Relationship between the Einstein-Laub electromagnetic force and the Lorentz force on free charge

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An electromagnetic force density expression that is consistent with a development attributed to Einstein and Laub appears to be able to describe optical force experiments done to date with homogenized media. However, a major question that has persisted for about one century relates to the apparent discrepancy with the usual interpretation of the force description due to Lorentz in magnetized media. Specifically, it had appeared that the Einstein and Laub force density incorporated only the free-space permeability in relation to the force on the electric current density. It is shown here that the Einstein and Laub force density is consistent with the Lorentz picture in the static limit. This resolves a key impediment in establishing a unified force density description for electromagnetic waves interacting with matter.

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I. INTRODUCTION

Understanding the force due to electromagnetic fields is fundamental and of importance in applications like optical tweezers [1,2] and the study of optical traps [3]. Consequently, the pertinent theory has received substantial attention during the past century (see, for example, [4–22]). However, a key and apparently open question has been the description of the electromagnetic force density in magnetic media [6]. In particular, the theory attributed to Einstein and Laub [4], used to explain important experiments [23,24], has appeared inconsistent with the description due to Lorentz [5] in magnetized media, which is supported by experiments done by Hall [25] and work on electron beams in magnetized media [26,27]. It is shown here that the Einstein and Laub theory is consistent with that of Lorentz in magnetized media in the static limit, thereby providing a unified description of the force density in homogenized media.

II. LORENTZ FORCE

The Lorentz force density (N/m^3) is commonly written as

$$\mathbf{f}_L = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}, \tag{1}$$

where ρ (C/m^3) is the free electric charge density, $\mathbf{E}$ (V/m) is the electric field intensity, $\mathbf{J}$ (A/m^2) is the electric current density, and $\mathbf{B}$ ($T = Wb/m^2$) is the magnetic flux density. Equation (1) stems from a static picture of stationary charges and steady-state currents. It has no information related to photon momentum or the wave character of light, although of course an electromagnetic wave can establish a charge density and a current density in a material.

A focus of Lorentz was on charges moving in vacuum [5], where in (1)

$$\mathbf{B} = \mu_0\mathbf{H}, \tag{2}$$

with μ_0 (H/m) being the free-space permeability and $\mathbf{H}$ (A/m) being the magnetic field intensity. However, in homogenized magnetic media having magnetization $\mathbf{M}$ (A/m), one has [28]

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}). \tag{3}$$

Early evidence for the use of (3) in (1) came from the resistivity experiments of Hall in ferromagnetic media, and he provided an explanation in terms of magnetization [25]. This has been understood as a spin-orbit effect and described with $\mathbf{H}$ and $\mathbf{M}$ contributions [29]. The related anomalous Hall effect is the subject of substantial recent interest [30]. While some experiments gave conflicting pictures [31] (see the introduction), a result showing the deflection of mesons by magnetized iron also suggested that use of (3) in (1) is correct [31]. Electron microscopy has been used to determine the distribution of magnetism [32], and those experiments also support use of magnetization $\mathbf{M}$ in evaluating the force on a beam and indicate that in the situations considered the applied field $\mathbf{H}$ was unimportant. Subsequent work indicated sensitivity to magnetized material states [26,27], and the characterization method became known as Lorentz microscopy. Therefore, as is broadly understood, experimental evidence overwhelmingly supports use of (3) in (1). This established background is projected against an apparent discrepancy with a rigorous theory for the electromagnetic force density that presumably should approach (1) in the static limit.

III. EINSTEIN AND LAUB FORCE

The particular challenge of relevance here has been that $\mathbf{J} \times \mu_0\mathbf{H}$ appears in the electromagnetic force density attributed to Einstein and Laub [4], and the explanation has presented an open issue for a very long time (see, for example, [6,33]). It is shown here that in fact the Einstein and Laub force density expression is in agreement with Lorentz and (1) together with (3) in the static limit.

The force density attributed to Einstein and Laub [4], $\mathbf{f}_{EL}$ (N/m^3), and used by many [6,13,16–19,22,34], is

$$\mathbf{f}_{EL} = \frac{\partial \mathbf{P}}{\partial t} \times \mu_0\mathbf{H} - \frac{\partial \mu_0\mathbf{M}}{\partial t} \times \epsilon_0\mathbf{E} + \rho\mathbf{E} + \mathbf{J} \times \mu_0\mathbf{H} + (\mathbf{P}\cdot\nabla)\mathbf{E} + \mu_0(\mathbf{M}\cdot\nabla)\mathbf{H}, \tag{4}$$

where $\mathbf{P}$ (C/m^3) is the polarization and ϵ_0 (F/m) is the free-space permittivity. The corresponding stress tensor is [24]

$$\mathbf{T}_{EL} = \tfrac{1}{2}(\epsilon_0 E^2 + \mu_0 H^2)\mathbf{I} - \mathbf{DE} - \mathbf{BH}, \tag{5}$$

where $\mathbf{I}$ is the identity matrix, $\mathbf{D}$ (C/m^2) is the electric flux density, and, for example, $\mathbf{DE}$ is a dyadic product of two vectors with elements $(\mathbf{ab})_{ij} = a_i b_j$ [6]. This stress tensor is arrived at through incorporation of coupled electromagnetic, polarization, and magnetization systems [24]. For general material arrangements, $(\mathbf{P}\cdot\nabla)\mathbf{E}$ and $(\mathbf{M}\cdot\nabla)\mathbf{H}$ in (4) can be nonzero, and the identities

$$-(\nabla\cdot\mathbf{P})\mathbf{E} = -\nabla\cdot(\mathbf{PE}) + (\mathbf{P}\cdot\nabla)\mathbf{E}, \tag{6}$$

$$-(\nabla\cdot\mathbf{M})\mathbf{H} = -\nabla\cdot(\mathbf{MH}) + (\mathbf{M}\cdot\nabla)\mathbf{H} \tag{7}$$

indicate that $\mathbf{PE}$ and $\mathbf{MH}$ provide a contribution in (5).

IV. ELECTROMAGNETIC FORCE IN THE STATIC LIMIT

In vacuum and for the static limit, (4) produces $\rho\mathbf{E} + \mathbf{J}\times\mu_0\mathbf{H}$, the Lorentz result in (1), providing confidence. The relevant point related to the credibility of (4) addressed here is the static limit in magnetic materials and consistency with the work of Lorentz (1). Equation (4) has $\mathbf{J}\times\mu_0\mathbf{H}$, while static experiments suggest that $\mathbf{J}\times\mathbf{B}$ is the appropriate term. Introducing an electron beam into a sample, where we make the assumption of uniform velocity, would appear to be captured in $\mathbf{J}$. Therefore, at least superficially, there appears to be an issue with consistency between (4) and (1). It has been proposed before that the root of the issue is that (4) applies to local, homogenized media, and the situation of charges passing through material is more complicated and may not conform [6]. However, $\mathbf{J}$ can be viewed as an impressed current (a mathematical source) that can encompass Ohm's law and the steady current when $\partial\mathbf{P}/\partial t \to 0$ or is an equivalent current (in Huygen's sense). A legitimate electromagnetic force theory should capture this as the circular temporal frequency ω approaches zero, i.e., in the static limit. Therefore, consider more carefully the static limit for (4), where terms involving a time derivative are removed.

Use of the identity $(\nabla\mathbf{b})\cdot\mathbf{a} = (\mathbf{a}\cdot\nabla)\mathbf{b} + \mathbf{a}\times(\nabla\times\mathbf{b})$ with tensor operation $(\nabla\mathbf{b})_{ij} = \partial b_j/\partial x_i$ gives

$$(\nabla\mathbf{H})\cdot\mathbf{M} = (\mathbf{M}\cdot\nabla)\mathbf{H} + \mathbf{M}\times(\nabla\times\mathbf{H}). \tag{8}$$

Using (8), with reference to (4),

$$\begin{aligned}\mu_0(\mathbf{M}\cdot\nabla)\mathbf{H} &= \mu_0(\nabla\mathbf{H})\cdot\mathbf{M} - \mu_0\mathbf{M}\times(\nabla\times\mathbf{H})\\ &= \mu_0(\nabla\mathbf{H})\cdot\mathbf{M} + \mathbf{J}\times\mu_0\mathbf{M},\end{aligned} \tag{9}$$

with application of Ampere's law for magnetostatics, neglecting displacement current [to investigate the relationship between (4) for the *static case* and the Lorentz result]. With (9) and considering the two relevant terms in (4),

$$\begin{aligned}&\mathbf{J}\times\mu_0\mathbf{H} + \mu_0(\mathbf{M}\cdot\nabla)\mathbf{H}\\ &\quad= \mathbf{J}\times\mu_0\mathbf{H} + \mu_0(\nabla\mathbf{H})\cdot\mathbf{M} + \mathbf{J}\times\mu_0\mathbf{M}\\ &\quad= \mathbf{J}\times\mathbf{B} + \mu_0(\nabla\mathbf{H})\cdot\mathbf{M}\\ &\quad= \mathbf{J}\times\mathbf{B},\end{aligned} \tag{10}$$

where the approximation assumes that the local mean field is constant over the length scale of interest ($\nabla\mathbf{H} = 0$), appropriate because this is a force density. The force density here is a macroscopic quantity and is applicable with spatial averages (mean field) over a length scale appreciable relative to interatomic distances [35,36]. The assumption of a constant local field is also compatible with the local homogenization of materials [36] (and a more recent example from metamaterials [37]). Equation (10) indicates that (4) is consistent with the result from Lorentz for the static force on a free-current density $\mathbf{J}$, expressed in (1). This key point does not seem to have been recognized previously.

Consider now the electrostatic situation. Using the same vector identity that led to (8),

$$\begin{aligned}(\nabla\mathbf{E})\cdot\mathbf{P} &= (\mathbf{P}\cdot\nabla)\mathbf{E} + \mathbf{P}\times(\nabla\times\mathbf{E})\\ &= (\mathbf{P}\cdot\nabla)\mathbf{E}\\ &= 0,\end{aligned} \tag{11}$$

because the electrostatic field is conservative and hence has zero curl, with the assumption of a locally constant field, giving $\nabla\mathbf{E} = 0$, so from (11), $(\mathbf{P}\cdot\nabla)\mathbf{E} = 0$.

With use of (10) and (11) in (4) and in the static limit,

$$\lim_{\omega\to 0}\mathbf{f}_{EL} = \rho\mathbf{E} + \mathbf{J}\times\mathbf{B} = \mathbf{f}_L, \tag{12}$$

in agreement with (1). Equation (12), showing the force density on a free-charge distribution, is the central result. Using $\nabla\mathbf{H} = 0$ and $\nabla\mathbf{E} = 0$, (12) can also be obtained from a previous development based on (4) [33].

V. DISCUSSION

In part, the point of this paper is to draw attention to the meaning of the terms in (4) in relation to the material properties and the force description in (1) due to Lorentz. This has been considered a major problem, i.e., that there is not an adequate picture linking the two forms [6,33]. An electric current source or an equivalent boundary representation is captured by $\mathbf{J}$ in (4). Free- and bound-charge motion are rigorously incorporated in a temporal Fourier representation of $\mathbf{P}(t)$ and $\mathbf{M}(t)$ in (4), so in a simple isotropic situation the complex susceptibilities provide the complete material description: $\mathbf{P}(\omega) = \epsilon_0(\chi'_E + i\chi''_E)\mathbf{E}(\omega)$ and $\mathbf{M}(\omega) = (\chi'_H + i\chi''_H)\mathbf{H}(\omega)$. Taking (4) into the frequency domain, we can legitimately represent the complex electric susceptibility as a complex conductivity ($\sigma = \sigma' + i\sigma''$), thereby forming the equivalence between $\partial\mathbf{P}/\partial t\times\mu_0\mathbf{H}$ and $\mathbf{J}\times\mu_0\mathbf{H}$. For a locally homogeneous medium, the fields are divergenceless (so for the isotropic material case, $\nabla\cdot\mathbf{E} = 0$ and $\nabla\cdot\mathbf{H} = 0$), and a plane-wave superposition can be used. In this situation, the "free-charge" density within the material is $\nabla\cdot\mathbf{D} = \rho = 0$. Current density and charge density are, of course, linked through the conservation of charge requirement in the continuity condition. In the static case, $\mathbf{J}$ describes the conduction current density (free charge), $\mathbf{P}$ describes the bound charge, and the two terms can be separated exactly. In the context of homogeneous media, $\mathbf{J}$ and ρ in (4) describe the free or introduced charged particles, the e-beam for instance. This is precisely the framework to understand how (4) maps to (1).

Let us review what $\nabla\mathbf{H} = 0$ and $\nabla\mathbf{E} = 0$ in (10) and (11) mean in the broader physical context in arriving at (12). First $\nabla\mathbf{H}$ and $\nabla\mathbf{E}$ do not appear in (1), either because only the local field is relevant (and not its spatial variation at each point in space) or something has been ignored, perhaps electrostriction and magnetostriction, in applying (1) in materials. The fact that an enormous body of experimental work in dielectric and magnetic materials with static fields supports (1) suggests that it captures the relevant physics thus far. The issue then becomes the basis of any approximation. The case was made that the mean local field that is built into Maxwell's equations implies that only the field amplitude and not the spatial variation should be used in the force density impacting free-charge motion in a material. For a linear time-invariant problem, superposition can be applied. Each elemental component of a beam has weights $\mathbf{J}(\mathbf{r}')$ and $\rho(\mathbf{r}')$ at some point $\mathbf{r} = \mathbf{r}'$ in space. Thus, an element of the current density is $\mathbf{J} = \hat{\mathbf{j}} J(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}')$, where $\hat{\mathbf{j}}$ is a unit vector and $\delta(\cdot)$ is the Dirac delta function. In this mathematical Dirac limit and a pointwise spatial representation, clearly, $\nabla\mathbf{H} = 0$ and $\nabla\mathbf{E} = 0$ at each point where the force (density) on the charged particle beam is determined, considering the differential limit for a continuous field. Integration over each differential volume gives the local force on the current and charge.

Consider again the physical picture behind (12). The description is of the force on free charge (the e-beam, for instance) in a material. With this perspective, material deformation is not being represented. In this sense, electrostriction and magnetostriction are irrelevant. Earlier work considering the relationship between the Lorentz and Einstein-Laub forces (see [23,33], for example) considered the force on the material rather than directly considering the force on the free-charge density. Hall measurements monitor current and hence conform to this picture. Although straightforward, this point is important because it allows the decomposition of the Einstein-Laub force into a form that is exactly equivalent to the Lorentz picture, as widely used and as it must.

VI. CONCLUSION

Equation (12) indicates that the Einstein-Laub formulation for force density is consistent with the accepted static form of the Lorentz force on free charges in static fields. This means that the material magnetization will influence the local force density in (4), just as in the expression from Lorentz that has been supported by experiments. Equation (4) has been used to explain key optical force experiments, including those due to Jones and Leslie [38] (see [24]), as well as Ashkin and Dziedzic [39] (see [23]). It might optimistically be concluded that legitimate interpretations of (4) may be able to explain all macroscopic experiments. Whether other theories hold, at least approximately, in certain situations represents another set of questions. Finally, lacking in force experiments has been verification of the influence of the dispersive material response that is incorporated into (4) through $\partial\mathbf{P}/\partial t$ and $\partial\mu_0\mathbf{M}/\partial t$.

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