Formation of new phase inclusions in the system of quasiequilibrium magnons of high density

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The paper studies the spatial variation of the magnetization in a nonconducting magnetic sample with an excess number of magnons in comparison to the equilibrium. The phenomenon is considered using the Landau-Lifshits equation with additional terms describing the longitudinal relaxation of the magnetization, the magnon diffusion, and the magnon creation by external pumping. The free energy of the system is presented in the mean-field approximation. It is shown that, if the pumping exceeds some critical value, regions of a new phase arise where the magnetic moments are oriented opposite to the magnetization of the magnetic sample. The phenomenon is similar to the appearance of droplets of condensed phase in a supersaturated vapor. The appearance of a new phase either in the form of a single domain or a periodical lattice is demonstrated. The studied process is a competitor to the process of the Bose-Einstein condensation of magnons.

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I. INTRODUCTION

Spatial structures in nonequilibrium nonlinear systems have been studied widely and successfully during the last half century [1]. The processes of formation of such structures are referred to as nonequilibrium phase transitions. There are various nonequilibrium phase transitions and in their majority differ from the equilibrium phase transitions. There is a class of specific nonequilibrium phase transitions in systems of particles (or quasiparticles in crystals) that have finite values of lifetime. If combining particles produces an energy gain and the lifetime of particles is much larger than the time of interparticle collisions, the particles may form a condensed phase. The formation of drops of electron-hole liquid in germanium or silicon with high density of excitons created by light is a classical example of phase transitions in systems of unstable particles [2,3]. The stationary state of the condensed phase of unstable particles may exist only in the presence of a source which creates new particles instead of the disappearing ones. When the value of the lifetime is large, the parameters of the condensed phase (density, critical temperature of the phase transition, etc.) are only slightly modified compared to the same parameters for the infinite value of the lifetime. However, for the range of parameters where the gas and the condensed phase coexist, the spatial distribution of finite lifetime particles has particular features which will be discussed later.

Magnons in magnetic materials are a classical example of particles with a finite lifetime. The chemical potential of magnons is equal to zero in the equilibrium state because their number is not conserved due to the magnon-magnon and magnon-phonon interactions. In the presence of an external pumping, additional magnons appear besides the equilibrium magnons. As a result, the chemical potential is not equal to zero. Magnons are Bose particles and when their concentration exceeds some threshold value, an appearance of Bose-Einstein condensation (BEC) could be expected together with its interesting manifestations: accumulation of particles at a certain level, superfluidity, and so on. An interesting effect was observed in Refs. [4,5] in which the authors investigated magnons in the yttrium-iron-garnet (YIG) films. The magnons were excited by the parametric longitudinal pumping. The analysis of the magnon spectra was carried out using Brillouin light scattering (BLS) spectroscopy. The authors showed from the analysis of experiments the manifestation of the Bose-Einstein condensation of magnons. In particular, an increase of the magnon concentration in the state with the wave vector that corresponded to the minimum of the magnon band was observed [4,5]. The appearance of the spontaneous coherence in BLS by magnons was observed if the pumping exceeded a critical value [6]. The emergence of a periodical variation of the magnon density at a high level of magnon excitations was also demonstrated [7]. The latter phenomenon was explained by the presence of two minima in the magnon dispersion law leading to the formation of two condensates and to the interaction between the condensates.

Since then the investigation of the magnon condensation in YIG obtained further development in numerous works which further advanced the explanation of the phenomenon observed in Refs. [4–7] and suggested different other effects related to Bose-Einstein condensation. Thus the stability of BEC in the high-density magnon system in YIG was analyzed in Ref. [8]. The microwave emission from the uniform mode generated by BEC was studied in Ref. [9]. The spatial structure of interacting bosons with two minima in the dispersion law was investigated in Refs. [10–12]. The dramatic peak in the density of the proposed condensed magnons after switching off the pumping was observed in Ref. [13]. It is interesting that the time it took for the peak to increase coincided with the time of the magnon decay and the time it took for the peak to decrease was much larger than the magnon decay time. The problem of the spin current in the system of an isolator and a conductor was theoretically studied under the condition of BEC of magnons in the isolator in Ref. [14]. In Refs. [15,16], the Josephson oscillations in the magnon density between two spatially separated magnon clouds were calculated and also the methods of the magnon current measurement were analyzed. In Ref. [17], a temporal decrease of the magnon condensate density in YIG in a gradient of temperature, created by a laser after the pumping shutdown, was observed. The authors explained this effect by the appearance of a supercurrent in the condition of BEC. However, there are

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works in which doubt is expressed about the correctness of the interpretation of the results observed in Refs. [4,5]. The authors of Ref. [18] showed that the reason for the accumulation of particles created by pumping at the lowest state may be caused by the peculiarities of the Bose-Einstein condensation of quasiparticles. In Ref. [19], the authors described the time evolution of the magnon condensate under pumping by the classical stochastic Landau-Lifshits-Gilbert equation including magnon-phonon hybridization and came to the conclusion that the phenomenon observed in Refs. [4,5] has a purely classical nature.

In the current paper, we present another version of the processes in a ferromagnet with the magnon density exceeding the equilibrium value. We show that, if there are additional magnons, created by an external pumping, the evolution of the system may choose a scenario alternative to the Bose-Einstein condensation. This new scenario is the formation of regions in the ferromagnetic material where the magnetic moments are oriented opposite to the orientation of the magnetic moment of the sample. Similar to the Bose-Einstein condensation, this phenomenon appears in crystals with a magnon density higher than the equilibrium value.

The system of magnons is in a way equivalent to a gas of particles. If the concentration of the particles exceeds the equilibrium value, the gas is referred to as "oversaturated" or "supersaturated." Processes of precipitation of regions of a new phase are known to occur in oversaturated gas. Similar systems arise also in mixtures of liquids or solids after rapid cooling. There are two popular models that describe processes of the unmixing of mixtures from one thermodynamic phase to form two coexisting phases: the model of the spinodal decomposition [20,21] and the model of the nucleation and growth [22]. The subject of our interest, a system with magnon pumping, is oversaturated with magnons. So, during the relaxation, individual magnons would cluster forming inclusions of the new phase. Within these inclusions, the orientation of the magnetic moments would be opposite to the magnetic moment of the crystal. A qualitative picture for the dynamics of the magnon system is shown in Fig. 1.

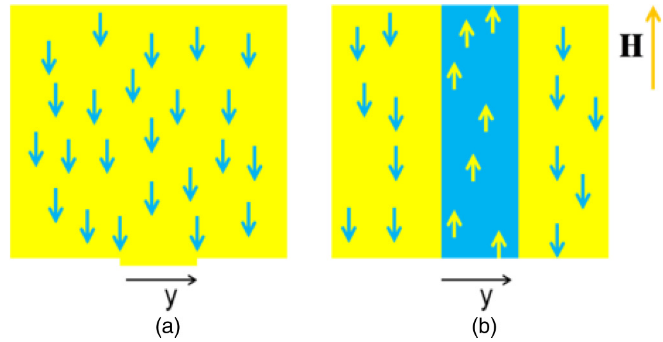

FIG. 1. The distribution of magnon magnetic moments in a crystal. The continuous regions present the areas with the majority of lattice cell moments oriented along the field (light) and opposite to the field (dark). The individual lattice cell moments are not shown because there are too many of them. The arrows show the directions of the moments of magnons that are opposite to the direction of the majority of cell moments. The left part (a) presents the uniform distribution of magnons at which the Bose-Einstein condensation may occur if the magnon density is higher than some threshold value. The right part (b) shows an alternative distribution of the magnetization, in which the state with a domain (the dark strip) having the moments of the lattice cells oriented opposite to the magnetic field arises. The light arrows in the dark region show the magnetic moments of the magnons in the domain.

Let us assume that the magnon state of Fig. 1(a) presents a uniform quasiequilibrium magnon distribution, which arises due to both the thermal excitation and the external pumping. There are two scenarios for the further development of the uniform magnon distribution. According to the first scenario, the Bose-Einstein condensation shall occur if the magnon concentration exceeds the critical value. Such process is investigated in Refs. [4,5]. However, the second scenario, according to which the formation of the new phase occurs in the oversaturated magnon system, is also realistic. There is a strong short-range interaction between magnons. When magnons are collected in a cluster, the energy per magnon decreases by a value of order of 0.1 eV [23], which significantly exceeds the thermal energy at the room temperature. The equilibrium state of the system is determined by the minimum of the free energy and not by the minimum of the energy. Also, if the magnon concentration exceeds the equilibrium one, magnon clusters have to form with an opposite orientation of their magnetic moments to the magnetic moment of the other part of the crystal. The magnon clusters are inclusions of regions of the magnon condensed phase. However, it is not a Bose-Einstein condensation, it is a conventional condensation in the coordinate space due to the interaction between particles. The clusters of the new condensed phase may be shaped variously. A structure in the form of a single domain is drawn in Fig. 1(b). The present paper investigates the processes of formation of the regions of a new phase in the supersaturated magnon gas.

Because the magnon lifetime is finite, the arising structures may exist only during the continuous magnon pumping. Similar studies of the phase transitions in systems of particles with finite lifetimes have been carried out for different types of quasiparticles: for the radiation defects [24–26] and for the excitons [27–32]. The theories of these works have modified and generalized the stochastic model of the nucleation and growth (Lifshiz-Slyosov [22]) and the model of the spinodal decomposition (Cahn-Hillert [20,21]) to make them applicable to systems of particles with a finite lifetime. The theories have been successful in explaining the unconventional experimental results obtained by different authors (mainly by the Timofeev's [33] and Butov's [34] groups) during investigations of the light emission by excitons from double quantum well heterostructures in semiconductors at low temperature. Excitons were created by lasers and, at high density, formed islands of the excitonic condensed phase. Sometimes the islands were localized periodically in space. (For more information on the application of the theory of phase transitions in systems of unstable particles for the explanation of experiments with excitons see Refs. [31,32,35] and references therein.) The formation of a new phase in a system of unstable particles has distinct features compared to the phase transition in a system of stable particles. The distinctions include the following: (1) the size of the regions of the new phase is restricted, (2) there is a correlation between the regions of the new phase, which may cause the appearance of periodical structures, and (3) the regions of the new phase exist only with external pumping. The structures are the results of self-organization processes in nonequilibrium systems.

In the present paper, we apply the approach developed in the above papers (see Ref. [35] and references therein), devoted to the study of the phase transitions in systems of unstable particles, to a many-magnon system. We shall show the possibility of an appearance of a new phase inclusion in a magnetic supersaturated magnon gas. The qualitative analysis, given in the last section, argues that effects caused by the new inclusions may be similar to the effects that were observed in Refs. [4–7] and explained by the manifestation of BEC.

II. TAKING INTO ACCOUNT MAGNON DIFFUSION AND MAGNON PUMPING INTO EQUATION FOR MAGNETIZATION

We shall consider a nonconducting magnetic crystal in a magnetic field oriented along the crystalline axis $0Z$. We assume that the nonequilibrium magnons are excited in the system by the two-magnon longitudinal pumping and the number of magnons is larger than the equilibrium one. Due to the strong magnon-magnon and magnon-phonon interactions, the magnons are in a quasiequilibrium state. Our aim is to determine the spatial variation of the magnetization $\mathbf{M}$. To this end, we shall study the clustering of magnons into a new phase with the creation of regions that have the magnetic moment oriented opposite to the orientation of the main magnetization. After the creation of regions of the new inverse phase, the magnetization is nonuniform, though the initial system and external fields (the static magnetic field, the pumping) are assumed to be uniform. The processes of the self-organization in the system spontaneously break the symmetry. In describing the nonuniform system we assume that the principle of local equilibrium holds. In this case, in a small vicinity of some spatial point, the thermodynamic functions are the same functions of the local microscopic variables (magnon density, temperature) as in the equilibrium system. This assumption allows the introduction of the free energy in a nonequilibrium system. The local free energy depends on the magnon density and the magnon density depends on the spatial coordinates. The principle of the local equilibrium is conventionally used in the majority cases when considering the self-organization problems in nonequilibrium systems [1].

We shall use the phenomenological approach for the solution of the problem. Let us analyze the phenomenological equation for the magnetization $\mathbf{M}$, the solution of which will be investigated in this paper. The equation for the change of the magnetization in the unit time contains the dynamic and the relaxation parts. The relaxation terms of the Landau-Lifshits (LL) and the Landau-Lifshits-Gilbert (LLG) equations cannot be used in our paper because they require the conservation of the magnetization. The pumping creates magnons and decreases the absolute value of the magnetization. The LL and LLG equations do not describe the equilibration of the magnetization after the pumping is switched off. Taking into account the processes of the establishment of the equilibrium state is important in a description of the nonequilibrium system. The equation for the evolution of the magnetization, which does not require the conservation of the absolute value of the magnetization, is given in the monograph of Akhiezer, Baryakhtar, and Peletminskii [36]. But neither the equation in Ref. [36] and nor the LL and LLG equations take into account the diffusion of magnons. The diffusion processes induce a spatial redistribution of magnons due to the interaction between them, which may be a reason for the formation of the new phase. The diffusion is important in the formation of the new phase at spinodal decomposition processes [20–22]. So, we presented the main equation for the magnetization in the form given in Ref. [36] and added to its right-hand side the terms describing the magnon diffusion $(\partial\mathbf{M}/\partial t)_D$ and the pumping $\mathbf{P}$:

$$\left(\frac{\partial \mathbf{M}}{\partial t}\right) = -\gamma[\mathbf{M},\mathbf{H}_{\text{eff}}] + \mathbf{R} + \mathbf{P}, \tag{1}$$

where γ is the gyromagnetic ratio, $\mathbf{H}_{\text{eff}}$ is the effective magnetic field determined as the variational derivative of the free energy with respect to the magnetization

$$\mathbf{H}_{\text{eff}} = -\frac{\delta F}{\delta \mathbf{M}}, \tag{2}$$

$\mathbf{R}$ is the relaxation term

$$\mathbf{R} = -\gamma_{R1}[\mathbf{n}_{\mathbf{M}},[\mathbf{n}_{\mathbf{M}},\mathbf{H}_{\text{eff}}]] + \gamma_R \mathbf{H}_{\text{eff}} + \left(\frac{\partial \mathbf{M}}{\partial t}\right)_D, \tag{3}$$

γ_R and γ_{R1} are the relaxation rates, and $\mathbf{n}_M = \mathbf{M}/M$.

The two first terms in the right-hand side of Eq. (3) determine the relaxation in Ref. [36]. The first term in Eq. (3) is equal to the relaxation term of the LL equation. For such a relaxation, the conservation of the absolute value of the magnetization holds. The LLG equation for the magnetization is equivalent to the LL equation: the LLG equation may be obtained from the LL equation by redefining the parameters [37]. Therefore the first term cannot describe the magnetization in the condition of external magnon pumping. The second and the third terms in Eq. (3) are important for the description of the development of the nonuniform variation of the magnetization under pumping. The third term in Eq. (3), responsible for the magnon diffusion, is absent from the LL, LLG, and Akhiezer, Baryakhtar, and Peletminskii [36] equations.

To describe the contribution of the exchange interaction to the free energy, we use the method of the self-consistent field and present the free energy in the form

$$F(\mathbf{M},\nabla\mathbf{M}) = \frac{aM^2}{2} + \frac{bM^4}{4} + \frac{K}{2}(\nabla\mathbf{M})^2 - MH\cos\theta - \frac{1}{2}\mathbf{M}\cdot\mathbf{H}^{(m)} + K_{\text{an}}\sin^2\theta, \tag{4}$$

where K_{an} is the anisotropy constant, θ is the angle between the magnetic field and the crystal axis, $\mathbf{H}^{(m)}$ is the magnetic field created by the magnetic moment $\mathbf{M}$, a, b, and K are parameters depending on the temperature and not depending on the spatial coordinates. The first three terms describe the exchange interaction. The free energy may be given in the form of Eq. (4) at a temperature close to the critical temperature of the phase transition. Such case will be studied in this paper.

As mentioned, the regions of the new phase, appearing due to the pumping, may have different shapes. We shall consider the formation of a magnetic domain with the orientation of the magnetic moments opposite to the orientation of the magnetic moment of the crystal and parallel to its easy axis. Domains of such type are simple and widespread defects in magnetics.

In the transition area, where one orientation of the magnetic moments change to another, the exchange energy increases. The transition may occur either by a rotation of the magnetic moment, which preserves its absolute value, or by changing the value of the moment. Usually, the first way dominates due to the high magnitude of the exchange interaction. However, in the vicinity to the phase transition, the exchange interaction decreases and the second way is plausible (see the textbook problem in Ref. [38]). Since we study the processes nearby the critical temperature, the transition between the domain and the matrix is considered by changing the value of the magnetic moment without its rotation.

In the framework of the chosen approach, when the normal to the domain plane is perpendicular to the magnetic field $\mathbf{H}$, the orientation of the magnetic moments has the form presented schematically in Fig. 1. The magnetic moment inside the domain has the single component M_z and depends on the single variable y. Therefore $\mathbf{M} \longrightarrow M_z(y)$ and $\mathbf{H}^{(m)} = 0$. For a such orientation of the magnetic moment, the angle θ in Eq. (4) is zero and both the first term in the right-hand side of Eq. (1) and the first term in the right-hand side of Eq. (3) disappear.

Let us apply the phenomenological approach to determine how the diffusion contributes to the time evolution of the magnetic moment. The density of the magnon current in the nonhomogeneous system at the uniform distribution of the temperature may be expressed by the gradient of the chemical potential μ,

$$\mathbf{j} = -K_M \nabla \mu, \tag{5}$$

where the coefficient K_M (mobility) depends on the temperature. K_M may be evaluated from the Boltzmann equation for the magnon distribution function taking into account the magnon scattering on magnons, phonons, and impurities.

Let us introduce the tensor of magnetization current, Π_{Mik}, that describes the density current of the ith component of the magnetic moment when the magnons are moving along the axis k. It is equal to the product of the ith component of the magnetic moment of a single magnon and the kth component of the magnon current density,

$$\Pi_{Mik} = m_i j_k. \tag{6}$$

A single magnon has the following magnetic moment:

$$\mathbf{m} = -g\mu_B \mathbf{n}_{\mathbf{M}}. \tag{7}$$

The vector $\mathbf{n}_{\mathbf{M}}$ is oriented along the axis z. It may assume two values, ± 1. The sign "+" takes place in the matrix where the moment is oriented along the magnetic field. The sign "-" is realized in the domain. The rate of the change of the magnetization is equal to

$$\left(\frac{\partial \mathbf{M}}{\partial t}\right)_{Di} = -\frac{\partial \Pi_{Mik}}{\partial x_k}. \tag{8}$$

The magnon current creates the magnetic moment current

$$\Pi_{Mik} = -g\mu_B n_{Mi} j_k. \tag{9}$$

In the considered case, the tensor of the magnetization current has a single nonzero component Π_{Mzy}. The contribution of the magnetic moment current to the rate of the magnetization change is equal to

$$\left(\frac{\partial M_z}{\partial t}\right)_{Dy} = -\frac{\partial \Pi_{Mzy}}{\partial y} = g\mu_B \frac{\partial}{\partial y}\left(n_{Mz} K_M \frac{\partial \mu}{\partial y}\right). \tag{10}$$

The chemical potential may be determined via the free energy by the formula $\mu = \delta F/\delta n$ at a constant temperature, where n is the magnon density. The magnon density may be related to the magnetization by the expression

$$\mathbf{M} = \mathbf{n}_{\mathbf{M}}(M_s - g\mu_B n), \tag{11}$$

where the saturation magnetization $M_s = N\mu_B$, and N is the number of the crystal cells in a unit volume. Equation (11) does not take into account Walker's modes. However, since we consider the high-temperature case, the main contribution to the decrease of the magnetization comes from the magnons with a high value of wave vectors. The authors of Ref. [39] showed that the notion of magnons as quasiparticles may hold almost up to the temperature of the phase transition.

Using the free energy of Eq. (2) and Eqs. (4) and (11), we obtain the chemical potential

$$\begin{aligned}\mu &= \frac{\delta F}{\delta \mathbf{M}} \frac{\partial \mathbf{M}}{\partial n} = -\mathbf{H}_{\text{eff}}(-g\mu_B \mathbf{n}_{\mathbf{M}}) \\ &= -g\mu_B(aM + bM^3 - n_{Mz}H - K\mathbf{n}_{\mathbf{M}}\triangle \mathbf{M}).\end{aligned} \tag{12}$$

As seen from Eq. (12), the chemical potential may be presented as the interaction energy of the effective magnetic field with the magnetic moment of a magnon $(-m_B g \mathbf{n}_{\mathbf{M}})$. Without the pumping ($\mathbf{P} = 0$), Eq. (1) has the solution $\mathbf{H}_{\text{eff}} = 0$ and, therefore, $\mu = 0$, as follows from Eq. (12).

Let us consider now the contribution of the pumping into Eq. (1). The rate of the magnon injection into the sample depends on the method of the injection and numerous other conditions: the frequency and amplitude of the external field, the state of the magnetization of the sample, the magnon spectrum and so on [23,40]. We shall not consider the processes of the magnon creation by the external microwave field. We assume only that the value of P, which determines the change of the magnetization in the unit time and the unit volume due to the pumping, does not depend on time or spatial position. The magnons created by the pumping come very quickly to the quasiequilibrium state due to the magnon-magnon and the magnon-phonon interactions and the dynamics of system is determined by the total number of magnons. It is the major assumption of the BEC studies as well. The pumping causes the decrease of the magnetization. So, the value of P should be negative with respect to the crystal magnetization and positive in the domain, where the magnetization changes sign, and should be equal to zero at the point where $M_z = 0$. We shall present the pumping in a simplified form. Its value will be approximated by the formula

$$P = -qM_z, \tag{13}$$

where q is a positive phenomenological parameter. Its value is deduced from the experimental decrease of the magnetization due to the pumping.

Using Eqs. (10), (12), and (13) in Eq. (1) and taking into account that the dynamic term in the considered case is zero, we obtain the equation for the magnetization

$$\frac{\partial M_z}{\partial t} = R_z + P, \tag{14}$$

where

$$Rz = \gamma_R H_{\text{eff}} - g\mu_B K_M \frac{\partial^2 H_{\text{eff}}}{\partial y^2}, \tag{15}$$

$$H_{\text{eff}} = H - aM_z - bM_z^3 + K\frac{\partial^2 M_z}{\partial y^2}. \tag{16}$$

Since the dynamic term in Eq. (14) for the magnetization is zero in the considered system, its right-hand side consists of the relaxation term given by Eq. (15) and the pumping. It is seen that dissipative term in Eq. (15) contains the component with the second derivative of the effective magnetic field. The form of the dissipative term R_z coincides with the form of the dissipative term R_{Bar} obtained for the magnetization in the Landau-Lifshits-Baryakhtar equation [41–43],

$$\left(\frac{\partial \mathbf{M}}{\partial t}\right) = -\gamma[\mathbf{M},\mathbf{H}_{\text{eff}}] + \mathbf{R}_{\mathbf{Bar}}, \tag{17}$$

where

$$\mathbf{R}_{\mathbf{Bar}} = \lambda_1 \mathbf{H}_{\text{eff}} - \lambda_2 \Delta \mathbf{H}_{\text{eff}}, \tag{18}$$

where λ_1 and λ_2 are tensor coefficients in the general case.

Comparing the relaxation term of Eq. (15) with the Baryakhtar's expression of Eq. (18), we obtain the values of the coefficients: $\lambda_1 = \gamma_R$, $\lambda_2 = g\mu_B K_M$ for our particular case of the magnon diffusion. Baryakhtar built the equation for the magnetization using the Onsager kinetic equations and the crystal symmetry. As shown in a number of works, the terms additional to the LL and the LLG equations are important for the explanation of many processes in ferromagnets: the dynamics of domain walls [41,44], the dynamics of solitons [43], the damping of the spin waves with high values of the wave vectors [42,44], the anisotropic damping in the feromagnetics [45], and the temporal spin evolution in the magnetic heterostructures disturbed by femtosecond laser pulses [46]. The general theory [41] does not specify the numerical value of the coefficients. Their values are determined for particular systems. For example, the inclusion of the conductivity electrons in the magnetization dynamics of conducting ferromagnets [44,47] gives additional terms that depend on the conductivity. The value of the coefficient we obtained is determined by the diffusion of magnons.

Let us introduce the dimensionless variables $\tilde{y} = y/l_0$, $l_0 = (K/(-a))^{1/2}$, $\tilde{M} = M/M_{00}$, $\tilde{H} = H/H_{00}$, $\tilde{t} = Kt/(a^2 g \mu_B K_M)$, $\tilde{q} = qK/(a^2 g\mu_B K_M)$, and $\tilde{\gamma_R} = \gamma_R K/((-a)g^2 \mu_B^2 K_M)$, where $M_{00} = ((-a)/b)^{1/2}$ and $H_{00} = (-a)(-a/b)^{1/2}$ are the value of the magnetization and the effective magnetic field, respectively, in the absence of the magnetic field and pumping.

Equation (14) for the magnetization in the dimensionless variables takes the form (the diacritic "∼" above the notations for M_z, H, q, t, and y will be omitted from now on)

$$\frac{\partial M_z}{\partial t} = \frac{\partial^2}{\partial y^2}\left(-M_z + M_z^3 - H - \frac{\partial^2 M_z}{\partial y^2}\right) - \tilde{\gamma}_R\left(-M_z + M_z^3 - H - \frac{\partial^2 M_z}{\partial y^2}\right) - qM_z. \tag{19}$$

Equation (19) determines the variation of the magnetization in the region of the phase transition in the presence of the magnon pumping. The first and the second terms in its right-hand side describe dissipative processes. The first term originates from the processes of the magnon diffusion. The second term describes the relaxation of the magnetization to the equilibrium value.

Let us consider expressions that may be used for the estimation of the numerical values of parameters. The mobility K_M may be related to the magnon diffusion coefficient by the Einstein's relation $K_M = Dn/\kappa T$, where T is the temperature. The diffusion coefficient may be obtained from the solution of the Boltzmann equation for magnons. According to Eq. (11), $n = (M_s - M)/g\mu_B$. In the vicinity of the phase transition, where $M \ll M_s$, we have $n \sim M_s/g\mu_B$. In the mean-field approximation, the parameters of the free energy [48] are

$$a = -\Delta t\frac{\kappa T_c}{M_s \mu_B}, \quad b \sim \frac{1}{3}\frac{\kappa T_c}{M_s^3 \mu_B}, \quad K \sim \frac{\kappa T_c}{M_s \mu_B} d^2, \tag{20}$$

where T_c is the phase transition temperature, $\Delta t = (T_c - T)/T_c$, d is the period of the crystal lattice. In the approach of Eq. (20) in dimensionless variables, the length unit is equal to $l_o = d/(\Delta t)^{1/2}$, the magnetization unit is $M_{00} = (\Delta t)^{1/2}\sqrt{3}M_s$, and the magnetic field unit is equal to $H_{00} = M_s(\Delta t)^{3/2}\sqrt{3}(\kappa T_c)/(\mu_B M_s)$. Let us do the estimations of the values of the parameters in the dimensionless units. We consider the parameters of the yttrium-iron-garnet crystal, in which $T_c = 560$ K and $M_s = 140$ Gs. We assume that $D = 100$ cm^2/s. Since the dimensionless magnetization is the ratio of the magnetization to the value of the magnetization of the sample without the magnetic field and pumping, its magnitude is of order of unity or smaller ($\tilde{M} \leqslant 1$). The magnetic fields, which vary in the experiments [4–6] from 0 and to 1000 Oe, are described in the dimensionless units by the values that are less than the unity ($\tilde{H} \ll 1$). For example, a magnetic field of 1000 Oe in dimensionless units is equal to 0.00077 and 0.0022 at temperatures $\Delta t = 0.2$ and $\Delta t = 0.1$, correspondingly. We shall choose the parameter of the pumping q in Eq. (13) in a way that ensures a small decrease of the magnetization due to the pumping (less than several percent of the magnetization value). The dimensionless coefficient $\tilde{\gamma}_R$ is found to be very small. For $\Delta t = 0.1$, varying the parameter γ_R from 10^{-6} to 10^{-3} s^{-1} leads to the decrease of the dimensionless coefficient $\tilde{\gamma_R}$ from 10^{-6} to 10^{-9}. Therefore, in dimensionless units, the new magnon diffusion related term in the LL equation has a coefficient which is large compared to the typically used relaxation term. However, because this term contains derivatives of a higher (second) order than other terms, its effect becomes important only in the case of the strongly nonuniform states. For example, for Walker's modes in a sample with the size of order of 1 mm, the presence of the second derivative in the first term in the left part of Eq. (19)

decreases this term, presented, in dimensionless units, by 13 orders of magnitude and it becomes negligible.

Equation (19) describes the domains in the ZOX plane and a magnetization oriented along the z axis and depending on y ($M \equiv M_z(y)$). Let us consider another orientation of the domain plane. Using the free energy of Eq. (4), we may describe the domain in the $X0Y$ plane with a magnetization depending on z [$M \equiv M_z(z)$] with an equation that may be obtained from Eqs. (14)–(16) by the transformations $M_z(y) \longrightarrow M_z(z), H \longrightarrow H - 4\pi M_z(z)$. The new term $4\pi M_z(z)$ may be combined with the term aM_z from the free energy. As the condition $a \gg 2\pi$ holds, the equation for the magnetization for the alternative XOY orientation of the domain plane will assume the form of Eq. (18). Naturally, the resulting solutions will be similar. Therefore we shall study only Eq. (18) [Eq. (19) in dimensionless units].

III. INVESTIGATION OF STABILITY OF UNIFORM MAGNETIZATION

At the uniform steady pumping, Eq. (19) has a uniform steady state solution which determines the stationary magnetization M_0 that satisfies the following equation:

$$M_0^3 - M_0(1 + q/\tilde{\gamma}_R) - H = 0. \tag{21}$$

The solution of this equation in the absence of the pumping ($q = 0$) determines the equilibrium magnetization M_{eq}.

In order to investigate the stability of the uniform solution, let us put $M = M_0 + \delta M \exp(\lambda(k)t + iky)$, where $\delta M \ll M_0$. The decrement of the damping obtained from Eq. (19) is equal to

$$\lambda(k) = \left(1 - 3M_0^2 - k^2\right)(k^2 + \tilde{\gamma}_R) - q. \tag{22}$$

Equation (22) implies that the decrement $\lambda(k)$ is negative in the absence of the pumping for every value of k, meaning that the uniform state is stable. The dependence of the damping decrement on k is presented in Fig. 2 for the different values of the pumping q. As seen from Fig. 2, the damping decrement $\lambda(k)$ becomes positive with increasing pumping at a certain wave number $k = k_c$. At $q < 3.3424 \times 10^{-8}$, the uniform state is stable. At the increased pumping $q > 3.34261 \times 10^{-8}$, the uniform solution becomes unstable with respect to the creation of the periodical magnon density variation with the wave number k_c. The instability occurs in the region of the spinodal decomposition, where the second derivative of the free energy with respect to the order parameter (M_z in the considered case) changes sign ($\partial^2 F/\partial M_z^2 = 0$). But the numerical value of the drop of the magnetization M_z caused by the pumping needed to reach the instability is very large. For the example considered in Fig. 2, the magnetization M_z changes from 1.0005 (in the dimensionless units) at $q = 0$ to 0.577244 at $q = q_s$. Such a decrease of the magnetization requires an extremely strong pumping, which is likely to change the temperature state of the crystal. We shall not consider the region in the vicinity to the spinodal decomposition and such strong pumping.

So, at $\lambda(k) < q_s$, the uniform state is stable even in the presence of the pumping and for the finite lifetime of the nonequilibrium state. However, nonuniform stable states may coexist with the uniform state even at $\lambda(k) < q_s$ in a certain range of the pumping intensity. These states arise when the parameters of the system belong to the region between the binodal and the spinodal. For a magnetic system, the magnetization in this region is restricted by the conditions imposed on $\partial F/\partial M = 0$ and $\partial^2 F/\partial M^2 = 0$ (this region is slightly affected by the pumping and the finite value of the particle's lifetime). The condition $\partial F/\partial M = -H_{\text{eff}} = 0$ is realized in the equilibrium state and determines the equilibrium magnetization. The region between $\partial F/\partial M = 0$ and $\partial^2 F/\partial M^2 = 0$ arises at a magnetization smaller than the equilibrium value, which can be caused by the pumping, which creates the magnons and decreases the magnetization. The total number of magnons in this region exceeds the equilibrium value. The system is supersaturated with magnons. The gas of the particles in the state between the binodal and the spinodal may remain in the uniform supersaturated state or could transfer due to fluctuations to a state with nuclei of the new condensed phase. Subsequently, these nuclei grow with time. In the case of stable particles, the growth of the condensed phase nuclei slows down with time because their number in the matrix is limited. To the contrary, in a system of constantly created particles with a finite lifetime, the spatially localized regions of the new phase may be stabilized by the interplay of the steady generation and decay. Stationary localized islands of the condensed phase of indirect excitons created by light in double quantum well heterostructures were studied in Refs. [49,50] in parameter ranges at which the uniform and the nonuniform solutions are stable simultaneously. The localized solutions are called either the autosolitons (static solitons) according to the classification by Ref. [51] or the breathers according to Ref. [52]. In the next section, we shall use Eq. (19) to study the nonuniform nonequilibrium stationary states in a magnetic sample at steady pumping.

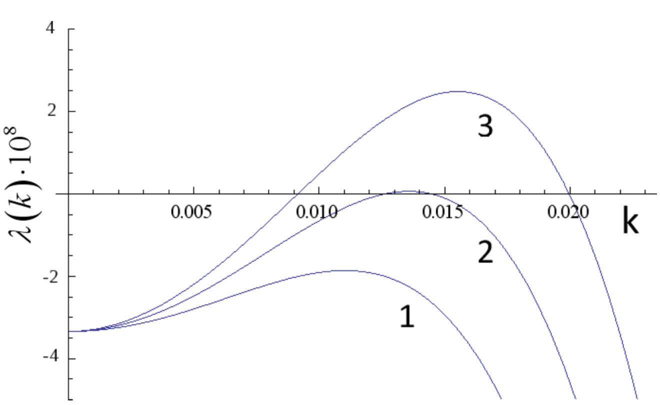

FIG. 2. Dependence of the damping decrement $\lambda(k)$ on the wave number of the uniform solution fluctuation k at the relaxation rate $\tilde{\gamma}_R = 5 \times 10^{-8}$, magnetic field $H = 0.001$ for different values of the pumping parameter q: (1) 3.3424×10^{-8}, (2) 3.34261×10^{-8}, and (3) 3.3428×10^{-8}. The values in this figure and in the subsequent figures are given in dimensionless units .

IV. FORMATION OF A SINGLE DOMAIN

We shall consider the steady state solutions of Eq. (19) for the magnetization. The nonuniform states studied in the paper are domains with the magnetic moment oriented opposite to the magnetization of the matrix. As we already mentioned, we shall consider the solutions in the region on the magnetization diagram where the uniform solution is stable, but the nonuniform solutions appear as well due to the magnon created by pumping. In the search of the nonhomogeneous solutions, we follow the procedure applied in Refs. [49,50]. Two approaches are used. In the first approach, we solve Eq. (19) for a certain value of the pumping choosing the initial magnetization in the form of the nonhomogeneous function $M_z,(y,t)_{t=0} = M_{\rm in}(y)$ depending on some parameters, and the value of q is given. In the second method, the external pumping is presented in the form of $q \to q + Q_i(y,t)$, where $Q_i(y,t)$ is a function containing some parameters and tends to zero at $t \to \infty$, the initial magnetization being uniform. Thus the solution of the evolution equation converges with time to the solution of Eq. (19) with a given value of q. By varying the functions $M_{\rm in}(y)$ and $Q_i(y,t)$, different nonuniform stationary solutions for the magnetization may be obtained. These solutions give $M(y,t) \to M_{eq}$ at $q \to 0$. The solutions are stable because they do not change at $t \to \infty$. They do not change with the variation of the functions of $M_{\rm in}(y)$ and $Q_i(y,t)$ in some limited region of parameters of the functions. In other words, for every solution there is a region of parameters of the functions $M_{\rm in}(y)$ and $Q_i(y,t)$ in which the solution is the same. This region determines "the attraction basin" for the given solution.

In the absence of pumping ($q = 0$), the solution of Eq. (19) is uniform. Nonuniform solutions are possible at $q \neq 0$. Firstly, let us find one such nonuniform solution localized in the center of the sample. It may be obtained by choosing the initial magnetization in the form

$$M_{\rm in}(y) = M_{\rm in0} \exp[-(y - L/2)^2/s^2], \tag{23}$$

where L is the length of the sample, and $M_{\rm in0}$ and s are parameters.

If the function given by Eq. (23) with certain values of parameters belongs to the attraction basin of some solution of Eq. (19), it converges to this solution with time. We consider such a solution in the time limit $t \to \infty$ as one of the desired solutions. An example of the magnetization variation obtained in such a way is presented in Fig. 3. In the center of the sample, a region is seen where the orientation of the magnetic moment is opposite to the orientation of the magnetization of the remaining part of the sample. The change of the parameters $M_{\rm in0}$ and s of the trial function (23) within some limits gives the same solution of Eq. (19), presented in Fig. 3. This confirms that the applied method for the solution (19) is correct. Figure 3 is the manifestation of the scenario shown in Fig. 1. The inverted spins (magnons) created by the pumping are clustered into a domain. The applied method allows determination of the possible states in the system. The realization of a certain state depends on the boundary initial condition and on fluctuations. The calculations showed that the time of the establishment of the steady size of the inclusions is very long (much longer than magnon lifetime). This fact should be taken into account when one performs an investigation of the magnon distribution at a pulse excitation.

The stationary state of the domain may exist if there is an additional inflow of magnons from outside. There are minimal values of the domain thickness and the sample thickness L at which the domain could develop. The domain thickness grows if the value of L rises, since the region from which the domain harvests magnons increases. However, if L becomes greater than the diffusion length, the size of the single domain reaches the limit for the fixed value of the pumping rate. This occurs at $L > (1/\gamma_R)^{1/2}$. In this limit, the results would not depend on the boundary conditions. We studied the magnetization for two types of the boundary conditions: for the periodical conditions and for the fixed magnetization at the boundary.

The domain size grows with increasing pumping. Figure 4 presents the domain thickness as a function of the pumping rate. It is seen from Fig. 4 that there is a threshold of the pumping rate for the domain creation. The pumping in Fig. 4 leads to the decrease of the magnetization. The pumping is such that the relative change of the uniform magnetization is equal to 5×10^{-3} and 3×10^{-2} at $q = 10^{-8}$ and 6×10^{-8}, correspondingly. As seen from Fig. 4, the threshold q_c for pumping is 0.8×10^{-8} for the given value of the magnetic field. Only the uniform solution exists at $q < q_c$. The threshold grows with increasing magnetic field. In dimensional units, at $T_c/(T_c - T) = 10$, $q = 5 \times 10^{-8}$, $D = 100$ cm^2/s, the thickness of the domain reaches 1.2 mμ. The size of the domains increases with decreasing the parameter $\tilde{\gamma}_R$, i.e., with increasing the magnon lifetime.

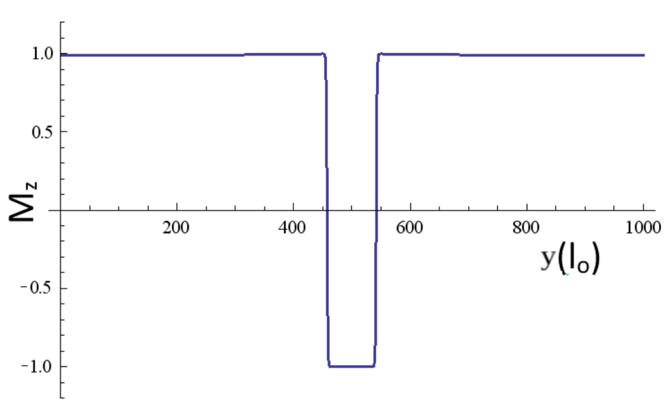

FIG. 3. Spatial dependence of the magnetization at the pumping parameter $q = 0.000003$, the magnetic field $H = 0.001$, and the relaxation rate $\tilde{\gamma}_R = 0.0001$.

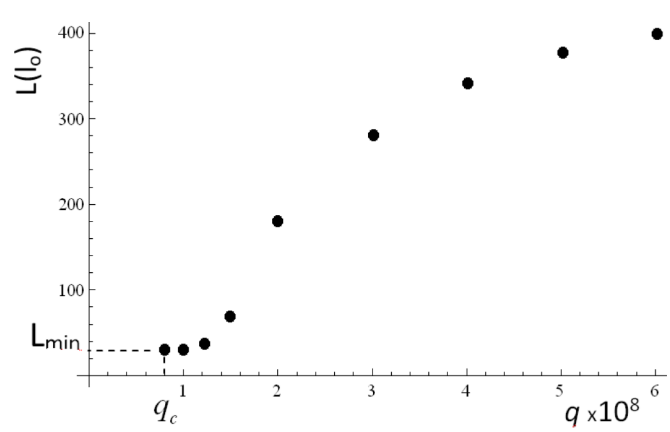

FIG. 4. Dependence of the thickness of the domain L on the parameter of the pumping q. q_c is the pumping threshold of the domain creation and $L_{\rm min}$ is the minimal thickness of the domain. The parameters of the system are the relaxation rate $\tilde{\gamma}_R = 10^{-6}$ and the magnetic field $H = 0.01$.

If the decrease of the magnetization is such that the state of the system is between the binodal and the spinodal, the nuclei of a new phase (for example, a domain) are created due to fluctuations. Only those nuclei survive that overcome a barrier. Therefore, in order to describe this process of the domain formation mathematically, we solved the main equation (23) with a given initial trial nonhomogeneous magnetization. If the width of the initial nonhomogeneous magnetization increases [the value of s in Eq. (23) decreases], a solution with two parallel domains arises. The two domains move apart. The repulsion may be explained in the following way. Since the magnons in domains relax, the domains exist due to the magnon inflow from outside. The region between the domains is common for both domains and, as long as the number of the magnons is restricted, their density is not sufficient to support a stationary state of two domains close to each other. As a result, the domains tend to be situated at a certain distance from each other.

V. SUPERLATTICE OF DOMAINS

Comparing Fig. 5, Fig. 6 and Fig. 7, one can see that with increasing the pumping rate the domains widen.

Solutions of Eq. (19) with a periodical variation of the magnetization are also possible. Some of them are presented in Figs. 5 and 6. In Figs. 5(b) and 6(b), the enlarged regions of one of the domains are given separately. Comparing Figs. 5–7, one can see that with increasing pumping rate the domains widen. The appearance of a periodical structure of the magnetization was observed in Ref. [7] in YIG films at external pumping.

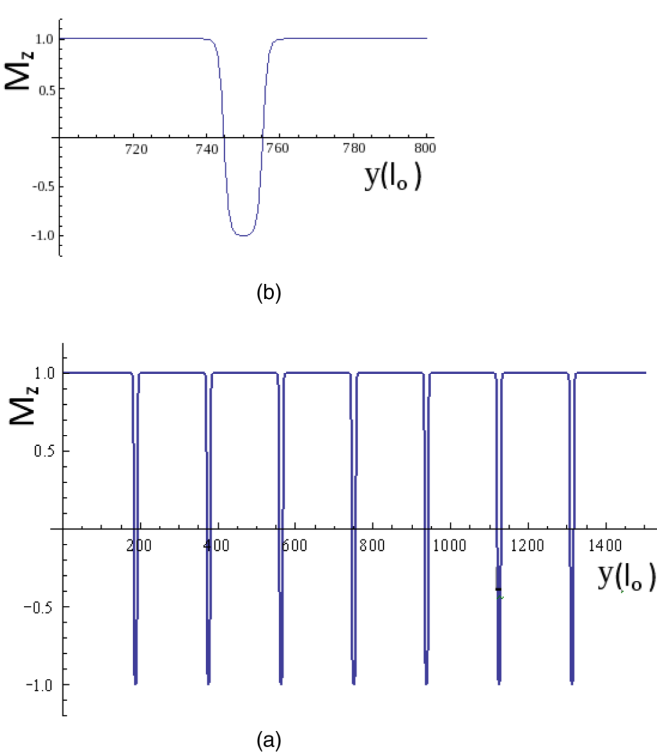

FIG. 5. Spatial dependence of the magnetization at the pumping parameter $q = 8 \cdot 10^{-9}$, the magnetic field $H = 0.01$, the relaxation rate $\tilde{\gamma}_R = 10^{-5}$ (a). A single domain is given separately in (b).

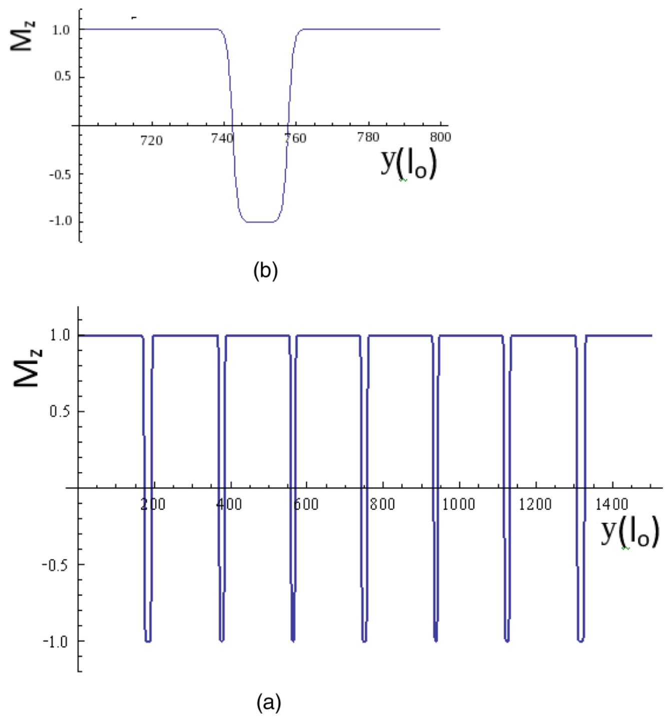

FIG. 6. Spatial dependence of the magnetization at the pumping parameter $q = 1.5 \cdot 10^{-8}$, the magnetic field $H = 0.01$, the relaxation rate $\tilde{\gamma}_R = 10^{-5}$.($a$). A single domain is given separately in (b).

Some comments about the effect of fluctuations are possible. The problem of the fluctuations of the density of unstable particles was investigated in the Refs. [28–30] by solving the Fokker-Planck functional equation for the free energy in the Landau-Ginzburg form. Both the intrinsic fluctuations and the fluctuations of the pumping were taken into account. The studies showed the appearance of a second maximum in the one-point distribution function if the pumping rate exceeded a certain value. This maximum corresponded to the development of the second phase. The separation of two phases may occur if the particle lifetime is greater than a certain threshold value. The two-point correlation function was obtained. The Fourier transform of the two-point correlation function calculated in Ref. [28] has a sharp maximum at some value of the wave

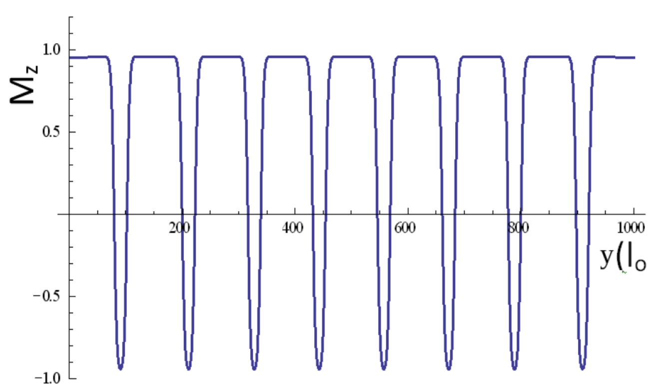

FIG. 7. Periodical distribution of the magnetization at pumping parameter $g = 0.018$, the magnetic field $H = 0.005$, the relaxation rate $\tilde{\gamma}_R = 0.2$.

vector that corresponds to the oscillations of the correlation function as a function of coordinates. So, periodical structures appear in the system. The problem that was solved in Ref. [28] differs from the problem in the current work in the presentation of the lifetime of particles: the lifetime in Ref. [28] is constant, it is a function of the spatial coordinate [the second term in the right-hand side of Eq. (19)]. However, the first term in the right-hand side of Eq. (19) plays the main role in the formation of a nonuniform structure. Using these results, we may argue that the fluctuations do not destroy the periodical structures that arise in the considered ferromagnetic system if the magnon lifetime is much greater than the times of magnon-magnon and magnon-phonon collisions and the quasiequilibrium state is formed in the system.

VI. DISCUSSION AND CONCLUSION

We have shown that, in a system with a high density of quasiequilibrium magnons, nonuniform structures may arise similar to the formation of inclusions of a condensed phase in a supersaturated gas or to the separation of a new phase in a crystal supersaturated with impurities. To perform a calculation, we used the Landau presentation of the free energy of magnons, which is valid in the vicinity to the phase transition temperature. The work explored the simplest model, namely, we considered the appearance of simple defects in the form of domains. However, in our opinion, some of the obtained results give insight into the general behavior in more complicated cases. Besides the considered simplest one-dimensional structures, the system may develop other types of nonuniformities having different shapes and being restricted in all directions (disks, balls, ellipsoids, and so on). What new features may be expected when more complex regions of the new phase are formed? Let us make some qualitative analysis of the possible manifestations of the development of nonuniform structures. The following list names the most important of them.

(1) The appearance of regions of nonuniform magnetization causes additional scattering of the electromagnetic waves. The intensity of the light scattering by clusters of magnons at some frequencies is greater than the one caused by the same number of individually independent particles.

(2) Since the nonuniform inclusions are small, the magnon levels in them will be quantized. The lowest states vary slowly in space and manifest themselves intensively in the scattering and absorption of the electromagnetic waves. The levels with small quantum numbers will display themselves most strongly. This may cause an increase in the lowest part of the scattering spectra similar to the effect observed at the Bose-Einstein condensation. It was shown that a stationary state becomes established if the time of the pumping pulse duration is much longer that the magnon lifetime. The size of the inclusions will grow with increasing duration of the pulse. This will lower the quantized levels of magnons in the inclusions and will shift the scattering spectra of electromagnetic waves to lower frequencies.

(3) The spin orientation in an inclusion of the condensed phase is determined by the strong exchange interaction. In order to change the orientation of its spin to the orientation of the matrix spins, a magnon should jump out of the inclusion, where it is bound by the strong exchange interaction. There is another mechanism of the spin relaxation in the inclusion. Firstly, magnons (light arrows in dark regions of Fig. 1) are exited in the domain and then leave the inclusion. This is a two-stage process and therefore it has small probability. As a result, the inclusions should live long.

(4) Let us compare the contributions to the thermal conductivity of independent magnons and the inclusions of the condensed phase. Since the drawing forces acting on the inclusions in the temperature gradient depend on the volume of the inclusion, and the friction forces depend on its surface, the gathering of the magnons into inclusions of the condensed phase enhances the resulting force. This would be observed experimentally as an increase of the thermal conductivity coefficient.

(5) The regions of the condensed phase in systems supersaturated with magnons may arrange themselves into structures. These structures would form a periodical distribution of domains with parallel planes. The normal to the domain plane may be oriented either along the external magnetic field or perpendicular to the field. Such picture was observed in Ref. [7].

All these effects, which may be expected due to the inclusions of the new phase, were observed in experiments [4–7], and explained as a manifestation of the Bose-Einstein condensation. We cannot object to that attribution because our study is carried out in somewhat different conditions. The majority of experiments [4–6] were carried out using magnon excitation by pulses. Our calculations studied the steady states. Yet we would like to underline that, besides the Bose-Einstein condensation, another scenario should be considered when studying the many-magnon systems. A search for the inclusions of the new phase and evaluation of their role in the physical processes should be performed.

Thus the presented paper studied a magnetic sample with a large magnon concentration created by the external pumping. Due to the long lifetimes, magnons are in the quasiequilibrium state. It is shown that besides the Bose condensation, formation of inclusions of a new condensed magnon phase may occur similarly to the development of the condensed phase in a supersaturated gas. The exchange interaction promotes the combination of separate magnons into inclusions. The orientation of the magnetic moments in each inclusion is opposite to the magnetic moment of the magnetic matrix. Therefore the inclusions may exist only in the conditions of external pumping. Summarizing, inclusions are dissipative structures that arise as a result of self-organization processes in nonequilibrium conditions [1]. The presented study considered the inclusions of the condensed phase shaped as individual domains or periodical supperlattices of domains. The possibility of the formation of inclusions of a new phase should be taken into account side by side with the process of Bose-Einstein condensation when analyzing experiments.

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