Decay of Bogoliubov excitations in one-dimensional Bose gases

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We study the decay of Bogoliubov quasiparticles in one-dimensional Bose gases. Starting from the hydrodynamic Hamiltonian, we develop a microscopic theory that enables one to systematically study both the excitations and their decay. At zero temperature, the leading mechanism of decay of a quasiparticle is disintegration into three others. We find that low-energy quasiparticles (phonons) decay with the rate that scales with the seventh power of momentum, whereas the rate of decay of the high-energy quasiparticles does not depend on momentum. In addition, our approach allows us to study analytically the quasiparticle decay in the whole crossover region between the two limiting cases. When applied to integrable models, including the Lieb-Liniger model of bosons with contact repulsion, our theory confirms the absence of the decay of quasiparticle excitations. We account for two types of integrability-breaking perturbations that enable finite decay: three-body interaction between the bosons and two-body interaction of finite range.

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I. INTRODUCTION

At low temperatures three-dimensional Bose gas undergoes Bose-Einstein condensation, characterized by macroscopic occupation of the zero-momentum state. This feature enabled Bogoliubov in 1947 to develop a mean field theory of weakly-interacting Bose gas [1,2]. In this theory, the excitation spectrum acquires the so-called Bogoliubov form:

$$\varepsilon_q = \sqrt{v^2 q^2 + \left(\frac{q^2}{2m}\right)^2}. \tag{1}$$

Here v is the sound velocity, m denotes the mass of bosonic particles, while q is the momentum. At low momenta, $q \ll mv$, Bogoliubov quasiparticles are phonons with linear spectrum. At high momenta, $q \gg mv$, the quasiparticle energy (1) reproduces the quadratic spectrum of the physical particles forming the Bose gas.

Bogoliubov's mean field approach neglects the residual interaction between the quasiparticles. As a result of these interactions, quasiparticles are not entirely free and eventually decay. In three dimensions, the leading mechanism is the decay of a quasiparticle into two others. For quasiparticle excitations of low momenta, $q \ll mv$, the decay rate at zero temperature was found in 1958 by Beliaev [2,3]. It scales with the fifth power of the quasiparticle momentum.

The decay of quasiparticles is reflected in the dynamic structure factor of interacting bosons. It does not have the form of an infinitely sharp delta function, but rather that of a peak with the width determined by the decay rate. Alternatively, the decay rate can be probed by measuring the cross section for collisions of a quasiparticle with the particles of the condensate. The latter technique was used recently [4] (see also Ref. [5]) to confirm the predictions of the Beliaev theory in three-dimensional Bose-Einstein condensates.

In contrast to the three-dimensional case, bosons in one dimension do not condense due to the enhanced role of quantum fluctuations. Therefore, the Bogoliubov mean-field approach cannot be applied. Instead, Lieb and Liniger [6] studied the model of one-dimensional bosons with contact repulsion, which allows an exact solution. This enabled them to study both the ground state properties of the system [6] and its elementary excitations [7]. Importantly, unlike the three-dimensional case, there are two branches of elementary excitations, see Fig. 1. Excitation of type I behaves qualitatively similar to the Bogoliubov mode in three dimensions, and in the limit of weak interaction has been shown [8] to have the dispersion (1). The second, type II excitation exists in the limited range of momenta determined by the density and describes the so-called dark soliton [2,8]. At the lowest momenta the two branches approach each other, having the common linear part of the spectrum, see Fig. 1.

The type II branch bends down and thus represents the lowest energy state of the system for a given momentum. Therefore, at zero temperature these excitations cannot decay. On the other hand, momentum and energy conservation laws do not forbid the decay of the excitation of type I. A simple analysis shows that these excitations still cannot decay into two others, but decay into three quasiparticles is allowed. In addition to momentum and energy, integrable models possess a macroscopic number of additional conserved quantities. This prevents any quasiparticle decay. On the other hand, in practice no system is exactly integrable, and even the smallest deviation from integrability leads to a finite decay of quasiparticles.

Decay of quasiparticle excitations in one-dimensional quantum liquids is a subject of great current interest [9–19]. In this paper we study the decay of Bogoliubov quasiparticles in a system of weakly-interacting bosons. In the limit of high energy of the initial quasiparticle, $q \gg mv$, this problem was addressed in Ref. [11]. The integrability of the Lieb-Liniger model was broken by the addition of weak three-body interaction [20,21]. It was shown that this perturbation leads to a finite decay rate that does not depend on the quasiparticle momentum. Unlike Ref. [11], our theory enables one to study analytically the decay of quasiparticles of arbitrary momenta. Furthermore, in addition to the effects of the three-body interaction, we study another integrability-breaking perturbation, which accounts for a finite range of two-body interaction. This complementary term turns out to be an important factor that also affects the decay rate. A summary of our results for the decay of quasiparticles of

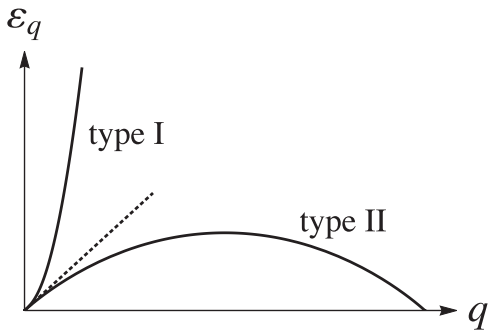

FIG. 1. Two branches of excitations in a one-dimensional system of bosons with contact repulsion. At small momenta the excitations on both branches are characterized by the linear spectrum, $\varepsilon_q = v|q|$, represented by the dotted line. At weak interaction, the dispersion of type I excitations deviates from linearity as $|q|^3$, while for type II as $|q|^{5/3}$. Such form of the deviation is actually true above the very small quantum crossover momentum, as we discuss further below.

small momenta, $q \ll mv$, has been reported in Ref. [17], where we relied on certain phenomenological properties of one-dimensional quantum liquids. The approach of the present paper is fully microscopic and enables us to find the decay rate of Bogoliubov quasiparticles in the whole range of momenta. In the cases $q \gg mv$ and $q \ll mv$, we recover the results of Refs. [11,17].

The description of the excitation spectrum of weakly interacting Bose gas in terms of Bogoliubov quasiparticles and dark solitons is applicable only at sufficiently high momenta, $q \gg q^*$, where $q^* \sim (mv)^{3/2}(\hbar n_0)^{-1/2} \ll mv$ and n_0 is the mean particle density [22–24]. Below the momentum scale q^* the excitations are effective fermions [23,25], with type I and type II branches corresponding to quasiparticles and quasiholes, respectively. At zero temperature, fermionic quasiparticles decay with the rate that scales as the eighth power of momentum [9,16]. We apply the results of Ref. [16] to evaluate this rate in our system, thereby presenting a complete theory of the decay of type I excitations at zero temperature.

The paper is organized as follows. In Sec. II we present the hydrodynamic description of the system of weakly interacting bosons. We discuss various terms in the gradient expansion and split the Hamiltonian into a harmonic part describing the Bogoliubov quasiparticles and the anharmonic part that accounts for their interactions. In Sec. III we calculate and analyze the scattering matrix describing the decay of Bogoliubov quasiparticles with momenta $q \gg q^*$. The rate of decay is evaluated in Sec. IV. In Sec. V we obtain the rate of decay of fermionic quasiparticles at momenta $q \ll q^*$. We discuss our results in Sec. VI. Some technical details of our work are presented in the appendices.

II. HAMILTONIAN OF WEAKLY INTERACTING BOSONS

A. Microscopic model

In the representation of second quantization, the system of interacting bosons in one dimension is described by the Hamiltonian

$$H = H_{\rm kin} + H_{\rm int}, \tag{2}$$

where

$$H_{\rm kin} = \frac{\hbar^2}{2m}\int dx(\nabla\Psi^\dagger)(\nabla\Psi), \tag{3}$$

$$H_{\rm int} = \frac{1}{2}\int dx dx'\, g(x-x')n(x)n(x'). \tag{4}$$

Here Eq. (3) is the kinetic energy, while Eq. (4) describes the interaction between the bosons. By $\Psi(x)$ and $\Psi^\dagger(x)$ we denote the bosonic single particle field operators that satisfy the standard commutation relation $[\Psi(x),\Psi^\dagger(x')] = \delta(x-x')$. The mass of bosonic particles is m. The repulsive two-particle interaction in Eq. (4) is described by the short-ranged function $g(x)$, while $n = \Psi^\dagger\Psi$ denotes the density of particles. In the following we consider the case of weak interaction. This regime is defined by the condition

$$\int dx g(x) \ll \frac{\hbar^2 n_0}{m}, \tag{5}$$

where n_0 denotes the mean density.

The Hamiltonian H provides a microscopic description for an arbitrary system of bosons in one dimension interacting via a pairwise interaction. In some special cases Eqs. (2)–(4) describe the so-called integrable models. Throughout this paper, we will be particularly interested in the Lieb-Liniger model, which is defined by the contact interaction $g(x) = g\delta(x)$. The integrability of this model allows an exact solution by means of the Bethe ansatz technique [6,7]. On the other hand, because of integrability there is no decay of quasiparticle excitations in this model. In this paper we consider leading corrections to the Lieb-Liniger model that break the integrability and thus ensure the decay of quasiparticles. Since there is no well established way to develop perturbation theory starting with Bethe ansatz, here we develop an alternative theoretical description. It is based on the microscopic hydrodynamic approach that enables us to study both the excitations and their decay. Unlike Bethe ansatz, this approach is limited to weak interactions, but it has the advantage that its applicability is not limited to integrable models.

Experimentally, the system of one-dimensional bosons can nowadays be routinely realized with cold atomic gases [26]. Starting from the three-dimensional system of bosons, one applies an external potential to confine the particle motion to one dimension. At energies smaller than the intersubband spacing of the confining potential, one effectively obtains a one-dimensional system of interacting bosons. In such situations, making use of the Hamiltonian in the form (2)–(4) to describe the system is *a priori* not justified. Instead, one must carefully derive the corresponding one-dimensional model. For a typical experimental situation of bosonic atoms in a harmonic confining potential interacting via a short-range potential [27], the effective one-dimensional model is derived in several papers [11,20,21]. The kinetic energy in the effective model of bosons is still described by Eq. (3). However, the interaction term takes a more complicated form

$$H'_{\rm int} = \frac{1}{2}\int dx dx'\, g(x-x')n(x)n(x') - \frac{\hbar^2}{m}\alpha\int dx\, n^3. \tag{6}$$

In Refs. [11,20,21], the two-body interaction in Eq. (6) was found to be of the contact type, $g(x) = g\delta(x)$. In comparison

to Eq. (4), the last term in Eq. (6) is new and has the meaning of effective three-body interaction. It was obtained [11,20,21] by accounting for the effect of virtual transitions of bosons into higher radial modes.

An important property of the last term in the interaction Hamiltonian (6) is that it breaks the integrability of the Lieb-Liniger model and thus enables the decay of quasiparticles. In addition, we modify the interaction Hamiltonian by assuming that the two-body interaction potential $g(x)$ has finite width, which amounts to adding another integrability-breaking perturbation. In the following we refer to H as defined by $H = H_{\rm kin} + H'_{\rm int}$ and treat both perturbations on equal footing.

B. The density-phase representation

The Hamiltonian of the system of weakly interacting bosons, given by Eqs. (2), (3), and (6), is expressed in terms of the bosonic field operators $\Psi(x)$ and $\Psi^\dagger(x)$. For our purposes it is convenient to apply the hydrodynamic approach [28–30], in which the field operators are expressed in terms of the particle density $n(x)$ and its conjugate field $\theta(x)$ that can be thought of as the superfluid phase. In the regime of weak interaction the resulting Hamiltonian is then naturally expressed as a sum of the contribution H_0 that is quadratic in the new fields and the higher-order perturbations V_3, V_4, etc. In this representation H_0 naturally accounts for the Bogoliubov quasiparticles, while the perturbations describe the interactions between quasiparticles that enable their decay.

We start by expressing the bosonic field operators in terms of the density and phase fields using the so-called Madelung representation [28,29]

$$\Psi = e^{-i\theta}\sqrt{n}, \quad \Psi^\dagger = \sqrt{n}\, e^{i\theta}. \tag{7}$$

The operators $\Psi(x)$ and $\Psi^\dagger(x)$ expressed in this fashion have the usual bosonic commutation relations provided $[n(x),\theta(x')] = -i\delta(x-x')$. Substituting Eq. (7) into the kinetic energy (3) of the Hamiltonian, one obtains [28]

$$H_{\rm kin} = \frac{\hbar^2}{2m}\int dx\left[n(\nabla\theta)^2 + \frac{(\nabla n)^2}{4n}\right]. \tag{8}$$

The next step is to express the density as

$$n = n_0 + \frac{1}{\pi}\nabla\varphi, \tag{9}$$

where n_0 is the mean particle density and the new bosonic field φ satisfies the commutation relation

$$[\nabla\varphi(x),\theta(x')] = -i\pi\delta(x-x'). \tag{10}$$

The hydrodynamic approach is applicable as long as the length scale associated with the density fluctuations is large compared with the distance between particles n_0^{-1}. In this regime the density fluctuations are small, $|\nabla\varphi| \ll n_0$, and the square root in Eq. (7) is real.

We now take advantage of the smallness of $|\nabla\varphi|/n_0$ and expand the Hamiltonian in powers of bosonic fields φ and θ. The expansion starts with quadratic contributions. The standard Luttinger liquid form

$$H_{LL} = \int dx\left[\frac{\hbar^2 n_0}{2m}(\nabla\theta)^2 + \frac{g}{2\pi^2}(\nabla\varphi)^2\right] \tag{11}$$

is obtained from the first term in the kinetic energy (8) and the first term in Eq. (6). Here $g = g_0$, where $g_q = \int dx e^{-iqx/\hbar} g(x)$ denotes the Fourier transform of the interaction potential.

Apart from Eq. (11), there are a number of additional quadratic terms in the Hamiltonian. First, the three-body interaction in Eq. (6) upon substitution of Eq. (9) generates the contribution

$$-\frac{3\alpha\hbar^2 n_0}{\pi^2 m}\int dx(\nabla\varphi)^2. \tag{12}$$

Second, the so-called quantum pressure, given by the second term in Eq. (8), and the two-particle interaction term in Eq. (6) give rise to

$$\frac{\chi^2\hbar^2}{8\pi^2 m n_0}\int dx(\nabla^2\varphi)^2, \tag{13}$$

where

$$\chi^2 = 1 + 2mn_0\frac{d^2 g_q}{dq^2}\bigg|_{q=0}. \tag{14}$$

For contact interaction, $g_q = \text{const}$, i.e., $\chi^2 = 1$. In this special case the two-particle interaction does not contribute to Eq. (13). Finally, for noncontact interactions the first term in Eq. (6) generates contributions proportional to $(\nabla^3\varphi)^2$, $(\nabla^4\varphi)^2$, etc. Such contributions become important only at very high momenta and therefore will be neglected.

Collecting the terms of Eqs. (11), (12), and (13), we obtain the quadratic Hamiltonian

$$H_0 = \frac{\hbar v}{2\pi}\int dx\left\{K(\nabla\theta)^2 + \frac{1}{K}\left[(\nabla\varphi)^2 + \frac{2\chi^2\hbar^2}{q_0^2}(\nabla^2\varphi)^2\right]\right\}. \tag{15}$$

In Eq. (15), the sound velocity v satisfies

$$v^2 = \frac{gn_0}{m} - \frac{6\alpha\hbar^2 n_0^2}{m^2}, \tag{16}$$

the crossover momentum q_0 is introduced as

$$q_0 = \sqrt{8}mv, \tag{17}$$

while the Luttinger liquid parameter is defined as

$$K = \frac{\pi\hbar n_0}{mv}. \tag{18}$$

The regime of weak interactions considered in this paper corresponds to $K \gg 1$, cf. Eq. (5).

The strength of the three-particle interaction is quantified by the dimensionless coupling constant α [see Eq. (6)]. In this paper we will require this perturbation to have only a weak effect on the physical properties of the Bose gas. It is instructive to consider the effect of the three-particle interaction on the sound velocity. From Eq. (16) we conclude that the correction to v is small provided

$$A = K^2\alpha \ll 1. \tag{19}$$

Since $K \gg 1$, this condition is more restrictive than the naive expectation $\alpha \ll 1$. We will see below that other physical quantities of interest are also controlled by the parameter A rather than α.

In addition to H_0, the original hydrodynamic Hamiltonian contains a number of higher order in φ and θ contributions that describe the interactions between quasiparticles. The cubic correction to H_0 is

$$V_3 = \frac{\hbar^2}{m}\int dx\left[a_1(\nabla\varphi)(\nabla\theta)^2 - \frac{a_2}{n_0^2}(\nabla^2\varphi)^2(\nabla\varphi) - \frac{\alpha}{\pi^3}(\nabla\varphi)^3\right], \tag{20}$$

where for convenience we introduced

$$a_1 = \frac{1}{2\pi}, \quad a_2 = \frac{1}{8\pi^3}. \tag{21}$$

The first term in Eq. (20) arises from the first term in the kinetic energy (8). The second term in Eq. (20) emerges from the expansion of the second term in Eq. (8). The last term in Eq. (20) originates from the second term in Eq. (6).

In order to evaluate the decay rate of excitations with momenta $q \sim q_0$ one has to account for the quartic in φ and θ contributions to the Hamiltonian. We write the corresponding term as

$$V_4 = \frac{\hbar^2}{mn_0}\int dx\left[\frac{a_3}{n_0^2}(\nabla^2\varphi)^2(\nabla\varphi)^2 + \beta(\nabla\varphi)^4\right], \tag{22}$$

where

$$a_3 = \frac{1}{8\pi^4}, \quad \beta = 0. \tag{23}$$

The first term in Eq. (22) appears from the expansion of the quantum pressure term in Eq. (8). The second term in V_4 is not generated in the formal expansion of the Hamiltonian given by Eqs. (6) and (8). We added it to Eq. (22) with a formally vanishing coefficient for completeness and future convenience (see Appendix E).

So far we have expanded our Hamiltonian to the fourth order in the bosonic fields. The terms V_3 and V_4 will be used to evaluate the decay rate of Bogoliubov quasiparticles with momenta of order q_0, where the crossover from linear to quadratic behavior of the quasiparticle dispersion (1) occurs. To understand why the subsequent higher-order terms can be neglected, one can analyze the low-energy scaling of the Hamiltonian. Such an analysis is performed in Appendix A, where we show that our expansion of the Hamiltonian in powers of the bosonic fields φ and θ is in fact expansion in small parameter $1/\sqrt{K}$. In particular, we find $V_3 \propto 1/\sqrt{K}$ and $V_4 \propto 1/K$.

C. Normal mode expansion

Our next goal is to obtain Bogoliubov quasiparticles as normal modes of the quadratic Hamiltonian (15). To this end we express the fields φ and θ in terms of bosonic quasiparticle operators b_q and $b_q^\dagger$ via the relations

$$\nabla\varphi(x) = \sum_q \sqrt{\frac{\pi^2 n_0}{2Lm\varepsilon_q}}|q|e^{iqx/\hbar}(b_{-q}^\dagger + b_q), \tag{24}$$

$$\nabla\theta(x) = \sum_q \sqrt{\frac{m\varepsilon_q}{2L\hbar^2 n_0}}\,\mathrm{sgn}(q)e^{iqx/\hbar}(b_{-q}^\dagger - b_q). \tag{25}$$

Here L denotes the system size. As a result, the Hamiltonian (15) takes the diagonal form

$$H_0 = \sum_q \varepsilon_q b_q^\dagger b_q, \tag{26}$$

with the excitation spectrum given by

$$\varepsilon_q = \sqrt{v^2q^2 + \chi^2\left(\frac{q^2}{2m}\right)^2}. \tag{27}$$

For the Lieb-Liniger model, we have $\chi = 1$, and the spectrum coincides with the well known expression (1), Ref. [8].

Deviation of the spectrum (27) from the form (1) appears in the case of nonvanishing range of interactions between the bosons. This deviation is most important at high momenta $q \gg q_0$, where $\varepsilon_q \simeq \chi q^2/2m$ rather than $q^2/2m$. The latter expression represents the energy of a highly excited boson, which essentially does not interact with other bosons because of its high momentum q. This physics is not captured by the hydrodynamic theory, which is applicable only at $q \ll \hbar n_0$.

As we show in Appendix A, the anharmonic terms (20) and (22) represent corrections to the quadratic Hamiltonian H_0 that are small as $1/\sqrt{K}$ and $1/K$, respectively. As a result, they do not affect the excitation spectrum significantly. On the other hand, they represent the residual interactions between the quasiparticles that enable finite decay rate. Using the normal mode representation (24) and (25), the cubic anharmonic term (20) becomes

$$V_3 = \frac{\pi v^2}{\sqrt{8Lmn_0}}\sum_{q_1,q_2,q_3}\frac{|q_1q_2q_3|}{\sqrt{\varepsilon_{q_1}\varepsilon_{q_2}\varepsilon_{q_3}}}\delta_{q_1+q_2+q_3,0} \times\left[\frac{1}{3}f_+(q_1,q_2,q_3)(b_{q_1}^\dagger b_{q_2}^\dagger b_{q_3}^\dagger + \text{H.c.}) + f_-(q_1,q_2,q_3)(b_{q_1}^\dagger b_{q_2}^\dagger b_{-q_3} + \text{H.c.})\right], \tag{28}$$

where the dimensionless functions are

$$f_\pm(q_1,q_2,q_3) = \frac{a_1}{v^2}\left(\frac{\varepsilon_{q_1}\varepsilon_{q_2}}{q_1q_2} \pm \frac{\varepsilon_{q_1}\varepsilon_{q_3}}{q_1q_3} \pm \frac{\varepsilon_{q_2}\varepsilon_{q_3}}{q_2q_3}\right) + \frac{8\pi^2 a_2}{q_0^2}(q_1q_2 + q_1q_3 + q_2q_3) - \frac{3A}{\pi^3}. \tag{29}$$

Similarly, the quartic anharmonic term (22) transforms to

$$V_4 = \frac{\pi^2 v^2}{4Lmn_0}\sum_{q_1,q_2,q_3,q_4}\left[f(q_1,q_2,q_3,q_4)\delta_{q_1+q_2+q_3+q_4,0} \times\prod_{i=1}^4\frac{|q_i|}{\sqrt{\varepsilon_{q_i}}}(b_{q_i}^\dagger + b_{-q_i})\right], \tag{30}$$

where

$$f(q_1,q_2,q_3,q_4) = -\frac{4\pi^2 a_3}{3q_0^2}(q_1q_2 + q_1q_3 + q_1q_4 + q_2q_3 + q_2q_4 + q_3q_4) + B, \tag{31}$$

where $B = K^2\beta$. We will now apply the results (28)–(31) to the evaluation of the decay rate of Bogoliubov quasiparticles.

III. SCATTERING MATRIX ELEMENT

The spectrum of a Bogoliubov quasiparticle in a weakly interacting Bose gas is given by Eq. (27). The presence in the Hamiltonian of weak anharmonic perturbations, such as V_3 and V_4, means that the quasiparticles are weakly interacting. This generally leads to their decay. Our goal is to study the decay of a state with a single quasiparticle as a function of its momentum Q.

For one-dimensional particles with the spectrum (27), decay into two quasiparticles is incompatible with simultaneous conservation of energy and momentum. The simplest allowed decay process corresponds to three particles in the final state, see Fig. 2. It will become clear below that this is the dominant decay channel in a weakly interacting Bose gas.

We start our evaluation by considering the scattering matrix element $\mathcal{A}_{fi}$ for the decay of the initial state $|i\rangle = b_Q^\dagger|0\rangle$ into the final one $|f\rangle = b_{q_1}^\dagger b_{q_2}^\dagger b_{q_3}^\dagger|0\rangle$. $\mathcal{A}_{fi}$ is defined in terms of the T matrix as

$$\mathcal{A}_{fi} = \langle 0|b_{q_1}b_{q_2}b_{q_3}|T|b_Q^\dagger|0\rangle. \tag{32}$$

Such a matrix element can be obtained in a number of ways. The simplest contribution is in the first order in the quartic term V_4 [Eq. (30)] that allows for the direct transition between the initial and final states. Alternatively, the same transition can be accomplished in second order in the cubic perturbation V_3 [Eq. (28)]. In a weakly interacting Bose gas, i.e., at $K \gg 1$, the two perturbations are small, $V_3 \propto 1/\sqrt{K}$ and $V_4 \propto 1/K$. As a result, the two contributions to the matrix element (32) appear in the same order, $\mathcal{A}_{fi} \propto 1/K$. A straightforward argument shows that higher-order anharmonic perturbations to the Hamiltonian H_0 give rise to parametrically smaller contributions to the matrix element (32). Accounting only for the leading contributions, we find

$$\mathcal{A}_{fi} = \langle f|V_4|i\rangle + \sum_m \frac{\langle f|V_3|m\rangle\langle m|V_3|i\rangle}{\varepsilon_Q - E_m}. \tag{33}$$

Here the summation is over the intermediate states $|m\rangle$, whose energies are denoted by E_m.

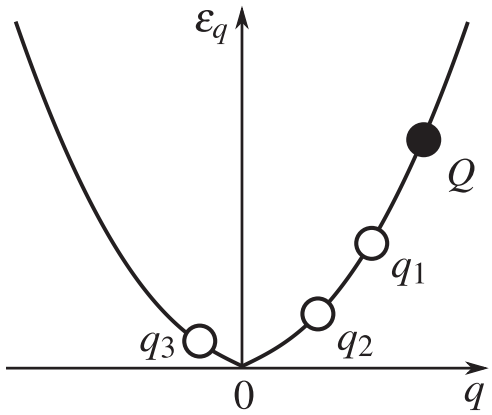

FIG. 2. In a one dimensional Bose gas, a quasiparticle excitation of momentum Q decays into three excitations with momenta q_1,q_2, and q_3. Using the conservation laws, one finds that two quasiparticles in the final state propagate in the direction of the initial quasiparticle, unlike the remaining one.

The contribution to the scattering matrix element due to the quartic anharmonic term (30) arises from the combinations of operators in V_4 that contain three creation and one annihilation operator. There are four such terms. After a simple calculation one obtains

$$\langle f|V_4|i\rangle = \frac{6\pi^2 v^2}{Lmn_0}\frac{|Qq_1q_2q_3|}{\sqrt{\varepsilon_Q\varepsilon_{q_1}\varepsilon_{q_2}\varepsilon_{q_3}}} \times f(Q,-q_1,-q_2,-q_3)\delta_{Q,q_1+q_2+q_3}, \tag{34}$$

where the function f is defined in Eq. (31).

The calculation of the contribution to the scattering matrix element (33) that arises from V_3 is more involved and deferred to Appendix B. Accounting for Eq. (34), the final result for the scattering matrix element (33) is

$$\mathcal{A}_{fi} = \frac{\pi^2 v^2}{2Lmn_0}\frac{|Qq_1q_2q_3|}{\sqrt{\varepsilon_Q\varepsilon_{q_1}\varepsilon_{q_2}\varepsilon_{q_3}}}\big[F(Q,q_1,q_2,q_3) + F(Q,q_2,q_1,q_3) + F(Q,q_3,q_2,q_1) + 12f(-Q,q_1,q_2,q_3)\big]\delta_{Q,q_1+q_2+q_3}, \tag{35}$$

where we introduced the dimensionless function

$$F(q_1,q_2,q_3,q_4) = \frac{v^2(q_1-q_2)^2}{\varepsilon_{q_1-q_2}} \times\left[\frac{f_-(q_4,q_3,-q_3-q_4)f_-(q_1-q_2,q_2,-q_1)}{\varepsilon_{q_1}-\varepsilon_{q_2}-\varepsilon_{q_1-q_2}} - \frac{f_-(q_1,-q_1+q_2,-q_2)f_+(-q_3-q_4,q_3,q_4)}{\varepsilon_{q_3}+\varepsilon_{q_4}+\varepsilon_{q_1-q_2}}\right]. \tag{36}$$

Here ε_q and the functions $f_\pm$ are defined by Eqs. (27) and (29), respectively.

The scattering matrix element (35) has some important general properties. Since $F(q_1,q_2,q_3,q_4) = F(q_1,q_2,q_4,q_3)$, the matrix element (35) is symmetric with respect to the exchanges of the momenta of excitations in the final state. This is a manifestation of the fact that Bogoliubov quasiparticles obey bosonic statistics. More importantly, one can show that at

$$A = B = 0, \quad \chi = 1 \tag{37}$$

the result (35) vanishes, provided that q_1, q_2, q_3, and Q satisfy conservation laws of momentum and energy. This is because under the conditions (37) our theory describes the weakly interacting Lieb-Liniger model. The latter is integrable, and its quasiparticles do not decay. We now simplify the scattering matrix element (35) in the regimes of small and large momenta.

A. Small momentum region

At small momentum of the initial excitation, $Q \ll q_0$, the other three momenta are also small compared to q_0. In this regime we have been able to simplify the expression (35) considerably, as discussed in Appendix C. The final result takes the form

$$\mathcal{A}_{fi} = \frac{\Lambda}{2Lmn_0}\sqrt{|Qq_1q_2q_3|}\delta_{Q,q_1+q_2+q_3}, \tag{38}$$

where the momentum independent Λ is given by

$$\Lambda = 12\pi^2 B - 6\pi^2 a_1^2 + \frac{24\pi^4 a_1 a_2}{\chi^2} - \frac{A}{\pi}\left(18a_1 + \frac{24\pi^2 a_2}{\chi^2}\right). \quad (39)$$

Using the values of a_1, a_2, and $\beta \equiv B/K^2$ given by Eqs. (21) and (23), in the leading order in small $1-\chi$ we obtain

$$\Lambda = -\frac{3\Omega}{\pi^2}, \quad (40)$$

where we defined

$$\Omega = 4A - \pi^2(1-\chi). \quad (41)$$

We observe again that in the Lieb-Liniger limit (37) the scattering matrix element vanishes.

B. Large momentum region

At large momentum of the initial excitation, $Q \gg q_0$, we have also been able to considerably simplify the matrix element (35). The main steps are described in Appendix C, resulting in

$$\mathcal{A}_{fi} = \frac{2mv^2}{Ln_0}\Xi\delta_{Q,q_1+q_2+q_3}, \quad (42)$$

where

$$\Xi = 12\pi^2 B - \frac{23\pi^2 a_1^2}{8} + \frac{13\pi^4 a_1 a_2}{\chi^2} + \frac{26\pi^6 a_2^2}{\chi^4} - \frac{4\pi^4 a_3}{\chi^2} - \frac{A}{\pi}\left(\frac{21}{2}a_1 + \frac{30\pi^2 a_2}{\chi^2}\right). \quad (43)$$

Substituting the specific values of the parameters of our Hamiltonian from Eqs. (21) and (23), in the leading order in small $1-\chi$ we find

$$\Xi = -\frac{9\Omega}{4\pi^2}. \quad (44)$$

As expected, in the Lieb-Liniger case (37) the scattering matrix element $\mathcal{A}_{fi} = 0$.

IV. DECAY RATE

Let us now evaluate the rate of decay of a quasiparticle of momentum $Q > 0$ at zero temperature. The dominant decay process is illustrated in Fig. 2. The corresponding rate of decay is given by the Fermi golden rule expression

$$\frac{1}{\tau} = \frac{2\pi}{\hbar}\sum_{q_1,q_2,q_3}{}' |\mathcal{A}_{fi}|^2 \delta(\varepsilon_Q - \varepsilon_{q_1} - \varepsilon_{q_2} - \varepsilon_{q_3}). \quad (45)$$

The matrix element $\mathcal{A}_{fi}$ describing the decay of the initial quasiparticle excitation of momentum Q into three quasiparticles with momenta q_1, q_2, and q_3 is given by Eq. (35). The prime symbol in Eq. (45) denotes the summation over distinct final states.

The conservation laws of energy and momentum

$$Q = q_1 + q_2 + q_3, \quad (46)$$

$$\varepsilon_Q = \varepsilon_{q_1} + \varepsilon_{q_2} + \varepsilon_{q_3}, \quad (47)$$

ensure that out of three new quasiparticles two propagate in the same direction as the initial quasiparticle, $q_1, q_2 > 0$, while the third one is counterpropagating, $q_3 < 0$, see Fig. 2. Conditions (46) and (47) enable us to express the momentum of the counterpropagating quasiparticle as a function of Q and one of the two remaining momenta, for example, q_1. We denote it as $q_3 \equiv q_3(Q,q_1)$. With the help of the two conservation laws we now easily perform two summations in Eq. (45), yielding

$$\frac{1}{\tau} = \frac{L^2}{4\pi\hbar^3}\int_0^Q dq_1 \frac{|\mathcal{A}(Q,q_1,Q-q_1-q_3,q_3)|^2}{\left|\varepsilon'_{Q-q_1-q_3} - \varepsilon'_{q_3}\right|}, \quad (48)$$

where

$$\varepsilon'_q = v\,\mathrm{sgn}(q)\frac{1+4\chi^2\frac{q^2}{q_0^2}}{\sqrt{1+2\chi^2\frac{q^2}{q_0^2}}}. \quad (49)$$

In the following we use Eq. (48) to evaluate the quasiparticle decay rate as a function of Q.

A. Regime of low momenta

Let us first consider the case of low momentum of the initial excitation, $Q \ll q_0$, where we recall the definition (17). In this regime the excitation spectrum is almost linear and thus the denominator in Eq. (48) simplifies into $2v$. Using the conservation laws (46) and (47) we find the leading order result for the momentum of the counterpropagating excitation

$$q_3 = -\frac{3Qq_1}{2q_0^2}(Q - q_1). \quad (50)$$

Substituting it in Eq. (48) with the matrix element given by (38), after integration we obtain

$$\frac{1}{\tau} = \frac{9\sqrt{2}}{5\pi}\frac{\Omega^2}{K^4}\frac{T_d}{\hbar}\left(\frac{Q}{q_0}\right)^7. \quad (51)$$

Here we introduced the quantum degeneracy temperature $T_d = \hbar^2 n_0^2/m$.

In the limit of contact interaction, $\chi = 1$, and the decay rate (51) becomes

$$\frac{1}{\tau} = \frac{144\sqrt{2}}{5\pi}\alpha^2\frac{T_d}{\hbar}\left(\frac{Q}{q_0}\right)^7. \quad (52)$$

This result was found earlier in Ref. [17] using a phenomenological approach, in which the phonon is treated as a mobile impurity. Here we rederived that result fully microscopically and generalized it to the case of noncontact interaction.

B. Regime of high momenta

Now we consider the case of large momentum of the initial excitation, $Q \gg q_0$. The conservation laws (46) and (47) can be easily solved when all quasiparticles are in the quadratic

part of the spectrum. One finds

$$q_2 = \frac{1}{2}\Big[Q - q_1 + \sqrt{(Q-q_1)(Q+3q_1)}\Big], \tag{53}$$

$$q_3 = \frac{1}{2}\Big[Q - q_1 - \sqrt{(Q-q_1)(Q+3q_1)}\Big]. \tag{54}$$

The latter expressions enable us to simplify the denominator in Eq. (48), which becomes $2\sqrt{2}v\sqrt{(Q-q_1)(Q+3q_1)}/q_0$. Here we take into account the leading order result in small $1-\chi$. Using the matrix element (42) and the expression

$$\int_0^Q \frac{dq_1}{\sqrt{(Q-q_1)(Q+3q_1)}} = \frac{2\sqrt{3}\pi}{9}, \tag{55}$$

we obtain the decay rate of quasiparticles of large momenta:

$$\frac{1}{\tau} = \frac{9\sqrt{3}}{8}\frac{\Omega^2}{K^4}\frac{T_d}{\hbar}. \tag{56}$$

We note that the expression (48) contains regions of integration where the momentum q_1 is either close to zero or Q. In these regions two quasiparticles of the final state are in the linear part of the spectrum, where the approximations (53) and (54) fail. We checked that the contributions arising from these boundary regions give only a subleading correction to the decay rate (56).

In the limit of contact interaction $\chi = 1$. Equation (56) then reduces to

$$\frac{1}{\tau} = 18\sqrt{3}\alpha^2\frac{T_d}{\hbar}. \tag{57}$$

This result was obtained earlier in Ref. [11] using a different approach.

C. The crossover regime

In the regime of intermediate momenta, $Q \sim q_0$, complete analytical evaluation of the decay rate (48) is a challenging problem. However, we are able to express it in the form

$$\frac{1}{\tau} = \frac{\Omega^2}{K^4}\frac{T_d}{\hbar}\mathcal{F}\left(\frac{Q}{q_0}\right), \tag{58}$$

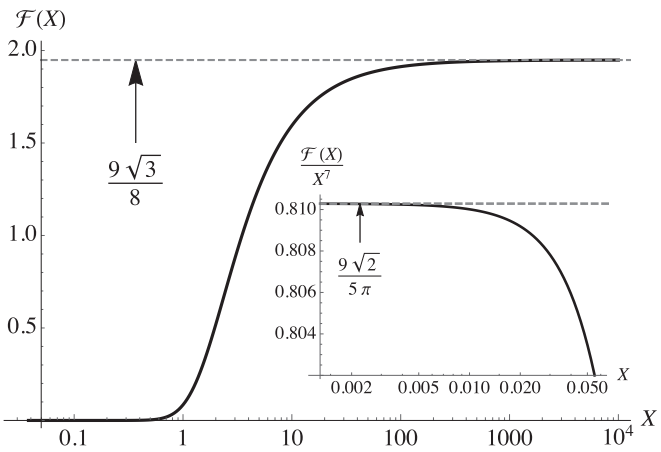

FIG. 3. Plot of the function $\mathcal{F}(X)$ given by Eqs. (E16)–(E19) that enters the relaxation rate (58). The inset shows the limiting behavior of $\mathcal{F}(X)$ at $X \to 0$.

where Ω is given by Eq. (41). The analytical form for the function $\mathcal{F}$ is given by Eqs. (E16)–(E19) of Appendix E. It has the asymptotic behavior

$$\mathcal{F}(X) = \begin{cases} \frac{9\sqrt{2}}{5\pi}X^7, & X \ll 1, \\ \frac{9\sqrt{3}}{8}, & X \gg 1. \end{cases} \tag{59}$$

The latter result is in agreement with already calculated decay rates in the limiting cases of low [Eq. (51)] and high [Eq. (56)] momenta. In Fig. 3 we plot the function $\mathcal{F}$.

V. DECAY OF FERMIONIC EXCITATIONS AT LOW ENERGIES

Description of elementary excitations of weakly interacting Bose gas in terms of phonons with Bogoliubov dispersion (27) is applicable only at sufficiently high momenta. Indeed, the correction to the linear spectrum $\varepsilon_q = v|q|$ in Eq. (27) is due to the term proportional $(\nabla^2\varphi)^2$ in the Hamiltonian (15). At $q \to 0$ the relative significance of a perturbation in the Hamiltonian is determined by its scaling dimension, which for the operator $(\nabla^2\varphi)^2$ is four. On the other hand, perturbations $\nabla\varphi(\nabla\theta)^2$ and $(\nabla\varphi)^3$ of lower scaling dimension three are also present in the Hamiltonian, see Eq. (20). At the lowest energies, the latter perturbations control the physics of the elementary excitations and their spectrum [25]. Specifically, the excitations at $q \to 0$ are fermions with spectrum

$$\varepsilon_q = v|q| + \frac{q^2}{2m^*} + \frac{1}{6}\lambda^*|q|^3 + \dots. \tag{60}$$

Most importantly, unlike the Bogoliubov dispersion (27), the leading correction is quadratic, with finite effective mass m^*.

To determine m^* and λ^* it is sufficient to consider the low-momentum part of the hydrodynamic Hamiltonian, accounting for the right-moving excitations only. This is accomplished by substituting

$$\varphi = \frac{\sqrt{K}}{2}(\phi^L + \phi^R), \tag{61}$$

$$\theta = \frac{1}{2\sqrt{K}}(\phi^L - \phi^R) \tag{62}$$

into Eqs. (15), (20), and (22) and limiting oneself to terms containing only the right-moving field ϕ^R. The leading operator of this form

$$\tilde{H}_{LL} = \frac{\hbar v}{4\pi}\int dx(\nabla\phi^R)^2 \tag{63}$$

is simply the right-moving part of the Luttinger liquid Hamiltonian (11). It has scaling dimension two and is responsible for the linear part of the excitation spectrum in Eq. (60). The terms of scaling dimensions three and four can be combined into

$$H_{\mathrm{KdV}} = \frac{\hbar^2}{12\pi m^*}\int dx[(\nabla\phi^R)^3 + a^*(\nabla^2\phi^R)^2], \tag{64}$$

where

$$\frac{1}{m^*} = \frac{1}{m}\frac{3}{4\sqrt{K}}\left(1 - \frac{2}{\pi^2}A\right), \tag{65}$$

$$a^* = \frac{\hbar\chi^2\sqrt{K}}{2mv}\left(1 - \frac{2}{\pi^2}A\right)^{-1}. \tag{66}$$

The Hamiltonian (64) describes one of the possible realizations of the quantum KdV problem [31,32]. The spectrum of elementary excitations in this model has been recently studied in detail in Ref. [33]. At $q \to 0$ it has Taylor expansion (60) with $\lambda^* = \chi^2/4m^2v$. The crossover from fermionic excitations to phonons with Bogoliubov dispersion occurs at momentum scale $q^* \sim \hbar/a^* \sim q_0/\sqrt{K} \ll q_0$.

At $Q \ll q^*$ type I and type II excitations (see Fig. 1) correspond to fermionic quasiparticles and quasiholes, respectively. In the absence of integrability, quasiparticles can decay at zero temperature, with the rate that scales as the eighth power of momentum [9,16],

$$\frac{1}{\tau} = \frac{3}{5120\pi^3}\frac{\tilde{\Lambda}^2 Q^8}{\hbar^5 m^* v^2}. \tag{67}$$

A general expression for the coefficient $\tilde{\Lambda}$ in terms of the parameters v, m^* and λ^* was obtained in Ref. [16]. At weak interactions, $K \gg 1$, the expression for $\tilde{\Lambda}$ simplifies significantly,

$$\tilde{\Lambda} = -\frac{2\pi}{3m^*}\frac{\partial}{\partial n_0}(a^*\sqrt{K}). \tag{68}$$

This result was recently obtained for a one-dimensional Wigner crystal [34], whose low-energy excitations are also described by the Hamiltonian in the form of Eqs. (63) and (64).

In the integrable case of the Lieb-Liniger model achieved at $A = 0$ and $\chi = 1$ one easily sees that $a^*\sqrt{K}$ does not depend on particle density n_0, and the decay rate vanishes. Taking into consideration the integrability breaking perturbations described by parameters A and $1 - \chi$ that both scale linearly with n_0 [see Eqs. (19) and (14)], we obtain

$$\tilde{\Lambda} = -\frac{2\hbar^2\Omega}{3m^*m^2v^2}. \tag{69}$$

Substituting this expression into Eq. (67) we find the decay rate of the fermionic quasiparticle in the form

$$\frac{1}{\tau} = \frac{9}{20\pi}\frac{\Omega^2}{K^{7/2}}\frac{T_d}{\hbar}\left(\frac{Q}{q_0}\right)^8. \tag{70}$$

Reassuringly, at the crossover between Bogoliubov phonons and fermions, i.e., at $Q \sim q_0/\sqrt{K}$, both the expressions (51) and (70) predict a very small rate $\tau^{-1} \sim \Omega^2(T_d/\hbar)K^{-15/2}$.

VI. DISCUSSION

In this paper we studied the decay of type I excitations in a one-dimensional system of weakly interacting bosons at zero temperature. The approach we used was based on the hydrodynamic description of the system, which limits the momenta of the bosons to $Q \ll \hbar n_0$. Two additional momentum scales play important roles in this system. First, the momentum $q_0 = \sqrt{8}mv \sim \hbar n_0/K$ determines the crossover between the linear and quadratic dependences of the excitation energy (27) on momentum. Second, at the momentum scale $q^* \sim q_0/\sqrt{K}$ the nature of type I excitations changes from fermionic quasiparticles at $Q \ll q^*$ to phonons at $Q \gg q^*$. We note that at weak interactions the Luttinger liquid parameter $K \gg 1$, thus $q^* \ll q_0 \ll \hbar n_0$.

Our main result (58) applies in the region $q^* \ll Q \ll \hbar n_0$ and accurately describes the crossover region $Q \sim q_0$. In addition, we obtained the decay rate of the fermionic quasiparticles at $Q \ll q^*$. Although we are not able to describe the crossover at $Q \sim q^*$, our results (51) and (70) for $Q \gg q^*$ and $Q \ll q^*$, respectively, give the decay rate of the same order of magnitude when extrapolated to $Q \sim q^*$. This strongly indicates that no additional crossover regions remain unexplored.

It is instructive to compare our result (58) to those in the earlier work on weakly interacting bosons. In the case of contact two-body repulsion the system is described by the Lieb-Liniger model, in which case the integrability prevents decay of excitations. A small perturbation commonly added to the system to break integrability is the three-body interaction given by the second term in Eq. (6). In this case the regimes $Q \gg q_0$ and $q^* \ll Q \ll q_0$ were studied in Refs. [11] and [17], respectively. Our main result (58) recovers the corresponding expressions (57) and (52) for the decay rate and accurately describes the crossover between them.

An alternative way to break the integrability of the Lieb-Liniger model is by considering two-body interaction of small but finite range. Our theory incorporates this perturbation on equal footing with the three-body interactions. The relative significance of the two perturbations depends on the specific model of interacting bosons. In the case of atoms confined to one dimension by a trap, we expect the three-body interaction to dominate [35]. On the other hand, noncontact interactions in a purely one-dimensional model should generate the three-body interactions in the effective low-energy theory, in which case both perturbations may be of the same order of magnitude.

To illustrate this point, we have considered the hyperbolic Calogero-Sutherland model in the regime of weak short-range interaction. It is defined by the two-body interaction of the form [36]

$$g(x) = \frac{\hbar^2}{m}\frac{\lambda(\lambda-1)\kappa^2}{\sinh^2(\kappa x)}. \tag{71}$$

In the limit when $\kappa \to +\infty$ and $\lambda \to +0$, such that $c = 2\kappa\lambda$ is kept fixed, the scattering matrix of the potential (71) coincides with that of the potential $g(x) = (\hbar^2 c/m)\delta(x)$ [36]. Therefore, in this limit the model (71) is equivalent to the Lieb-Liniger model. We then obtained the excitation spectrum of the model (71) at large but finite κ, see Appendix D. Using the latter, we have found the values of parameters α and χ that quantify the two integrability-breaking perturbations:

$$\alpha = -\frac{\pi^2 c^2}{24\kappa^2}, \quad \chi = 1 + \frac{\pi^2 c n_0}{6\kappa^2}. \tag{72}$$

We observe that for the integrable model (71), the combination (41) becomes

$$\Omega = 4K^2\alpha - \pi^2(1-\chi) = 0. \tag{73}$$

We therefore conclude that the two perturbations give comparable contributions to the scattering amplitude corresponding to the decay process, which for the model (71)

cancel each other. This cancellation was, of course, expected, as the hyperbolic Calogero-Sutherland model (71) is integrable for any κ and λ [36].

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APPENDIX A: SCALING ANALYSIS OF THE HYDRODYNAMIC HAMILTONIAN

Our main goal is to study the decay rate of Bogoliubov quasiparticles at momenta of the order of the crossover value $q_0 \sim mv$, assuming that the interactions are weak. The latter condition can be expressed as $q_0 \ll \hbar n_0$ or $K \gg 1$, cf. Eq. (18). To this end we apply the following procedure to the hydrodynamic Hamiltonian given by Eqs. (15), (20), and (22).

We rescale the lengths by the scale determined by q_0, i.e., introduce

$$\widetilde{x} = x q_0/\hbar. \tag{A1}$$

Correspondingly, the derivative transforms as

$$\nabla = \frac{q_0}{\hbar}\widetilde{\nabla}. \tag{A2}$$

At the same time, we rescale the bosonic fields according to

$$\varphi = \sqrt{K}\,\widetilde{\varphi}, \quad \theta = \frac{\widetilde{\theta}}{\sqrt{K}}. \tag{A3}$$

Note that the above procedure preserves the commutation relations of the bosonic fields, $[\widetilde{\nabla}\widetilde{\varphi}(\widetilde{x}),\widetilde{\theta}(\widetilde{x}')] = -i\pi\delta(\widetilde{x}-\widetilde{x}')$. In rescaled variables the contributions (15), (20), and (22) to the Hamiltonian become

$$H_0 = \frac{vq_0}{2\pi}\int d\widetilde{x}[(\widetilde{\nabla}\widetilde{\theta})^2 + (\widetilde{\nabla}\widetilde{\varphi})^2 + 2(\widetilde{\nabla}^2\widetilde{\varphi})^2], \tag{A4a}$$

$$V_3 = \frac{\sqrt{2}vq_0}{\pi}\frac{1}{\sqrt{K}}\int d\widetilde{x}\Big[(\widetilde{\nabla}\widetilde{\varphi})(\widetilde{\nabla}\widetilde{\theta})^2 - 2(\widetilde{\nabla}^2\widetilde{\varphi})^2(\widetilde{\nabla}\widetilde{\varphi}) - \frac{2A}{\pi^2}(\widetilde{\nabla}\widetilde{\varphi})^3\Big], \tag{A4b}$$

$$V_4 = \frac{8vq_0}{\pi}\frac{1}{K}\int d\widetilde{x}[(\widetilde{\nabla}^2\widetilde{\varphi})^2(\widetilde{\nabla}\widetilde{\varphi})^2 + \pi^2 B(\widetilde{\nabla}\widetilde{\varphi})^4]. \tag{A4c}$$

Here we introduced

$$B = K^2\beta \tag{A5}$$

and substituted the values (21) and (23) of the constants a_1, a_2, and a_3.

The scaling procedure (A1)–(A3) enables one to estimate the relative significance of the various contributions to the Hamiltonian describing the physics of the system at the momentum scale q_0 and the respective energy scale vq_0. Contributions H_0, V_3, and V_4 represent the first three terms of the expansion of the Hamiltonian in small parameter $1/\sqrt{K}$. The terms of higher orders in the bosonic fields continue this trend. Indeed, all such terms emerge from the expansion of the density n in the denominator of the quantum pressure term in Eq. (8) with the help of Eq. (9). Rewriting the latter expression in rescaled variables, we obtain

$$n = n_0\left(1 + \sqrt{\frac{8}{K}}\,\widetilde{\nabla}\widetilde{\varphi}\right). \tag{A6}$$

Thus each additional order in the bosonic field $\widetilde{\varphi}$ is accompanied by a small coefficient of order $1/\sqrt{K}$.

APPENDIX B: SECOND ORDER PERTURBATION THEORY FOR THE SCATTERING MATRIX ELEMENT

In this appendix we present some details of the evaluation of the matrix element defined by Eq. (33). We first consider the contribution arising from the cubic perturbation V_3 of Eq. (28). Using the identity

$$\frac{1}{E+i\delta} = \frac{1}{i\hbar}\int_0^\infty dt e^{it(E+i\delta)/\hbar}, \quad \delta > 0, \tag{B1}$$

we perform the summation over the intermediate states, reexpressing the scattering matrix element that arises due to V_3 as

$$\sum_m \frac{\langle f|V_3|m\rangle\langle m|V_3|i\rangle}{\varepsilon_Q - E_m + i\delta} = \int_0^\infty dt\frac{e^{-t\delta/\hbar}}{i\hbar}\langle f|V_3(0)V_3(-t)|i\rangle. \tag{B2}$$

Here we use the operators in the Heisenberg representation $V_3(t) = e^{itH_0/\hbar}V_3e^{-itH_0/\hbar}$, where H_0 is the quadratic part of the Hamiltonian, see Eqs. (15) and (26). The creation and annihilation operators in this picture are

$$b_q(t) = e^{-it\varepsilon_q/\hbar}b_q, \quad b_q^\dagger(t) = e^{it\varepsilon_q/\hbar}b_q^\dagger. \tag{B3}$$

Direct inspection on Eq. (28) reveals that out of sixteen possible terms, only three of them may give nonzero contribution in $\langle f|V_3(0)V_3(-t)|i\rangle$, because they contain an equal number of creation and annihilation operators. One of them nullifies after performing Wick contractions due to the momentum conservation, while the remaining terms are

$$\langle f|V_3(0)V_3(-t)|i\rangle = \frac{\pi^2v^4}{8Lmn_0}\sum_{\substack{p_1,p_2,p_3\\ p_1',p_2',p_3'}}\delta_{p_1+p_2+p_3,0}\delta_{p_1'+p_2'+p_3',0}\frac{|p_1p_2p_3p_1'p_2'p_3'|}{\sqrt{\varepsilon_{p_1}\varepsilon_{p_2}\varepsilon_{p_3}\varepsilon_{p_1'}\varepsilon_{p_2'}\varepsilon_{p_3'}}}$$
$$\times\Big[\frac{1}{3}f_-(p_1,p_2,p_3)f_+(p_1',p_2',p_3')\big\langle b_{q_3}b_{q_2}b_{q_1}b_{-p_3}^\dagger b_{p_2}b_{p_1}b_{p_1'}^\dagger b_{p_2'}^\dagger b_{p_3'}^\dagger b_Q^\dagger\big\rangle e^{-it(\varepsilon_{p_1'}+\varepsilon_{p_2'}+\varepsilon_{p_3'})/\hbar}$$
$$+ f_-(p_1,p_2,p_3)f_-(p_1',p_2',p_3')\big\langle b_{q_3}b_{q_2}b_{q_1}b_{p_1}^\dagger b_{p_2}^\dagger b_{-p_3}b_{p_1'}^\dagger b_{p_2'}^\dagger b_{-p_3'}^\dagger b_Q^\dagger\big\rangle e^{-it(\varepsilon_{p_1'}+\varepsilon_{p_2'}-\varepsilon_{p_3'})/\hbar}\Big]. \tag{B4}$$

We now use Wick theorem [37,38] to evaluate the expression (B4). Denoting

$$C_1 = \langle b_{q_3} b_{q_2} b_{q_1} b^\dagger_{-p_3} b_{p_2} b_{p_1} b^\dagger_{p_1'} b^\dagger_{p_2'} b^\dagger_{p_3'} b^\dagger_Q \rangle, \tag{B5}$$

we note that $b^\dagger_Q$ must not be contracted with any of b_{q_1}, b_{q_2}, or b_{q_3} because in this case the energy and momentum conservation would imply zero value for the remaining two momenta in the final state $|f\rangle$, which is not the scattering process we consider. Thus, we contract $b^\dagger_Q$ with, for example, b_{p_1} and account for a factor of 2 because $f_-(p_1,p_2,p_3) = f_-(p_2,p_1,p_3)$. Then the other operator b_{p_2} must be contracted with either $b^\dagger_{p_1'}$, $b^\dagger_{p_2'}$, or $b^\dagger_{p_3'}$. Because $f_+(p_1',p_2',p_3')$ is fully symmetric with respect to the permutations of its arguments, we arbitrary select, e.g., $b^\dagger_{p_1'}$ and account for a factor of 3 in this choice. We then obtain $C_1 = 6\delta_{Q,p_1}\delta_{p_2,p_1'}\langle b_{q_3} b_{q_2} b_{q_1} b^\dagger_{-p_3} b^\dagger_{p_2'} b^\dagger_{p_3'}\rangle$. We note that the last expression is symmetric with respect to the permutations of q_1, q_2, and q_3. Therefore, the six remaining contractions we write in a compact notation introducing the symmetrization operator $\hat{\mathcal{S}}$ that denotes the summation over all permutations of q_1, q_2, and q_3:

$$C_1 = 6\delta_{Q,p_1}\delta_{p_2,p_1'}\hat{\mathcal{S}}\big(\delta_{-p_3,q_1}\delta_{p_2',q_2}\delta_{p_3',q_3}\big). \tag{B6}$$

We note that there will actually be only three different terms in the final result for the matrix element, since the momenta p_2' and p_3' under the symmetrization operator in the last expression enter symmetrically because $f_+(p_1',p_2',p_3')$ is already symmetric, see Eq. (B4).

The other combination of the operators in Eq. (B4) we denote by

$$C_2 = \langle b_{q_3} b_{q_2} b_{q_1} b^\dagger_{p_1} b^\dagger_{p_2} b_{-p_3} b^\dagger_{p_1'} b^\dagger_{p_2'} b_{-p_3'} b^\dagger_Q \rangle. \tag{B7}$$

Clearly, $b^\dagger_Q$ must be contracted with $b_{-p_3'}$. Next, b_{-p_3} must be contracted with either $b^\dagger_{p_1'}$ or $b^\dagger_{p_2'}$. We select $b^\dagger_{p_1'}$ and account for a factor of 2, which yields $C_2 = 2\delta_{Q,-p_3'}\delta_{-p_3,p_1'}\langle b_{q_3} b_{q_2} b_{q_1} b^\dagger_{p_1} b^\dagger_{p_2} b^\dagger_{p_2'}\rangle$. The last expression we rewrite using the symmetrization operator

$$C_2 = 2\delta_{Q,-p_3'}\delta_{-p_3,p_1'}\hat{\mathcal{S}}\big(\delta_{p_1,q_1}\delta_{p_2,q_2}\delta_{p_2',q_3}\big). \tag{B8}$$

Similarly as in C_1, in C_2 we eventually have only three distinct contributions since $f_-(p_1,p_2,p_3)$ is symmetric with respect to the interchange of p_1 and p_2.

Substituting those results and performing the summation, the matrix element (B4) becomes

$$\langle f|V_3(0)V_3(-t)|i\rangle = \frac{\pi^2 v^4}{4Lmn_0}\frac{|Qq_1q_2q_3q_4|}{\sqrt{\varepsilon_Q\varepsilon_{q_1}\varepsilon_{q_2}\varepsilon_{q_3}}}\hat{\mathcal{S}}\Bigg(\frac{(q_2+q_3)^2}{\varepsilon_{q_2+q_3}} f_-(Q,q_1-Q,-q_1) f_+(-q_2-q_3,q_2,q_3) e^{-it(\varepsilon_{q_2+q_3}+\varepsilon_{q_2}+\varepsilon_{q_3})/\hbar}$$
$$+ \frac{(Q-q_3)^2}{\varepsilon_{Q-q_3}} f_-(q_1,q_2,-q_1-q_2) f_-(Q-q_3,q_3,-Q) e^{-it(\varepsilon_{Q-q_3}+\varepsilon_{q_3}-\varepsilon_Q)/\hbar}\Bigg)\delta_{Q,q_1+q_2+q_3}. \tag{B9}$$

After performing the integration over t, see Eq. (B2), in the limit $\delta \to +0$, we obtain the final result contained in the expression (35).

APPENDIX C: EVALUATION OF THE SCATTERING MATRIX ELEMENT

In this Appendix we provide some details of the evaluation of the scattering matrix element (35) in the regimes of small and large momenta. Due to the conservation laws, we select the momenta to satisfy $Q > q_1,q_2 > 0$ and $q_3 < 0$, see Fig. 2.

The function F that determines the amplitude is defined by Eq. (36). We conveniently split it as $F = F_1 - F_2$, where

$$F_1(q_1,q_2,q_3,q_4) = \frac{v^2(q_1-q_2)^2}{\varepsilon_{q_1-q_2}} \times \frac{f_-(q_4,q_3,-q_3-q_4) f_-(q_1-q_2,q_2,-q_1)}{\varepsilon_{q_1}-\varepsilon_{q_2}-\varepsilon_{q_1-q_2}}, \tag{C1}$$

$$F_2(q_1,q_2,q_3,q_4) = \frac{v^2(q_1-q_2)^2}{\varepsilon_{q_1-q_2}} \times \frac{f_-(q_1,-q_1+q_2,-q_2) f_+(-q_3-q_4,q_3,q_4)}{\varepsilon_{q_3}+\varepsilon_{q_4}+\varepsilon_{q_1-q_2}}. \tag{C2}$$

1. Small momentum region

At small momenta, $Q,q_1,q_2,|q_3| \ll q_0$, in F_2 terms we can safely linearize the spectrum at low momenta, i.e., we can use $\varepsilon_q = v|q|$. It yields

$$F_2(Q,q_1,q_2,q_3) = F_2(Q,q_2,q_1,q_3) = F_2(Q,q_3,q_2,q_1) = \frac{1}{2}\left(a_1 + \frac{3A}{\pi^3}\right)^2 + \ldots, \tag{C3}$$

where the ellipsis denotes the subleading terms that tend to zero at small momenta.

In the F_1 part we have to keep the spectrum nonlinearity because of the energy denominator. We use the momentum conservation to replace Q by $q_1+q_2+q_3$, and then we use the expression for the smallest momentum given by

$$q_3 = -\frac{3q_1q_2}{2q_0^2}(q_1+q_2). \tag{C4}$$

Finally, we expand the obtained expression at small $q_1,q_2 \ll q_0$, keeping the ratio q_1/q_2 fixed.

We obtain

$$F_1(Q,q_1,q_2,q_3) = -a_1^2\frac{6\hat{q}_0^2+13q_1^2+22q_1q_2+7q_2^2}{6q_1(q_1+q_2)} + \frac{8\pi^2a_1a_2}{\chi^2}\frac{q_1^2+q_1q_2-2q_2^2}{3q_1(q_1+q_2)} - \frac{a_1A}{\pi^3}\frac{6\hat{q}_0^2+17q_1^2+26q_1q_2+11q_2^2}{3q_1(q_1+q_2)} + \frac{A^2}{\pi^6}\frac{6\hat{q}_0^2+5q_1^2+14q_1q_2-q_2^2}{2q_1(q_1+q_2)} + \frac{8a_2A}{\pi\chi^2}\frac{q_1^2+q_1q_2+2q_2^2}{q_1(q_1+q_2)}+\ldots, \quad \text{(C5)}$$

where for notational convenience we introduced $\hat{q}_0 = q_0/\chi$. The terms in the ellipsis contain the subleading terms.

The expression for $F_1(Q,q_2,q_1,q_3)$ is trivially obtained from Eq. (C5) by exchanging momenta q_1 and q_2, both of which assumed to be positive. The remaining term $F_1(Q,q_3,q_2,q_1)$ cannot be directly inferred from Eq. (C5) because q_3 enters the expression in a special way, and is negative by the initial assumption. By repeating the above procedure and using the conservation laws to rewrite $1/(\varepsilon_Q - \varepsilon_{q_3} - \varepsilon_{Q-q_3})$ as $1/(\varepsilon_{q_1}+\varepsilon_{q_2}-\varepsilon_{q_1+q_2})$ and again using Eq. (C4) to remove q_3, we eventually expand at small q_1 and q_2. We obtain

$$F_1(Q,q_3,q_2,q_1) = a_1^2\frac{6\hat{q}_0^2+7q_1^2+q_1q_2+7q_2^2}{6q_1q_2} + \frac{8\pi^2a_1a_2}{\chi^2}\frac{2q_1^2+5q_1q_2+2q_2^2}{3q_1q_2} + \frac{a_1A}{\pi^3}\frac{6\hat{q}_0^2+11q_1^2+5q_1q_2+11q_2^2}{3q_1q_2} + \frac{A^2}{\pi^6}\frac{-6\hat{q}_0^2+q_1^2+7q_1q_2+q_2^2}{2q_1q_2} - \frac{8a_2A}{\pi\chi^2}\frac{2q_1^2+3q_1q_2+2q_2^2}{q_1q_2}+\ldots. \quad \text{(C6)}$$

Alternatively, if we do not use the conservation law to transform the energy denominator, we need to find the subleading correction in the result for q_3 [Eq. (C4)], which is a more involved, but an equivalent way to obtain Eq. (C6). As expected, Eq. (C6) is symmetric with respect to the exchange of momenta q_1 and q_2.

When we sum all the terms, recalling $F = F_1 - F_2$, we obtain the leading contribution at low momenta

$$F(Q,q_1,q_2,q_3)+F(Q,q_2,q_1,q_3)+F(Q,q_3,q_2,q_1) = -6a_1^2+\frac{24\pi^2a_1a_2}{\chi^2}-\frac{A}{\pi^3}\left(18a_1+\frac{24\pi^2a_2}{\chi^2}\right). \quad \text{(C7)}$$

2. Large momentum region

In the regime of large momentum of the initial excitation, $Q \gg q_0$, at least one momentum of quasiparticles in the final state is of the same order as Q. Let us denote it by q_1. At such high momenta, the Bogoliubov dispersion (27) can be approximated as $\varepsilon_q = \chi q^2/2m + mv^2/\chi$. Using the latter expression one can easily solve the conservation laws of the momentum and energy to find q_2 and q_3:

$$q_2 = \frac{1}{2}\left[Q-q_1+\sqrt{(Q-q_1)(Q+3q_1)}\right] - \frac{2m^2v^2}{\chi^2\sqrt{(Q-q_1)(Q+3q_1)}}, \quad \text{(C8)}$$

$$q_3 = \frac{1}{2}\left[Q-q_1-\sqrt{(Q-q_1)(Q+3q_1)}\right] + \frac{2m^2v^2}{\chi^2\sqrt{(Q-q_1)(Q+3q_1)}}. \quad \text{(C9)}$$

Substituting the latter expressions in Eq. (35), after some algebra one obtains the final result given by Eqs. (42) and Eq. (43). Let us comment that in Eq. (43) we neglected small terms that are momentum dependent. The leading one in that expansion is proportional to $[(1-\chi^2)Q/\chi q_0]^2$. Since $1-\chi$ is small, the latter term imposes the condition

$$Q \ll \frac{q_0}{\sqrt{1-\chi}}. \quad \text{(C10)}$$

It does not affect the decay rate in a wide region around q_0, since $1-\chi$ is the small parameter.

APPENDIX D: HYPERBOLIC CALOGERO-SUTHERLAND MODEL

In Sec. III we found the general expression for the scattering matrix element for the decay process of a Bogoliubov quasiparticle shown in Fig. 2. At low momenta, it is given by Eq. (38), while the expression at high momenta is Eq. (42). In this Appendix we take advantage of integrability of the hyperbolic Calogero-Sutherland model to find the parameters α [cf. Eq. (20)] and χ [cf. Eqs. (15) and (27)] that enter the prefactors of the scattering matrix element.

We consider the two-body interaction potential for the hyperbolic Calogero-Sutherland model [36]

$$g(x) = \frac{\hbar^2}{m}\frac{\lambda(\lambda-1)\kappa^2}{\sinh^2(\kappa x)}. \quad \text{(D1)}$$

A many-body problem of bosons interacting with the potential (D1) is exactly solvable by Bethe ansatz. An important quantity for this technique is the two-particle scattering phase shift, which is given by [36]

$$\theta_+(k) = i\log\left(\frac{\Gamma(1+ik/2\kappa)\Gamma(\lambda-ik/2\kappa)}{\Gamma(1-ik/2\kappa)\Gamma(\lambda+ik/2\kappa)}\right). \quad \text{(D2)}$$

This complicated function has an important yet simple limiting case. Namely, at $\kappa\to+\infty$, $\lambda\to+0$, such that $c = 2\kappa\lambda$ is kept fixed, the phase shift (D2) coincides with the one of the Lieb-Liniger model [36]:

$$\theta_+(k) \underset{\substack{\kappa\to+\infty\\ \lambda\to+0}}{\longrightarrow} \theta_{\mathrm{LL}}(k) = -2\arctan\left(\frac{k}{c}\right). \quad \text{(D3)}$$

The phase shift (D3) corresponds to the interaction potential $g(x) = g\delta(x)$, where $c = mg/\hbar^2$.

The relation between the two integrable models enables us to consider corrections to the Lieb-Liniger model caused by finite interaction range without losing the integrability. To this

end, we account for the leading deviation in Eq. (D3) due to large but finite κ. While Eq. (D3) is valid at any c, here we are interested in the limit of weak interaction. Therefore, we expand $\theta_+(k) - \theta_{\rm LL}(k)$ in linear order at small c and obtain

$$\theta_+(k) = \theta_{\rm LL}(k) + \frac{\pi c}{2\kappa}\left[\coth\left(\frac{\pi k}{2\kappa}\right) - \frac{2\kappa}{\pi k}\right]. \tag{D4}$$

The correction terms in this expression account for the deviation of the Calogero-Sutherland model from the Lieb-Liniger model at weak interaction due to large but finite κ. The phase shift (D4) contains necessary information to find the excitation spectrum of the hyperbolic Calogero-Sutherland model in this particular limit of small interaction range.

At small wave vectors, $k \ll \kappa$, Eq. (D4) contains a nontrivial correction to the phase shift of the Lieb-Liniger model. The leading order expression in the large-κ limit is

$$\theta_+(k) = -2\arctan\left(\frac{k}{c}\right) + \frac{\pi^2 ck}{12\kappa^2}. \tag{D5}$$

The knowledge of the phase shift (D5) suffices to find the excitation spectrum at not too high momenta. It is parametrically given by

$$p(k) = 2\pi\hbar\int_{k_0}^{k} dq\rho(q), \quad \varepsilon(k) = \int_{k_0}^{k} dq\sigma(q), \tag{D6}$$

where k_0 is the Fermi rapidity. We consider particlelike excitations, so we study the case $k > k_0$. In the previous equation $\rho(k)$ is the density of rapidities. It is determined by the Lieb's integral equation

$$\rho(k) + \frac{1}{2\pi}\int_{-k_0}^{+k_0} dq\theta'_+(k-q)\rho(q) = \frac{1}{2\pi} \tag{D7}$$

and normalized as

$$\int_{-k_0}^{k_0} dq\rho(q) = n_0. \tag{D8}$$

The other density function $\sigma(k)$ in Eq. (D6) satisfies a similar equation

$$\sigma(k) + \frac{1}{2\pi}\int_{-k_0}^{+k_0} dq\theta'_+(k-q)\sigma(q) = \frac{\hbar^2 k}{m}. \tag{D9}$$

The two integral equations can be treated in the limit of large κ by iterations. Their solution at $k > k_0$ can be expressed as

$$\rho(k) = \left(1 - \frac{\pi^2 k_0^2}{48\kappa^2}\right)\frac{1}{2\pi}\frac{d}{dk}\sqrt{k^2 - k_0^2}, \tag{D10}$$

$$\sigma(k) = \frac{\hbar^2}{6m}\frac{d^2}{dk^2}(k^2 - k_0^2)^{3/2}. \tag{D11}$$

Substituting them in Eq. (D6) we find the spectrum

$$\varepsilon_p = \sqrt{\frac{\hbar^2 k_0^2 p^2}{4m^2\mathcal{N}^2} + \frac{p^4}{4m^2\mathcal{N}^4}}, \quad \mathcal{N} = 1 - \frac{\pi^2 k_0^2}{48\kappa^2}. \tag{D12}$$

The normalization condition (D8) leads to the Fermi rapidity

$$k_0 = 2\sqrt{cn_0}\left(1 + \frac{\pi^2 cn_0}{24\kappa^2}\right). \tag{D13}$$

This result requires the knowledge of $\rho(k)$ function at momenta below k_0: $\rho(k) = \mathcal{N}\sqrt{k_0^2 - k^2}/2\pi c$. The density functions below and above k_0 are connected by the Lieb's integral equation.

Substituting the value of k_0 in Eq. (D12) we easily obtain linear spectrum at low momenta, $\varepsilon_p = v|p|$, where

$$v = \frac{\hbar}{m}\sqrt{cn_0}\left(1 + \frac{\pi^2 cn_0}{8\kappa^2}\right). \tag{D14}$$

The latter expression is the sound velocity of the hyperbolic Calogero-Sutherland model at large κ. As expected, at $\kappa \to +\infty$ one obtains the familiar expression for the sound velocity of the Lieb-Liniger model [6]. We notice that the sound velocity can be also found from the thermodynamic expression

$$v = \sqrt{\frac{L}{mn_0}\frac{\partial^2 E_0}{\partial L^2}}, \tag{D15}$$

where

$$E_0 = \frac{\hbar^2 L}{2m}\int_{-k_0}^{k_0} dk\, k^2\rho(k) \tag{D16}$$

denotes the ground state energy.

Detailed knowledge of the sound velocity as a function of the density is sufficient to find the parameters α [cf. Eq. (20)] and β [cf. Eq. (22)] in the hydrodynamic Hamiltonian. They can be expressed as [39]

$$\alpha = -\frac{m^2}{6\hbar^2}\frac{d}{dn_0}\left(\frac{v^2}{n_0}\right), \quad \beta = \frac{m^2 n_0}{24\pi^4\hbar^2}\frac{d^2}{dn_0^2}\left(\frac{v^2}{n_0}\right). \tag{D17}$$

Substituting the velocity (D14) in the latter expression enables us to find at $1/\kappa^2$ order

$$\alpha = -\frac{\pi^2 c^2}{24\kappa^2}, \quad \beta = 0. \tag{D18}$$

The expression (D12) for the energy of excitations obtained in leading order in $1/\kappa$ has the form (27). This enables us to obtain the expression for the parameter χ in the hyperbolic Calogero-Sutherland model

$$\chi = 1 + \frac{\pi^2 cn_0}{6\kappa^2}. \tag{D19}$$

Comparing Eqs. (D18) and (D19) we find the following relation between $A = K^2\alpha$ and χ,

$$\frac{A}{1-\chi} = \frac{\pi^2}{4}. \tag{D20}$$

When the latter condition is satisfied and $\beta = 0$, the matrix element (35) nullifies at all momenta. This observation is consistent with the expected absence of decay of excitations in integrable models.

We note that the parameter χ could also be obtained from the definitions (14) and (D1). Formally, Fourier transform of the potential (D1) diverges. However, the second derivative at zero momentum, which enters Eq. (14), is finite:

$$\left.\frac{d^2 g_q}{dq^2}\right|_{q=0} = -\frac{1}{\hbar^2}\int_{-\infty}^{+\infty} dx x^2 g(x) = -\frac{\pi^2\lambda(\lambda-1)}{3m\kappa}. \tag{D21}$$

The limiting procedure $\kappa \to +\infty, \lambda \to +0$ with fixed $c = 2\kappa\lambda$ reproduces Eq. (D19).

APPENDIX E: TRANSFORMATION OF THE MATRIX ELEMENT USING LAGRANGIAN VARIABLES

1. Lagrangian description of interacting one-dimensional bosons

Our derivation of the hydrodynamic Hamiltonian in Sec. II B was based on Eqs. (7) and (9) that replace the bosonic operators of the particles constituting the Bose gas with new bosonic fields $\varphi(x)$ and $\theta(x)$, which describe the state of the fluid at point x. Alternatively, one can consider a uniform reference state of density n_0 and develop hydrodynamics in terms of the fields that are functions of the coordinate y of the fluid element in that state [40]. This approach corresponds to using Lagrangian variables in the classical fluid dynamics, as opposed to the standard Eulerian ones [41]. The physical position x of the fluid element is obtained by adding its displacement u to the position y in the reference state,

$$x = y + u(y). \tag{E1}$$

Using Eq. (E1) one obtains the expression for the particle density

$$n(y) = \frac{n_0}{1 + u'(y)}, \tag{E2}$$

where prime denotes the derivative with respect to y. Comparing this expression with Eq. (9) we obtain the relation

$$\frac{1}{\pi}\nabla\varphi(x) = -\frac{u'(y)}{1 + u'(y)} \tag{E3}$$

between the fields $\varphi(x)$ and $u(y)$.

In addition to the displacement $u(y)$, one can introduce the momentum density $p(y) = mn_0v(y)$ of the fluid. The two fields obey the standard commutation relation

$$[u(y), p(y')] = i\hbar\delta(y - y'). \tag{E4}$$

A relation between $p(y)$ and the Eulerian field $\theta(x)$ is found by comparing the above definition of $p(y)$ with the expression for the velocity of the fluid $v(x) = -(\hbar/m)\nabla\theta(x)$, see Ref. [40]. This yields

$$\nabla\theta(x) = -\frac{1}{\hbar n_0}\, p(y). \tag{E5}$$

One can now substitute Eqs. (E1), (E3), and (E5) into the hydrodynamic Hamiltonian approximated by Eqs. (15), (20), and (22) and obtain an equivalent theory in Lagrangian variables.

For our needs it is helpful to first rescale the new bosonic fields u and p as follows:

$$u(y) = -\frac{1}{\pi n_0}\,\Phi(y), \tag{E6}$$

$$p(y) = -\hbar n_0 \nabla\Theta(y), \tag{E7}$$

where $\nabla = d/dy$. The new fields satisfy the commutation relation

$$[\Phi(y), \nabla\Theta(y')] = i\pi\,\delta(y - y'). \tag{E8}$$

The convenience of the bosonic fields Φ and Θ manifests itself in the equivalence of the commutation relations (10) and (E8) upon replacing

$$x \to y, \quad \varphi \to \Phi, \quad \theta \to \Theta. \tag{E9}$$

Combining the Eqs. (E1), (E3), (E5), (E6), and (E7) we obtain the following relations between the bosonic fields in Eulerian and Lagrangian variables:

$$x = y - \frac{1}{\pi n_0}\Phi(y), \tag{E10a}$$

$$\nabla\varphi(x) = \frac{\nabla\Phi(y)}{1 - (1/\pi n_0)\nabla\Phi(y)}, \tag{E10b}$$

$$\nabla\theta(x) = \nabla\Theta(y). \tag{E10c}$$

One should stress that ∇ here denotes the derivative with respect to the appropriate variable, i.e., $\nabla = d/dx$ and d/dy in the left- and right-hand sides of Eq. (E10), respectively.

Substituting Eq. (E10) into the hydrodynamic Hamiltonian given by Eqs. (15), (20), and (22), we find that, up to contributions of higher than quartic order in the bosonic fields, the Hamiltonian retains its general form, provided that the parameters $\{a_1, a_2, a_3, \alpha, \beta\}$ are replaced with $\{a_1^L, a_2^L, a_3^L, \alpha^L, \beta^L\}$ given by

$$a_1^L = a_1 - \frac{1}{2\pi}, \tag{E11a}$$

$$a_2^L = a_2 - \frac{5\chi^2}{8\pi^3}, \tag{E11b}$$

$$a_3^L = a_3 - \frac{6a_2}{\pi} + \frac{15\chi^2}{8\pi^4}, \tag{E11c}$$

$$\alpha^L = \alpha - \frac{\pi^2}{2K^2}, \tag{E11d}$$

$$\beta^L = \beta - \frac{2\alpha}{\pi^4} + \frac{1}{2\pi^2K^2}. \tag{E11e}$$

Because the physics of the system cannot be sensitive to our choice of Eulerian or Lagrangian variables used to describe it, all the physically observable quantities should be invariant with respect to the change of parameters from $\{a_1, a_2, a_3, \alpha, \beta\}$ to $\{a_1^L, a_2^L, a_3^L, \alpha^L, \beta^L\}$. In particular, it is easy to check that the expressions for Λ and Ξ given, respectively, by Eqs. (39) and (43) are invariant with respect to this transformation.

Using the values of the constants given by Eqs. (21) and (23), we eventually obtain

$$a_1^L = 0, \tag{E12a}$$

$$a_2^L = \frac{1 - 5\chi^2}{8\pi^3}, \tag{E12b}$$

$$a_3^L = -\frac{5}{8\pi^4}(1 - 3\chi^2), \tag{E12c}$$

$$A^L = A - \frac{\pi^2}{2}, \tag{E12d}$$

$$B^L = -\frac{2A}{\pi^4} + \frac{1}{2\pi^2}. \tag{E12e}$$

Use of Lagrangian variables greatly simplifies the following calculations.

2. Evaluation of the scattering matrix element (35)

We now apply the hydrodynamic theory in terms of Lagrangian variables to the evaluation of the matrix element (35) responsible for the decay of Bogoliubov excitations. The main advantage of using the Lagrangian variables is that the coefficient a_1 entering the definitions (29) of functions $f_\pm$ now vanishes, see Eq. (E12a). One should keep in mind that in this case the constant B takes a nonvanishing value (E12e), which somewhat complicates the expression (31) for the function f.

We have been able to show that to first order in A and $1-\chi$ the matrix element (35) nullifies for any set of the four momenta that satisfy the conservation laws (46) and (47), provided that

$$\Omega \equiv 4A - \pi^2(1-\chi) = 0. \tag{E13}$$

This observation enables us to simplify Eq. (35) using a fixed value $\chi = 1$ by collecting terms linear in A. This calculation benefits greatly from using Lagrangian variables. Upon restoring nonzero $1-\chi$ in the final expression, we obtain

$$\mathcal{A}_{fi} = -\frac{3\Omega v^2}{8\pi^2 Lmn_0}\frac{|Qq_1q_2q_3|}{\sqrt{\varepsilon_Q\varepsilon_{q_1}\varepsilon_{q_2}\varepsilon_{q_3}}}\big[8 + f_L(Q,q_1,q_2,q_3) + f_L(Q,q_2,q_1,q_3) + f_L(Q,q_3,q_1,q_2)\big]\delta_{Q,q_1+q_2+q_3}, \tag{E14}$$

where

$$f_L(Q,q_1,q_2,q_3) = m^2\left[\frac{(\varepsilon_Q+\varepsilon_{q_1})^2-\varepsilon_{Q-q_1}^2}{Q^2q_1^2} + \frac{(\varepsilon_{q_2}-\varepsilon_{q_3})^2-\varepsilon_{q_2+q_3}^2}{q_2^2q_3^2}\right]. \tag{E15}$$

The latter expression assumes the dispersion $\varepsilon_q = \sqrt{v^2q^2+q^4/4m^2}$, i.e., one should replace $\chi = 1$ in Eq. (27). The expression in brackets in Eq. (E14) interpolates between 4 at $Q \ll mv$ and 3 at $Q \gg mv$. We thus recover the results (38) and (42) for $B = 0$.

The decay rate (48) can be conveniently evaluated using Eq. (E14). After introducing the dimensionless momenta $X = Q/q_0$, $x = q_1/q_0$, $y = q_2/q_0$, and $z = q_3/q_0$, the decay rate assumes the form (58), with the crossover function given by

$$\mathcal{F}(X) = \frac{9\sqrt{2}X^2}{128\pi\epsilon_X}\int_0^X dx\frac{x^2y^2z^2}{\epsilon_x\epsilon_y\epsilon_z}\left[64 + \frac{(\epsilon_X+\epsilon_x)^2-\epsilon_{X-x}^2}{X^2x^2} + \frac{(\epsilon_X+\epsilon_y)^2-\epsilon_{X-y}^2}{X^2y^2} + \frac{(\epsilon_X+\epsilon_z)^2-\epsilon_{X-z}^2}{X^2z^2} + \frac{(\epsilon_y-\epsilon_z)^2-\epsilon_{y+z}^2}{y^2z^2} + \frac{(\epsilon_x-\epsilon_z)^2-\epsilon_{x+z}^2}{x^2z^2} + \frac{(\epsilon_x-\epsilon_y)^2-\epsilon_{x+y}^2}{x^2y^2}\right]^2\left(\frac{1+4y^2}{\sqrt{1+2y^2}} + \frac{1+4z^2}{\sqrt{1+2z^2}}\right)^{-1}. \tag{E16}$$

In this formula we introduced the dimensionless energy as $\epsilon_x = \sqrt{x^2+2x^4}$, while the dimensionless momentum $y = X - x - z$ is fixed by the momentum conservation. Finally, z is obtained as a negative solution of the energy conservation equation that takes the form $\epsilon_X = \epsilon_x + \epsilon_{X-x-z} + \epsilon_z$. It can be expressed as

$$z = \frac{X-x}{2} - \frac{1}{4}\sqrt{R_1+\sqrt{R_2}}, \tag{E17}$$

where

$$R_1 = \left(\frac{\epsilon_X-\epsilon_x}{X-x}\right)^2 - 2(2+3X^2+3x^2-2Xx), \tag{E18}$$

$$R_2 = \left(\frac{\epsilon_X-\epsilon_x}{X-x}\right)^4 + 4\left(\frac{\epsilon_X-\epsilon_x}{X-x}\right)^2(2+11X^2+11x^2-18Xx) - 4(X+x)^2(4+7X^2+7x^2-2Xx). \tag{E19}$$

At small momenta, $X,x \ll 1$, from Eq. (E17) we recover the expression (50). Similarly, at $X,x \gg 1$, Eq. (E17) leads to the result (C9).

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