is at the heart of the complexity of many-body quantum states, challenging all realms of quantum physics, from relativistic quantum field theory to atomic/molecular physics and quantum condensed matter. At the same time, quantum fluctuations and correlations (going beyond entanglement [6]) are by now recognized as essential ingredients for the supremacy of quantum devices over classical ones in the context of quantum metrology [7–9] and potentially also for quantum information processing [6].

Despite their fundamental as well as practical importance, quantum coherence and quantum correlations remain very hard to both quantify theoretically and to measure experimentally. Quantum uncertainty of an observable and quantum entanglement between two subsystems are generically defined only for pure states [1,5]. In the case of generic, real-life mixed states, the most widespread concept of quantum coherence is related to the thermal de Broglie wavelength (TdBWL) [10], expressing the spatial extent of coherent quantum fluctuations for a single quantum particle in free space; but this concept does not even extend to systems as simple as a particle in a potential. More recently, several definitions of mixed-state quantum coherence have been put forward [11–17], which nonetheless share the generically prohibitive requirement of knowing the full density matrix of the state, and they are therefore limited to few-body systems. As for the entanglement of mixed states, one can only provide sufficient conditions (witnesses) for the presence of entanglement between the components of the system [18–20]. Yet, even for nonentangled mixed states, it has been recognized that quantum correlations may exist, associated with the violation of classical information-theory identities, and quantified via the so-called quantum discord [6]; alternatively quantum correlations can be captured via the minimum quantum uncertainty (quantified by the skew information [11] or quantum Fisher information [7]) of local observables [13,21]. Despite their deep conceptual meaning, entanglement and quantum correlations of mixed states remain in general information-theoretical objects, generically accessible (to calculations and measurements alike) only when defined between two (or a few) elementary quantum degrees of freedom [21–27].

Here we show that both quantum coherent fluctuations and quantum correlations in generic quantum states can be quantified in terms of elementary physical concepts. The variance of fluctuations in generic mixed states possesses in fact an *additive* structure, in which the incoherent/thermal part can be separated from the coherent/quantum part, which we name *quantum variance* (QV). The QV is defined in terms of the violation of a classical, static fluctuation-dissipation relation, and as such it is fully computable and measurable for generic systems. The QV of a given observable is a measure of its genuine quantum uncertainty in mixed states, and, in the case of bipartite fluctuations, it represents a measure of correlated quantum uncertainties, namely of quantum correlations. Remarkably, the QV is convex (namely it *decreases* upon incoherent mixing of states with the same QV), and it gives a tight lower bound to both the Wigner-Yanase skew information [11,12] and to the quantum Fisher information [28], which are widely accepted, yet generically prohibitive measures (for both calculations and experiments) of the quantumness of observables and of correlations [13,14,21,29–31].

The structure of the paper is as follows. Section II introduces the separation between coherent and incoherent fluctuations, and the definition of QV. Section III reviews the fundamental properties of the QV in connection with known measures of "quantumness" of observables, and as a measure of quantum coherence. Section IV describes the volume-law scaling of QV for generic extensive observables. Section V illustrates the fundamental separation of scales between thermal and quantum fluctuations at a thermal critical point. Section VI discusses the area-law scaling of the QV of bipartite fluctuations. Section VII illustrates the link between the QV and other measures of quantum correlations in the case of free fermions. Section VIII proposes a realistic experimental setup to measure the QV in cold-atom quantum simulators of correlated lattice models; and finally, Sec. IX elaborates on the general link between QV on the one side, and entanglement and quantum correlations on the other; and on the experimental measurement of the QV with cold-atom quantum simulators. The technical aspects are kept to a minimum level in the main text, and they are postponed to the appendices.

II. SEPARATING CLASSICAL AND QUANTUM FLUCTUATIONS

Let us first show that, given a density matrix $\hat\rho$ such that $\langle\ldots\rangle = \mathrm{Tr}[\hat\rho(\ldots)]/\mathrm{Tr}\hat\rho$, and a generic Hermitian operator $\hat O$, the fluctuations of the latter can be written as

$$\langle\delta^2\hat O\rangle = \langle\hat O^2\rangle - \langle\hat O\rangle^2 = \langle\delta^2\hat O\rangle_T + \langle\delta^2\hat O\rangle_Q, \tag{1}$$

where $\langle\delta^2\hat O\rangle_T$ represents thermal/incoherent fluctuations, while $\langle\delta^2\hat O\rangle_Q$ represent quantum/coherent fluctuations. In the following we shall focus our attention on thermal equilibrium states, but the whole treatment is readily generalizable to arbitrary density matrices (see Appendix B). If $\hat\rho = e^{-\beta\hat{\mathcal H}}/\mathcal Z$ ($\mathcal Z = \mathrm{Tr}(e^{-\beta\hat{\mathcal H}})$) is the thermal density matrix of a system of Hamiltonian $\hat{\mathcal H}$ at temperature $k_BT = 1/\beta$, and $[\hat O,\hat{\mathcal H}] = 0$, it is well known that the fluctuations of $\hat O$ satisfy a (classical) fluctuation-dissipation theorem

$$\langle\delta^2\hat O\rangle = \chi_O k_B T, \tag{2}$$

where $\chi_O = -\partial^2F/\partial h^2|_{h=0}$ is the susceptibility associated with the application of a term $-h\hat O$ to the Hamiltonian, and $F = -k_BT\ln\mathcal Z$ is the free energy. On the other hand, if $[\hat O,\hat{\mathcal H}] \neq 0$, the quantum uncertainty on the value of $\hat O$ adds up to the thermal fluctuations, and, as a result

$$\langle\delta^2 O\rangle \geqslant \chi_O k_B T = \frac{1}{\beta}\int_0^\beta d\tau\,\langle\delta\hat O(\tau)\delta\hat O(0)\rangle =: \langle\delta^2\hat O\rangle_T, \tag{3}$$

where $\hat O(\tau) = e^{\tau\hat{\mathcal H}}\hat O e^{-\tau\hat{\mathcal H}}$ is the operator evolved in imaginary time. Equation (3) shows that thermal fluctuations do not exhaust the total fluctuations of the observable. It is then natural to define the QV for the observable $\hat O$ as the residual fluctuations, or as the violation of the classical fluctuation-dissipation relation of Eq. (2):

$$\langle\delta^2\hat O\rangle_Q = \langle\delta^2\hat O\rangle - \chi_O k_B T. \tag{4}$$

The QV has a particularly suggestive interpretation in the context of a path-integral representation of the partition function of the system, using a basis of Hilbert space which diagonalizes the $\hat O$ operator [see Fig. 1(a)]. As discussed in Appendix B, this allows one to cast the partition function in the form

$$\mathcal Z = \int \mathcal D[O(\tau)]e^{-S[O(\tau),\partial_\tau O(\tau),\ldots]}, \tag{5}$$

where $O(\tau)$ is a periodic trajectory ($O(0) = O(\beta)$) in the space of eigenvalues of $\hat O$, associated with a Feynman path in the basis diagonalizing $\hat O$, and S is the associated action weighting the trajectory. When assigning a path-integral expression to each of the terms in Eq. (4), one can easily find that (see Appendix B)

$$\langle\delta^2\hat O\rangle_Q = \left\langle\frac{1}{\beta}\int d\tau(O(\tau) - \bar O)^2\right\rangle_S, \tag{6}$$

where $\langle\ldots\rangle_S$ is the average over the ensemble of paths $O(\tau)$ weighed by the action S, and

$$\bar O = \bar O[O(\tau)] = \frac{1}{\beta}\int d\tau\,O(\tau) \tag{7}$$

is the centroid of the path [32]. Equation (6) shows that the QV represents the (squared) amplitude of the imaginary-time fluctuations of the trajectory $O(\tau)$ around the path centroid. Clearly such fluctuations have a genuine quantum origin [32–34]. If $\hat O$ is the position $\hat x$ of a one-dimensional particle, in Appendix C, we show that $\langle\delta^2\hat x\rangle_Q \sim \lambda_{\mathrm{dB}}^2$, namely, the QV is tightly related to the quantum uncertainty on the position expressed by the TdBWL λ_{dB}. When moving to higher dimensions and generic quantum systems, the QV generalizes therefore the concept of TdBWL (or quantum coherence length) to the space of eigenvalues of any Hermitian operator. And, most remarkably, it does so in a computable and measurable manner, being expressed as the difference between a fluctuation property and a response function.

III. PROPERTIES OF THE QUANTUM VARIANCE

A. Quantum variance as a lower bound to skew and quantum Fisher information

The QV represents a physically measurable lower bound to fundamental quantities in quantum information. The Dyson-Wigner-Yanase (DWY) skew information [35]

$$I_\alpha(\hat O,\hat\rho) = -\tfrac{1}{2}\mathrm{Tr}\{[\hat O,\hat\rho^\alpha][\hat O,\hat\rho^{1-\alpha}]\} \tag{8}$$

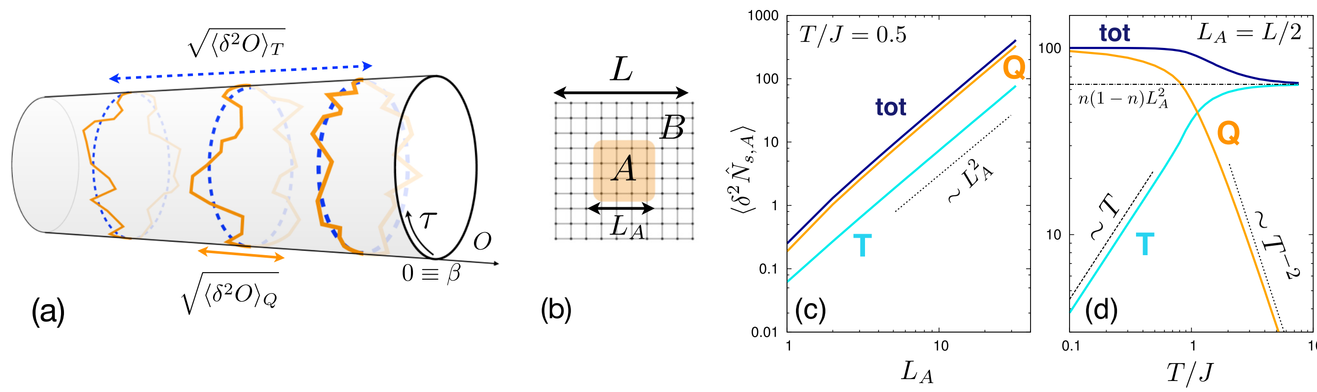

FIG. 1. Thermal vs quantum fluctuations. (a) Different imaginary-time paths $O(\tau)$ in the space of eigenvalues of the observable $\hat{O}$ are shown, associated with the path-integral representation of a generic mixed state $\hat{\rho}$. While the thermal/incoherent fluctuations $\langle\delta^2 O\rangle_T$ are associated with the fluctuations of the centroids of the paths (dashed blue lines), the quantum/coherent fluctuations $\langle\delta^2 O\rangle_Q$ are associated with the fluctuations of the paths around their centroids. (b) Geometry of the $A - B$ bipartition of an extended quantum system used in the text. (c) Scaling of the total (tot), thermal (T) and quantum (Q) fluctuations of the staggered particle number $N_{s,A}$ on a subsystem A of size L_A for hardcore bosons on a square lattice at temperature $T/J = 0.5$ (the system is defined on a $L \times L$ torus with $L = 32$). All fluctuation terms exhibit volume-law scaling. Here and in the following graphs, the error bar is smaller than or comparable to the line thickness. (d) Temperature dependence of the $N_{s,A}$ fluctuations for a subsystem of linear size $L_A = L/2$. The dashed line indicates the infinite temperature limit, in which each lattice site fluctuates independently, with a shot-noise variance $n(1-n)$ where $n = 1/2$ is the lattice filling.

with $\alpha \in [0,1]$, generalizing the Wigner-Yanase skew information ($\alpha = 1/2$) [11], probes the quantum uncertainty of $\hat{O}$ stemming from its noncommutativity with $\hat{\rho}$. As shown in Appendix D, the QV is simply

$$\langle\delta^2 \hat{O}\rangle_Q[\hat{\rho}] = \int_0^1 d\alpha\, I_\alpha. \tag{9}$$

From the convexity of I_α [35] follows the convexity of the QV. Moreover one can prove that

$$\langle\delta^2 \hat{O}\rangle_Q[\hat{\rho}] \leqslant I_{1/2}(\hat{O},\hat{\rho}) \tag{10}$$

(the equality holding for pure states). Finally, the quantum Fisher information [28]

$$F_Q(\hat{O};\hat{\rho}) = 2\sum_{nm} |\langle n|\hat{O}|m\rangle|^2 (p_n - p_m)^2/(p_n + p_m) \tag{11}$$

(where $\hat{\rho} = \sum_n p_n |n\rangle\langle n|$) expresses the sensitivity of the density matrix to a unitary transformation $\hat{U}(h) = e^{-ih\hat{O}}$ generated by the observable $\hat{O}$, and it quantifies the fundamental metrological gain in using the state $\hat{\rho}$ to estimate the parameter h [7]. As shown in Appendix E,

$$\langle\delta^2 \hat{O}\rangle_Q[\hat{\rho}] \leqslant F_Q(\hat{O};\hat{\rho})/4. \tag{12}$$

The inequality becomes an equality for pure states. As discussed later, the inequalities satisfied by the QV have considerable implications concerning its importance for entanglement witnessing and metrological applications. Conversely, the computability and measurability of QV gives unprecedented insight into the skew and quantum Fisher information for quantum many-body systems.

B. Quantum variance as a measure of coherence

Several studies have recently concentrated their attention on proper measures of quantum coherence of a given observables [14–17]. An indisputable signature of quantum coherence of a given observable $\hat{O}$ associated with a density matrix $\hat{\rho}$ is the existence of off-diagonal matrix elements of $\hat{\rho}$ on the basis of eigenstates of $\hat{O}$—in fact, this is the very definition of coherence of an observable. Given two nondegenerate eigenstates $|O_1\rangle$ and $|O_2\rangle$ of $\hat{O}$, it is immediate to see that the existence of coherence between the two eigenstates in the density matrix $\hat{\rho}$ implies the existence of imaginary-time fluctuations in a path integral representation of ρ built on a basis of eigenstates that diagonalizes $\hat{O}$, as in Eq. (5). Indeed, the coherence of the density matrix reads

$$\rho_{O_1,O_2} = \langle O_1|\hat{\rho}|O_2\rangle = \frac{\mathcal{Z}(O_1,O_2)}{\mathcal{Z}} = \frac{1}{\mathcal{Z}}\int_{O(0)=O_1,O(\tau)=O_2} \mathcal{D}[O(\tau)]e^{-S}, \tag{13}$$

namely, it is the ratio of two partition functions: the ordinary one $\mathcal{Z}$, which is a sum over periodic $O(\tau)$ paths ($O(0) = O(\beta)$) as in Eq. (5), and a modified one $\mathcal{Z}(O_1,O_2)$, which is the sum over constrained paths $O(0) = O_1$, $O(\beta) = O_2$. It is obvious that, while the ordinary partition function is nonzero even in the absence of imaginary-time evolution (namely $O(\tau) = \text{const}$ for all paths), the modified partition function is nonzero iff the paths $O(\tau)$ display a nontrivial imaginary-time evolution. This means that imaginary-time fluctuations of the observable $\hat{O}$, whose amplitude is expressed by the QV, imply the existence of coherence between different eigenvectors of $\hat{O}$, and viceversa. The above property applies to thermal as well as nonthermal states alike, as in the latter case one can always cast the density matrix in the form of an equilibrium one (Appendix B).

Beyond the physical intuition provided by the above discussion, one can easily prove that the QV verifies fundamental requirements for a measure of quantum coherence, thanks to its relationship to the DWY skew information as in Eq. (9). We already mentioned convexity as an important mathematical requirement. Moreover, as shown in Ref. [36], the DWY skew information related to the observable $\hat{O}$ cannot grow under closed-system or dissipative dynamics which is *translationally invariant* (TI) with respect to translations generated by $\hat{O}$. More formally, if $\mathcal{E}$ is a completely positive, trace-preserving map such that

$$\mathcal{E}(e^{-i\hat{O}t}\hat{\rho}e^{i\hat{O}t}) = e^{-i\hat{O}t}\mathcal{E}(\hat{\rho})e^{i\hat{O}t} \tag{14}$$

then $I_\alpha(\hat{O},\hat{\rho}) \geqslant I_\alpha(\hat{O},\mathcal{E}(\hat{\rho}))$ for all α. This property of monotonicity under TI dynamics is straightforwardly inherited by the QV via Eq. (9).

As we shall later discuss, the QV of thermal equilibrium states decreases generically (but not always) when increasing the temperature. Monotonicity under thermal mixing is a different requirement than convexity. Indeed, if higher energy states have a larger QV than lower energy ones, it can happen that QV increases upon increasing the temperature. Convexity only states that the thermal QV at a given temperature is lower than the average QV of the energy eigenstates, but it does even guarantee that the thermal QV is lower than that of the ground state. It is also different from monotonicity under TI dynamics. Indeed, thermal mixing can be seen as the following operation: starting from $\hat{\rho}(\beta) = e^{-\beta\hat{\mathcal{H}}}/\mathcal{Z}(\beta)$, one has that $\hat{\rho}(\beta') = \mathcal{E}(\hat{\rho}(\beta)) = \hat{A}^\dagger\hat{\rho}(\beta)\hat{A}$ where

$$\hat{A} = \sqrt{\frac{\mathcal{Z}(\beta)}{\mathcal{Z}(\beta')}}e^{-\frac{1}{2}(\beta'-\beta)\hat{\mathcal{H}}}. \tag{15}$$

Evidently, if $[\hat{O},\mathcal{H}] \neq 0$, any variation of the temperature (either increasing or decreasing) is not a TI operation. A slight thermal increase in the QV can be observed close to thermal critical points (see Sec. V), and it implies that the same behavior applies to the DWY skew information (at least for some values of the α parameter) via Eq. (9); therefore it is not in contradiction with the QV being a measure of coherence.

IV. QUANTUM VARIANCE OF A GLOBAL OBSERVABLE AND VOLUME LAW

Due to its inherent quantum nature, the QV exhibits very special size and temperature dependencies. In the following, we shall concentrate on thermal equilibrium states, and we start our analysis with the case of a generic, macroscopic observable $\hat{O}$ that does *not* commute with the Hamiltonian of the system. As an example we consider the case of two-dimensional hardcore bosons on the square lattice:

$$\hat{\mathcal{H}} = -J\sum_{\langle ij\rangle}(\hat{b}_i^\dagger\hat{b}_j + \text{H.c.}), \tag{16}$$

where $\hat{b}_i, \hat{b}_i^\dagger$ are hardcore boson operators, satisfying the (anti)-commutation relations $\{b_i,b_i^\dagger\} = 1$ and $[b_i,b_j] = [b_i,b_j^\dagger] = 0$ $(i \neq j)$. We treat this model with a numerically exact quantum Monte Carlo algorithm based on the stochastic series expansion approach [37], which allows us to investigate the imaginary-time dynamics of many-body observables [38]. The Hamiltonian $\hat{\mathcal{H}}$ does not commute with any finite-wave-vector Fourier component of the density profile, and in particular with the staggered particle number

$$\hat{N}_\text{s} = \sum_i(-1)^i\hat{b}_i^\dagger\hat{b}_i. \tag{17}$$

To investigate the scaling of fluctuations (both thermal and quantum) we isolate a subsystem A of linear size L_A in a larger system [of linear size L, see Fig. 1(b)], and we investigate the scaling of local observables/fluctuations in A with the size of the A region itself. This approach allows one to extract scaling properties while using a single simulation box, and it is also directly applicable to experiments giving access to local properties, such as those based on quantum-gas microscopy [39].

Figure 1(c) shows that both the thermal and the quantum contribution to fluctuations obey a *volume-law* scaling in the example at hand:

$$\langle\delta^2\hat{N}_{s,A}\rangle_T, \langle\delta^2\hat{N}_{s,A}\rangle_Q \sim L_A^d, \tag{18}$$

where $\hat{N}_{\text{s,A}} = \sum_{i\in A}(-1)^i\hat{b}_i^\dagger\hat{b}_i$. A volume-law scaling of quantum fluctuations is generically expected in systems with short-range interactions/hopping (hereafter called local systems/Hamiltonians) when the observable of interest is extensive, and its Heisenberg's uncertainty is the result of the noncommutativity between an extensive set of terms in the Hamiltonian and in the observable in question. The separation between thermal and quantum fluctuations gives rise to a very meaningful result when tracking the temperature dependence of the fluctuations on a subsystem of fixed size ($L_A = L/2$ in this case). As shown in Fig. 1(c), the thermal component grows linearly with T at low T, while the quantum component decreases monotonically with T starting from the zero-point fluctuations. Most remarkably, in the example at hand quantum fluctuations are found to dominate the total fluctuations, and they lead to a *monotonically* decreasing behavior of $\langle\delta^2\hat{N}_\text{s}\rangle$, in complete contradiction with the classical expectation that fluctuations should grow with temperature at low T.

V. QUANTUM VARIANCE DOES NOT GO CRITICAL AT A THERMAL TRANSITION

Having shown that QV generically obeys a volume law for extensive non-conserved observables in local systems, one can naturally ask what is the fate of QV at a thermal critical point, at which thermal fluctuations of the order parameter become superextensive. If the QV only captures the quantum mechanical part of fluctuations of the order parameter, one would naturally expect that its scaling is *not* modified at a thermal transition, given that the latter is purely driven by thermal fluctuations. To answer to this question, we consider a quantum many-body model exhibiting a thermal phase transition with an order parameter not commuting with the Hamiltonian; this is readily obtained by generalizing the hardcore-boson Hamiltonian of Eq. (16) to include a nearest-neighbor repulsion V:

$$\hat{\mathcal{H}} = -J\sum_{\langle ij\rangle}(\hat{b}_i^\dagger\hat{b}_j + \text{H.c.}) + V\sum_{\langle ij\rangle}\left(\hat{n}_i - \frac{1}{2}\right)\left(\hat{n}_j - \frac{1}{2}\right). \tag{19}$$

When $V > 2J$, the model has an Ising phase transition at finite temperature, marking the onset of a checkerboard density wave, with an order parameter given by the staggered density $\hat{N}_s$. Hence, as in the previous section, it is meaningful to investigate the temperature and size scaling of the fluctuations of the local staggered density $\hat{N}_s$. In particular, to mimic the

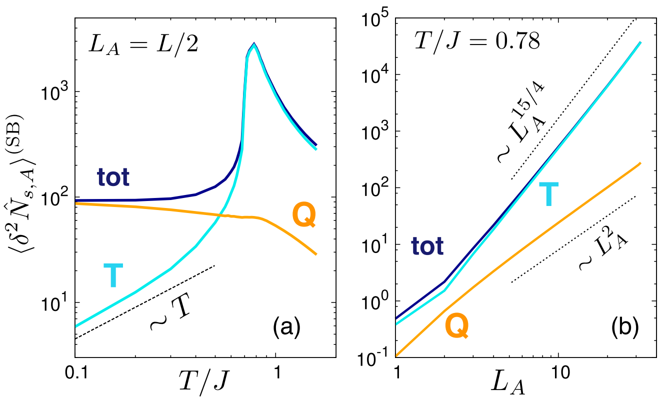

FIG. 2. Critical thermal fluctuations and noncritical quantum fluctuations. (a) Temperature dependence of the order parameter fluctuations at the Ising transition of $2d$ hardcore bosons with nearest-neighbor repulsion $V = 2.1J$; the sharp peak marks the transition at $T_c \approx 0.78J$. (b) Scaling of fluctuations close to the critical temperature: the total and thermal fluctuations are found to scale as $L_A^{d+\gamma/\nu}$ with $\gamma = 7/4$ and $\nu = 1$ for the $2d$ Ising universality class.

behavior in the thermodynamic limit (in which $\langle \hat{N}_{s,A} \rangle \neq 0$), we focus on the fluctuations around a finite-size estimate of the order parameter in the symmetry-breaking (SB) phase, given by $\langle |\hat{N}_{s,A}| \rangle$:

$$\langle \delta^2 N_{s,A} \rangle^{(\mathrm{SB})} = \langle \hat{N}_{s,A}^2 \rangle - \langle |\hat{N}_{s,A}| \rangle^2. \tag{20}$$

Figure 2(a) shows that the total and thermal fluctuations of the order parameter exhibit a sharp peak in correspondence with the Ising transition temperature, while the QV remains smooth at the transition. Not only is the QV not exhibiting any singularity, but it generally decreases with temperature (exhibiting an almost horizontal inflection around the transition[1]). A closeup on the scaling close to the critical point [Fig. 2(b)] finds that the total and thermal fluctuations exhibit the critical super-extensive scaling $L_A^{d+\gamma/\nu}$, where γ and ν are the critical exponents for the susceptibility and correlation length. On the other hand, the quantum variance maintains a volume-law scaling as in the noncritical regime. Therefore a critical point marks a net separation of scales between thermal and quantum fluctuations of the order parameter, the latter being essentially irrelevant in the thermodynamic limit. This observation substantiates the common wisdom that quantum mechanics is irrelevant for the universal properties at thermal critical points, and it shows that order parameters close to a critical point have the nature of emergent classical observables.

VI. QUANTUM VARIANCE OF BIPARTITE FLUCTUATIONS AND AREA LAW

The scaling properties of the QV change drastically when considering the case of bipartite fluctuations of an otherwise globally conserved quantity, such as the particle number $\hat{N}$. Such fluctuations have been the subject of several recent studies in view of their relationship to entanglement in the case of pure states [40–43], as well as at finite temperature, for which a suggestion of how to extract the quantum contribution to fluctuations has been proposed in Ref. [41]. In the case of mixed states, $[\hat{N},\hat{\mathcal{H}}] = 0$ implies automatically that $\langle \delta^2 \hat{N} \rangle_Q = 0$. Taking then *any* bipartition of the system into A and B subsystems, imaginary-time fluctuations of the local particle numbers N_A and N_B are perfectly anticorrelated, so that the QV in each subsystem is the same, $\langle \delta^2 \hat{N}_A \rangle_Q = \langle \delta^2 \hat{N}_B \rangle_Q$. Perfect correlation in the quantum uncertainties of N_A and N_B suggests that the QV captures genuine quantum correlations between A and B whenever applied to bipartite fluctuations of globally conserved quantities. Remarkably, Figs. 3(a)–3(c) shows that, for local systems, the QV of bipartite fluctuations scales like the *boundary* between A and B, thereby obeying a so-called area law

$$\langle \delta^2 \hat{N}_A \rangle_Q \sim L_A^{d-1}, \tag{21}$$

namely, the extensive (volume-law) part of bipartite fluctuations is entirely of incoherent origin. This strongly suggests that the QV captures the fluctuations associated with coherent particle exchanges at the boundary between A and B. For the hardcore-boson problem at hand, such fluctuations obey a logarithmically corrected area law at $T = 0$ (when all fluctuations are quantum) [42,43],

$$\langle \delta^2 \hat{N}_A \rangle_Q \sim L_A^{d-1} \ln L_A, \tag{22}$$

turning then into an area-law scaling at finite T. Nonetheless, a logarithmic violation can still be observed at sufficiently low temperature and for small sizes of the A region—namely, smaller than the thermal correlation length ξ for density fluctuations.[2] Interestingly, the area-law scaling of the QV (either straight or logarithmically violated) is found to dominate the scaling of total fluctuations at sufficiently small sizes L_A of the subsystem A, as shown in Figs. 3(a)–3(c). This makes the (logarithmically violated) area law of bipartite quantum fluctuations observable under experimentally realistic conditions.

VII. QUANTUM VARIANCE OF BIPARTITE FLUCTUATIONS PROVIDES QUANTUM CORRELATIONS

The area-law scaling of bipartite QV of mixed states suggests a link to the similar scaling exhibited by entanglement entropy in ground states of local Hamiltonians [45]. Yet measures of entanglement at finite temperature (such as the

[1] A closer inspection into our data shows that the QV of $\hat{N}_{s,A}$ is always a monotonically decreasing function of temperature when $L_A \gtrsim L/2$ up to $L_A = L$, whereas for A sufficiently small (typically $L_A \lesssim L/2$) the QV displays a tiny maximum close to the transition. This observation is unique to the QV of the order parameter in the presence of a transition, while the QV of generic quantities is indeed monotonically decreasing for any size of A.

[2] Indeed, for hardcore bosons, the density-density correlation function at finite temperature exhibits a finite correlation length, $\langle \delta n_i \delta n_j \rangle \sim \exp(-|\boldsymbol{r}_i - \boldsymbol{r}_j|/\xi)$, even though the phase correlations exhibit a critical behavior below the Kosterlitz-Thouless temperature. This was already noticed in Ref. [61].

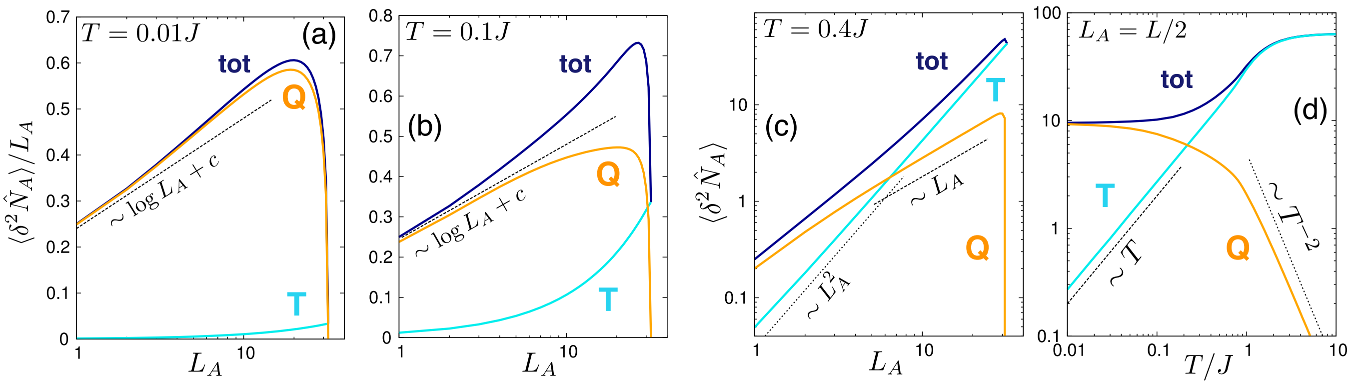

FIG. 3. Bipartite fluctuations. (a)–(c) Scaling of local particle-number fluctuations in a subsystem A for square-lattice hardcore bosons ($V = 0$) at three different temperatures ($T/J = 0.01$, 0.1, and 0.4). Other parameters as in Fig. 1(c). (d) Temperature dependence of the particle-number fluctuations; same parameters as in Fig. 1(d).

negativity [19]) do not admit a simple physical interpretation in terms of entropy of quantum fluctuations (but see below for further discussion on QV and entanglement). As already pointed out, QV of bipartite fluctuations is rather connected to quantum correlations, a more general concept than entanglement. Quantum correlations between A and B may be defined via the disturbance that a measurement on B has on the state of A, in which case they are captured by the quantum discord [6]. The latter quantity is given by the difference between the quantum mutual information, $I(A:B) = S_A + S_B - S_{AB}$ (or the entropy subextensivity due to correlations between A and B) and the classical mutual information $J(A:B)$ (or the maximum amount of information gained on A by performing measurements on B). Here $S_{A(B)} = -\mathrm{Tr}\hat{\rho}_{A(B)} \ln \hat{\rho}_{A(B)}$ is the entropy of the reduced density matrix of subsystem $A(B)$, and S_{AB} is the total entropy. The operation of maximization implicit in the definition of quantum discord makes it generically noncomputable when A and B are extended subsystems of a quantum many-body system.

On the other hand, in some special systems, the existence of quantum correlations is witnessed by more accessible quantities. Indeed, we argue that, in the case of an *ideal* gas, any form of correlation stems from quantum statistics, while it is trivially absent in the classical limit. The existence of correlations between A and B is generically captured by the quantum mutual information, whose nonzero value is then a direct proof of quantum correlations existing in the system.[3] In the case of an ideal lattice gas, the existence of correlations between A and B stems physically from the coherent exchange of particles at the $A - B$ boundary, and hence it is tightly linked to the quantum fluctuations of particle numbers. In the following, we shall particularly focus on the case of a free Fermi gas on a lattice at half-filling, for which the mutual information and QV of particle-number fluctuations can be easily calculated via exact diagonalization [46].

The quantum mutual information of many-body systems exhibits in general an area law at finite temperature [47] from which a finite-temperature area law descends for quantum discord as well [48]. Figure 4 shows that the area law of mutual information and of QV of particle-number fluctuations are tightly related, as the prefactors of the thermal area laws, $I(A:B)/2 \approx a_I(T) L_A$ and $\langle \delta^2 \hat{N}_A \rangle_Q \approx a_N(T) L_A$, are proportional at all T, $a_I \approx \frac{\pi^2}{3} a_N$. Remarkably, this is the same relationship holding between the particle-number variance and the entanglement entropy for free fermions at $T = 0$ [Fig. 4(a)] [41,49], and between total entropy and variance in a degenerate Fermi gas [Fig. 4(b)]. Hence the particle-number

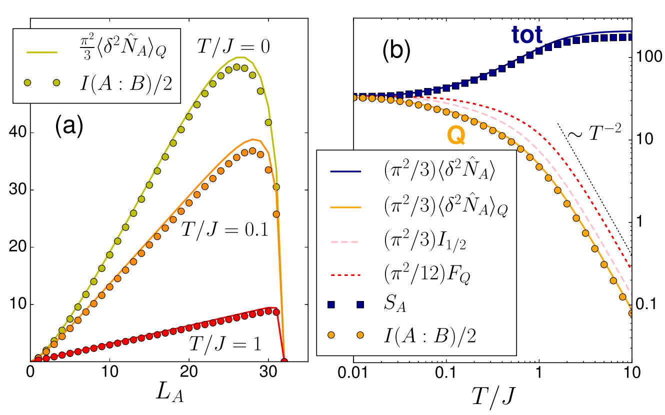

FIG. 4. Quantum correlations vs quantum mutual information. (a) Scaling of the quantum variance of bipartite particle-number fluctuations and of the quantum mutual information in a system of free fermions on a $L \times L$ square lattice ($L = 32$) at half-filling for three different temperatures (J is the hopping amplitude); and (b) temperature dependence of the same two quantities, along with the total entropy S_A, the total fluctuations $\langle \delta^2 \hat{N}_A \rangle$, the Wigner-Yanase skew information $I_{1/2}(\hat{N}_A, \hat{\rho})$, and the quantum Fisher information $F_Q(\hat{N}_A; \hat{\rho})$, for an A region with linear size $L_A = L/2$. The T^{-2} decay of the mutual information at high temperature has been proven rigorously for free fermions in Ref. [44], and it is proven for the quantum variance, skew information and quantum Fisher information in Appendix F.

[3]Even though the quantum discord of an ideal gas is not equal to the mutual information [62], the vanishing of quantum discord in the classical limit is the trivial result of both $I(A:B)$ and $J(A:B)$ being vanishing quantities. Hence the existence of a finite $I(A:B)$ and $J(A:B)$ is already a proof of quantum correlations, and the further aspect that they are not identical—leading to quantum discord—is a generic feature of quantum systems.

QV provides experimental access to the mutual information of free fermions at finite T, as much as the total variance of particle-number fluctuations gives access to the entanglement entropy in the ground state. Moreover, as shown in Fig. 4(b) the QV provides a meaningful lower bound to both the Wigner-Yanase skew information and to the quantum Fisher information; in particular, at high temperatures, we find that $\langle\delta^2\hat{N}_A\rangle_Q \approx \frac{2}{3}I_{1/2}(\hat{N}_A,\hat{\rho})$ and $\langle\delta^2\hat{N}_A\rangle_Q \approx \frac{1}{3}(F_Q(\hat{N}_A;\hat{\rho})/4)$.

VIII. MEASUREMENT OF BIPARTITE QUANTUM VARIANCE WITH QUANTUM-GAS MICROSCOPES

Owing to its definition in terms of fully measurable quantities [fluctuations and response function, see Eq. (4)], the QV is readily accessible to state-of-the-art experiments. All the requirements for the measurement of the QV, and in particular of its scaling in a bipartite setting, are met by trapped-ion experiments [50] as well as quantum-gas microscope experiments [39], enabling access to local degrees of freedom. As an example, in microscopy experiments recent progress [51] has demonstrated the ability to resolve different single-site occupation numbers ($n = 0,\ldots,3$) in an optical lattice, providing access to local fluctuations. A concrete proposal to measure the quantum variance of bipartite particle-number fluctuations in the context of ultra-cold quantum gases is illustrated in Fig. 5. To access the quantum variance of the particle number N_A in the subsystem A one needs to measure the total variance of fluctuations $\langle\delta^2 N_A\rangle$, as well as the response function

$$\chi_{N_A} \approx \frac{\langle N_A\rangle(\mu_A + \delta\mu_A) - \langle N_A\rangle(\mu_A)}{\delta\mu_A}, \tag{23}$$

where μ_A is the local chemical potential in region A, coupling to the particle number N_A. The two quantities $\langle\delta^2 N_A\rangle$ and χ_{N_A} need to be measured in the same conditions of temperature and (offset) chemical potential. A way to achieve this in cold-atom experiments is to use a "multiplexing" setup as in Fig. 5, in which one single trap geometry allows one to measure both quantities at once. Indeed monitoring fluctuations of N_A in region A allows one to extract $\langle\delta^2 N_A\rangle$ the total variance; on the other hand, a boxlike potential superimposed to the optical lattice creates a local increase in the chemical potential, giving

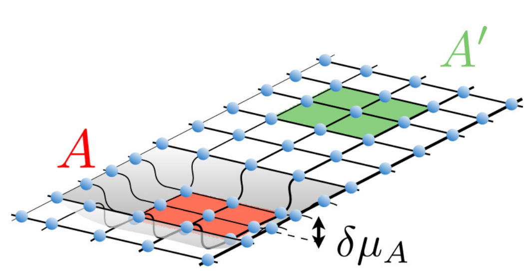

FIG. 5. Quantum-gas microscope setup to measure the quantum variance. Here we sketch a possible scheme to measure the quantum variance of the local particle number in region A by adding a boxlike potential to a two-dimensional optical lattice. This potential induces a local increase in the chemical potential, allowing one to probe the response function as the particle-number difference between region A' and A; supplementing this measurement with the one of particle-number variance in region A gives access to the quantum variance.

access to the response function as $(\langle N_{A'}\rangle - \langle N_A\rangle)/\delta\mu_A$. If the regions A and A' are built symmetrically around the (global) trap center, and if thermal equilibrium is established across the system, one is ensured that the two quantities are measured under the same thermodynamic conditions of temperature and offset chemical potential.

One may worry that in cold-atom experiments the total particle number has wide shot-to-shot fluctuations going well beyond a grand-canonical description, and that this may alter the estimate of the quantum variance, adding spurious contributions coming from experimental systematics. On the other hand, as already discussed in Sec. II and further elaborated in the appendices Appendixes A and B, all incoherent fluctuations (either stemming from the grand-canonical ensemble or from other sources) are systematically subtracted away in the quantum variance. This remains valid even when the total particle number obeys an arbitrary statistics, namely even when the density matrix takes the general form

$$\hat{\rho} = \frac{1}{\mathcal{Z}}\sum_N p_{\exp}(N)\hat{\mathcal{P}}_N e^{-\beta\hat{\mathcal{H}}}\hat{\mathcal{P}}_N, \tag{24}$$

where $p_{\exp}(N)$ is the experimental particle-number statistics, accounting for systematic shot-to-shot fluctuations, and $\hat{\mathcal{P}}_N$ is the projector onto the Fock subspace with N particles. The deformation of the Hamiltonian implied in Fig. 5 leads to the desired deformation of the density matrix probing the response function. Hence the quantum variance (and its peculiar size and temperature scaling) can be experimentally measured even without postelection of the measurement shots according to the total particle number, with the obvious caveat that one is not measuring properties of the grand-canonical ensemble but the ones of the artificial ensemble realized experimentally.

A similar setup, and similar considerations, can be applied to measure the quantum variance of the staggered particle number. In that case, one needs to shine a weak superlattice potential with twice the lattice spacing of the primary potential over the region A'. Varying the size of region A and A' allows then to probe the scaling of the QV with subsystem size, as illustrated in Figs. 1(c), 1(d), 3, and 4.

IX. CONCLUSIONS AND OUTLOOK

In conclusion, we have introduced the quantum variance of generic observables, generalizing the concept of quantum Heisenberg uncertainty to the case of mixed states—and acting as the "thermal de Broglie wavelength" in the space of eigenvalues of arbitrary observables. In the case of bipartite fluctuations, the QV expresses the quantum correlations among the two subsystems in arbitrary mixed states. The quantum uncertainty may dominate the fluctuations in quantum many-body systems, leading to a completely nonclassical behavior (fluctuations decreasing with temperature, scaling of fluctuations obeying area laws or logarithmically violated area laws, etc.).

At the theory level, the QV represents a most accessible way to assess quantum correlations, entanglement, and the metrological use of quantum many-body states. As proposed in Refs. [13,21], the minimal skew information and quantum Fisher information associated with local observables in a subsystem A are both measures of quantum correlations, and the latter dictates the minimal precision on the estimation of the parameter of an arbitrary local unitary operation; the QV offers a natural measurable lower bound to both quantities (see Appendixes D and E for further discussion). Moreover, both the skew information [29] and the quantum Fisher information [30,31] of collective spin variables witness entanglement among k qubits when exceeding a k-dependent bound: a similar violation of the bound by the QV is therefore an even stronger witness—see Appendix G for a detailed discussion.

The QV lends itself to analytical as well as to large-scale numerical simulations based, e.g., on quantum Monte Carlo—as shown in the present work. Hence its study can be readily extended to generic quantum many-body systems at equilibrium, including interacting fermions, quantum spin models, etc., as well as to nonequilibrium mixed states. While we have mostly focused our attention on bipartite correlations, an extension of our study to multipartite correlations can also be readily achieved by introducing the concept of quantum covariance or quantum correlation function, as developed in a recent work by one of us [52]. This opens the perspective of a complete characterization of quantum correlations in extended quantum systems, based on experimentally accessible quantities.

ACKNOWLEDGMENTS

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APPENDIX A: OPERATOR APPROACH TO COHERENT VERSUS INCOHERENT FLUCTUATIONS IN MIXED STATES: AREA LAW OF QUANTUM COHERENCE

As pointed out in the main text, in the case of mixed states described by a density matrix $\hat{\rho}$ there is a fundamental distinction between thermal/incoherent fluctuations and quantum/coherent fluctuations of any observable, which does not commute with $\hat{\rho}$. This distinction is best captured via the path-integral representation of the density matrix, as discussed in the following Appendix B. Here we give an alternative picture solely based on the operator picture of the density matrix. In the following we shall choose, as observable of interest, the particle number $\hat{N}_A$ in the region A of the system, capturing the quantum correlations between the region in question and its complement.

When leaving the ground state of local Hamiltonians, one expects to encounter states with the generic feature of possessing volume-law entanglement, and volume-law fluctuations of particle number [53]. Hence one may naively suspect that, when dealing with excited states, the quantum coherent fluctuations are stronger, and not weaker, than in the ground state. This is indeed true, but it is only meaningful provided that, in an experiment, one is able to deterministically prepare one and the same excited state, in order to probe its fluctuation properties over many shots of the experiment itself. This last requirement is generally prohibitive, as experiments on excited states generally probe the properties of *ensembles* (every shot of the experiment reproducing a different state). Whence the relevance of the concept of density matrices $\hat{\rho}$, not only in the context of systems coupled to dissipative baths, but also in the context of systems evolving uniquely under their own Hamiltonian dynamics.

In the latter case, let $|\Psi(t)\rangle$ be the instantaneous state of the system, and let Θ be a time window sufficiently long for time averages to equal ensemble averages (namely, averages over repeated shots of the experiment). Then the density matrix describing the ensemble is well described by

$$\hat{\rho} \approx \frac{1}{\Theta}\int_0^\Theta dt|\Psi(t)\rangle\langle\Psi(t)|. \tag{A1}$$

Despite the fact that each $|\Psi(t)\rangle$ state may exhibit volume-law entanglement and coherent fluctuations, the ensemble properties are quite different. Indeed, we can write $|\Psi(t)\rangle$ as

$$|\Psi(t)\rangle = \sum_{N_A}\sum_{\{n_i\}_{N_A}} c_{\{n_i\};N_A}(t)|\{n_i\},N_A\rangle, \tag{A2}$$

where $|\{n_i\};N_A\rangle$ is a Fock state $\{n_i\}$ characterized by having N_A particle in A. After time/ensemble averaging, the density matrix takes the form $\hat{\rho} = \hat{\rho}_{\rm D} + \hat{\rho}_{\rm OD}$, where

$$\hat{\rho}_D = \sum_{N_A}\sum_{\{n_i\}_{N_A}}\sum_{\{n'_i\}_{N_A}} \rho_{\{n_i\},N_A;\{n'_i\},N_A}|\{n_i\},N_A\rangle\langle\{n'_i\},N_A| \tag{A3}$$

is the diagonal part of the density matrix (in terms of the quantum number N_A), and

$$\hat{\rho}_{\rm OD} = \sum_{N_A\neq N'_A}\sum_{\{n_i\}_{N_A}}\sum_{\{n'_i\}_{N'_A}} \rho_{\{n_i\},N_A;\{n'_i\},N'_A}|\{n_i\},N_A\rangle\langle\{n'_i\},N'_A| \tag{A4}$$

is the off-diagonal part; here

$$\rho_{\{n_i\},N_A;\{n'_i\},N'_A} = \frac{1}{\Theta}\int_0^\Theta dt c_{\{n_i\},N_A}(t)c^*_{\{n'_i\},N'_A}(t). \tag{A5}$$

It is evident that $[\hat{\rho}_{\rm D},\hat{N}_A] = 0$, while $[\hat{\rho}_{\rm OD},\hat{N}_A] \neq 0$. Therefore the off-diagonal part, containing the coherence between configurations differing by the number of particles in A, is the part of $\hat{\rho}$ responsible for the quantum fluctuations of $\hat{N}_A$, captured by the quantum variance. The total, extensive fluctuations of N_A are given by the diagonal part, $\langle\delta^2 N_A\rangle = \langle\hat{N}_A^2\rangle_{\hat{\rho}_D} - \langle\hat{N}_A\rangle^2_{\hat{\rho}_D}$; as a consequence the quantum coherent contribution, which stems from the off-diagonal terms, remains hidden in this calculation, and it cannot be formally separated from the incoherent part. The proper separation between incoherent and coherent fluctuations is achieved within the path-integral formalism, as described in the main text and below in Appendix B. Nonetheless, the operator form of the density matrix provides further insight into the physical origin and spatial structure of coherent quantum fluctuations, as discussed below.

In local systems, the instantaneous coherence $c_{\{n_i\},N_A}(t)c^*_{\{n'_i\},N'_A}(t)$ connects states with $N_A - N'_A \sim \mathcal{O}(L_A^{d/2})$, as it is typical of excited states in Hilbert space. However, the time/ensemble-averaged coherence $\rho_{\{n_i\},N_A;\{n'_i\},N'_A}$ in Eq. (A5) has a much shorter range away from the diagonal. Indeed, assuming that $\{n_i\}$ and $\{n'_i\}$ are connected by moving m particles from sites $j_1,\ldots,j_m$ to sites $i_1,\ldots,i_m$, one has

$$\rho_{\{n_i\},N_A;\{n'_i\},N'_A} = \mathrm{Tr}[\hat\rho b^\dagger_{i_1}\cdots b^\dagger_{i_m} b_{j_1}\cdots b_{j_m}|\{n_i\},N_A\rangle\langle\{n_i\},N_A|], \quad \text{(A6)}$$

where $b_i,b_i^\dagger$ are the destruction/creation operators of the particles of interest (of arbitrary statistics). Hence, as one may have expected, the magnitude of $\rho_{\{n_i\},N_A;\{n'_i\},N'_A}$ is controlled by that of the $2m$-point correlation function, namely,

$$|\rho_{\{n_i\},N_A;\{n'_i\},N'_A}| \leqslant |\langle b^\dagger_{j_1}\cdots b^\dagger_{j_m} b_{j_1}\cdots b_{j_m}\rangle|. \quad \text{(A7)}$$

In general, such a correlation function will exhibit a fast decay with the (minimum) distances between pairs of points i_p and j_q. This in turn implies that, in order to have a sizable coherence [Eq. (A5)], two configurations $\{n_i\}$ and $\{n'_i\}$ should differ by particle moves, which, when occurring between A and its complement B, are localized (algebraically or exponentially) around the boundary between the two regions. This observation generally excludes a volume law for the coherent part of particle-number fluctuations, and it leaves an area law (up to multiplicative logarithmic corrections) as the only possibility.

APPENDIX B: PATH-INTEGRAL REPRESENTATION OF A GENERIC DENSITY MATRIX AND OF THE QUANTUM VARIANCE

In this section, we derive the path-integral representation for a generic density matrix, generalizing the discussion of the main text to arbitrary mixed states beyond thermal equilibrium. Moreover we derive the path-integral expression for the quantum variance.

Any (semipositive definite) density matrix $\hat\rho$ can always be cast in the form

$$\hat\rho = \frac{e^{-\beta\hat{\mathcal{H}}}}{\mathrm{Tr}(e^{-\beta\hat{\mathcal{H}}})}, \quad \text{(B1)}$$

namely in the form of a thermal density matrix with (effective) temperature $k_BT = 1/\beta$. For generic (nonthermal) mixed states, the specific value of β is completely irrelevant, and one could set in the following $\beta = 1$ in some convenient energy units; yet, in order to make contact with the case of thermal equilibrium, hereafter we will keep the inverse temperature β explicitly indicated. We consider a generic observable $\hat O$ which is diagonalized by a basis $|O_\alpha,\{\boldsymbol\lambda\}_\alpha\rangle$, where O_α is the eigenvalue for $\hat O$, and $\{\boldsymbol\lambda\}_\alpha$ are the other quantum numbers possibly labeling the state. The partition function $\mathcal{Z} = \mathrm{Tr}[\exp(-\beta\hat{\mathcal{H}})]$ can be cast in the form of the trace of the product of infinitesimal propagators between successive states $|O_{\alpha_i},\{\boldsymbol\lambda\}_{\alpha_i}\rangle$, namely,

$$\mathcal{Z} = \lim_{M\to\infty}\sum_{\{\alpha_i\}}\prod_{i=1}^{M-1}\langle O_{\alpha_i},\{\boldsymbol\lambda\}_{\alpha_i}|e^{-\frac{\beta}{M}\hat{\mathcal{H}}}|O_{\alpha_{i+1}},\{\boldsymbol\lambda\}_{\alpha_{i+1}}\rangle \quad \text{(B2)}$$

where $\sum_{\{\alpha_i\}}$ is a short-hand notation for the multiple sum over the quantum numbers $(O_{\alpha_i},\{\boldsymbol\lambda\}_{\alpha_i})$ labeling each state in the propagation sequence $\alpha_1,\alpha_2,\ldots,\alpha_M\equiv\alpha_1$. Summing over the $\boldsymbol\lambda$ quantum numbers, and taking the limit $M\to\infty$, one obtains formally the path-integral expression

$$\mathcal{Z} = \int_{O(0)\equiv O(\beta)}\mathcal{D}[O(\tau)]\, e^{-S[O(\tau),\partial_\tau O(\tau),\ldots]}, \quad \text{(B3)}$$

where $O(\tau)$ is the continuum limit of the sequence $\{O_{\alpha_1},O_{\alpha_2},\ldots,O_{\alpha_M}\}$, and

$$e^{-S} = \lim_{M\to\infty}\sum_{\{\boldsymbol\lambda_{\alpha_i}\}}\prod_{i=1}^{M-1}\langle O_{\alpha_i},\{\boldsymbol\lambda\}_{\alpha_i}|e^{-\frac{\beta}{M}\hat{\mathcal{H}}}|O_{\alpha_{i+1}},\{\boldsymbol\lambda\}_{\alpha_{i+1}}\rangle. \quad \text{(B4)}$$

Once the density matrix has been given the thermal form Eq. (B1), it is straightforward to deform the density matrix upon application of a field h coupling to $\hat O$,

$$\hat\rho(h) = \frac{e^{-\beta(\hat{\mathcal{H}}-h\hat O)}}{\mathrm{Tr}[e^{-\beta(\hat{\mathcal{H}}-h\hat O)}]}, \quad \text{(B5)}$$

which allows one to define the response function in the standard way as $\chi_O = \frac{\partial}{\partial h}\mathrm{Tr}[\hat\rho(h)\hat O]|_{h=0}$.

The path-integral representation of response function leads to the expression

$$\chi_O = \left\langle\int d\tau\,\delta O(\tau)\delta O(0)\right\rangle_S = \frac{1}{\beta}\left\langle\int d\tau\int d\tau'\,\delta O(\tau)\delta O(\tau')\right\rangle_S, \quad \text{(B6)}$$

where $\delta O(\tau) = O(\tau) - \langle O\rangle_S$, and we have invoked the periodicity of $O(\tau)$ paths in imaginary time. Here

$$\langle\ldots\rangle_S = \frac{1}{\mathcal{Z}}\int_{O(0)\equiv O(\beta)}\mathcal{D}[O(\tau)](\ldots)e^{-S} \quad \text{(B7)}$$

is the average over the space of $O(\tau)$ paths. Moreover, one has that

$$\langle\delta^2 O\rangle = \frac{1}{\beta}\left\langle\int d\tau(\delta O(\tau))^2\right\rangle_S. \quad \text{(B8)}$$

Combing Eqs. (B6) and (B8), one readily obtains the path-integral expression for the quantum variance

$$\langle\delta^2 O\rangle_Q = \langle\delta^2 O\rangle - \chi_O k_BT = \frac{1}{\beta}\left\langle\int d\tau\left[O(\tau) - \frac{1}{\beta}\int d\tau' O(\tau')\right]^2\right\rangle_S \quad \text{(B9)}$$

showing that it represents the average variance of fluctuations of $O(\tau)$ paths around their centroid.

We end this section by noticing that the deformation of the density matrix to Eq. (B5) is a physically meaningful operation for thermal states—as it can be obtained by turning on the perturbation $-h\hat O$ in the Hamiltonian within an isothermal setting—see Sec. VIII. Hence in the case of thermal states, neither the measurement nor the calculation of the quantum variance requires the full knowledge of the density matrix. On the other hand, for generic mixed states the deformation of $\hat\rho$ should be thought of in general as a mathematical operation. Devising physical (namely, experimentally realistic) operations that can lead to the deformation of a generic density matrix as in Eq. (B5) is an outstanding task, which we postpone to future investigations.

APPENDIX C: QUANTUM VARIANCE AS GENERALIZED DE BROGLIE WAVELENGTH

1. Quantum variance for simple models

The analytical calculation of the quantum variance of the position operator is illuminating in the case of simple models, namely the free particle and the harmonic oscillator in one dimension. It is convenient to start from the second one, and to obtain the free-particle result as a limiting case. In the case of the harmonic oscillator, the position fluctuations are readily obtained from the diagonal part of the density matrix $\langle x|e^{-\beta\hat{\mathcal{H}}}|x\rangle/\mathcal{Z}$ [32], while the susceptibility $\chi_x = \frac{\partial\langle x\rangle}{\partial h}$ to a displacement of the harmonic oscillator potential $\frac{1}{2}m\omega^2x^2 \to \frac{1}{2}m\omega^2x^2 - hx$ is readily obtained by the linear displacement of the average, $\langle x\rangle \to \langle x\rangle - h/(m\omega^2)$. As a result the quantum variance takes the form

$$\langle\delta^2 x\rangle_Q = \frac{a_{\rm ho}^2}{2}\left[\frac{\sinh(1/\theta)}{\cosh(1/\theta)-1} - 2\theta\right] \quad \text{(harm. osc.)}, \tag{C1}$$

where $a_{\rm ho} = \sqrt{\hbar/(m\omega)}$ and $\theta = k_BT/\hbar\omega$. In the limit $T \to 0$, one recovers Heisenberg's uncertainty in the ground state, $\langle\delta^2 x\rangle_0 = a_{ho}^2/2$.

On the other hand, taking the limit $\omega \to 0$ gives the result for the free particle, which, after careful expansion of Eq. (C1), gives

$$\langle\delta^2 x\rangle_Q = \frac{1}{24\pi}\lambda_{dB}^2(T) \quad \text{(free particle)}, \tag{C2}$$

where $\lambda_{dB}(T) = \sqrt{2\pi\hbar^2/(mk_BT)}$ is the thermal de Broglie wavelength. The link between the quantum variance and the de Broglie wavelength shows that the quantum variance of the particle position gives the characteristic (squared) amplitude of coherent quantum fluctuations at finite temperature [32]. In fact, one may interpret the quantum variance of the position as a generalization of the thermal de Broglie wavelength to the case of a particle in an external potential, such as the case of the harmonic oscillator. In Fig. 6, the quantum variance of the position for the two models discussed above shows the expected monotonic decrease with temperature, due to the shrinking of the imaginary-time "duration" of coherent quantum fluctuations.

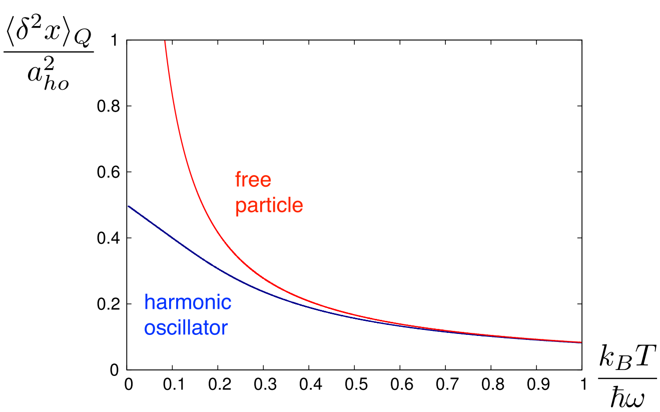

FIG. 6. Quantum variance of the position of a $1d$ particle. The plot shows the quantum variance of the position for a one-dimensional harmonic oscillator, as well as for a free particle as a function of temperature. In the case of the free particle, the frequency ω is to be understood as an arbitrary constant setting the energy scale $\hbar\omega$ and length scale $a_{\rm ho}$.

2. Imaginary-time fluctuations as quantum coherent fluctuations

As seen in Appendix B, the quantum variance of a generic observable gives the characteristic amplitude of fluctuations for such an observable along the imaginary-time dynamics of the system. On the other hand, in Appendix C, we have established a direct relationship between the quantum variance of a free particle and the thermal de Broglie wavelength, namely the characteristic width of wave packets at finite temperature. In this section, we bring the two observations together to argue that the quantum variance generalizes the concept of thermal de Broglie wavelength, or finite-temperature coherence length, to the space of eigenvalues of *any* observable (not only the position operator) and for *any* quantum system.

The relationship between the quantum variance and the de Broglie wavelength is very natural when considering the fundamental link existing between the wave function of a pure state and the statistics of Feynman paths. Thanks to the parametrization in Eq. (B1), a pure state can always be thought of as the $\beta \to \infty$ limit of a mixed-state density matrix, and hence represented in the form of a path integral. In the case of a one-dimensional quantum particle, the path integral for a pure state with wave function $\psi(x)$ runs over infinitely long trajectories $x(\tau)$, whose fluctuations δx around the average position $\langle x\rangle = \langle\psi|\hat{x}|\psi\rangle$ obey the statistics dictated by the modulus of the square function [54], namely, taking an arbitrary time τ:

$$\begin{aligned} P(\delta x) &= \frac{1}{\mathcal{Z}}\int \mathcal{D}[x(\tau)]\delta(x(\tau) - \langle\hat{x}\rangle - \delta x)e^{-S} \\ &= |\psi(\langle x\rangle + \delta x)|^2. \end{aligned} \tag{C3}$$

In particular, any infinite trajectory contributing to the path integral has the same statistical properties as the whole ensemble, so that the centroid of the path $\bar{x} = \bar{x}[x(\tau)]$ must correspond to the expectation value $\langle\hat{x}\rangle$. Hence, as depicted in Fig. 7(a), the imaginary-time fluctuations span the support of the wave function, and the (squared) amplitude of fluctuations of the path $x(\tau)$ around its centroid $\bar{x}$—the quantum variance—is the same as the (squared) width of the wavefunction, giving Heisenberg's uncertainty. In the case of free particles, the thermal de Broglie wavelength generalizes Heisenberg's uncertainty on the position to the case of thermal states. Therefore it is not too surprising that the quantum variance follows the de Broglie wavelength at finite temperature, as shown in Appendix C.

The concept of quantum variance extends all the above considerations to generic observables and generic quantum systems. The quantum variance provides the width of the probability distribution for the fluctuations of generic observables around the path centroid [see Fig. 7(b)], namely,

$$P(\delta O) = \frac{1}{\mathcal{Z}}\int \mathcal{D}[O(\tau)]\delta(O(\tau) - \bar{O} - \delta O)e^{-S}, \tag{C4}$$

where $\bar{O} = \bar{O}[O(\tau)]$ is the centroid. As seen in the case of the position of a one-dimensional particle, for a pure state

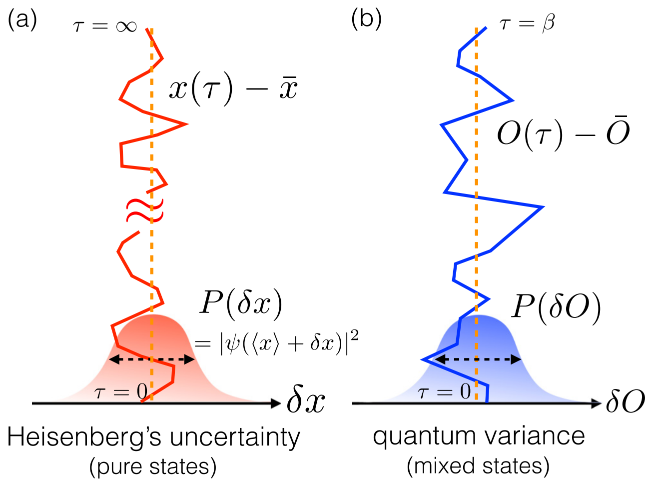

FIG. 7. Quantum variance as Heisenberg's uncertainty for mixed states. (a) The path-integral representation of the partition function for a single one-dimensional particle at $T\to 0$ (or for a generic pure state) is a sum over infinitely long paths in imaginary time $x(\tau)$; on average, each path visits the region $[x,x+dx]$ a number of times proportional to $|\psi(x)|^2dx$, where $\psi(x)$ is the ground-state (or, more generally, the pure-state) wave function. The width of the wave function's square modulus gives the Heisenberg's uncertainty, namely the amplitude of coherent quantum fluctuations. (b) In the case of mixed states of a generic many-body system, the probability distribution for the coherent quantum fluctuations of a generic observable $\hat{O}$ is instead given by the probability that the (imaginary-time) instantaneous value of $O(\tau)$ differs from the path centroid $\bar{O}=\beta^{-1}\int d\tau\, O(\tau)$.

the width of $P(\delta O)$ expresses the Heisenberg's uncertainty on the observable $\hat{O}$. When applied to a mixed state, the quantum variance generalizes therefore Heisenberg's uncertainty, expressing the (squared) amplitude of coherent quantum fluctuations of the observable.

APPENDIX D: QUANTUM VARIANCE VERSUS SKEW INFORMATION AND LOCAL QUANTUM UNCERTAINTY

In this section, we shall discuss the relationship between the quantum variance and the skew information [11], the latter being a widespread concept in quantum information to quantify the quantum uncertainty of an observable. In particular, we shall show that the quantum variance provides a tight lower bound, based on physical observables, to the otherwise abstract skew information. Moreover the discussion of the relationship between the skew information and the quantum variance allows one to conclude on the convexity of the latter. Finally, we will see how the quantum variance relates to the recently introduced "local quantum uncertainty" [13], which is advocated as a measure of quantum correlations.

1. Wigner, Dyson, Lieb and the convexity of quantum variance

The Dyson-Wigner-Yanase (DWY) *skew information* [11,35]

$$I_\alpha(\hat{O},\hat{\rho})=-\tfrac{1}{2}\mathrm{Tr}\{[\hat{O},\hat{\rho}^\alpha][\hat{O},\hat{\rho}^{1-\alpha}]\}\quad(0\leqslant\alpha\leqslant 1)\tag{D1}$$

probes the quantum uncertainty of the observable $\hat{O}$ due to its noncommutativity with the density matrix of the system. Replacing $\hat{O}$ by $\delta\hat{O}=\hat{O}-\langle\hat{O}\rangle$ does not alter the above definition.

Writing again the generic density matrix $\hat{\rho}$ as a thermal state, $\hat{\rho}=e^{-\beta\hat{\mathcal{H}}}/\mathcal{Z}$ (with an arbitrary effective inverse temperature β), one can immediately show that the DWY skew information can be expressed as an imaginary-time correlation function

$$I_\alpha(O,\rho)=\langle\delta^2\hat{O}\rangle-\langle\delta\hat{O}(\tau=\alpha\beta)\delta\hat{O}(0)\rangle.\tag{D2}$$

Hence, clearly, the DWY skew information I_α expresses the amount by which the imaginary-time correlation function $\langle\delta\hat{O}(\tau)\delta\hat{O}(0)\rangle$ at a time $\tau=\alpha\beta$ has decreased with respect to the equal-time $(\tau=0)$ value. Hence the DWY skew information probes the imaginary-time fluctuations in a similar manner to quantum variance. As a consequence a link between the two quantities can be expected, and it is straightforwardly established in the form

$$\langle\delta^2\hat{O}\rangle_Q=\langle\delta^2\hat{O}\rangle-\frac{1}{\beta}\int_0^\beta d\tau\langle\delta\hat{O}(\tau)\delta\hat{O}(0)\rangle$$
$$=\int_0^1 d\alpha\, I_\alpha(\hat{O},\hat{\rho}),\tag{D3}$$

namely, the quantum variance is equal to the *average DWY skew information*. In particular, the Wigner-Yanase (WY) skew information [11], $I_{\alpha=1/2}$, is an upper bound to the DWY skew information

$$I_\alpha(\hat{O},\hat{\rho})\leqslant I_{\alpha=1/2}(\hat{O},\hat{\rho})\tag{D4}$$

as it is easy to prove due to the monotonic decay of imaginary-time correlation functions, $\langle\delta\hat{O}(\tau)\delta\hat{O}(0)\rangle\geqslant\langle\delta\hat{O}(\tau=\beta/2)\delta\hat{O}(0)\rangle$. As a consequence one readily obtains that the quantum variance is always lower than the WY skew information

$$\langle\delta^2\hat{O}\rangle_Q\leqslant I_{\alpha=1/2}(\hat{O},\hat{\rho}).\tag{D5}$$

Lieb [55] proved that the DWY skew information is *convex* for any value of α, namely,

$$I_\alpha(\hat{O},\lambda_1\hat{\rho}_1+\lambda_2\hat{\rho}_2)\leqslant\lambda_1 I_\alpha(\hat{O},\hat{\rho}_1)+\lambda_2 I_\alpha(\hat{O},\hat{\rho}_2)\tag{D6}$$

for any real numbers λ_1, λ_2. Using Eq. (D3), the property of convexity is immediately inherited by the quantum variance. Convexity is a fundamental figure of merit to assess the quantum variance as a probe of quantum coherent fluctuations: if $\hat{\rho}_1$ and $\hat{\rho}_2$ have the same quantum variance, any linear incoherent superposition of the two has necessarily a lower quantum variance.

2. Quantum variance versus local quantum uncertainty

Given an $A-B$ bipartition of an extended quantum system, Ref. [13] has introduced the concept of *local quantum uncertainty* (LQU)

$$\mathcal{U}_A^\Lambda(\hat{\rho})=\min_{\hat{O}_A^\Lambda}I_{1/2}(\hat{O}_A^\Lambda,\hat{\rho})\tag{D7}$$

as the minimum of the WY skew information over all local observables O_A^Λ in A having a given spectrum Λ. Reference

[13] argues that this observable-independent (but spectrum-dependent) quantity acts as a measure of quantum correlations between A and B. In order to capture quantum correlations among *all*, equally weighted degrees of freedom of A and those of B, it is obvious to request that the observable O_A be an extensive one, namely the sum of local observables $\hat{O}_j$,

$$\hat{O}_A^{(\text{macro})} = \sum_{j\in A} \hat{O}_j. \tag{D8}$$

This ensures that quantum fluctuations of all degrees of freedom in A are taken into account into the skew information. As the size of A grows, the spectrum of the operator $\hat{o}_A^{(\text{micro})} = \hat{O}_A^{(\text{macro})}/L_A^d$ becomes a continuous one, and it is contained in a finite interval $[\lambda_{\min},\lambda_{\max}]$. This behavior applies to all extensive operators of the kind of $\hat{O}_A^{(\text{macro})}$, and their spectrum can easily be reduced to one and the same Λ by a simple shift and rescaling in the definition of the operator. Hence, in the sense of Ref. [13], one can define a *macroscopic* LQU $\mathcal{U}_A^{(\text{macro})}(\rho)$ defined as a minimum over all operators $\hat{O}_A^{(\text{macro})}$, which is arguably a most appropriate definition of quantum correlations among *all* degrees of freedom of A and those in B. We assume that A and B interact with a coherent Hamiltonian term, leading to an exchange of energy, and possibly also particle, or magnetization, etc,.... Hence, in the minimization procedure, we explicitly exclude the possible existence of local conserved quantities $[\hat{O}_A^{(\text{macro})},\hat{\rho}] = 0$, which would trivially lead to a vanishing macroscopic LQU.

It follows immediately from Eq. (D5) that the macroscopic LQU is lower-bounded by the minimum quantum variance of macroscopic observables

$$\begin{aligned}\mathcal{U}_A^{(\text{macro})}(\hat{\rho}) &= \min_{\hat{O}_A^{(\text{macro})}} I_{1/2}(\hat{O}^{(\text{macro})},\hat{\rho})\\ &\geqslant \min_{\hat{O}_A^{(\text{macro})}} \big\langle \delta^2 \hat{O}_A^{(\text{macro})}\big\rangle_Q.\end{aligned} \tag{D9}$$

The minimization implied by Eq. (D9) is readily performed for the quantum variance: the minimum quantum variance of macroscopic observables is realized by bipartite fluctuations of an otherwise conserved quantity, namely $\hat{O}_A$ such that $[\hat{O}_A + \hat{O}_B,\rho] = 0$. For general quantum systems, the above requirement applies to the local energy, and, in the presence of a continuous symmetry, to the local particle number (for particle models) or to the local magnetization (for spin models), etc., assuming that the latter quantities are not conserved. In the case of equilibrium states of local Hamiltonians, we have shown in this work that the quantum variance of bipartite fluctuations obeys an area law: as a consequence, Eq. (D9) implies that the macroscopic LQU obeys *at least* an area law. On the other hand, in the ground state, the WY skew information reduces to the variance of the operator

$$I_{1/2}\big(\hat{O}_A^{(\text{macro})},\hat{\rho}\big) \underset{T=0}{=} \big\langle \delta^2 O_A^{(\text{macro})}\big\rangle, \tag{D10}$$

and the scaling of the minimum variance of local macroscopic operators in the ground state of local Hamiltonians satisfies a (logarithmically violated) area law [43]. Even though the temperature dependence of the WY skew information is not generally known in the literature, one can assume that it is maximized at $T = 0$ (this is the case of the free-fermion example studied explicitly in Sec. VII). Under this assumption, and given that the WY skew information at $T = 0$ coincides with the total variance, we obtain the inequalities

$$\begin{aligned}\min_{\hat{O}_A^{(\text{macro})}} \big\langle \delta^2 \hat{O}_A^{(\text{macro})}\big\rangle_Q &\leqslant \mathcal{U}_A^{(\text{macro})}(\hat{\rho})\\ &\leqslant \min_{\hat{O}_A^{(\text{macro})}} \langle \delta^2 O^{(\text{macro})}\rangle(T=0)\end{aligned} \tag{D11}$$

implying that the macroscopic LQU obeys *at most* a logarithmically violated area law, namely

$$\mathcal{O}\big(L_A^{d-1}\big) \leqslant \mathcal{U}_A^{(\text{macro})}(\hat{\rho}) \leqslant \mathcal{O}\big(L_A^{d-1}\log L_A\big). \tag{D12}$$

APPENDIX E: QUANTUM VARIANCE VERSUS QUANTUM FISHER INFORMATION: QUANTUM CORRELATIONS AND METROLOGY

In this section, we focus on the relationship between the quantum variance and the quantum Fisher information [28], a central quantity in quantum metrology due to its link with the maximum precision achievable in the estimation of the parameter of a given unitary transformation. Similarly to the skew information, the quantum variance offers a lower bound to the quantum Fisher information; we shall exploit this fact in the context of the recently introduced "interferometric power" [21] to explore the importance of the quantum variance of bipartite fluctuations both for metrology and for quantum correlations. Further implications of this bound in the context of entanglement witnessing will be discussed in Appendix G.

1. Quantum variance as a lower bound to the quantum Fisher information

The quantum Fisher information (QFI) [28] expresses the "distinguishability" (in the sense of the Bures distance) between two density matrices $\hat{\rho}(h)$ and $\hat{\rho}(h+\delta h)$, belonging to a family $\hat{\rho}(h)$ continuously parametrized by the parameter h. If the family of density matrices is obtained via a unitary transformation generated by an Hermitian operator $\hat{O}$, $\hat{\rho}(h) = e^{-i\hat{O}h}\hat{\rho}(h=0)e^{i\hat{O}h}$, the QFI takes the explicit form $F_Q(\hat{O},\hat{\rho}) = \sum_{nm} G_{\text{F}}(p_n,p_m)|\langle n|\delta\hat{O}|m\rangle|^2$, where p_n and $|n\rangle$ are eigenvalues and eigenvectors of the density matrix, and

$$G_{\text{F}}(p_n,p_m) = 2\frac{(p_n-p_m)^2}{p_n+p_m}. \tag{E1}$$

This is to be compared with the expression of the quantum variance, namely, $\langle\delta^2 O\rangle_Q = \sum_{nm} G_{\text{QV}}(p_n,p_m)|\langle n|\delta O|m\rangle|^2$, where

$$G_{\text{QV}}(p_n,p_m) = \frac{p_n+p_m}{2} - \frac{p_n-p_m}{\ln(p_n)-\ln(p_m)}. \tag{E2}$$

Comparing the two functions, it is easy to realize that

$$\frac{G_{\text{F}}(x,y)}{4} \geqslant G_{\text{QV}}(x,y) \quad 0 \leqslant x,y \leqslant 1, \tag{E3}$$

whence the announced inequality

$$\frac{F_Q(\hat{O},\hat{\rho})}{4} \geqslant \langle\delta^2\hat{O}\rangle_Q. \tag{E4}$$

Incidentally, we notice that the WY skew information admits a similar expression $I_{1/2}(\hat{O},\hat{\rho}) = \sum_{nm} G_{\mathrm{I}_{1/2}}(p_n,p_m)\, |\langle n|\delta\hat{O}|m\rangle|^2$ with

$$G_{\mathrm{I}_{1/2}}(p_n,p_m) = \frac{p_n + p_m}{2} - \sqrt{p_n p_m}. \tag{E5}$$

Direct inspection into the G functions reveals the inequality chain:

$$\frac{G_{\mathrm{F}}(x,y)}{4} \geqslant G_{\mathrm{I}_{1/2}}(x,y) \geqslant G_{\mathrm{QV}}(x,y)\ 0 \leqslant x,y \leqslant 1, \tag{E6}$$

whence the ensuing hierarchy:

$$\frac{F_Q(\hat{O},\hat{\rho})}{4} \geqslant I_{1/2}(\hat{O},\hat{\rho}) \geqslant \langle\delta^2\hat{O}\rangle_Q. \tag{E7}$$

The inequality relating the QFI and WY skew information was already proven in Ref. [56]. The quantum variance further explicits this relationship by finding a common, nontrivial lower bound for both quantities.

2. Metrological implications: the interferometric power

In a similar manner to the definition of local quantum uncertainty discussed in Appendix D 2, Ref. [21] has introduced the concept of *interferometric power* (IP) of an observable O_A^Λ in a bipartite $(A+B)$ system with density matrix $\hat{\rho}$ as

$$\mathcal{P}_A^\Lambda(\hat{\rho}) = \tfrac{1}{4}\min_{\hat{O}_A^\Lambda} F_Q\big(\hat{O}_A^\Lambda,\hat{\rho}\big), \tag{E8}$$

where $F_Q(\hat{O}_A^\Lambda,\hat{\rho})$ is the quantum Fisher information associated with a unitary transformation generated by a local observable $\hat{O}_A^\Lambda$ acting on A, and with spectrum Λ. The IP has a direct metrological meaning: it expresses the worst-case-scenario uncertainty (in the sense of the Cramér-Rao bound [57]) that one can achieve in the estimation of the parameter of a unitary transformation generated by an arbitrary observable which is local in A and has a given spectrum Λ. Reference [21] argues that the IP is another measure of quantum correlations between A and B, leading to the conclusion that quantum correlations are a resource for metrology.

It is immediate to see that the above conclusions carry automatically over to the case of the quantum variance. Using the inequality Eq. (E4), one immediately has that

$$\mathcal{P}_A^\Lambda(\rho) \geqslant \min_{\hat{O}_A^\Lambda}\langle\delta^2\hat{O}_A^\Lambda\rangle_Q. \tag{E9}$$

In Appendix D 2, we argued that macroscopic observables $\hat{O}_A^{(\mathrm{macro})}$ in A, having an extensive spectrum Λ, capture the quantum correlations between all degrees of freedom in A and those in B. In the case of such observables, one can perform the minimization immediately for the right-hand side of Eq. (E9), identifying the $O_A^{(\mathrm{macro})}$ operator with the one satisfying the condition $[\hat{O}_A^{(\mathrm{macro})} + \hat{O}_B^{(\mathrm{macro})},\hat{\rho}] = 0$ (again, as in Appendix D 2 we are excluding local conserved quantities from the minimization). Hence the quantum variance of bipartite fluctuations provides a lower bound on the IP of macroscopic observables, and on the quantum correlations and metrological resource that this quantity expresses. Similarly to Eq. (D12), this lower bound allows one to establish an area law scaling (with at most logarithmic corrections) to the IP of macroscopic observables, under the assumption (verified, e.g., by free fermions as in Sec. VII) that the QFI is maximised at $T=0$. In particular, this bound is very instructive in terms of the metrological utility of many-body states: the maximum quantum variance of bipartite fluctuations, and hence the maximum IP, is achieved for states exhibiting power-law correlations, and specifically in the vicinity of quantum critical points—see also [58] for a recent calculation of the quantum Fisher information in exactly solvable models of quantum-critical points, which confirms this conclusion.

APPENDIX F: QUANTUM VARIANCE, SKEW INFORMATION, AND QUANTUM FISHER INFORMATION OF BIPARTITE FLUCTUATIONS FOR FREE FERMIONS

1. Quantum variance

In this section, we calculate the quantum variance of local particle-number fluctuations in the case of free fermions on a d-dimensional hypercubic lattice at half-filling. The density-density correlation function is given by

$$\langle\delta\hat{n}_i(\tau)\delta\hat{n}_j(0)\rangle = \frac{1}{L^{2d}}\sum_{\boldsymbol{k},\boldsymbol{k}'} e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)}\, e^{(\epsilon_{\boldsymbol{k}}-\epsilon'_{\boldsymbol{k}})\tau} f_{\boldsymbol{k}}(1-f_{\boldsymbol{k}'}), \tag{F1}$$

where $f_{\boldsymbol{k}} = [\exp(\beta\epsilon_{\boldsymbol{k}})+1]^{-1}$ is the Fermi distribution, and $\epsilon_{\boldsymbol{k}} = -2J\sum_{\alpha=x,y,\ldots}\cos(k_\alpha)$ is the dispersion relation. Integrating the correlation function to get $\langle\delta^2\hat{N}_A\rangle$ and $\langle\delta^2\hat{N}_A\rangle_T$, one obtains the quantum variance in the form

$$\langle\delta^2\hat{N}_A\rangle_Q = \frac{1}{L^{2d}}\sum_{\boldsymbol{k},\boldsymbol{k}'}\sum_{i,j\in A} e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)} f_{\boldsymbol{k}}(1-f_{\boldsymbol{k}'}) \times\left[1+\frac{1-e^{\beta(\epsilon_{\boldsymbol{k}}-\epsilon_{\boldsymbol{k}'})}}{\beta(\epsilon_{\boldsymbol{k}}-\epsilon_{\boldsymbol{k}'})}\right]. \tag{F2}$$

In the high-temperature limit $\beta\to 0$, the quantum variance reduces to

$$\langle\delta^2\hat{N}_A\rangle_Q = \frac{\beta^2}{48}\frac{1}{L^{2d}}\sum_{\boldsymbol{k},\boldsymbol{k}'}\sum_{i,j\in A} \times e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)}(\epsilon_{\boldsymbol{k}}-\epsilon_{\boldsymbol{k}'})^2 + \mathcal{O}(\beta^3). \tag{F3}$$

One observes that the term linear in β vanishes, so that the dominant temperature dependence goes like T^{-2}.

Assuming for A the geometry of a hypercube of side L_A, the double sum over the A region can be performed exactly, leading to

$$\langle\delta^2\hat{N}_A\rangle_Q = \frac{\beta^2}{48}\left(\frac{L_A}{L}\right)^{2d}\sum_{\boldsymbol{k},\boldsymbol{k}'} \times\left(\prod_{\alpha=x,y,\ldots}\frac{\mathrm{sinc}^2[(k_\alpha-k'_\alpha)L_A/2]}{\mathrm{sinc}^2[(k_\alpha-k'_\alpha)/2]}\right)(\epsilon_{\boldsymbol{k}}-\epsilon_{\boldsymbol{k}'})^2 + \mathcal{O}(\beta^3). \tag{F4}$$

In the limit $L_A\to\infty$ one has that

$$L_A\frac{\mathrm{sinc}^2[(k_\alpha-k'_\alpha)L_A/2]}{\mathrm{sinc}^2[(k_\alpha-k'_\alpha)/2]} \approx 2\sum_{n=-\infty}^{\infty}\delta(k_\alpha-k'_\alpha-2\pi n). \tag{F5}$$

One sees immediately that this limit would highlight a term scaling as L_A^d (volume law), but that this term is actually vanishing because $\epsilon_{\boldsymbol{k}} = \epsilon_{\boldsymbol{k}+2\pi n \boldsymbol{e}_\alpha}$. Hence one is left with an area-law scaling term.

2. Wigner-Yanase skew information

Using Eq. (D2) leads immediately to the following expression for the skew information of bipartite particle-number fluctuations of free fermions

$$I_{1/2}(\hat{N}_A,\hat{\rho}) = \frac{1}{L^{2d}} \sum_{\boldsymbol{k},\boldsymbol{k}'} \sum_{i,j\in A} e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)} \times f_{\boldsymbol{k}}(1-f_{\boldsymbol{k}'})[1-e^{\beta(\epsilon_{\boldsymbol{k}}-\epsilon_{\boldsymbol{k}'})/2}], \tag{F6}$$

which, when expanded at high temperature, leads to the behavior

$$I_{1/2}(\hat{N}_A,\hat{\rho}) = \frac{\beta^2}{32}\frac{1}{L^{2d}} \sum_{\boldsymbol{k},\boldsymbol{k}'} \sum_{i,j\in A} \times e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)}(\epsilon_{\boldsymbol{k}}-\epsilon_{\boldsymbol{k}'})^2 + \mathcal{O}(\beta^3). \tag{F7}$$

The above expression is very similar to Eq. (F4) for the quantum variance, confirming that the two quantities have the same high-temperature behavior (as well as the same zero-temperature value). In particular, when $\beta \to 0$:

$$\langle \delta^2 \hat{N}_A\rangle_Q = \tfrac{2}{3} I_{1/2}(\hat{N}_A,\hat{\rho}) + \mathcal{O}(\beta^3). \tag{F8}$$

3. Quantum Fisher information

Finally, when considering the QFI for free fermions [58], one finds the following expression for bipartite particle-number fluctuations:

$$F_Q(\hat{N}_A,\hat{\rho}) = \frac{4}{L^{2d}} \sum_{\boldsymbol{k},\boldsymbol{k}'} \sum_{i,j\in A} e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)} \times f_{\boldsymbol{k}}(1-f_{\boldsymbol{k}'})\tanh^2\left[\frac{\beta(\epsilon_{\boldsymbol{k}}-\epsilon_{\boldsymbol{k}'})}{2}\right], \tag{F9}$$

which leads to the high-temperature behavior

$$F_Q(\hat{N}_A,\hat{\rho}) = \frac{\beta^2}{4}\frac{1}{L^{2d}} \sum_{\boldsymbol{k},\boldsymbol{k}'} \sum_{i,j\in A} \times e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)}(\epsilon_{\boldsymbol{k}}-\epsilon_{\boldsymbol{k}'})^2 + \mathcal{O}(\beta^3). \tag{F10}$$

Comparing again with Eq. (F4), one can conclude that, at high temperatures,

$$\langle \delta^2 \hat{N}_A\rangle_Q = \frac{1}{3}\frac{F_Q(\hat{N}_A,\hat{\rho})}{4} + \mathcal{O}(\beta^3). \tag{F11}$$

4. Discussion

Hence, as anticipated in the main text, the quantum fluctuations captured by the quantum variance, the WY skew information or the QFI display the same high-temperature behavior up to a global prefactor. This leads to a coherent picture for bipartite quantum fluctuations of free fermions. While the calculation of the quantum variance is easily extended to arbitrary many-body systems, which can be treated with state-of-the-art numerics, the same is generally not true for the WY skew information nor the QFI—although, unlike the QFI, the WY skew information lends itself to path-integral Monte Carlo approaches probing imaginary-time correlation functions. On the experimental side, the WY skew information, being an imaginary-time correlation function, is not accessible to experiments as such. As for the QFI, Ref. [58] has recently shown that it is potentially accessible to experiments when cast as a frequency integral involving the dynamic susceptibility; in this respect, the quantum variance has the advantage of being expressed solely in terms of static correlations and response functions.

APPENDIX G: QUANTUM VARIANCE AS MULTIPARTICLE ENTANGLEMENT WITNESS

Let us consider a system of N qubits, with collective spin operators $\hat{\boldsymbol{J}} = \sum_{i=1}^N \hat{\boldsymbol{S}}_i$. A pure state $|\psi\rangle$ is said to be k-producible [29,59,60] if it can be written as

$$|\psi_{k\text{-prod}}\rangle = \otimes_{l=1}^M |\psi_{N_l}\rangle, \tag{G1}$$

where $|\psi_{N_l}\rangle$ is an (entangled) state of a block of $N_l \leqslant k$ spins, with the constraint that $\sum_l N_l = N$. A mixed state is then said to be k-producible if it is an incoherent superposition of k_s-producible states with $k_s \leqslant k$

$$\hat{\rho}_{k\text{-prod}} = \sum_s p_s |\psi_{k_s\text{-prod}}\rangle\langle\psi_{k_s\text{-prod}}|. \tag{G2}$$

Using Eq. (E4) and the results of Refs. [30,31], one can prove that for k-producible states the quantum variance of the collective spin components $\hat{J}^\alpha$, and the QFI associated to transformation generated by the $\hat{J}^\alpha$, satisfy the inequality:

$$4\langle\delta^2 J^\alpha\rangle_Q \leqslant F_Q(J^\alpha;\rho_{k\text{-prod}}) \leqslant nk^2 + (N-nk)^2, \tag{G3}$$

where $n = [N/k]$ is the integer part of N/k. In fact, the exact same bound as in the last inequality of Eq. (G3) holds for the WY skew information [29], and again it carries over to the quantum variance thanks to the inequality in Eq. (D5). The inequality of Eq. (G3) can be readily generalized to more general degrees of freedom than qubits, namely to collective operators $\hat{C} = \sum_i \hat{c}_i$, where c_i is an operator with a bounded spectrum contained in the interval $[c_{\min},c_{\max}]$. In that case the inequality takes the form [7]

$$4\langle\delta^2\hat{C}\rangle_Q \leqslant F_Q(\hat{C};\rho_{k\text{-prod}}) \leqslant (c_{\max}-c_{\min})^2[nk^2+(N-nk)^2]. \tag{G4}$$

Hence, similarly to what was already found for the WY skew information [29] and the QFI [7,30,31], a violation of the inequalities in Eqs. (G3) or (G4) or for the quantum variance is a strong indication of the existence of multiparticle entanglement among a least $(k+1)$ degrees of freedom. The condition of violation is actually very strong, as the bound of the last inequality in is rather loose for thermal states. Indeed, the bound of Eq. (G3) is valid for k-producible pure states and mixed states alike, but, given that all the quantities in question (WY skew information, QFI, and quantum variance) are expected to decrease under thermal mixing, the bound is much looser for thermal states, and all the more so the higher the temperature.

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