

**Emergent quasicrystals in strongly correlated systems**Eran Sagi<sup>1</sup> and Zohar Nussinov<sup>1,2</sup><sup>1</sup>*Department of Condensed Matter Physics, Weizmann Institute of Science, Rehovot 76100, Israel*<sup>2</sup>*Department of Physics, Washington University, St. Louis, Missouri 63130, USA*

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Commensurability is of paramount importance in numerous strongly interacting electronic systems. In the fractional quantum Hall effect, a rich cascade of increasingly narrow plateaux appear at larger denominator filling fractions. Rich commensurate structures also emerge, at certain filling fractions, in high temperature superconductors and other electronic systems. A natural question concerns the character of these and other electronic systems at *irrational* filling fractions. Here we demonstrate that quasicrystalline structures naturally emerge in these situations, and trigger behaviors not typically expected of periodic systems. We first show that irrationally filled quantum Hall systems cross over into quasiperiodically ordered configuration in the thin-torus limit. Using known properties of quasicrystals, we argue that these states are unstable against the effects of disorder, in agreement with the existence of quantum Hall plateaux. We then study analogous physical situations in a system of cold Rydberg atoms placed on an optical lattice. Such an experimental setup is generally disorder free, and can therefore be used to detect the emergent quasicrystals we predict. We discuss similar situations in the Falicov-Kimball model, where known exact results can be used to establish quasicrystalline structures in one and two dimensions. We briefly speculate on possible relations between our theoretical findings and the existence of glassy dynamics and other features of strongly correlated electronic systems.

DOI: [10.1103/PhysRevB.94.035131](https://doi.org/10.1103/PhysRevB.94.035131)**I. INTRODUCTION**

The effects of commensurability appear in an extensive set of strongly correlated systems that, among many others, includes the high-temperature cuprate superconductors [1–6] and the fractional quantum Hall systems [7,8]. These effects often arise from an intricate interplay between the inherent length scales of the system and restrictions, e.g., those concerning magnetization or total particle number. Depending on such externally imposed constraints, distinct phases may arise. This dependence may have strong consequences for the nature of excitations and criticality in such systems.

To motivate the quintessential physics investigated in this work, consider  $N$  strongly interacting particles placed on the sites of a periodic lattice of  $M$  sites. In a continuum rendition of such a theory (i.e., one in which the particles do not need to occupy lattice sites), when the interactions are long ranged and repulsive, homogeneous Wigner-crystal type [9] structures will be energetically preferred; the periodicity of such a Wigner lattice will be set by the particle density and interactions.

In this study, the particles will be constrained to reside on the discrete sites of another spatial structure—the underlying periodic crystal. The latter lattice defining the theory may be incommensurate relative to the basic periodicity of the ideal Wigner lattice. The mismatch between the externally imposed lattice spacing and the energetically favored Wigner lattice length scale may spawn complex superlattice structures, which may spontaneously break the original lattice translational symmetries (as well as rotational and other point group symmetries).

In systems where the average number ( $f = N/M$ ) of particles per site is irrational, simple periodic order is prohibited. Nevertheless, long-range interactions may still favor the formation of structures with some form long-range correlations, even in the absence of periodicity.

As the above arguments hint, strongly interacting systems at irrational filling fractions may exhibit rich structures (and ensuing physical characteristics). Indeed, even deceptively simple-looking Ising and other models [10–20] harbor a plethora of highly nontrivial ground states and dynamics [21] with devil staircase structures. Frustration effects between incommensurate length scales were additionally shown to induce incommensurately modulated crystals in various classical systems [22,23]. It was further demonstrated experimentally that similar effects stabilize incommensurate composite crystals [24–26]. Additionally, it was recently shown [27] that one-dimensional (1D) quasicrystals may emerge due to the coupling between two incommensurate charge ordered subsystems. Related effects appear also in quantum systems, e.g., the frustration between the lattice and magnetic length scales in the Hofstadter problem [28] gives rise to a fractal spectrum [29].

As we will describe in this work, naturally occurring strongly correlated electronic systems and other quantum theories having only ubiquitous kinetic hopping, Coulomb, and spin exchange interactions may, quite generally, exhibit largely unexplored emergent quasicrystalline structures for irrational filling fractions (without spin-orbit terms explored in interesting recent studies [30–33]). For instance, as we will demonstrate, even on periodic one- and two-dimensional ionic lattices, interactions may render the underlying electronic structures to be quasicrystalline. The peculiar phenomenon of emergent quasicrystals (QCs) may trigger behaviors not typically expected of translationally invariant systems.

These predicted *emergent* electronic (and other) quantum quasicrystalline structures notably differ from quasicrystals discovered long ago in metallic alloys [34–36] and intensely studied in the decades since. In the celebrated metallic alloy quasicrystalline systems, the underlying ionic structure is, on its own, already quasiperiodic and may be further stabilized, in some cases, by electronic effects [37]. By contrast, in the

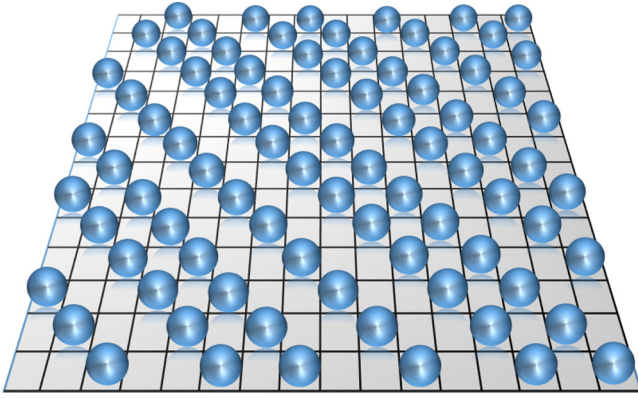


FIG. 1. A finite patch of the electronic quasicrystal that emerges in the two-dimensional Falicov-Kimball model. Notice that while the particles reside on a periodic lattice, the resulting structure must be aperiodic due to the irrational filling of the system. However, the strong interactions trigger a long-range ordered quasiperiodic structure, as revealed by the pointlike diffraction pattern. Our results indicate that such quasicrystalline structures appear quite broadly in clean strongly correlated systems which exhibit the type of irrational frustration we describe.

systems studied here an effective quasicrystalline electronic (or other) structure emerges on periodic ionic or optical lattices (see, e.g., Fig. 1).

In general, QCs are aperiodic structures with well defined Bragg peaks [34–36]. The Fourier transform of the density takes the form

$$\rho(\mathbf{k}) = \sum_{\mathbf{G}} \rho_{\mathbf{G}} \delta(\mathbf{k} - \mathbf{G}), \quad (1)$$

where the reciprocal vectors are combinations of  $D$  basis vectors,  $\mathbf{G} = \sum_{i=1}^D m_i \mathbf{b}_i$ , and the coefficients  $m_i$  are integer valued. QCs differ from periodic crystals in that  $D$  exceeds the spatial dimension  $d$  [38,39].

A direct consequence of the above is that these structures have  $D$  global  $U(1)$  symmetries,  $\rho(\mathbf{G}) \rightarrow \rho(\mathbf{G}) e^{2\pi i \chi(\mathbf{G})}$ , satisfying  $\chi(\mathbf{G} = \sum_{i=1}^D m_i \mathbf{b}_i) = \sum_{i=1}^D m_i \chi(\mathbf{b}_i)$ . Rigid translations are described by  $d$  linear combinations of  $\chi(\mathbf{b}_i)$ . The remaining  $(D - d)$  independent phases describe additional global rearrangements that generate distinct QCs with identical statistical characteristics. These symmetry operations, having no analogs in periodic crystals, are called phason symmetries [40]. Phason symmetries will play a key role in our investigation.

## II. QUANTUM HALL SYSTEMS AT IRRATIONAL FILLING FACTORS

To concretely describe our results, we first study the quantum Hall effect at an irrational filling and demonstrate the emergence of quasiperiodic structures. We predict that while these states are unstable against disorder, as evident from the observation of quantum Hall plateaux, one may, nonetheless, see signatures of the underlying quasiperiodic structure by looking at increasingly cleaner samples. Furthermore, as we will explain, similar phenomena appear in a system of cold Rydberg atoms, where disorder is absent

and emergent quasicrystalline structures can be more crisply observed.

We focus on the so called “thin-torus limit,” in which the planar quantum Hall (QH) system is mapped onto a one-dimensional (1D) classical problem [41–46]. The original two-dimensional (2D) fractional quantum Hall (FQH) Hamiltonian admits a natural 1D description within the guiding center representation, in which the single-electron wave functions  $\psi_k(\vec{r})$  are labeled by a single momentum index  $k$ . In the Landau gauge, for example, the momentum  $k$  in the  $x$  direction controls the average position in the  $y$  direction.

Remarkably, when placing the FQH system on a torus for which the circumference associated with one of its directions (the “ $y$  direction”) far exceeds the system size along the other transverse (“ $x$ ”) direction, the overlap between adjacent wave functions is small and the 1D model becomes classical. Specifically, the remnant nonvanishing terms lead to

$$H = \sum_{j_1 j_2} V(j_1 - j_2) n_{j_1} n_{j_2}, \quad (2)$$

where the natural numbers  $n_j$  denote the occupancies of respective states  $\psi_{\frac{2\pi j}{L_x}}$ . The projected interactions take the form

$$V(j_1 - j_2) = \frac{1}{2} \int \int d^2 r_1 d^2 r_2 [|\psi_{\frac{2\pi j_1}{L_x}}(\vec{r}_1)|^2 V_c(\vec{r}_1 - \vec{r}_2) \times |\psi_{\frac{2\pi j_2}{L_x}}(\vec{r}_2)|^2], \quad (3)$$

where  $V_c(\vec{r}_1 - \vec{r}_2)$  is the Coulomb interaction.

Notwithstanding the formal simplicity of Eq. (2), much nontrivial physics is captured by this classical Hamiltonian. Fortunately, a general solution for this problem exists. Assuming a general repulsive interaction  $V(j)$  which satisfies  $V(j+1) + V(j-1) \geq 2V(j)$  and vanishes as  $j \rightarrow \pm\infty$ , a prescription for generating the ground state configuration corresponding to any rational filling  $f = \frac{p}{q}$  was presented in Ref. [47]. This general recipe illustrates that the ground states are periodic, with a unit cell of size  $q$ . Examples of the ground state configurations corresponding to various rational filling fractions  $f$  are provided in Table I, the third column of which presents the pattern of consecutive  $n_j$  values in a unit

TABLE I. The ground state configurations corresponding to 1D systems whose filling factors are given by the sequence  $f_a = F_a/F_{a+2}$ , where  $F_a$  is the  $a$ th Fibonacci number. The first and second columns present the index  $a$  and the corresponding value of  $f_a$ . The third column presents the unit cell of the ground state configurations in the occupation basis, and the fourth column presents these in the compact notation  $S = 10$  and  $L = 100$ .

$a$	$f_a$	Configuration	Sequence
1	$\frac{1}{2} = 0.5$	10	$S$
2	$\frac{1}{3} = 0.3333 \dots$	100	$L$
3	$\frac{2}{5} = 0.4$	10100	$SL$
4	$\frac{3}{8} = 0.375$	10010100	$LSL$
5	$\frac{5}{13} \approx 0.38461 \dots$	1010010010100	$SLLSL$
6	$\frac{8}{21} \approx 0.38095 \dots$	100101001010010010100	$LSLSLLSL$

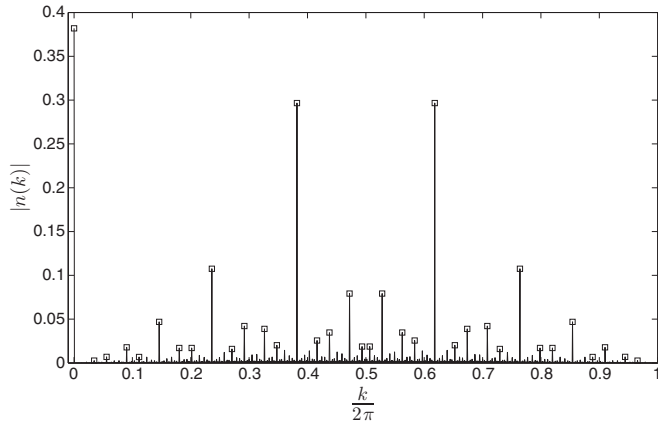


FIG. 2. The Fourier transform of the occupation in the ground state configuration corresponding to  $a = 15$ , whose unit cell is of size 1597. For comparison, the squares represent the Fourier components of the Fibonacci quasicrystal.

cell. We observe that in general, FQH states cross over into classical periodic states as we approach the thin-torus limit. Remarkably, some of the topological properties of the original FQH states, such as the fractional charges, are encoded in those periodic structures [41–46].

Thus far we largely reviewed the properties of the ground state in the thin-torus limit. We now explicitly turn in more detail to our results associated with irrational filling factors  $f$ . As an illustrative example specifically associated with Table I, we set  $f$  equal to an archetypal irrational number  $f = 1 - \tau = \frac{3-\sqrt{5}}{2} \approx 0.38197$ , where  $\tau$  is the reciprocal of the golden ratio  $\tau = \frac{\sqrt{5}-1}{2}$ .

A sequence of rational numbers that converges to  $(1 - \tau)$  is provided by  $f_a = \frac{F_a}{F_{a+2}}$ , where  $F_a$  is the  $a$ th Fibonacci number ( $F_a = F_{a-1} + F_{a-2}$  with  $F_1 = F_2 = 1$ ). Using the general prescription of Ref. [47], we may then generate the periodic ground state configurations for any such fraction. The unit cells corresponding to  $a = 1, 2, \dots, 6$  are presented in Table I. Perusing Table I, one observes that two adjacent occupied sites (sites  $j$  at which  $n_j = 1$ ) are always separated by either one or two empty sites ( $n_j = 0$ ). We verified that this persists for higher values of  $a$  as well. The ground state configurations can therefore be compactly encoded by combinations of the strings  $S = 10$  and  $L = 100$  as presented in the last column of Table I.

Interestingly, we find that the  $(a + 1)$ th unit cell may be iteratively generated from the  $a$ th cell via the inflation rules  $S \rightarrow L$ ,  $L \rightarrow SL$ . Remarkably, these are the very same inflation rules defining the Fibonacci QC [48,49]. To verify that the ground state configuration tends to that of the Fibonacci QC in the irrational  $f_a$  limit, we numerically compute the Fourier transform of the function  $n(j)$  corresponding to  $a = 15$  (for which the unit cell is of length 1597). In Fig. 2 these Fourier weights are contrasted with the Fourier components of the Fibonacci QC (whose Bragg peaks are located at  $k = 2\pi[(\tau l) \bmod 1]$ , where  $l$  is an integer, and Fourier weights can easily be calculated analytically [50]). Aside from small amplitude fluctuations, which asymptotically tend to zero as  $a$

increases, the two diffraction patterns coincide to a very good approximation.

The above numerical evidence indicates that in the  $a \rightarrow \infty$  limit, the ground state configuration coincides with the Fibonacci QC. The realization of a QC structure has immediate physical consequences. This is so as QCs exhibit the earlier noted continuous  $U(1)$  phason symmetries. These symmetries become transparent when writing the Fourier components of the density in the form

$$n(k) = \sum_{l=-\infty}^{\infty} \delta(k - 2\pi l \tau) n_l e^{2\pi i l \chi}, \quad (4)$$

where the phason symmetry is manifest as a  $U(1)$  invariance under changes of the phase  $\chi$ .

We now describe the low lying excitations about QC ground states for disparate potentials  $V$ . Such excitations result from effective long-range spatial variations of the phase  $\chi$ . However, as  $\chi$  is a globally defined quantity, any such description poses a fundamental difficulty. Heuristically, however, we may partition the system into large patches whose linear spatial size is still much smaller than the scale associated with the change of  $\chi$ . As  $\chi$  is essentially a constant on the scale of a single patch, it can be defined locally by calculating the Fourier transform of the density in that region. This intuitive idea can be implemented formally with the aid of the local Fourier transform (LFT) [50],

$$n(k, j) = \frac{1}{A} \sum_m w_\sigma(j - m) n(m) e^{-ikm}, \quad (5)$$

where  $w_\sigma(m)$  is a weight function that is equal to unity in a region of linear size  $\sigma$  centered around the origin, and vanishes otherwise. The parameter  $A$  is defined as  $A = \sum_j w_\sigma(j)$ . Pictorially,  $n(k, j)$  indeed describes the Fourier transform of  $n(j)$  inside a patch of size  $\sigma$ , centered around the point  $j$ . In terms of the LFT, we can write the low energy excitations as

$$n(j, k) = \sum_{l=-\infty}^{\infty} \delta(k - 2\pi l \tau) n_l e^{2\pi i l \chi(j)}, \quad (6)$$

where  $\chi(j)$  is a slowly varying function  $\sigma \partial_j \chi \ll 1$ . Writing the Hamiltonian in terms of the local Fourier components, and invoking Eq. (6), we obtain

$$H = \sum_m V(m) \sum_{l, j} |n_l|^2 e^{2\pi i m l [\partial_j \chi(j) + \tau]}. \quad (7)$$

Here we approximated  $\chi(j + m) - \chi(j) = m \partial_j \chi(j)$ . The sum over  $l$  can be performed analytically, as shown in the Supplemental Material [51].

For a given interaction  $V$ , the energy associated with the low-energy configurations can be evaluated directly from the resulting expression. To illustrate this, we plot in Fig. 3 the energy corresponding to  $V(m) = V_0 m^{-3}$ . For this interaction (and other potentials of the form  $V(m) = V_0 m^{-\alpha}$  [51]), we obtain, for small  $\partial_j \chi$ , a nonanalytic energy dependence of the form

$$H = \sum_j |\partial_j \chi(j)| \{a + b \Theta[\partial_j \chi(j)]\}. \quad (8)$$

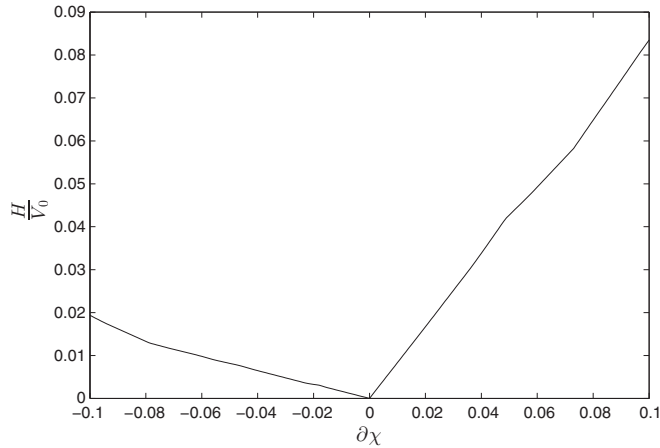


FIG. 3. Phason excitation energies as a function of  $\partial_x \chi$ , for  $V(m) = V_0 m^{-3}$ .

Thus, in the limit of strong interactions the system is gapless, in agreement with the Goldstone theorem.

As our primary model is one dimensional, the quantum fluctuations caused by inflating the torus inhibit spontaneous breaking of the continuous phason symmetry. Instead, the system may exhibit algebraic correlations of the form  $\langle e^{2\pi i[\chi(j) - \chi(0)]} \rangle \propto |j|^{-\eta}$ . Such a behavior was observed numerically in Ref. [50] in an analogous 2D classical system.

More generally, as our results indicate that the FQH states at irrational filling are gapless, these states are unstable to disorder. Such an instability is consistent with the existence of quantum Hall plateaux. In fact, we can easily use the above framework to establish the hierarchy of quantum Hall states, in which the gap (and therefore, the region of stability) of quantum Hall states of filling  $\nu = p/q$  monotonically decreases with  $q$ . This is shown in details in the Supplemental Material [51].

Despite the above instability with respect to disorder, it is clear that as the quality of samples improves, additional quantum Hall plateaux emerge, serving as approximants to the underlying quasicrystalline patterns.

### III. QUASICRYSTALS IN AN ULTRACOLD ATOMIC SYSTEM

Following Refs. [52,53], we now study a system of cold Rydberg atoms placed on an optical lattice (of arbitrary spatial dimensionality). These disorder free systems are natural candidates for observing the emergent quasicrystalline structures that we find. We assume that each site contains exactly one particle. Using an external laser, a transition from the ground state to an excited Rydberg state is enabled, making each lattice site a two-level system. We note that a realization of such a setup was reported in Ref. [54], where the authors use  $^{87}\text{Rb}$  atoms, and the excited Rydberg state is the  $43S_{1/2}$  state.

We label the states by a pseudospin  $S_z = \uparrow / \downarrow$  (where  $\downarrow$  represents the ground state and  $\uparrow$  represents the excited state). In terms of the spin-1/2 degrees of freedom, the Hamiltonian

can be written in the form [52]

$$H = \sum_{\mathbf{R}, \mathbf{R}'} J(|\mathbf{R} - \mathbf{R}'|) \left( S_{\mathbf{R}}^z + \frac{1}{2} \right) \left( S_{\mathbf{R}'}^z + \frac{1}{2} \right) - \frac{J_{\perp}}{2} \sum_{(\mathbf{R}, \mathbf{R}')} (S_{\mathbf{R}}^+ S_{\mathbf{R}'}^- + \text{H.c.}) + \sum_{\mathbf{R}} (\Omega S_{\mathbf{R}}^x - \Delta S_{\mathbf{R}}^z). \quad (9)$$

Here  $\Omega$  is the Rabi frequency,  $\Delta$  is the detuning,  $J(|\mathbf{R}|) = \frac{J_0}{|\mathbf{R}|^{\alpha}}$  represents the repulsive interactions between excited atoms (which can be, e.g., of the van der Waals type, for which  $\alpha = 6$ ), and the parameter  $J_{\perp}$  quantifies the hopping of excitations. Remarkably, the strength  $J_0$  of van der Waals interactions scales as  $n^{11}$  [55], where  $n$  is the principle quantum number [56]. This makes the van der Waals interactions between two Rydberg atoms a dominant effect over length scales as large as a few micrometers.

As indicated above, in realistic experimental setups the largest scale is  $J_0$ , prompting us to start by neglecting all other terms. Once we do that, it is clear that the ground state is polarized, with  $S_{\mathbf{R}}^z = -1/2$  for all  $\mathbf{R}$ . If we introduce a nonzero positive  $\Delta$ , it becomes energetically preferable to have a finite density of up-spins.

In fact, for any rational number  $f = p/q$ , we can find a finite range of  $\Delta$  for which the density of up-spins is  $f$  in the ground state. Notice, however, that the size of this range diminishes as  $q$  increases. Taking the long-range interactions into account, and starting from the 1D situation for simplicity, the considerations used to study quantum Hall systems can be replicated *mutatis mutandis*, showing that the up-spins form periodic structures with a unit cell of size  $q$  (see Table I).

In particular, specializing in the sequence  $f_a$  of densities, the resulting unit cell is of size  $F_{a+2}$ , and converges again to the Fibonacci quasicrystal as the index  $a$  increases. For any value of  $a$ , there are  $F_{a+2}$  distinct ground states differing by translations. For convenience, the different ground states are labeled by the parameter  $\chi \equiv (d \frac{F_{a+1}}{F_{a+2}}) \bmod 1$ , where  $d$  is an integer describing the translation with respect to some reference ground state. The parameter  $\chi$  can take the values  $0, \frac{1}{F_{a+2}}, \frac{2}{F_{a+2}}, \dots, \frac{F_{a+2}-1}{F_{a+2}}$ , and uniformly covers the segment  $[0, 1)$  in the limit  $a \rightarrow \infty$ . This notation is useful as in the limit  $a \rightarrow \infty$ , the parameter  $\chi$  represents the phason symmetry of the Fibonacci quasicrystals. We represent a ground state configuration as  $u_a^{\chi} u_a^{\chi} u_a^{\chi} u_a^{\chi} u_a^{\chi} u_a^{\chi} \dots$ , where  $u_a^{\chi}$  is the unit cell (for example, for  $f = 1/2$ , we get  $u_1^0 = 10$  and  $u_1^{1/2} = 01$ ).

For any finite  $a$ , the low energy excitations include configurations that differ from a ground state by a set of domain walls. These can generally be represented as  $u_a^{\chi_1} u_a^{\chi_2} u_a^{\chi_3} u_a^{\chi_4} u_a^{\chi_5} \dots$ , where the label  $\chi$  varies in space. Following Eq. (8), we model the energy of such a configuration as  $H_C = \sum_n \phi(\chi_n - \chi_{n+1})$ , where  $\phi$  is a periodic function which coincides with Eq. (8) for small changes. Next, we can consider the effects of quantum fluctuations induced, e.g., by a nonzero  $J_{\perp}$ .

We verified numerically that the smallest change in  $\chi$  within a unit cell is reduced in real space to an exchange of spin between adjacent sites. Consequently, the term multiplying  $J_{\perp}$  connects states of different  $\chi$ . Assigning a state written in the basis  $\{|\chi = 0\rangle, |\chi = \frac{1}{F_{a+2}}\rangle, |\chi = \frac{2}{F_{a+2}}\rangle, \dots, |\chi = \frac{F_{a+2}-1}{F_{a+2}}\rangle\}$  in each unit cell, we get a term of the form  $H_Q = \frac{J_{\perp}}{2} \sum_n \sigma_n +$

H.c., where the operators  $\sigma_n$  are defined such that  $\sigma_n|\chi_n\rangle = |(\chi_n + \frac{1}{F_{a+2}}) \bmod 1\rangle$ . We refer to the resulting 1D model (given by  $H_C + H_Q$ ) as a modified quantum  $F_{a+2}$ -state clock model, in which the energy associated with spatial variations of the spin is determined by the function  $\phi$ .

This model is in the universality class of the 2D classical clock model, which evolves into that of the 2D classical XY model in the limit  $a \rightarrow \infty$ . We conclude that arbitrarily weak quantum fluctuations demote the quasicrystalline ground state into a quasi-long-range ordered phase with algebraic correlations of the form  $\langle \exp[2\pi i(\chi_n - \chi_{n+m})] \rangle \propto m^{-\eta(J_\perp, J_0)}$ . However, following the numerical results presented in Ref. [50], we may speculate that  $\eta$  is typically parametrically small. Therefore, while true quasicrystalline long-range order cannot exist in 1D, Bragg peaks can still be observed in small systems.

In higher spatial dimensions ( $D > 1$ ), quasicrystalline long-range order is expected to survive the introduction of quantum fluctuations. Indeed, in what follows we turn to study the Falicov-Kimball model, in which exact results may be used to demonstrate the validity of our arguments beyond 1D.

#### IV. THE FALICOV-KIMBALL MODEL

We now use similar considerations to study the irrationally filled Falicov-Kimball model. The Hamiltonian [57]

$$H = -t \sum_{(\mathbf{R}, \mathbf{R}')} (f_{\mathbf{R}}^\dagger f_{\mathbf{R}'} + \text{H.c.}) + U \sum_{\mathbf{R}} c_{\mathbf{R}}^\dagger c_{\mathbf{R}} f_{\mathbf{R}}^\dagger f_{\mathbf{R}} \quad (10)$$

portrays interactions between spineless itinerant and localized electrons. Here  $f_{\mathbf{R}}$  ( $c_{\mathbf{R}}$ ) is the annihilation operator of itinerant (localized) electrons at site  $\mathbf{R}$ . Alternatively, the  $c$  (or  $f$ ) fermions may portray positively (negatively) charged ions (electrons) with an attractive (i.e.,  $U < 0$ ) interaction. We assume that the total electron and ion numbers are equal and fixed,  $\frac{1}{N} \sum_{\mathbf{R}} f_{\mathbf{R}}^\dagger f_{\mathbf{R}} = \frac{1}{N} \sum_{\mathbf{R}} c_{\mathbf{R}}^\dagger c_{\mathbf{R}} = f$ .

In the limit of large negative  $U$ , the second term drives localized electron-ion bound states. The second term, on the other hand, favors electron delocalization, and therefore acts qualitatively as repulsive interactions between bound states. In 1D, for any rational  $f = p/q$ , and sufficiently large  $|U|$  [58], the ions form a periodic lattice with a unit cell of size  $q$  [59]. In two dimensions (2D) [60,61], a periodic arrangement of diagonal stripes emerges for rational  $f \in [\frac{1}{3}, \frac{2}{5}]$ , wherein the stripe locations assume configurations identical to those in the corresponding ground states of the 1D system. A prescription for constructing the 1D ground state configurations was provided in [59].

These earlier rigorous results pave the way for our exact study of the irrationally filled model in 1D (and 2D). As before, we set  $\frac{1}{3} < f = (1 - \tau) < \frac{2}{5}$  in the large  $|U|$  regime [62]. Approximating  $f$  with high denominator elements of the rational number sequence  $\{f_a\}$ , as we have in the systems described above, and invoking the results of Ref. [59] (and [60,61]), we discover structures identical to those found in the Fibonacci QC. Putting all of the pieces together, *quasicrystalline type order* may emerge for large denominator approximants to irrational particle densities  $f$ , in *two-dimensional* electronic systems. A sketch is provided in Fig. 1.

As we argued above, in this case, quantum fluctuations are not expected to destabilize the quasicrystalline nature of the model. Thus, the electronic structure may, similar to a periodic crystal, reveal sharp Bragg peaks for momenta parallel to the direction of the stripes and concomitantly exhibit more intricate quasicrystalline features for momenta transverse to the stripe direction. If quantum and/or thermal fluctuations or disorder partially suppress the quasicrystalline features then the resulting momentum space patterns may be qualitatively similar to those anticipated for “electronic liquid crystals” (in particular for those of the nematic type) [63,64].

#### V. CONCLUSIONS

We demonstrated that QC type ground states and associated gapless excitations appear in a broad set of one- and two-dimensional strongly interacting systems. Clearly disorder, fluctuations, and other effects may stabilize more standard commensurate orders (or destroy these altogether). One may, nevertheless, expect to find imprints of the underlying quasiperiodic structures even when these are destroyed, e.g., in the form of stable approximants with a finite (but large) unit cell as experimentally appears elsewhere [65]. Alternatively, these effects may result in nonhomogeneous systems containing puddles of approximate quasiperiodic structures.

As an immediate consequence of our results, we expect these systems to be associated with slow dynamics, due to the exceptionally long relaxation of the phason degrees of freedom. In particular, one may postulate that the very slow dynamics observed in some correlated electronic systems via NMR and NQR [66–68] may be rationalized by phason-type excitations.

In general, quasicrystals may exhibit certain features similar to those of structural glasses such as stretched exponential type dynamics [69]. Specifically, the structural relaxation in an equilibrated quasicrystal is composed of an initial rapid (so-called  $\beta$  type) relaxation, which is followed by a slower ( $\alpha$  type) relaxation with a stretched exponential behavior as in glasses. This behavior is generally associated with the phason degrees of freedom. In particular, when supercooled from high temperatures, a system that is a quasicrystal in equilibrium might become quenched into a glass just as more common supercooled liquids do. Furthermore, in metallic liquids, compositions that lead to glasses and quasicrystals often lie in close proximity to each other [70,71]. In fact, certain theories consider glasses to be aperiodic crystals [72].

Taken together, all of the above suggest that systems exhibiting quasicrystalline ground states may be unstable to (i) commensurate lock-in effects (possibly to high order approximants in clean systems) or, as underscored above, (ii) an inherent susceptibility towards glassy dynamics and aperiodic structures. Indeed, in certain strongly correlated electronic and disorder free magnetic systems, stretched exponential decay and other features of glassy (or possible other extremely slow) dynamics appear [66,73–76].

The fate of the emergent quasicrystals that we found theoretically and imprints thereof including, notably, possible relations between our prediction of electronic and atomic QCs to experimental findings remain to be tested by numerics.

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- [1] J. M. Tranquada, B. J. Sternlieb, J. D. Axe, Y. Nakamura, and S. Uchida, *Nature (London)* **375**, 561 (1995).
- [2] K. Yamada, C. H. Lee, K. Kurahashi, J. Wada, S. Wakimoto, S. Ueki, H. Kimura, Y. Endoh, S. Hosoya, G. Shirane, R. J. Birgeneau, M. Greven, M. A. Kastner, and Y. J. Kim, *Phys. Rev. B* **57**, 6165 (1998).
- [3] J. Zaanen and O. Gunnarsson, *Phys. Rev. B* **40**, 7391 (1989).
- [4] H. J. Schulz, *Phys. Rev. Lett.* **64**, 1445 (1990).
- [5] M. Kato, K. Machida, H. Nakanishi, and M. Fujita, *J. Phys. Soc. Jpn.* **59**, 1047 (1990).
- [6] V. Emery and S. Kivelson, *Physica C* **209**, 597 (1993).
- [7] D. C. Tsui, H. L. Stormer, and A. C. Gossard, *Phys. Rev. Lett.* **48**, 1559 (1982).
- [8] R. B. Laughlin, *Phys. Rev. Lett.* **50**, 1395 (1983).
- [9] E. Wigner, *Phys. Rev.* **46**, 1002 (1934).
- [10] R. J. Elliott, *Phys. Rev.* **124**, 346 (1961).
- [11] P. Bak and J. von Boehm, *Phys. Rev. B* **21**, 5297 (1980).
- [12] M. E. Fisher and W. Selke, *Phys. Rev. Lett.* **44**, 1502 (1980).
- [13] W. Selke, *Phys. Rep.* **170**, 213 (1988).
- [14] A. Gendiar and T. Nishino, *Phys. Rev. B* **71**, 024404 (2005).
- [15] P. Bak and R. Bruinsma, *Phys. Rev. Lett.* **49**, 249 (1982).
- [16] L. Chayes, V. Emery, S. Kivelson, Z. Nussinov, and G. Tarjus, *Physica A* **225**, 129 (1996).
- [17] A. Giuliani, J. L. Lebowitz, and E. H. Lieb, *Phys. Rev. B* **74**, 064420 (2006).
- [18] S. Chakrabarty and Z. Nussinov, *Phys. Rev. B* **84**, 144402 (2011).
- [19] A. Giuliani, E. H. Lieb, and R. Seiringer, *Phys. Rev. B* **88**, 064401 (2013).
- [20] A. Giuliani and R. Seiringer, *Commun. Math. Phys.* (2016).
- [21] H. van Beijeren, *Fundamental Problems in Statistical Mechanics VII* (North-Holland Publishing Company, Amsterdam, 1990).
- [22] J. Frenkel and T. Kontorova, *Phys. Z. Sowjetunion* **13**, 1 (1938).
- [23] T. Janssen, in *Les Houches Lectures, Beyond Quasicrystals*, edited by F. Axel and D. Gratias (Springer, Berlin, 1995), pp. 75–140.
- [24] R. Lefort, J. Etrillard, B. Toudic, F. Guillaume, T. Breczewski, and P. Bourges, *Phys. Rev. Lett.* **77**, 4027 (1996).
- [25] S. V. Smaalen and K. D. M. Harris, *Proc. R. Soc. London Ser. A* **452**, 677 (1996).
- [26] T. Weber, H. Boysen, M. Honal, F. Frey, and R. B. Neder, *Z. Kristallogr.* **211**, 238 (1996).
- [27] F. Flicker and J. van Wezel, *Phys. Rev. Lett.* **115**, 236401 (2015).
- [28] D. R. Hofstadter, *Phys. Rev. B* **14**, 2239 (1976).
- [29] C. R. Dean, L. Wang, P. Maher, C. Forsythe, F. Ghahari, Y. Gao, J. Katoch, M. Ishigami, P. Moon, M. Koshino, T. Taniguchi, K. Watanabe, K. L. Shepard, J. Hone, and P. Kim, *Nature (London)* **497**, 598 (2013).
- [30] S. Gopalakrishnan, I. Martin, and E. A. Demler, *Phys. Rev. Lett.* **111**, 185304 (2013).
- [31] R. Lifshitz, *Phys. Rev. Lett.* **113**, 079602 (2014).
- [32] M. Sandbrink, J. Roth, and M. Schmiedeberg, *Phys. Rev. Lett.* **113**, 079601 (2014).
- [33] S. Gopalakrishnan, I. Martin, and E. A. Demler, *Phys. Rev. Lett.* **113**, 079603 (2014).
- [34] D. Shechtman, I. Blech, D. Gratias, and J. W. Cahn, *Phys. Rev. Lett.* **53**, 1951 (1984).
- [35] D. Levine and P. J. Steinhardt, *Phys. Rev. Lett.* **53**, 2477 (1984).
- [36] D. Levine and P. J. Steinhardt, *Phys. Rev. B* **34**, 596 (1986).
- [37] I. Martin, S. Gopalakrishnan, and E. A. Demler, *Phys. Rev. B* **93**, 235140 (2016).
- [38] N. Mermin, *Rev. Mod. Phys.* **64**, 3 (1992).
- [39] R. Lifshitz, *Isr. J. Chem.* **51**, 1156 (2011).
- [40] P. Bak, *Phys. Rev. Lett.* **54**, 1517 (1985).
- [41] R. Tao and D. J. Thouless, *Phys. Rev. B* **28**, 1142 (1983).
- [42] D. J. Thouless, *Phys. Rev. B* **31**, 8305 (1985).
- [43] R. Tao, *Phys. Rev. B* **29**, 636 (1984).
- [44] E. J. Bergholtz and A. Karlhede, *Phys. Rev. Lett.* **94**, 026802 (2005).
- [45] A. Seidel, H. Fu, D.-H. Lee, J. M. Leinaas, and J. Moore, *Phys. Rev. Lett.* **95**, 266405 (2005).
- [46] E. J. Bergholtz and A. Karlhede, *Phys. Rev. B* **77**, 155308 (2008).
- [47] J. Hubbard, *Phys. Rev. B* **17**, 494 (1978).
- [48] M. Senechal, *Quasicrystals and Geometry* (Cambridge University Press, Cambridge, 1996), p. 308.
- [49] R. Lifshitz, *J. Alloys Compd.* **342**, 186 (2002).
- [50] E. Sagi and E. Eisenberg, *Phys. Rev. E* **90**, 012105 (2014).
- [51] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevB.94.035131> for the general analytic form of the energy of phasons (for general interactions), and examples demonstrating the universal low-energy form (Eq. 8) for specific interactions. In addition, we present an explicit calculation of the energy gap for rational approximants.
- [52] E. Sela, M. Punk, and M. Garst, *Phys. Rev. B* **84**, 085434 (2011).
- [53] H. Weimer and H. P. Büchler, *Phys. Rev. Lett.* **105**, 230403 (2010).
- [54] P. Schauf, M. Cheneau, M. Endres, T. Fukuhara, S. Hild, A. Omran, T. Pohl, C. Gross, S. Kuhr, and I. Bloch, *Nature (London)* **491**, 87 (2012).
- [55] R. M. W. van Bijnen, Quantum engineering with ultracold atoms, Ph.D. thesis, Eindhoven University of Technology, 2013.
- [56] More specifically, assuming the Rydberg state is an  $S$  orbital, we get that the van der Waals interactions are of the order of  $J_0 \sim 10E_H a_0^6 n^{11}$ , where  $E_H = 4.3597 \times 10^{-18}$  J is the Hartree energy and  $a_0 = 5.29 \times 10^{-11}$  m is the Bohr radius.
- [57] L. M. Falicov and J. C. Kimball, *Phys. Rev. Lett.* **22**, 997 (1969).
- [58] Following [59], in 1D, the threshold minimal value ( $|U_{\min}|$ ) of  $|U|$  required to stabilize period  $q$  ground state is bounded by  $|U_{\min}| < c \times 4^q$  with  $c$  a constant.

- [59] P. Lemberger, *J. Phys. A* **25**, 715 (1992).
- [60] K. Haller and T. Kennedy, *J. Stat. Phys.* **102**, 15 (2001).
- [61] T. Kennedy, *J. Stat. Phys.* **91**, 829 (1998).
- [62] As, formally, the minimal  $|U_{\min}|$  required to enable a proof of the period  $q$  stripe order may diverge as  $q \rightarrow \infty$ , we may consider the electron density  $f$  to be a highly incommensurate fraction with a large denominator  $q$  that corresponds to one of the  $a \gg 1$  elements in the set of rational sequence approximants  $f_a = \frac{F_a}{F_{a+2}}$ , with  $F_a$  the  $a$ th Fibonacci number.
- [63] S. A. Kivelson, E. Fradkin, and V. J. Emery, *Nature (London)* **393**, 550 (1998).
- [64] K. Fujita, M. H. Hamidian, S. D. Edkins, C. K. Kim, Y. Kohsaka, M. Azuma, M. Takano, H. Takagi, H. Eisaki, S.-I. Uchida, A. Allais, M. J. Lawler, E.-A. Kim, S. Sachdev, and J. C. S. Davis, *Proc. Natl. Acad. Sci. USA* **111**, E3026 (2014).
- [65] A. I. Goldman and R. F. Kelton, *Rev. Mod. Phys.* **65**, 213 (1993).
- [66] N. J. Curro, P. C. Hammel, B. J. Suh, M. Hücker, B. Büchner, U. Ammerahl, and A. Revcolevschi, *Phys. Rev. Lett.* **85**, 642 (2000).
- [67] G. Allodi, M. C. Guidi, R. De Renzi, A. Caneiro, and L. Pinsard, *Phys. Rev. Lett.* **87**, 127206 (2001).
- [68] A. P. Dioguardi, M. M. Lawson, B. T. Bush, J. Crocker, K. R. Shirer, D. M. Nisson, T. Kissikov, S. Ran, S. L. Bud'ko, P. C. Canfield, S. Yuan, P. L. Kuhns, A. P. Reyes, H.-J. Grafe, and N. J. Curro, *Phys. Rev. B* **92**, 165116 (2015).
- [69] M. Dzugutov and J. Phillips, *J. Non-Cryst. Solids* **192-193**, 397 (1995).
- [70] K. Kelton, *J. Non-Cryst. Solids* **334-335**, 253 (2004).
- [71] O. S. Devinder Singh and R. S. Tiwari, *Metallic Glasses, Quasicrystals and their Nanocomposites: Synthesis and Characterization of Metallic Glasses, Quasicrystalline-Glass Composites in some Zr/Cu-Based Alloys* (LAP LAMBERT Academic, Saarbrücken, 2014).
- [72] V. Lubchenko and P. G. Wolynes, *Annu. Rev. Phys. Chem.* **58**, 235 (2007).
- [73] T. Park, Z. Nussinov, K. R. A. Hazzard, V. A. Sidorov, A. V. Balatsky, J. L. Sarrao, S.-W. Cheong, M. F. Hundley, J.-S. Lee, Q. X. Jia, and J. D. Thompson, *Phys. Rev. Lett.* **94**, 017002 (2005).
- [74] E. R. Kassner, A. B. Eyvazov, B. Pichler, T. J. S. Munsie, H. A. Dabkowska, G. M. Luke, and J. C. S. Davis, *Proc. Natl. Acad. Sci. USA* **112**, 8549 (2015).
- [75] J. Wu, J. S. Wildeboer, F. Werner, A. Seidel, Z. Nussinov, and S. A. Solin, *Europhys. Lett.* **93**, 67001 (2011).
- [76] M. Ovodov, D. Kalok, I. Tamir, S. Mitra, B. Sacépé, and D. Shahar, *Sci. Rep.* **5**, 13503 (2015).