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We propose a systematic magnetic-flux-free approach to detect, manipulate, and braid Majorana fermions in a semiconductor-nanowire-based topological Josephson junction by utilizing the Majorana spin degree of freedom. We find an intrinsic π -phase difference between spin-triplet pairings enforced by the Majorana zero modes (MZMs) at the two ends of a one-dimensional spinful topological superconductor. This π phase is identified to be a spin-dependent superconducting phase, referred to as the *spin phase*, which we show to be tunable by controlling spin-orbit coupling strength via electric gates. This electric controllable spin phase not only affects the coupling energy between MZMs but also leads to a fractional Josephson effect in the absence of any applied magnetic flux, which enables the efficient topological qubit readout. We thus propose an all-electrically controlled superconductor-semiconductor hybrid circuit to manipulate MZMs and to detect their non-Abelian braiding statistics properties. Our work on spin properties of topological Josephson effects potentially opens up a new thrust for spintronic applications with Majorana-based semiconductor quantum circuits.

DOI: [10.1103/PhysRevB.94.014511](https://doi.org/10.1103/PhysRevB.94.014511)**I. INTRODUCTION**

Spin is the fundamental electronic quantum degree of freedom in solid state materials. In superconductors, Cooper pairs, as composed of two spin-1/2 particles, can have spin-1 angular momentum, leading to spin-triplet pairings, in contrast to the usual spin-singlet pairing of opposite spins in the simplest *s*-wave superconductors. Recently, it has been shown that Majorana zero modes (MZMs), which may exist as stable localized zero-energy mid-gap excitations in topological-superconductor interfaces, can only have *s*-wave odd-frequency [1] spin-triplet correlations [2,3] at the boundary of topological superconductors (TSCs) [4–21]. Since the *s*-wave spin-triplet pairing is insensitive to nonmagnetic impurity scattering, spin-triplet pairs can be stabilized at the interface of a topological-superconductor/normal-metal (TSC/NM) hybrid system. Consequently, MZMs can assist the injection of pure stable spin-triplet Cooper pairs into the normal-metal region of a TSC/NM/TSC junction. Thus, a topological Josephson junction (JJ) is indeed a spin-triplet JJ which makes it possible to utilize the Majorana spin degree of freedom to detect and manipulate MZMs, and demonstrate their non-Abelian braiding statistics.

In this work, we theoretically study the spin-dependent current-phase relation and MZM coupling energy of topological JJs. We show that the MZM-induced spin-triplet pairing states [3] at the two ends of a realistic one-dimensional (1D) TSC (specifically, the nanowire-superconductor hybrid system of great current interest) have an intrinsic π -phase difference. We demonstrate that this π phase, originating from Majorana-enforced spin-triplet pairing, arises neither from a magnetic-flux-induced phase, referred to as the charge phase since magnetic flux is coupled to electrons' charge degree of freedom, nor from the Cooper pairs' orbital (e.g., *p* or *d* wave) effect, and can produce a fractional Josephson 0 junction

and π junction, which exhibit a Josephson phase of 0 and π in its ground state in the absence of any applied magnetic flux, respectively, in the N-shaped and U-shaped nanowires [Figs. 1(a) and 1(b)]. From this result, we unambiguously establish the presence of a spin-state-dependent phase, referred to as the *spin phase*, in spin-triplet pairings in this system. The spin phase difference across the topological JJ can be continuously tuned by spin-orbit coupling (SOC) in the normal part (black wire) of the JJ through a gate voltage [Fig. 2(a)] so as to turn on and off the MZM coupling energy for both time-reversal-invariant and time-reversal-symmetry-broken topological JJs leading to experimentally testable 4π -periodic Josephson current-phase relations in both charge phase and spin phase. In particular, the SOC-tunable spin phase can drive the time-reversal-symmetry-broken JJ to be a fractional Josephson φ_0 junction [22], which can exhibit a Josephson phase of φ_0 (neither 0 nor π) and a finite Josephson current in its ground state in the absence of any applied magnetic flux. The observation of this SOC-induced Josephson current would serve as a clear signal for topological superconductivity and MZMs. In addition, we show that the direction of the fractional Josephson current induced by the spin phase is locked to the fermion parity of the topological JJ. It is noted that the SOC, driving the nontrivial spin-phase-current relation, is inside the normal nanowire [black wire in Fig. 2(a)], which is not coupled to a superconductor, and thus readily tuned by an applied gate with the well-developed spintronic technique [23–28]. We thus combine Majorana physics and spintronics, and propose an all-electrically controlled superconductor-semiconductor hybrid circuit to manipulate and control MZMs, and detect their non-Abelian braiding statistics [13,15,17,29,30].

Our work also shows the conceptual depth and complexity of semiconductor-nanowire-based TSCs [8–11] being well beyond the scope of the prototypical spinless *p*-wave TSC

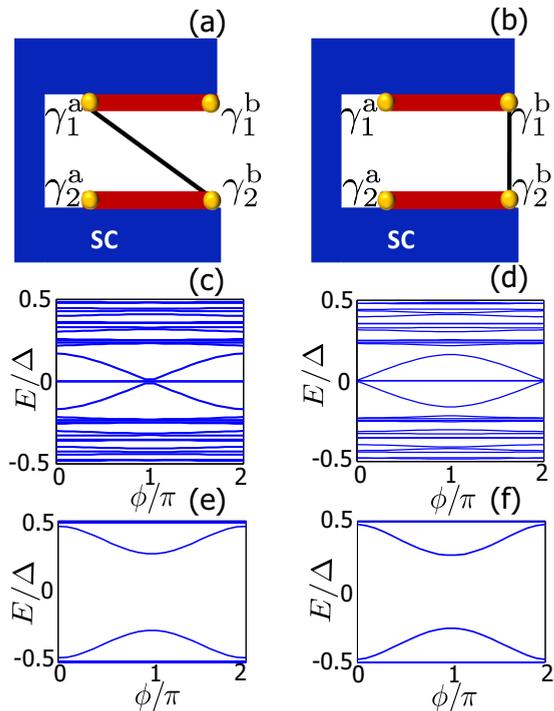


FIG. 1. (a) and (b) The Josephson junctions with N-shaped and U-shaped geometries. The red wires are the semiconductor wires attached to the s -wave superconductor. The yellow balls indicate the locations of MZMs. The black wires represent the normal metal. Panels (c) and (d) show the Andreev levels from the numerical calculations of the eigenenergy of the Josephson junctions in (a) and (b) with $t_{so} = \Delta$, $M = 8\Delta$, $\mu_s = -2\Delta$, $t_s = t_N = 10\Delta$, $\mu_N = 20\Delta$. Panels (e) and (f) show the Andreev levels of the Josephson junctions in (a) and (b) using the same set of parameters except changing M to $\Delta/2$.

models (e.g., the Kitaev 1D model [4]) since the spin degree of freedom plays no role in the latter type of manifestly spinless TSCs. The spintronic physics and the various spin-phase JJ physics being described in the current work simply do not exist within the 1D Kitaev (or for that matter, in any spinless p -wave TSC) model, showing conclusively that the topological superconductivity in the semiconductor-superconductor hybrid systems predicted in Refs. [8,9] is much richer than and goes far beyond the simple spinless p -wave TSC model often used in the literature. The Majorana spintronic physics described in the current work arises entirely from the interplay among spin-orbit coupling, Zeeman splitting, and s -wave superconductivity in the semiconductor-nanowire model, which leads not only to MZMs and TSCs, but also to very rich spin-phase JJ physics with manifest experimental consequences not present at all in the Kitaev model of spinless p -wave superconductivity. It is interesting that all of the rich spin-phase JJ physics being discussed in the current work in the context of TSC systems shows up only in the realistic semiconductor-nanowire systems and not at all in the idealized models of spinless p -wave TSCs, perhaps explaining why this important subject has so far been mostly ignored in the literature.

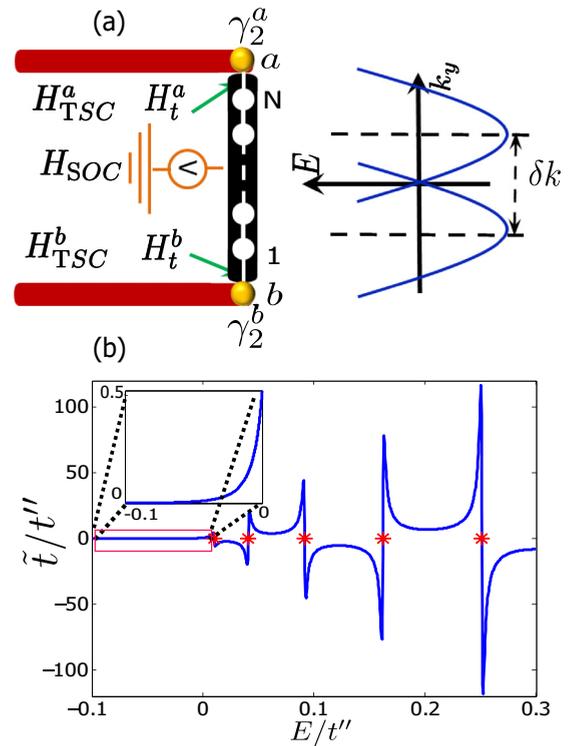


FIG. 2. (a) TSC/SOC-semiconductor/TSC junction. There are N sites (white dots) in the SOC wire (black wire). The red wires represent two TSCs. The yellow balls represent MZMs. The applied gate voltage can control the chemical potential and SOC inside the normal wire through standard technologies of the semiconductor spintronics. (b) The effective-coupling amplitude \tilde{t} as a function of energy E . Both horizontal and vertical axes are normalized by the hopping constant t'' of the SOC wire. The red stars indicate the eigenenergy $\epsilon_n = 2\sqrt{t''^2 + t_{so}^2} [1 - \cos(k_n d)]$ in the horizontal axis. The inset shows \tilde{t} with the energy inside the semiconductor band gap.

We clarify here the notations for the various JJs to be used throughout the article. We use the nomenclature 0 junction (i.e., conventional JJ), π junction, and φ_0 junction throughout this paper to refer to the JJs which exhibit the Josephson phase of 0 [31], π [32], and φ_0 (neither 0 or π) [22] in their ground states, respectively, in the absence of any applied magnetic flux. We use N junction and U junction to refer to the N-shaped and U-shaped geometries of the JJs shown in Figs. 1(a) and 1(b), respectively.

Our paper is organized as follows. In Sec. II, we demonstrate an intrinsic π -phase difference between the spin-triplet superconducting pairings enforced by MZMs localized at the opposite ends of a 1D TSC. This π -phase difference arises from neither a charge phase nor an orbital phase, and only exists in the topological-superconducting regime at the boundaries between topological and trivial phases (i.e., at interfaces or wire ends). We further establish that this π phase is indeed a spin phase and can lead to the implementation of fractional Josephson 0 and π junctions with N-shaped and U-shaped geometries [Figs. 1(a) and 1(b)]. In Sec. III, we show that the Majorana coupling and spin-phase difference across the topological JJ can be tuned by a gate-voltage controllable SOC in the normal nanowire, connecting the two topological

superconductors [a U-shaped topological JJ as shown in Fig. 2(a)], with exponential accuracy which leads to flux-free control of the MZM coupling energy and Josephson current, and gives rise to a fractional Josephson-junction. In particular, the direction of the spin-phase-driven topological Josephson current can measure the Fermion parity of a topological JJ. In Sec. IV, we propose an all-electrically controllable superconductor-semiconductor hybrid circuit to detect the non-Abelian nature of MZMs and present a discussion of the experimental feasibility of the proposed device. In Sec. V, we conclude with a summary of our results. (Some of the more complicated technical details are relegated to Appendices A and B although the results and equations from these Appendices are sometimes used in the main text.)

II. SPIN-STATE-DEPENDENT $0-\pi$ JOSEPHSON JUNCTION TRANSITION

In a SC/NM junction, Cooper pairs can tunnel into the NM which inherits certain superconducting properties such as supercurrent. If the tunneling between the superconductor and NM obeys spin rotation SU(2) symmetry, the phase of the induced superconducting condensate in the NM is normally expected to only depend on the superconducting charge phase and orbital phase. However we find that the spin phase, associated with the spin degree of freedom of Cooper pairs, may also play an important role in certain JJs, to be elaborated below.

We first consider the recently extensively studied time-reversal-symmetry-broken 1D TSC tight-binding model (a semiconductor nanowire with SOC coupling in the presence of a bulk superconductor and Zeeman spin splitting) whose Hamiltonian in the basis $(c_\uparrow, c_\downarrow, -c_\downarrow^\dagger, c_\uparrow^\dagger)^T$ has the form [8–11]

$$H_{\text{TS}} = [-2t_s \cos(kd) - \mu_s] \tau_z \otimes \sigma_0 - M \tau_0 \otimes \sigma_z + 2t_{\text{so}} \sin(kd) \tau_x \otimes \sigma_y + \Delta \tau_x \otimes \sigma_0, \quad (1)$$

where k and d are the wave vector and the lattice constant, respectively, t_s is the spin-independent hopping, t_{so} is the SOC strength, and μ_s is the chemical potential of the semiconductor nanowire with M the Zeeman coupling strength and Δ the proximity-induced superconducting gap. In the topological-superconducting regime of the semiconductor nanowire [8–11], there are two MZMs γ_1 and γ_2 located at the right and left ends, respectively [Figs. 1(a) and 1(b)]. In the strong Zeeman splitting limit, $M \gg \Delta$, the spin polarization of the two MZMs is almost antiparallel to the magnetization. Besides, the Hamiltonian in Eq. (1) commutes with the complex conjugation operator \mathcal{K} so that MZMs should be eigenfunctions of \mathcal{K} . Thus, the two MZMs at the right and left ends have the form

$$\gamma_{1\uparrow} = (c_\uparrow + c_\uparrow^\dagger), \quad \gamma_{2\uparrow} = i(c_\uparrow - c_\uparrow^\dagger),$$

which are even and odd under complex conjugation, respectively. Correspondingly, spin-triplet pairing coefficients [33] are $\psi_{\uparrow\uparrow,1} = -\psi_{\uparrow\uparrow,2} = 1$ (see details in Appendix A). The minus sign for $\psi_{\uparrow\uparrow,2}$ is from the square of the i in γ_2 . According to Eq. (A2) in our Appendix A, the anomalous density matrices

[3] for the MZMs at the two ends are

$$f_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}. \quad (2)$$

If the TSC respects time-reversal symmetry, we will have MZMs in the spin-down channel as well,

$$\gamma_{1\downarrow} = \hat{T} \gamma_{1\uparrow} \hat{T}^{-1} = c_\downarrow + c_\downarrow^\dagger, \\ \gamma_{2\downarrow} = \hat{T} \gamma_{2\uparrow} \hat{T}^{-1} = i(c_\downarrow - c_\downarrow^\dagger), \quad (3)$$

with the time-reversal operator $\hat{T} = -i\sigma_y \mathcal{K}$. The anomalous density matrices in this case are

$$f_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad f_2 = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (4)$$

with $\psi_{\downarrow\downarrow,1} = -\psi_{\downarrow\downarrow,2} = 1$. Based on Eqs. (2) and (4), the MZM-induced spin-triplet superconducting condensates $f_{1,2}$ have a π -phase difference, regardless of whether time-reversal symmetry is broken or not. As MZM-induced pairing is odd-frequency s -wave spin-triplet [1,3], the π -phase difference arises from neither a charge phase nor an orbital phase. We emphasize here that this π -phase difference is not just a mathematical construct, but has observable physical effects as shown below.

In the time-reversal-symmetry-broken TSC/NM/TSC JJs [Figs. 1(a) and 1(b)], the two identical nanowires (red wires) are proximity-induced topological superconductors. The normal-metal wire (black wire) connects the two TSCs in a different way which forms N and U junctions as shown in Figs. 1(a) and 1(b), respectively. The two TSC nanowires (red wires) are described in the minimal model by Eq. (1) and the normal wire (black wire) is described by

$$H_N = [-2t_N \cos(kd) - \mu_N] \tau_z \otimes \sigma_0, \quad (5)$$

where t_N and μ_N are the hopping energy and chemical potential in the normal-metal wire, respectively. The MZMs γ_1^a and γ_2^a (γ_1^b and γ_2^b) are located at the left and right ends of the a (b) wires [Figs. 1(a) and 1(b)]. In the strong Zeeman splitting limit $M \gg \Delta$, the spin direction of these MZMs is antiparallel to the magnetic field [2]. According to Eq. (2), the superconducting condensates induced by these two MZMs have a π -phase difference. It follows that the current-phase relation of the N junction [Fig. 1(a)] has a π -phase shift as compared to the coupling of γ_2^a and γ_2^b in the U junction [Fig. 1(b)]. To test this prediction, we set the Hamiltonian parameters in Eq. (1) to be in the topological-superconducting regime ($M > \Delta$, $\mu_s = 0$) and plot the eigenenergies as a function of the charge-phase difference ϕ in Figs. 1(c) and 1(d), which correspond to the N and U junctions, respectively. In Figs. 1(c) and 1(d), the Andreev levels cross at $\phi = \pi$ and $\phi = 0$, which indicates the 4π -periodic 0 and π JJs, respectively. The lines with constant $E = 0$ correspond to the MZMs at other uncoupled ends [Figs. 1(a) and 1(b)]. Moreover, we tune the superconductors into the topologically trivial regime by choosing $M = \Delta/2$ without changing other parameters and plot the eigenenergies of the N and U junctions as a function of ϕ in Figs. 1(e) and 1(f), respectively. The Andreev levels in both of the two JJs behave like the normal Josephson 0 junction whose minimal ground state is at $\phi = 0$ with

2π periodicity. Besides, we also calculate the eigenenergies for $M = 2\Delta, 4\Delta, 6\Delta$ (topologically nontrivial regime) and $M = 0, 0.3\Delta, 0.6\Delta$ (topologically trivial regime) in the two JJs. We find that in the topological trivial regime, the N-shaped and U-shaped junctions always have the same Andreev levels with a normal Josephson 0 junction with 2π periodicity. However, in the topologically nontrivial regime with MZMs located at the interface of the TSC/NM interface, besides the arising of the 4π periodicity of Andreev levels in both N and U junctions, only the latter becomes a π junction. These results confirm that the appearance of the π phase in the U junction only depends on the presence of the MZM-induced spin-triplet pairings. Thus this π phase is indeed a spin phase.

We also consider the time-reversal-invariant TSC/NM/TSC junction whose TSC Hamiltonian is given as [15]

$$H_{\text{TSC}} = [-2t_s \cos(kd) - \mu_s] \tau_z \otimes \sigma_0 + 2t_{s_0} \sin(kd) \tau_z \otimes \sigma_z + \Delta(kd) \tau_x \otimes \sigma_0, \quad (6)$$

where $\Delta(kd) = [\Delta_0 - \Delta_1 \cos(kd)]$. This spin-singlet superconducting gap involves both s_+ and s_- channels, and it vanishes at $\cos(k_0 d) = \Delta_0/\Delta_1$. The SOC in the semiconductor nanowire induces the spin splitting, and leads to two Fermi wave vectors k_{1f} and k_{2f} with $k_{1f} < k_{2f}$. For $k_{1f} < k_0 < k_{2f}$, the system is in a topological-superconducting regime [16], and the associated Andreev levels of the N and U junction are similar to those plotted in Figs. 1(c) and 1(d) except that there is a Kramers degeneracy in this time-reversal-symmetric case. Taking $\Delta_1 = 0$, the system is in the topologically trivial regime and the Andreev levels for the two JJ configurations are similar to those plotted in Figs. 1(e) and 1(f) except for the Kramers degeneracy. We thus conclude that for the time-reversal-symmetric topological JJ, the superconducting condensates at the opposite ends have a π -phase difference provided that there exist MZM-induced spin-triplet pairings. The π phase is related to the Cooper pair spin-triplet states, and thus belongs to the spin phase which is similar to the time-reversal-symmetry-broken case.

In a real semiconductor nanowire, which has a finite width in its transverse plane (x - y plane), the complex conjugation symmetry is broken in general, for example, by considering a SOC $-i\partial_y \sigma_z$. However, even in this case, we can still establish the MZM-related spin-phase difference in topological JJs, by theoretically treating the system as a multiband TSC, in the presence of the following mirror reflection symmetry:

$$\begin{aligned} \mathcal{M}_z H(k_x, k_y, k_z, \sigma_x, \sigma_y, \sigma_z) \mathcal{M}_z^{-1} \\ = H(k_x, k_y, -k_z, -\sigma_x, -\sigma_y, \sigma_z). \end{aligned}$$

In Appendix B, we show that the s -wave spin-triplet Cooper pairs described by a \mathbf{d} vector (which is a vector description of spin-triplet superconducting condensates introduced in Ref. [33] as defined in Eq. (A4) in our Appendix A) along the x or y direction are odd under the mirror reflection, \mathcal{M}_z . By contrast, the s -wave Cooper pairs of spin-singlet or spin-triplet with the \mathbf{d} vector along the z direction are even. Thus, with a \mathbf{d} vector along the x or y direction, the MZM-induced Cooper pairing at the two ends of the TSC are always opposite in sign, and thus have a π -phase difference, provided that the system respects the mirror reflection symmetry.

III. SOC-TUNABLE MAJORANA FERMION COUPLINGS AND UNCONVENTIONAL JOSEPHSON EFFECTS

Inspired by the π spin phase and its induced topological Josephson π junction, we expect to have a completely different technique to manipulate the Majorana fermion (MF) coupling in topological JJs from the method of using magnetic flux to control the phase of charge origin [34]. Because SOC, with the general form $(\nabla \mathbf{V} \times \boldsymbol{\sigma}) \cdot \mathbf{p}$, performs as a spin-dependent vector potential and can rotate the \mathbf{d} vector of spin-triplet Cooper pairs [35–37], its effect on the spin phase will affect the MF coupling and the current-phase relation in a topological JJ.

To explore the SOC effect in a topological JJ, we consider the U junction with the SOC normal wire along the y direction [Fig. 2(a)]. The Hamiltonian of this junction takes a general form

$$H = \begin{pmatrix} H_{\text{TSC}}^a & 0 & H_t^a \\ 0 & H_{\text{TSC}}^b & H_t^b \\ (H_t^a)^\dagger & (H_t^b)^\dagger & H_{\text{SOC}} \end{pmatrix}. \quad (7)$$

In this section, we consider both cases with and without time-reversal symmetry in the Hamiltonian $H_{\text{TSC}}^{a,b}$.

The Hamiltonian of the normal SOC wire H_{SOC} reads

$$\begin{aligned} H_{\text{SOC}} = \sum_{i,\sigma} \mu' c_{i\sigma}^\dagger c_{i\sigma} + \sum_{i,\sigma} -t' c_{i+1\sigma}^\dagger c_{i\sigma} \\ + \sum_{i,\sigma} i(-1)^\sigma t'_{\text{so}} c_{i+1\sigma}^\dagger c_{i\sigma} + \text{H.c.}, \end{aligned}$$

with μ' the chemical potential, t' the spin-independent hopping, and t'_{so} the strength of the experimentally accessible $p_y \sigma_z$ -type SOC [23–28], which is tunable by a gate voltage as shown in Fig. 2(a).

The couplings between the normal wire and superconductors in Eq. (7) are

$$H_t^a = \sum_{\sigma} t'' c_{a\sigma}^\dagger c_{1\sigma} + \text{H.c.}, \quad H_t^b = \sum_{\sigma} t'' c_{b\sigma}^\dagger c_{N\sigma} + \text{H.c.}$$

Here the indices 1 and N indicate the sites at the opposite ends of the SOC wire [Fig. 2(a)], and the coupling t'' models the electron tunneling across the normal and superconducting wires. According to Eq. (7), the two TSCs are indirectly coupled through the SOC in the normal wire. To analyze the SOC effect in the TSC/SOC-semiconductor/TSC junctions, we derive an effective coupling between the two TSCs. The normal SOC wire contributes an effective self-energy

$$\Sigma = -H_t G_{\text{SOC}} H_t^\dagger, \quad (8)$$

with $H_t = (H_t^a, H_t^b)^T$ and the Green's function G_{SOC} given by

$$G_{\text{SOC}}(y, y') = \sum_n \frac{|\psi_n(y)\rangle \langle \psi_n(y')|}{E - \epsilon_n + i\delta}. \quad (9)$$

Here ϵ_n is the eigenenergy, and $|\psi_n\rangle$ is the eigenfunction of H_{SOC} , which satisfies the boundary conditions $\psi_n(0) = 0$ and $\psi_n((N+1)d) = 0$ with d the lattice constant and N the number of lattice sites in the normal SOC wire. Due to the SOC, the two spin bands will be shifted oppositely in the k axis [Fig. 2(a)] by $\delta k = 2 \arcsin(t'_{\text{so}}/\sqrt{t'^2 + t'_{\text{so}}^2})/d$. The

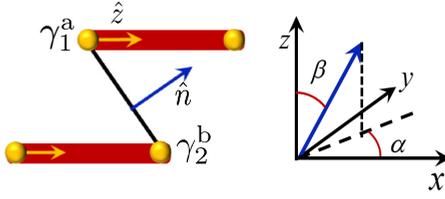


FIG. 3. A schematic plot of the TSC/SOC-semiconductor/TSC junction with misalignment between the spin quantization axis and SOC field direction. The yellow and blue arrows indicate the MF spin direction \hat{z} and SOC field direction \hat{n} . Given \hat{z} along the z axis, the SOC field direction can be described by the polar angle β and azimuthal angle α .

eigenfunctions for the two spin channels take the form

$$\begin{aligned}\psi_{k,\uparrow}(y) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\frac{\beta k}{2}y} \sin(ky), \\ \psi_{k,\downarrow}(y) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i\frac{\beta k}{2}y} \sin(ky)\end{aligned}\quad (10)$$

with $k_n = n\pi/(N+1)d$ according to the boundary condition. By integrating out the electron and hole degrees of freedom in the normal nanowire, we obtain the off-diagonal term

$$\begin{aligned}\Sigma_{ab} &= \Sigma_{ba}^\dagger = \tilde{t} e^{i\frac{\delta k(N-1)d}{2}\sigma_z} = \tilde{t} e^{i\frac{\theta}{2}\sigma_z}, \\ \tilde{t} &= t'^{1/2} \sum_k \frac{\sin(ka) \sin(kNd)}{E - 2\sqrt{t'^2 + t_{so}^2} [1 - \cos(kd)]},\end{aligned}\quad (11)$$

which gives rise to an effective spin-phase-dependent coupling between two TSCs, $\tilde{t} e^{i\frac{\theta}{2}\sigma_z}$. In Eq. (11), we neglect the

$$\begin{aligned}\tilde{H}_t &= \frac{\tilde{t}}{4} \begin{pmatrix} \gamma_1^a \\ \gamma_2^a \end{pmatrix}^T \begin{pmatrix} e^{i\frac{1}{2}(\phi\sigma_0 + \theta\hat{n}\cdot\sigma)} - e^{-i\frac{1}{2}(\phi\sigma_0 + \theta\hat{n}\cdot\sigma^*)} & i(e^{i\frac{1}{2}(\phi\sigma_0 + \theta\hat{n}\cdot\sigma)} + e^{-i\frac{1}{2}(\phi\sigma_0 + \theta\hat{n}\cdot\sigma^*)}) \\ -i(e^{i\frac{1}{2}(\phi\sigma_0 + \theta\hat{n}\cdot\sigma)} + e^{-i\frac{1}{2}(\phi\sigma_0 + \theta\hat{n}\cdot\sigma^*)}) & e^{i\frac{1}{2}(\phi\sigma_0 + \theta\hat{n}\cdot\sigma)} - e^{-i\frac{1}{2}(\phi\sigma_0 + \theta\hat{n}\cdot\sigma^*)} \end{pmatrix} \begin{pmatrix} \gamma_1^b \\ \gamma_2^b \end{pmatrix}, \\ \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} &= \sqrt{2}U \begin{pmatrix} c \\ c^\dagger \end{pmatrix}, \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_0 & \sigma_0 \\ -i\sigma_0 & i\sigma_0 \end{pmatrix},\end{aligned}\quad (15)$$

with $\gamma_1^a = (\gamma_{1\uparrow}, \gamma_{1\downarrow})^T$, $\hat{n} \cdot \sigma = \cos\beta\sigma_z + \sin\beta\cos\alpha\sigma_x + \sin\beta\sin\alpha\sigma_y$, β the polar angle, and α the azimuthal angle [Fig. 2(b)]. Equation (15) provides a general form of coupling between two TSCs through a SOC wire. In the following, we use it to study unconventional Josephson effects in both time-reversal-broken and time-reversal-invariant junctions. We refer to this description as the MF representation.

A. Time-reversal-symmetry-broken topological Josephson junction

For a time-reversal-symmetry-broken TSC/SOC-semiconductor/TSC junction, we assume (without loss of generality) that the MZMs at sites a and b are $\gamma_{1\uparrow}^a$ and $\gamma_{2\uparrow}^b$, respectively [Fig. 2(a)]. Then according to Eq. (15), the MF-coupling Hamiltonian reads

$$\tilde{H}_t = \frac{i\tilde{t}}{2} \left(\cos\frac{\theta}{2} \cos\frac{\phi}{2} - \sin\frac{\theta}{2} \sin\frac{\phi}{2} \cos\beta \right) \gamma_{1\uparrow}^a \gamma_{2\uparrow}^b. \quad (16)$$

contribution from the poles of G_{soc} in Eq. (10) if there is no eigenstate inside the superconducting gap. This is valid when the chemical potential of the SOC wire is in its semiconductor band gap, or the length of the SOC wire $L = (N+1)d$ is much smaller than the coherence length ξ so that around the Fermi surface, $\epsilon_{n+1} - \epsilon_n \gg \Delta$. We plot \tilde{t} as a function of energy E in Fig. 2(b). The effective-coupling amplitude changes sign with the energy E across the quantized eigenenergy ϵ_n and becomes a pure exponential decay inside the semiconductor band gap. If we also consider the charge phase by adding a vector potential $(A, 0, 0)^T$ in the SOC region, the tunneling Hamiltonian takes the form

$$\tilde{H}_t = \tilde{t} (c_a^\dagger e^{i\frac{1}{2}(\phi\sigma_0 + \theta\sigma_z)} c_b - c_a e^{-i\frac{1}{2}(\phi\sigma_0 + \theta\sigma_z)} c_b^\dagger), \quad (12)$$

where

$$\phi/2 = eAL/\hbar, \quad \theta = \delta kL \quad (13)$$

correspond to the charge phase and spin phase, respectively, for an electron traveling across the junction.

The effective-coupling Hamiltonian, Eq. (12), can be easily generalized to the case for the arbitrary SOC field direction \hat{n} (Fig. 3) as

$$\tilde{H}_t = \tilde{t} (c_a^\dagger e^{i\frac{1}{2}(\phi\sigma_0 + \theta\hat{n}\cdot\sigma)} c_b - c_a e^{-i\frac{1}{2}(\phi\sigma_0 + \theta\hat{n}\cdot\sigma^*)} c_b^\dagger). \quad (14)$$

As we are interested in the topological JJ, it is convenient to write the tunneling Hamiltonian in the Majorana representation by a unitary transformation [38]

The associated Andreev levels and Josephson currents are

$$E = \pm \frac{\tilde{t}}{2} \left(\cos\frac{\theta}{2} \cos\frac{\phi}{2} - \sin\frac{\theta}{2} \sin\frac{\phi}{2} \cos\beta \right), \quad (17a)$$

$$I = \mp \frac{\tilde{t}e}{\hbar} \left(\cos\frac{\theta}{2} \sin\frac{\phi}{2} + \sin\frac{\theta}{2} \cos\beta \cos\frac{\phi}{2} \right). \quad (17b)$$

For $\beta = \frac{\pi}{2}$, the MF spins are perpendicular to the z axis and the Andreev levels take the form

$$E = \pm \frac{\tilde{t}}{2} \cos\frac{\theta}{2} \cos\frac{\phi}{2}. \quad (18)$$

This indicates that the Andreev level crossing is always at $\phi = \pi$, the minimal ground state energy is always at $\phi = 0$, and the topological JJ is always a Josephson 0 junction. However the MF-coupling energy oscillates as a function of the spin phase θ , which is consistent with our previous study [3].

When the SOC field direction \hat{n} in the normal region is parallel to \hat{z} [along the two TSC wires (Fig. 3)] so that $\beta = 0$,

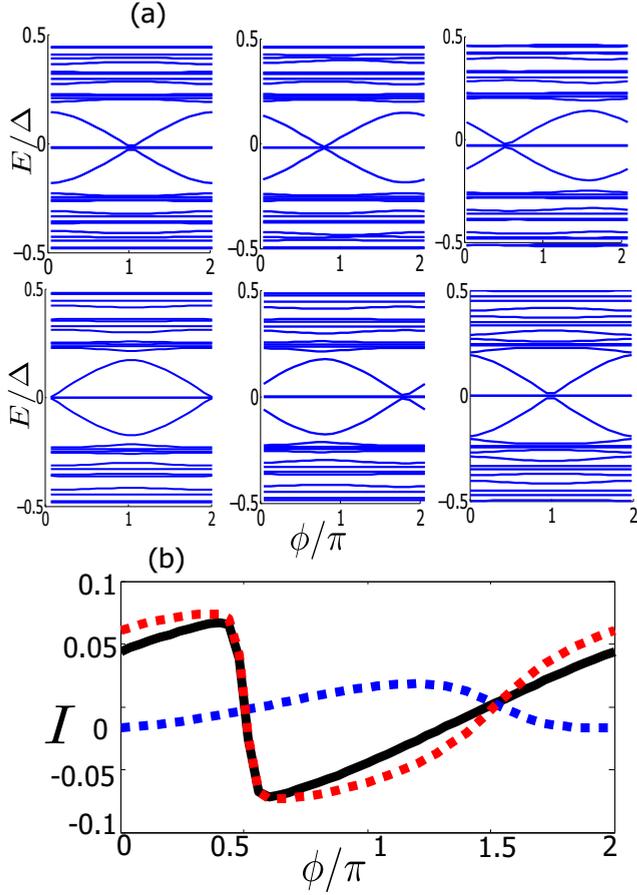


FIG. 4. (a) Andreev bound states with effective magnetic field of SOC perpendicular to the d vector of the MF at the end of TSCs. The horizontal axis is the phase difference of the two TSCs and the vertical axis is energy. The panels correspond to $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \pi, \frac{5\pi}{4},$ and 2π , respectively. (b) Josephson currents with the spin phase $\theta = \pi/2$. The red dashed line represents the contribution from the MF coupling. The blue dashed line represents the contribution from the bulk superconducting ground state below the gap. The black solid line is the net Josephson current.

the MF-coupling Hamiltonian is

$$\tilde{H}_\gamma = \frac{\tilde{t}}{2} i \gamma_{1\uparrow}^a \gamma_{2\uparrow}^b \cos\left(\frac{\phi + \theta}{2}\right). \quad (19)$$

The associated Andreev levels and Josephson currents have the form

$$\begin{aligned} E &= \pm \frac{\tilde{t}}{2} \cos\left(\frac{\phi + \theta}{2}\right), \\ I &= \pm \frac{|e|\tilde{t}}{h} \sin\left(\frac{\phi + \theta}{2}\right), \end{aligned} \quad (20)$$

which exhibit 4π periodicity in both charge phase ϕ and spin phase θ . Here $+$ ($-$) indicates the fractional Josephson current direction and is related to the fermion parity defined as $i\gamma_{1\uparrow}^a \gamma_{2\uparrow}^b = 1 - 2c^\dagger c$ with $c = \gamma_1^a - i\gamma_2^b$. The minimal ground state energy is shifted to $\phi = -\theta$. The obtained analytic results for the fractional Josephson relation have been confirmed in our numerical calculations (see Fig. 4). It is worth emphasizing that even given charge phase $\phi = 0$, we could have a finite spin-

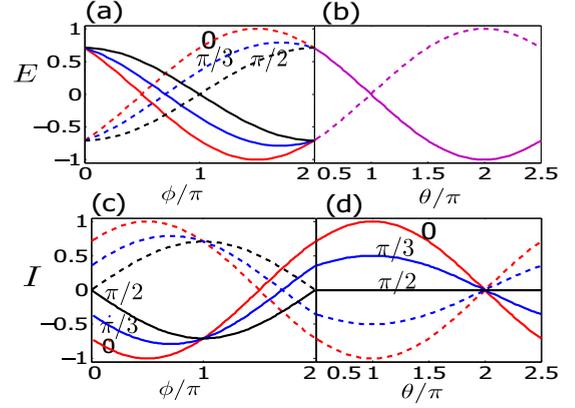


FIG. 5. (a) and (c) The plot of Andreev levels and Josephson currents as a function of charge phase ϕ with given spin phase $\theta = \pi/2$ and $\phi \in [-2\pi, 0]$. The red, blue, and black colors correspond to the misaligned angle $\beta = 0, \pi/3, \pi/2$, respectively. Energy and current are normalized by $\tilde{t}/2$ and $e\tilde{t}/h$, respectively, with e the electron charge and h the Plank constant. The solid (dashed) curves represent the $+$ ($-$) in Eq. (21). (b) and (d) The plot of Andreev levels and Josephson currents as a function of spin phase θ with fixed $\phi = 0$ and $\theta \in [\pi/2, 5\pi/2]$. In (b), the Andreev levels for $\beta = 0, \pi/3, \pi/2$ collapse into identical curves.

phase-driven fractional Josephson current whose direction measures the fermion parity in the topological JJ.

For an arbitrary SOC field direction, we find that the minimal ground state energy is generally shifted away from $\phi = 0$ due to the spin phase θ unless $\beta = 0$ as shown in Fig. 5(a). Consequently, the corresponding Josephson current has both $\cos(\phi/2)$ and $\sin(\phi/2)$ terms according to Eq. (17b) and therefore can be finite even at $\phi = 0$ [Fig. 5(c)]. In Figs. 5(a) and 5(c), the spin phase is set to be $\pi/2$, for which the SOC effect leads to two terms of equal weight on the right-hand sides of Eqs. (17a) and (17b).

To explicitly illustrate this unusual Josephson current phenomena, we focus on $\phi = 0$ and study the spin-phase-current relation. In this case, the Josephson current has the form

$$I = \mp \frac{\tilde{t}e}{h} \sin\frac{\theta}{2} \cos\beta,$$

which remains the 4π periodicity for arbitrary SOC field direction. We plot the above spin-phase-dependent Andreev levels and Josephson currents in Figs. 5(b) and 5(d) for $\beta = 0, \pi/3, \pi/2$. The solid (dashed) lines correspond to the $+$ ($-$) sign in Eq. (21). At $\phi = 0$, unless the polar angle $\beta = \pi/2$, the Josephson current can be turned on and off by varying the spin phase [Fig. 5(d)], and we have a φ_0 junction as introduced in Ref. [22] in a completely different context. To understand the appearance of this φ_0 junction in a time-reversal-symmetry-broken topological JJ, we compare the charge phase with the spin phase in the coupling Hamiltonian Eq. (14). For $\hat{n} \parallel \hat{z}$, the Hamiltonian in Eq. (14) is diagonal in spin space. If we focus on the spin-up channel, the spin phase plays exactly the same role as the charge phase so that it can be used to control the Josephson current. For $\hat{n} \perp \hat{z}$, the spin phase performs like the Zeeman coupling which rotates the MF spin direction without introducing a relative phase between the sites a and b

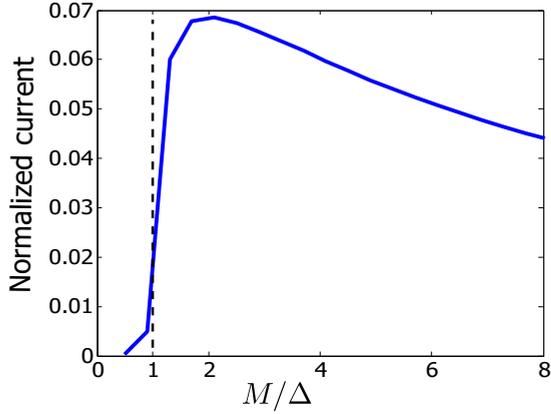


FIG. 6. Josephson current as a function of the magnetization with spin phase $\theta = \pi/2$ and charge phase $\phi = 0$. $M/\Delta = 1$ is the topological phase transition point.

[Fig. 2(a)]. In this case, the Josephson current cannot be turned on solely by the spin phase. Thus, as long as the SOC field has a finite component along the \hat{z} direction, this component will turn on the supercurrent in the topological JJ even at $\phi = 0$. When the superconductor is in the topologically trivial regime, say $0 \leq M < \Delta$ for $\mu = 0$, the magnetization-induced bulk spin-triplet pairing has no spin polarization [39] and thereby the spin-triplet Cooper pairs have the same amplitude in both spin-up and spin-down channels. Consequently, the Josephson current in the topologically trivial regime should be zero at $\phi = 0$. In Fig. 6, we plot the Josephson current as a function of magnetization M with $\theta = \pi/2$ and $\phi = 0$ for the topological Josephson junction illustrated in Fig. 2(a). The Josephson current drops down to zero sharply at the topological quantum phase transition point. Thus, the observation of this SOC-induced Josephson current would serve as a clear signal for topological superconductivity and MZMs. On the other hand, as the direction of the SOC-induced fractional Josephson current is solely determined by the fermion parity of the topological Josephson junction if the sign of the coupling amplitude \tilde{t} is fixed, this property can be used to detect the non-Abelian nature of MZMs which will be discussed in the next section (Sec. IV).

B. Time-reversal-invariant topological Josephson junctions

In the low-energy limit, $E \ll \Delta$, we consider the time-reversal-invariant TSC/SOC-semiconductor/TSC JJ with a pair of MZMs $(\gamma_{1(2)\uparrow}^{a(b)}, \gamma_{1(2)\downarrow}^{a(b)})^T$ located at site a (b) (Fig. 3). For simplicity, we first assume that the SOC field direction is parallel to the \hat{z} axis. According to Eq. (15), the tunneling Hamiltonian can be projected to the Hilbert space expanded by these four MZMs as

$$\begin{aligned} \tilde{H}_\gamma &= \frac{\tilde{t}}{4} \gamma_1^a i (e^{\frac{i}{2}(\phi\sigma_0 + \theta\sigma_z)} + e^{-\frac{i}{2}(\phi\sigma_0 + \theta\sigma_z)}) \gamma_2^b \\ &= \frac{\tilde{t}}{2} i \gamma_{1\uparrow}^a \gamma_{2\uparrow}^b \cos\left(\frac{\phi + \theta}{2}\right) + \frac{\tilde{t}}{2} i \gamma_{1\downarrow}^a \gamma_{2\downarrow}^b \cos\left(\frac{\phi - \theta}{2}\right). \end{aligned} \quad (21)$$

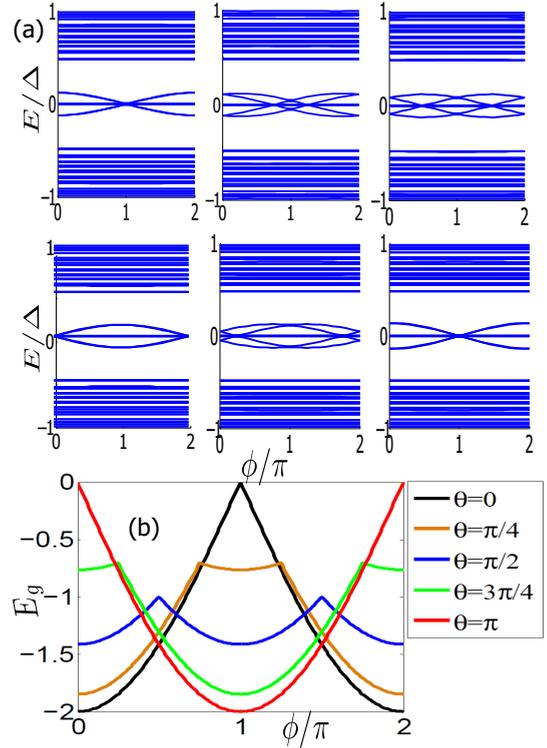


FIG. 7. (a) Andreev bound states when the effective magnetic field of SOC is parallel to the MF spins at the end of TSCs. The horizontal axis is the phase difference of the two TSCs and the vertical axis is energy. The panels correspond to $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \pi, \frac{5\pi}{4},$ and 2π , respectively. (b) The ground state energy E_g as a function of charge phase ϕ given spin phase $\theta = 0, \pi/4, \pi/2, 3\pi/4, \pi$. The ground state energy E_g is normalized by $\tilde{t}/2$.

According to Eq. (21), the MF coupling is not only determined by the charge phase ϕ but also depends on the spin phase θ . The associated Andreev levels and Josephson current have the form

$$E = \pm \frac{\tilde{t}}{2} \cos\left(\frac{\phi \pm \theta}{2}\right), \quad (22a)$$

$$I = \mp \frac{2e\tilde{t}}{h} \sin\frac{\phi}{2} \cos\frac{\theta}{2}. \quad (22b)$$

To confirm our analytical results, we numerically calculate the eigenenergies of the time-reversal-invariant TSC/SOC-semiconductor/TSC junction [see Fig. 7(a)]. The spin phase θ shifts the two branches of Andreev levels oppositely in the ϕ axis [Fig. 7(a)], which is consistent with our analytical results in Eq. (22a). Based on Eq. (22a), the ground state energy takes the form

$$E_g = -\frac{\tilde{t}}{2} \left(\left| \cos\frac{\phi + \theta}{2} \right| + \left| \cos\frac{\phi - \theta}{2} \right| \right), \quad (23)$$

whose minimum is located at $\phi = 0$ for $\theta \in (-\pi/2 + 2n\pi, \pi/2 + 2n\pi)$ and at $\phi = \pi$ for $\theta \in (\pi/2 + 2n\pi, 3\pi/2 + 2n\pi)$. At the transition points $\theta = (2n + 1)\pi/2$, the topological JJ has a double degeneracy at $\phi = 0$ and $\phi = \pi$. Therefore, SOC can lead to a transition between the Josephson 0 and π junction in this case. In Fig. 7(b), given $\theta =$

$0, \pi/4, \pi/2, 3\pi/4, \pi$, we plot the ground state energy as a function of ϕ based on Eq. (23). The blue line for $\theta = \pi/2$ shows the double degeneracy at $\phi = 0$ and $\phi = \pi$.

When the SOC field direction \hat{n} is perpendicular to the MF spin ($\beta = \pi/2$), the MF-coupling Hamiltonian depends on the azimuthal angle α , according to Eq. (15). With $\alpha = 0$, the Hamiltonian reads

$$\tilde{H}_t = i \frac{\tilde{t}}{2} \gamma_1^a \left(\cos \frac{\phi}{2} \cos \frac{\theta}{2} \sigma_0 - \sin \frac{\phi}{2} \sin \frac{\theta}{2} \sigma_x \right) \gamma_2^b,$$

whose eigenenergies are

$$E = \pm \frac{\tilde{t}}{2} \left(\cos \frac{\phi}{2} \cos \frac{\theta}{2} \pm \sin \frac{\phi}{2} \sin \frac{\theta}{2} \right) = \pm \frac{\tilde{t}}{2} \cos \frac{\phi \pm \theta}{2}, \quad (24)$$

the same as Eq. (22a) derived for the SOC field direction parallel with the MF spin.

For $\alpha = \pi/2$, the MF-coupling Hamiltonian is

$$\tilde{H}_t = \frac{i\tilde{t}}{2} \cos \frac{\phi}{2} (\gamma_{1\uparrow}^a \gamma_{2\uparrow}^b + \gamma_{1\downarrow}^a \gamma_{2\downarrow}^b), \quad (25)$$

with

$$\begin{aligned} \gamma_{2\uparrow}^b &= \cos \frac{\theta}{2} \gamma_{2\uparrow}^b + \sin \frac{\theta}{2} \gamma_{2\downarrow}^b, \\ \gamma_{2\downarrow}^b &= \cos \frac{\theta}{2} \gamma_{2\downarrow}^b - \sin \frac{\theta}{2} \gamma_{2\uparrow}^b. \end{aligned}$$

The associated Andreev levels and Josephson current are

$$E = \pm \frac{\tilde{t}}{2} \cos \frac{\phi}{2}, \quad I = \mp \frac{e\tilde{t}}{\hbar} \sin \frac{\phi}{2},$$

which behave exactly in the same manner as those in the JJ without SOC in the normal region. Moreover, the coupling strength is completely independent of the SOC.

To better understand the MF-coupling Hamiltonian in the three SOC field directions, we study the \mathbf{d} vector of MZM-induced superconducting condensates. In the normal region of a JJ, the \mathbf{d} vector of the spin-triplet pairing will experience a torque $\lambda \hbar k_{\uparrow} \hat{n} \times \mathbf{d}$ [35,36]. For the time-reversal-invariant case, at site a , the two MZMs $\gamma_{1\uparrow}^a, \gamma_{1\downarrow}^a$ induce the spin-triplet pairing amplitudes $\Psi_{\uparrow\uparrow} = \Psi_{\downarrow\downarrow} = 1$. According to Eq. (A5), the corresponding \mathbf{d} vector is $(0, i, 0)$, which is along the y axis. When the SOC field direction \hat{n} is along the z or x axis, this \mathbf{d} vector will precess in the x - y or y - z plane, respectively, with the same precession speed because of the same strength of the SOC-induced torque. This gives similar Andreev level shift in the ϕ axis due to the spin-phase θ [Fig. 7(a)]. When \hat{n} is along the y direction, the SOC-induced torque is zero, and the \mathbf{d} vector being $(0, i, 0)$ will not precess so that the JJ behaves like the ones without SOC in the normal region.

IV. ALL-ELECTRICAL CONTROL AND MAJORANA BRAIDING

In this section, using the spin-phase physics developed in the earlier sections of this paper, we construct a semiconductor circuit to braid MZMs of time-reversal-symmetry-broken TSCs by gate-voltage-tunable MF coupling with exponential sensitivity and detect their non-Abelian statistics by measuring the SOC-driven Josephson current (Fig. 8). We believe our

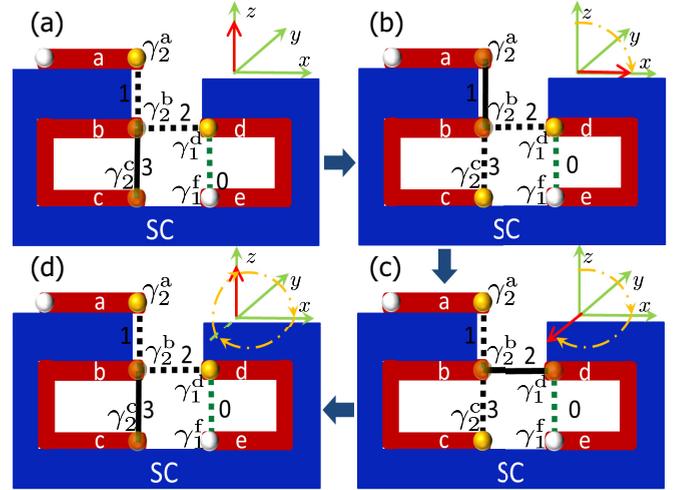


FIG. 8. Graphical representation of braiding MZMs and associated effective magnetic field. The three black wires labeled 1, 2, and 3 couple the four MZMs (yellow balls). Color white for the other two MZMs is used to distinguish them from the four MZMs which are involved in the braiding process. The solid (semitransparent) balls indicate that the MZMs are uncoupled (coupled). The solid (dashed) lines represent the “on” and “off” of the coupling through the wires. The red arrows indicate the effective magnetic field direction of the braiding Hamiltonian. The yellow arrows illustrate the trajectory of the effective magnetic field during the braiding process.

suggested JJ-based braiding experiment to be both conceptually the most straightforward and experimentally the most practical for semiconductor Majorana nanowire systems being extensively studied in many laboratories all over the world. The basic building block of this circuit is the topological Josephson junction which connects two MZMs (Fig. 8). The red wires represent semiconductor-wire-based topological superconductors [8,9] which are coupled to the same s -wave superconductor so that they have the same charge phase. The three black wires couple the four MZMs (indicated by the yellow balls in Fig. 8) and are used to braid γ_2^a and γ_1^d . The solid (dashed) lines represent turning on (off) the MF coupling which can be exponentially accurate if the gate-tunable chemical potential of these wires is inside the semiconductor band gap as shown in the inset of Fig. 2(b). The green line, coupling γ_1^d with γ_1^c (Fig. 8), is turned on before and after braiding to detect the non-Abelian statistics as being illustrated below (Fig. 9).

During the braiding process [Figs. 8(a)–8(d)] finite SOC is necessary to couple γ_2^b with γ_2^a and γ_2^c (wires 1 and 3 in Fig. 8) which form topological π JJs with the associated Hamiltonian

$$H_{1(3)} = i\tilde{t}_{1(3)} \gamma_2^{a(c)} \gamma_2^b \sin \frac{\theta}{2}. \quad (26)$$

Here $\tilde{t}_{1(3)}$ is the tunneling amplitude which is exponentially sensitive to the wire chemical potential inside the semiconductor band gap [inset of Fig. 2(b)] and θ is the spin phase defined in Eq. (13). For simplicity, θ is assumed to be the same in these two wires without loss of generality. The Zeeman

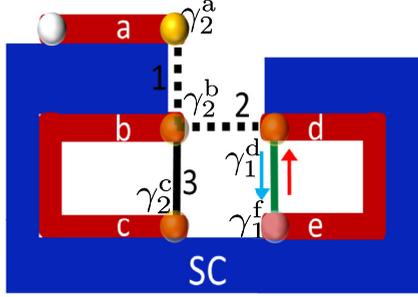


FIG. 9. Reading out the fermion parity by measuring the Josephson current direction. The green wire couples the MZMs γ_1^d and γ_1^e and is labeled “0”. The blue and red arrows illustrate that the Josephson current changes its direction before and after the braiding process due to the non-Abelian nature of MZMs.

term is parallel to the SOC direction in wires 0, 1, and 3, and thereby only effectively shifts the chemical potential of

$$H_{br} = H_1 + H_2 + H_3 = \begin{pmatrix} (\tilde{t}_2 s_y - \tilde{t}_1 s_x) \sin \frac{\theta}{2} - \tilde{t}_3 s_z & 0 \\ 0 & (\tilde{t}_2 s_y - \tilde{t}_1 s_x) \sin \frac{\theta}{2} + \tilde{t}_3 s_z \end{pmatrix}, \quad (28)$$

where $s_{x,y,z}$ are the three Pauli matrices acting on the subspace spanned by $(|00\rangle, |11\rangle)$ with even total fermion parity or $(|01\rangle, |10\rangle)$ with odd total fermion parity. The Hamiltonian H_{br} in Eq. (28) is block diagonal which indicates the conservation of the total fermion parity during the braiding process. In each block, the Hamiltonian is exactly the same as that of a spin- $\frac{1}{2}$ particle in a magnetic field (red arrows in Fig. 8). For simplicity, we assume $\tilde{t}_{1,2,3}$ are positive and $\theta \in (0, \pi)$. The red arrows in Fig. 8 illustrate the effective magnetic field in the block of the even total fermion parity with basis $|00\rangle$ and $|11\rangle$. The coupling of the four MZMs through the Hamiltonian Eq. (28) is equivalent to the tri-junction discussed extensively in the semiconductor Majorana circuit literature [40–42]. Accordingly, the braiding operation can be realized by turning on and off the coupling in wires 1, 2, and 3 sequentially as shown in Fig. 8.

In the beginning, only the coupling in wire 3 is on so that the effective magnetic field is along the z direction [Fig. 8(a)]. We first turn off the coupling in wire 3 and in the meantime turn on the coupling in wire 1 so that the MZM γ_2^a is transported to γ_2^c and the effective magnetic field is along the x direction [Fig. 8(b)]. Then we turn off the coupling in wire 1 and in the meantime turn on the coupling in wire 2. The MZM γ_1^d is then transported to γ_2^a and the effective magnetic field is along the $-y$ direction [Fig. 8(c)]. At last, we turn off the coupling in wire 2 and at the same time turn on the coupling in wire 3 so that the MZM is transported from γ_2^c to γ_1^d and the effective magnetic field comes back to its original direction [Fig. 8(d)]. During the braiding operation, the effective field encloses a solid angle $\pi/2$ as illustrated in Fig. 8. Consequently, the evolution operator takes the form [43]

$$U = \begin{pmatrix} \exp(-i\frac{\pi}{4}s_z) & 0 \\ 0 & \exp(-i\frac{\pi}{4}s_z) \end{pmatrix}, \quad (29)$$

electrons in the same spin channel without affecting the spin-phase across these wires. The coupling between γ_2^b and γ_1^d forms a topological 0 junction and SOC is not necessary for having a finite MZM coupling strength in wire 2 (Fig. 8). Accordingly, the Hamiltonian of wire 2 has the form

$$H_2 = i\tilde{t}_2 \gamma_2^b \gamma_1^d. \quad (27)$$

During the braiding process, the states corresponding to the four MZMs γ_2^a , γ_2^b , γ_2^c , and γ_1^d (yellow balls in Fig. 8) can be written in the Fock basis as

$$|00\rangle, |11\rangle = c_2^\dagger c_1^\dagger |00\rangle, \quad |01\rangle = c_2^\dagger |00\rangle, \quad |10\rangle = c_1^\dagger |00\rangle$$

with the occupation numbers of the two fermionic operators $c_1 = (\gamma_2^a - i\gamma_1^d)/2$ and $c_2 = (\gamma_2^c - i\gamma_2^b)/2$. Here $|0\rangle$ and $|1\rangle$ correspond to the $+$ and $-$ of the fermion parity $1 - 2c^\dagger c$, respectively. The total fermion parity of these four MZMs is $\mathcal{P}(t) = \langle t | i\gamma_2^a \gamma_1^d i\gamma_2^b \gamma_2^c | t \rangle$ with $\langle t | \dots | t \rangle$ the average of the state at time t . The MF-coupling Hamiltonians through the three black wires (Fig. 8) in the Fock basis have the form

and the MZMs γ_2^a and γ_1^d in the Heisenberg representation are transformed as

$$\begin{aligned} \gamma_2^a(T) &= U \gamma_2^a U^\dagger = -\gamma_1^d, \\ \gamma_1^d(T) &= U \gamma_1^d U^\dagger = \gamma_2^a, \end{aligned} \quad (30)$$

with T the braiding time.

To detect the non-Abelian braiding statistics, we connect γ_1^d with γ_1^e through the wire 0 (green wire in Fig. 9). According to the discussion in Sec. III A, given a finite spin phase, the fermion parity \mathcal{P}_0 is locked to the sign of the fractional Josephson current through the wire 0. As the braiding is operated in the absence of charge phase, we expect the normal Cooper pair tunneling with 2π periodicity will not contribute to the Josephson current. Thus, the initial fermion parity in wire 0, $\mathcal{P}_0(t=0) = \langle 0 | i\gamma_1^a \gamma_1^b | 0 \rangle$, can be detected by measuring the Josephson current through the wire 0. We then turn off the coupling in wire 0 and start to braid the MZMs γ_2^a and γ_1^d following the procedure in Fig. 8. After braiding the MZMs twice, according to Eq. (30), the evolution of MZMs in Heisenberg representation satisfies

$$\gamma_1^d(2T) = U^2 \gamma_1^d (U^\dagger)^2 = -\gamma_1^d.$$

At the same time, as the coupling in wire 0 is turned off during the process, the MZM γ_1^b commutes with the Hamiltonian H_{br} so that $\gamma_1^b(2T) = \gamma_1^b$. The fermion parity in wire 0 in this case is

$$\begin{aligned} \mathcal{P}_0(2T) &= \langle 2T | i\gamma_1^d \gamma_1^e | 2T \rangle = \langle 0 | (U^\dagger)^2 i\gamma_1^d \gamma_1^e U^2 | 0 \rangle \\ &= \langle 0 | i\gamma_1^d(2T) \gamma_1^e(2T) | 0 \rangle = -\langle 0 | i\gamma_1^d \gamma_1^e | 0 \rangle = -\mathcal{P}_0(0), \end{aligned}$$

which is opposite to its initial value. Consequently, when we turn on the coupling in wire 0 after braiding twice, the Josephson current direction should also be opposite to that before braiding. Therefore, the non-Abelian braiding statistics can

be directly probed simply by measuring the spin-phase-driven Josephson current direction in wire 0 before and after braiding.

Experimental feasibility. We note here that the U junction is the building block of our proposed braiding devices. Since the single-crystalline InSb nanowire networks [44] have been fabricated and semiconductor nanowires with SOC coupling, proximity-induced superconducting gap, and Zeeman spin splitting have been realized experimentally [45–48], there should be no significant technical obstacle to realize the proposed devices in Fig. 8 and Fig. 9. For the braiding process, our proposal only needs to tune the spin-orbit coupling through the gate voltages inside the normal nanowire region (black and green wires in Fig. 8 and Fig. 9) rather than the region of Majorana wires (red wires) in proximity to the superconductors. Thus the proposed gate control of spin-orbit coupling in our work can be achieved by standard technologies of the semiconductor spintronics [23–28]. The measured Josephson current, according to our calculation shown in Fig. 6, is about $0.07(2e\Delta/\hbar)$ with Δ the proximity-induced superconductor gap in the topological-superconducting nanowires (red wires in Figs. 8 and 9). By taking the proximity-induced hard gap in InAs as 0.2 meV [49], the maximal SOC current in our model is about 2 nA within the experimentally detectable regime. Thus our proposal has the advantage of experimental feasibility.

V. CONCLUSION

In conclusion, we have established the emergence of *spin phase* for MFs and demonstrated that it can be tuned by SOC in the normal region of the topological JJ. This spin phase has an important topological origin in the sense that it arises from an intrinsic Majorana-induced π -phase difference between opposite ends of a spinful one-dimensional TSC, which we have shown to exhibit crucial spin dependence. The spin-phase-based Majorana spin physics is not only conceptually novel, but also implies innovative applications in making robust gate-voltage-tunable fractional π and φ_0 junctions. We thus provide an all-electrically controlled semiconductor circuit to braid MZMs and to read out the topological information encoded by their non-Abelian statistical features without tuning any external magnetic flux, which opens the way to electrically controlled Majorana spintronics studies.

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APPENDIX A: SPIN-TRIPLET COOPER PAIRS

Considering a Cooper pair composed of two electrons, the wave function can be heuristically written as

$$\Psi = \psi_{\uparrow\uparrow}|\uparrow\uparrow\rangle + \psi_{\uparrow\downarrow}|\uparrow\downarrow\rangle + \psi_{\downarrow\uparrow}|\downarrow\uparrow\rangle + \psi_{\downarrow\downarrow}|\downarrow\downarrow\rangle, \quad (\text{A1})$$

where $\psi_{\uparrow\uparrow}(\uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow)$ are the coefficients of the corresponding spin states. However, Eq. (A1) is not convenient or

precise in describing superconductors lacking spin rotation symmetry. To get a more general and concise description of the spin-triplet Cooper pairs, we introduce a four-component spinor field $C(\hat{x}_+) = [c_{\uparrow}(\hat{x}_+), c_{\downarrow}(\hat{x}_+), -c_{\downarrow}^{\dagger}(\hat{x}_-), c_{\uparrow}^{\dagger}(\hat{x}_-)]^T$, where $\hat{x}_{\pm} = (\pm\epsilon, \mathbf{r})$ with ϵ and \mathbf{r} the energy and spatial coordinates, respectively. Then the density matrix in the second quantization language can be defined as

$$D_{ij}(\hat{x}_+) = -iG_{ij}^<(\hat{x}_+) = \langle C_j^{\dagger}(\hat{x}_+) \otimes C_i(\hat{x}_+) \rangle,$$

where $G^<$ is the lesser Green's function [50]. The anomalous density matrix [with elements taken from the upright 2×2 block of the 4×4 density matrix $D(\hat{x}_+)$] that corresponds to the pairing amplitudes can be written as

$$\begin{aligned} f(\hat{x}_+) &= \begin{pmatrix} -(c_{\downarrow}(\hat{x}_-)c_{\uparrow}(\hat{x}_+)) & (c_{\uparrow}(\hat{x}_-)c_{\uparrow}(\hat{x}_+)) \\ -(c_{\downarrow}(\hat{x}_-)c_{\downarrow}(\hat{x}_+)) & (c_{\uparrow}(\hat{x}_-)c_{\downarrow}(\hat{x}_+)) \end{pmatrix} \\ &= \begin{pmatrix} -\psi_{\downarrow\uparrow} & \psi_{\uparrow\uparrow} \\ -\psi_{\downarrow\downarrow} & \psi_{\uparrow\downarrow} \end{pmatrix}. \end{aligned} \quad (\text{A2})$$

Here $\langle \dots \rangle$ represents the thermal average. When electrons are coupled to a magnetic flux, as spin-up and spin-down electrons have the same charge e , they will acquire the same charge phase $\phi(\mathbf{r})/2$ as

$$\begin{pmatrix} c_{\uparrow}(\mathbf{r}) \\ c_{\downarrow}(\mathbf{r}) \end{pmatrix} \longrightarrow \begin{pmatrix} c_{\uparrow}(\mathbf{r}) \\ c_{\downarrow}(\mathbf{r}) \end{pmatrix} e^{i\phi(\mathbf{r})/2}.$$

According to Eq. (A2), the anomalous density matrix is transformed as

$$\begin{pmatrix} -\psi_{\downarrow\uparrow} & \psi_{\uparrow\uparrow} \\ -\psi_{\downarrow\downarrow} & \psi_{\uparrow\downarrow} \end{pmatrix} \longrightarrow \begin{pmatrix} -\psi_{\downarrow\uparrow} & \psi_{\uparrow\uparrow} \\ -\psi_{\downarrow\downarrow} & \psi_{\uparrow\downarrow} \end{pmatrix} e^{i\phi(\mathbf{r})}, \quad (\text{A3})$$

which indicates that all pairing states must have the same charge phase.

By using the anomalous density matrix, we can define a complex vector,

$$\mathbf{d} = \text{Tr} \left[\frac{1}{2} \sigma \mathbf{f} \right], \quad (\text{A4})$$

with $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ the Pauli matrices in spin space. From Eq. (A4), we have

$$d_x = \frac{\psi_{\uparrow\uparrow} - \psi_{\downarrow\downarrow}}{2}, \quad d_y = i \frac{\psi_{\uparrow\uparrow} + \psi_{\downarrow\downarrow}}{2}, \quad d_z = -\frac{\psi_{\uparrow\downarrow} - \psi_{\downarrow\uparrow}}{2}. \quad (\text{A5})$$

With the \mathbf{d} -vector representation, it is instructive to consider a spin-triplet state with coefficients $\psi_{\uparrow\uparrow} = e^{-i\theta}$, $\psi_{\uparrow\downarrow} = \psi_{\downarrow\uparrow} = 0$, $\psi_{\downarrow\downarrow} = e^{i\theta}$. The corresponding anomalous part of the density matrix is

$$f = \begin{pmatrix} 0 & e^{-i\theta} \\ -e^{i\theta} & 0 \end{pmatrix} = i(\cos\theta\sigma_y - \sin\theta\sigma_x), \quad (\text{A6})$$

with $(d_x, d_y, d_z)^T = i(-\sin\theta, \cos\theta, 0)^T$. We find that the relative phase between the states $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ corresponds to the azimuthal angle of the \mathbf{d} vector in the x - y plane. It is worthwhile to note that this state has no spin polarization although it is a spin-triplet state. This type of spin-triplet state is called the unitary state [33]. For any unitary spin-triplet state, its \mathbf{d} vector is always real up to an overall $U(1)$ phase [33] and

therefore it satisfies $|\mathbf{d} \cdot \mathbf{d}| = |d_x|^2 + |d_y|^2 + |d_z|^2$. There is yet another type of spin-triplet state whose pairing is restricted in one spin channel, and such states are fully spin polarized. For a state of this type, we can always choose a spin basis with $\psi_{\uparrow\uparrow} = 1$ and $\psi_{\uparrow\downarrow} = \psi_{\downarrow\uparrow} = \psi_{\downarrow\downarrow} = 0$. The corresponding anomalous density matrix reads

$$f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}\sigma_x + \frac{i}{2}\sigma_y, \quad (\text{A7})$$

and we have $(d_x, d_y, d_z) = (1/2, i/2, 0)$. The \mathbf{d} vector of this state satisfies the condition $\mathbf{d} \cdot \mathbf{d} = 0$.

APPENDIX B: $0-\pi$ TOPOLOGICAL JJ TRANSITION BEYOND THE STRONG ZEEMAN SPLITTING LIMIT

In this section, we consider the transformation of Cooper pairs under the mirror operation \mathcal{M}_z . Cooper pairs can be described by a block off-diagonal density matrix,

$$\begin{aligned} D^{\text{off}}(\mathbf{r}) &= \begin{pmatrix} 0 & f(\mathbf{r}) \\ \bar{f}(\mathbf{r}) & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & [d_0(\mathbf{r})\sigma_0 + d_i(\mathbf{r})\sigma_i]i\sigma_y \\ -i\sigma_y[d_0(\mathbf{r})^*\sigma_0 + d_i^*(\mathbf{r})\sigma_i] & 0 \end{pmatrix}, \end{aligned} \quad (\text{B1})$$

with $\mathbf{r} = (x, y, z)$ and $d_0(\mathbf{r})$ and $d_{i=x,y,z}(\mathbf{r})$ the amplitudes of s -wave spin-singlet Cooper pairs and spin-triplet triplet ones with the \mathbf{d} vector along the x , y , and z directions, respectively. Here, we assume that the spin-triplet pairs are induced by MZMs and are thereby s wave [1,3]. The mirror operator \mathcal{M}_z in electron-hole and spin spaces is

$$\mathcal{M}_z = \begin{pmatrix} i\sigma_z & 0 \\ 0 & -i\sigma_z \end{pmatrix}.$$

For a mirror (\mathcal{M}_z) symmetric system we have

$$\begin{aligned} \mathcal{M}_z D^{\text{off}}(\mathbf{r}) \mathcal{M}_z^{-1} &= \begin{pmatrix} 0 & i\sigma_z f(x, y, z) i\sigma_z \\ i\sigma_z \bar{f}(x, y, z) i\sigma_z & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & f(x, y, -z) \\ \bar{f}(x, y, -z) & 0 \end{pmatrix}, \end{aligned} \quad (\text{B2})$$

with

$$\begin{aligned} i\sigma_z f(\mathbf{r}) i\sigma_z &= [d_0(\mathbf{r})\sigma_0 + d_z(\mathbf{r})\sigma_z - d_x(\mathbf{r})\sigma_x - d_y(\mathbf{r})\sigma_y] i\sigma_y \\ &= f(x, y, -z) = [d_0(x, y, -z)\sigma_0 + d_z(x, y, -z)\sigma_z \\ &\quad + d_x(x, y, -z)\sigma_x + d_y(x, y, -z)\sigma_y] i\sigma_y. \end{aligned}$$

As the magnetic flux does not distinguish spin-up or spin-down, all of the four Cooper pairs should have the same flux-induced charge phase. However according to Eq. (B2), only those spin-triplet pairs with the \mathbf{d} vector along the x or y direction are opposite in sign at z and $-z$. In other words, they have a π -phase difference. Besides, the s -wave spin-singlet bulk superconducting gap in the semiconductor nanowire models [Eqs. (1) and (6)] is uniform in the entire wire. We therefore conclude here that this π -phase difference for the d_x and d_y spin-triplet Cooper pairs is not a charge phase but a spin phase.

We have demonstrated topological 0 and π JJs in the strong Zeeman splitting limit of a 1D BDI class topological superconductor model in Sec. II. Actually the conclusion of topological 0 and π JJs for the setups in Figs. 1(a) and 1(b) is valid for any multichannel D class topological superconductor model as long as it has mirror symmetry with mirror plane perpendicular to the wire direction.

We first consider the 1D BDI class topological superconductor model beyond the strong Zeeman splitting limit. The tight-binding model described by Eq. (1) in the continuous limit takes the form

$$H = \left[\left(-\frac{\hbar^2}{2m} \partial_z^2 - \mu \right) \sigma_0 + M_z \sigma_z + \lambda i \partial_z \sigma_y \right] \tau_z - \Delta \sigma_y \tau_y. \quad (\text{B3})$$

For the realistic semiconductor nanowire, the chemical potential $\mu \approx 0$ and the SOC energy λk_f is much smaller than the superconducting gap Δ and Zeeman energy M_z [45,51]. In this case, the right MZM of Eq. (B3) is [2,52]

$$\gamma_1(z) = (\hat{c} \quad \hat{c}^\dagger) \begin{pmatrix} u_1(z) \\ u_1(z) \end{pmatrix}, \quad (\text{B4})$$

where

$$\begin{aligned} \hat{c} &= \begin{pmatrix} c_\uparrow \\ c_\downarrow \end{pmatrix}, \quad u_1(z) = u_1^*(z) = e^{wz} \begin{pmatrix} \Delta + w\lambda \\ M_z - \mu - w^2 \end{pmatrix} \\ &\quad + j e^{vz} \begin{pmatrix} \Delta + v\lambda \\ M_z - \mu - v^2 \end{pmatrix} + j^* e^{v^*z} \begin{pmatrix} \Delta + v^*\lambda \\ M_z - \mu - v^{*2} \end{pmatrix}, \end{aligned}$$

$v \approx ik_{f,\text{eff}} + \delta$, and $w \approx k_{f,\text{eff}} + \delta$ with $k_{f,\text{eff}} \approx \sqrt{2m}(M_z^2 - \Delta^2)^{1/4}$ and $\delta \approx \lambda \Delta m / \sqrt{V_z^2 - \Delta^2}$. To satisfy the boundary condition $\gamma_1(z=0) = 0$, we have $j = -1/2 - i/2$.

As e^{wz} decays much faster than e^{vz} or e^{v^*z} , by neglecting the term containing e^{wz} in Eq. (B4), the MZM wave function inside the TSC can be simplified as

$$\begin{aligned} u_1(z) &= \cos(k_{f,\text{eff}} z) e^{\delta z} \begin{pmatrix} \Delta \\ M_z + \sqrt{M_z^2 - \Delta^2} \end{pmatrix} \\ &= s(z) \begin{pmatrix} \cos(\frac{\eta}{2}) \\ \sin(\frac{\eta}{2}) \end{pmatrix}, \end{aligned} \quad (\text{B5})$$

where its spin direction only depends on Δ and M_z with $\tan \eta = \Delta / (M_z + \sqrt{M_z^2 - \Delta^2})$. Here, because SOC energy is much smaller than the Zeeman energy and superconducting gap, we also neglect the terms containing SOC strength λ . As the system is invariant under the mirror operation \mathcal{M}_z , the left MZM γ_2 takes the form

$$\begin{aligned} \gamma_2(z) &= (\hat{c} \quad \hat{c}^\dagger) \begin{pmatrix} iu_2(z) \\ -iu_2(z) \end{pmatrix}, \\ u_2(z) &= i\sigma_z u_1(-z) = s(-z) \begin{pmatrix} \cos(\frac{\eta}{2}) \\ -\sin(\frac{\eta}{2}) \end{pmatrix}. \end{aligned}$$

For the topological JJ in Fig. 1(a), the Hamiltonian in Eq. (14) can be rewritten in the spin quantization basis of

γ_1^a and γ_2^b as

$$\begin{aligned}
\tilde{H}(t) &= \tilde{t} \begin{pmatrix} \cos(\frac{\eta}{2})c_{a,\uparrow}^\dagger \\ \sin(\frac{\eta}{2})c_{a,\downarrow}^\dagger \\ \cos(\frac{\eta}{2})c_{a,\uparrow} \\ \sin(\frac{\eta}{2})c_{a,\downarrow} \end{pmatrix}^T \begin{pmatrix} e^{\frac{i}{2}(\phi\sigma_0)} & 0 \\ 0 & -e^{-\frac{i}{2}(\phi\sigma_0)} \end{pmatrix} \begin{pmatrix} i \cos(\frac{\eta}{2})c_{b,\uparrow} \\ -i \sin(\frac{\eta}{2})c_{b,\downarrow} \\ i \cos(\frac{\eta}{2})c_{b,\uparrow} \\ -i \sin(\frac{\eta}{2})c_{b,\downarrow} \end{pmatrix} \\
&= \tilde{t} \begin{pmatrix} \cos(\frac{\eta}{2})c_{a,\uparrow}^\dagger \\ \sin(\frac{\eta}{2})c_{a,\downarrow}^\dagger \\ \cos(\frac{\eta}{2})c_{a,\uparrow}^\dagger \\ \sin(\frac{\eta}{2})c_{a,\downarrow}^\dagger \end{pmatrix}^T \hat{R}_1^\dagger \hat{R}_1 \begin{pmatrix} e^{\frac{i}{2}(\phi\sigma_0)} & 0 \\ 0 & -e^{-\frac{i}{2}(\phi\sigma_0)} \end{pmatrix} \hat{R}_2^\dagger \hat{R}_2 \begin{pmatrix} i \cos(\frac{\eta}{2})c_{b,\uparrow} \\ -i \sin(\frac{\eta}{2})c_{b,\downarrow} \\ -i \cos(\frac{\eta}{2})c_{b,\uparrow} \\ i \sin(\frac{\eta}{2})c_{b,\downarrow} \end{pmatrix} \\
&= \tilde{t} \begin{pmatrix} c_{a,\nearrow}^\dagger \\ 0 \\ c_{a,\nearrow} \\ 0 \end{pmatrix}^T \begin{pmatrix} e^{\frac{i}{2}(\phi\sigma_0+2\eta\sigma_y)} & 0 \\ 0 & -e^{-\frac{i}{2}(\phi\sigma_0-2\eta\sigma_y)} \end{pmatrix} \begin{pmatrix} ic_{b,\nwarrow} \\ 0 \\ -ic_{b,\nwarrow}^\dagger \\ 0 \end{pmatrix} \\
&= \tilde{t} \begin{pmatrix} c_{\nearrow}^\dagger \\ 0 \\ c_{\nearrow} \\ 0 \end{pmatrix}^T U^\dagger U \begin{pmatrix} e^{\frac{i}{2}(\phi\sigma_0+2\eta\sigma_y)} & 0 \\ 0 & -e^{-\frac{i}{2}(\phi\sigma_0-2\eta\sigma_y)} \end{pmatrix} U^\dagger U \begin{pmatrix} ic_{b,\nwarrow} \\ 0 \\ -ic_{b,\nwarrow}^\dagger \\ 0 \end{pmatrix} \\
&= \tilde{t} \begin{pmatrix} \gamma_{1,\nearrow}^a \\ 0 \\ 0 \\ 0 \end{pmatrix}^T \begin{pmatrix} e^{i\eta\sigma_y} i \sin \frac{\phi_0}{2} & i e^{i\eta\sigma_y} \cos \frac{\phi_0}{2} \\ -i e^{i\eta\sigma_y} \cos \frac{\phi_0}{2} & i e^{i\eta\sigma_y} \sin \frac{\phi_0}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \gamma_{2,\nwarrow}^b \\ 0 \end{pmatrix} = i \tilde{t} \cos \eta \cos \frac{\phi_0}{2} \gamma_{1,\nearrow}^a \gamma_{2,\nwarrow}^b, \tag{B6}
\end{aligned}$$

where

$$\hat{R}_1 = \exp\left(\frac{i}{2}\eta\sigma_y\right)\tau_0, \quad \hat{R}_2 = \exp\left(-\frac{i}{2}\eta\sigma_y\right)\tau_0$$

rotate the spin quantization axis of γ_1 and γ_2 to the z axis, respectively, and

$$\begin{aligned}
\gamma_{1,\nearrow} &= [\cos(\eta/2)c_{\uparrow} + \sin(\eta/2)c_{\downarrow}] + [\cos(\eta/2)c_{\uparrow}^\dagger \\
&\quad + \sin(\eta/2)c_{\downarrow}^\dagger], \\
\gamma_{2,\nwarrow} &= -i[\cos(\eta/2)c_{\uparrow} - \sin(\eta/2)c_{\downarrow}] + i[\cos(\eta/2)c_{\uparrow} \\
&\quad - \sin(\eta/2)c_{\downarrow}].
\end{aligned}$$

For the multichannel semiconductor wire, there is an additional SOC hopping term $\lambda \hat{p}_y \sigma_z$ which breaks the complex symmetry but respects the mirror symmetry with the mirror plane perpendicular to the wire direction. Thus the spin wave function of MZM γ_1 and γ_2 can be generally written as $(\cos(\eta/2), \sin(\eta)e^{-i\zeta})^T$ and $(i \cos(\eta/2), -i \sin(\eta)e^{-i\zeta})^T$,

respectively, with ζ the azimuthal angle of γ_1 spin polarization. Correspondingly, the rotation operators \hat{R}_1 and \hat{R}_2 are generalized to $\hat{R}_{1(2)} = \exp(\pm i \mathbf{n} \cdot \boldsymbol{\sigma})$ with $\mathbf{n} = -\sin(\zeta)\mathbf{e}_x + \cos(\zeta)\mathbf{e}_y$. By inserting the new spin wave functions and rotation operators into Eq. (B6), we find that the coupling Hamiltonian form is not affected by the azimuthal angle ζ and is thus valid for multichannel D class TSCs which are invariant under the mirror operation \mathcal{M}_z .

It is noted that effective coupling Hamiltonian [Eq. (B6)] is always a topological Josephson 0 junction and $\cos \eta$ is the inner product of the spin wave functions of $\gamma_{1,\nearrow}^a$ and $\gamma_{2,\nwarrow}^b$. When $\eta = 0$, Eq. (B6) returns to the MZM coupling Hamiltonian for the strong Zeeman splitting limit. For the topological JJ in Fig. 1(b), the two MZMs γ_1^a and γ_1^b have the same spin polarization so that their coupling Hamiltonian is always

$$H_t = i \sin \frac{\phi_0}{2} \gamma_{1,\nearrow}^a \gamma_{1,\nearrow}^b,$$

which corresponds to a topological Josephson π junction.

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