Fermi surface versus Fermi sea contributions to intrinsic anomalous and spin Hall effects of multiorbital metals in the presence of Coulomb interaction and spin-Coulomb drag

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Anomalous Hall effect (AHE) and spin Hall effect (SHE) are fundamental phenomena, and their potential for application is great. However, we understand the interaction effects unsatisfactorily, and should have clarified issues about the roles of the Fermi sea term and Fermi surface term of the conductivity of the intrinsic AHE or SHE of an interacting multiorbital metal and about the effects of spin-Coulomb drag on the intrinsic SHE. Here, we resolve the first issue and provide the first step about the second issue by developing a general formalism in the linear response theory with appropriate approximations and using analytic arguments. The most striking result is that even without impurities, the Fermi surface term, a non-Berry-curvature term, plays dominant roles at high or slightly low temperatures. In particular, this Fermi surface term causes the temperature dependence of the dc anomalous Hall or spin Hall conductivity due to the interaction-induced quasiparticle damping and the correction of the dc spin Hall conductivity due to the spin-Coulomb drag. Those results revise our understanding of the intrinsic AHE and SHE. We also find that the differences between the dc anomalous Hall and longitudinal conductivities arise from the difference in the dominant multiband excitations. This not only explains why the Fermi sea term such as the Berry-curvature term becomes important in clean and low-temperature case only for interband transports, but also provides the useful principles on treating the electron-electron interaction in an interacting multiorbital metal for general formalism of transport coefficients. Several correspondences between our results and experiments are finally discussed.

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I. INTRODUCTION

Anomalous Hall effect (AHE) and spin Hall effect (SHE) are fundamental phenomena and have great potential for application. The AHE [1–5] causes a charge current perpendicular to an external electric field even without an external magnetic field, and its spin-current version is the SHE [6–13]. Since the AHE and SHE are similar to usual Hall effect [14], an understanding of their properties develops our fundamental understanding of transport phenomena. Then, since we can control the magnitude and direction of the charge current of the AHE and spin current of the SHE in principle, the AHE and SHE may be utilized as useful devices [15,16].

For the fundamental understanding and efficient utilization of the AHE or SHE, we need to understand how its response depends on the detail of the electronic structure. Since the response may be affected by the differences in the band structure, the structure of doped impurities, and the strength of the electron-electron interaction, an understanding of their dependence of the response is helpful to understand the fundamental properties and find a good material for application.

The previous studies partially revealed the dependence of the response of the AHE or SHE on the detail of the electronic structure, and showed the potential of the intrinsic mechanism for a large response. First, the mechanisms of the AHE or SHE are categorized as either an intrinsic mechanism to the band structure [2,8,9,17–21] or an extrinsic mechanism due to the scattering of doped impurities [3,6,7,22,23]. Then, we can understand the intrinsic mechanisms for a lot of metals as acquiring the Aharanov-Bohm–type phase factor [24] by using the onsite spin-orbit coupling (SOC) and several hopping integrals [25] (for more details, see Appendix A). On the other hand, we can understand several extrinsic mechanisms [3,22,23] by considering a special scattering of doped nonmagnetic impurities. However, if their onsite scattering potential is small and the intrinsic term is non-negligible, the extrinsic term is less important than the intrinsic term. Actually, the extrinsic term completely vanishes in even-parity systems for the weak onsite scattering potential of dilute nonmagnetic impurities [17,19]. Furthermore, even in the absence of the inversion symmetry at an ab plane, the extrinsic term remains very small if orbital degrees of freedom exist and the hopping induced by the inversion-symmetry breaking is not large [26]. Since a lot of multiorbital metals have finite intrinsic terms [17–19,21,25] and the typical value of the scattering potential estimated in a first-principle calculation [27] is of the order of magnitude 0.1 eV, we may sufficiently analyze the AHE or SHE of a multiorbital metal by considering only the intrinsic mechanism. Actually, a systematic theoretical study [19] about the intrinsic SHE can qualitatively reproduce a chemical trend of the experimental responses [28] in several 4d or 5d transition metals. Since a multiorbital metal is more suitable than a semiconductor to obtain a large response [29], a theoretical research on the intrinsic AHE or SHE of a multiorbital metal may develop our fundamental understanding and the possibilities of application.

However, we have two issues about interaction effects, the effects of the electron-electron interaction, in the intrinsic AHE and SHE of a multiorbital metal. One is to clarify roles of the Fermi surface term and Fermi sea term of σ_{xy}^{C} or σ_{xy}^{S}, the intrinsic anomalous Hall or spin Hall conductivity, in the presence of the electron-electron interaction. Let us begin with noninteracting case with the weak onsite scattering potential of dilute nonmagnetic impurities at zero temperature. In that case, σ_{xy}^{C} or σ_{xy}^{S} consists of the Fermi surface term and Fermi sea

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term in general [17,19,30]. The Fermi surface term describes the excitations near the Fermi level, and the Fermi sea term describes the excitations in the Fermi sea. Then, those terms are affected by the nonmagnetic impurity scattering through changing the quasiparticle (QP) damping in $\sigma^{\rm C}_{xy}$ or $\sigma^{\rm S}_{xy}$ even if the extrinsic term is negligible [17,19]. If that QP damping goes to zero, $\sigma^{\rm C}_{xy}$ or $\sigma^{\rm S}_{xy}$ is given by the Berry-curvature term [18,20,31], part of the Fermi sea term [17,19], because of the cancellation between the other part of the Fermi sea term and the Fermi surface term [17,19]. As the QP damping increases due to an increase of the impurity concentration $n_{\rm imp}$, the dominant term of $\sigma^{\rm C}_{xy}$ or $\sigma^{\rm S}_{xy}$ becomes the Fermi surface term because of the cancellation between the two parts of the Fermi sea term [17,19]. This Fermi surface term qualitatively differs from the Berry-curvature term because only the former contains a retarded-advanced product of two single-particle Green's functions [17,19] [for the explicit comparison, for example, see Eqs. (54) and (76)]. Thus, only the Berry-curvature term is insufficient, and the Fermi surface term and Fermi sea term play important roles in discussing the intrinsic AHE or SHE of a noninteracting multiorbital metal. However, for discussions at finite temperatures, we should consider the electron-electron interaction because that may affect $\sigma^{\rm C}_{xy}$ or $\sigma^{\rm S}_{xy}$ through the inelastic scattering. Thus, it remains a challenging issue to clarify the roles of the Fermi surface term and Fermi sea term in an interacting multiorbital metal. Although this issue was discussed by Haldane [32], his proposal [32] did not resolve this because he assumed that only the Berry-curvature term is always dominant and did not analyze the roles of the non-Berry-curvature terms; his proposal is that part of the partial-integral term of the Berry-curvature term corresponds to the Fermi surface term which plays important roles in the Fermi liquid. Thus, we need to discuss this issue in a more elaborated method.

The other issue is to clarify effects of spin-Coulomb drag (SCD) on the intrinsic SHE. If the electron-electron interaction causes the scattering between spin-up and spin-down electrons with finite momentum transfer, the spin-up and -down components of the total momentum are not separately conserved [33,34] (see Fig. 1). This indicates the existence

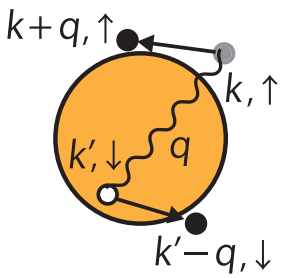

FIG. 1. Schematic picture of the scattering between spin-up and -down electrons due to the electron-electron interaction with momentum transfer $\boldsymbol{q}$. The wavy line represents the electron-electron interaction, the black circles represent the electrons after the scattering, and the yellow circle represents the Fermi sphere. This scattering conserves the sum of the total momentums of the spin-up and -down electrons (i.e., $\boldsymbol{k}+\boldsymbol{k}'=\boldsymbol{k}+\boldsymbol{q}+\boldsymbol{k}'-\boldsymbol{q}$), while the conservation of each total momentum is violated for $\boldsymbol{q}\neq\boldsymbol{0}$ (i.e., $\boldsymbol{k}\neq\boldsymbol{k}+\boldsymbol{q}$ and $\boldsymbol{k}'\neq\boldsymbol{k}'-\boldsymbol{q}$).

of the friction between spin-up and -down electrons, the SCD, even without the Umklapp scattering [33,34] because the momentum conservation results in the absence of the friction [35,36]. This is in contrast to case of the charge current because in that case the Umklapp scattering is essential to obtain the friction, which results in the finite resistivity [35–37]. Thus, the existence of the SCD is an important difference between spin transports and charge transports. Then, the SCD causes a correction [33,34], which is different from the mass enhancement and Fermi-liquid correction, and that effect on the spin-diffusion constant is experimentally observed in a two-dimensional electron gas [38]. In principle, the SCD may affect the intrinsic SHE [34], and its effects may lead to some differences between the SHE and AHE. Furthermore, since in contrast to an electron gas a multiorbital metal has a multiband structure, an interacting multiorbital metal may be a good target to deduce multiband effects in the SCD. However, the effects of the SCD on the intrinsic SHE have not been studied and remain unclear [34].

To improve this situation, we develop a general formalism of $\sigma^{\rm C}_{xy}$ or $\sigma^{\rm S}_{xy}$ of an interacting multiorbital metal using the linear response theory [39] with approximations appropriate for such metal, clarify the roles of the Fermi surface term and Fermi sea term and find a SCD-induced correction of $\sigma^{\rm S}_{xy}$. The former result resolves the first issue, and the latter provides the first step towards the complete resolution of the second issue. In particular, we find an interaction-driven mechanism of the damping dependence of $\sigma^{\rm C}_{xy}$ or $\sigma^{\rm S}_{xy}$ and crossover from damping-dependent to damping-independent intrinsic AHE or SHE. This highlights the emergence of the temperature dependence in high-temperature region of the intrinsic AHE or SHE even for clean systems. We also propose several experiments related to those results. Then, we clarify the origin of the differences between $\sigma^{\rm C}_{xy}$ and the longitudinal conductivity $\sigma^{\rm C}_{xx}$ and deduce the general principles in the formulations of transport coefficients including the interaction and the multiband effects. This origin is helpful to understand why the Fermi sea term such as the Berry-curvature term sometimes becomes important only for the interband transports such as the AHE, although only the Fermi surface term is always important for the intraband transports such as the resistivity. In addition, the obtained principles help guide further research of transports including the interaction effects and the multiband effects.

II. METHOD

In this section, we explain the method to analyze the intrinsic AHE and SHE of an interacting multiorbital metal. First, we show the Hamiltonian of our model, and argue its validity for their realistic analysis. Second, we explain how to treat each term of the Hamiltonian, and deduce several consequences of this treatment about the self-energy, the QP damping, and the irreducible four-point vertex function. Third, we show the exact expressions of $\sigma^{\rm C}_{xy}$ and $\sigma^{\rm S}_{xy}$ within the linear response of an external electric field. In addition, we explain several advantages of the linear response theory and an important remark about taking the limits such as $\lim_{\omega\to 0}$ and $\lim_{\boldsymbol{q}\to\boldsymbol{0}}$. In part of the derivations of those exact expressions, we use Appendix B. Hereafter, we set $\hbar=c=k_{\rm B}=1$.

A. Model

We consider a d-orbital Hubbard model [40] with the onsite SOC [19] and the weak onsite scattering potential [19] of dilute nonmagnetic impurities. Its Hamiltonian consists of four terms:

$$\hat{H} = \hat{H}_0 + \hat{H}_{\rm LS} + \hat{H}_{\rm int} + \hat{H}_{\rm imp}. \tag{1}$$

First, $\hat{H}_0$ represents the nonrelativistic noninteracting terms

$$\hat{H}_0 = \sum_{\boldsymbol{k}} \sum_{a,b} \sum_{s=\uparrow,\downarrow} \epsilon_{ab}(\boldsymbol{k}) \hat{c}^\dagger_{\boldsymbol{k}as} \hat{c}_{\boldsymbol{k}bs}, \tag{2}$$

where $\hat{c}^\dagger_{\boldsymbol{k}as}$ and $\hat{c}_{\boldsymbol{k}as}$ are a creation and an annihilation operator, respectively, of an electron at momentum $\boldsymbol{k}$, orbital a, and spin s, and $\epsilon_{ab}(\boldsymbol{k})$ is the noninteracting energy dispersion measuring from the chemical potential. Second, $\hat{H}_{\rm LS}$ represents the onsite SOC [19]

$$\hat{H}_{\rm LS} = \xi_{\rm LS} \sum_{\boldsymbol{j}} \hat{\boldsymbol{l}}_{\boldsymbol{j}} \cdot \hat{\boldsymbol{s}}_{\boldsymbol{j}}, \tag{3}$$

where $\boldsymbol{j}$ is site index, $\hat{\boldsymbol{l}}_{\boldsymbol{j}}$ and $\hat{\boldsymbol{s}}_{\boldsymbol{j}}$ are an orbital and a spin angular momentum operator, respectively [19], and $\xi_{\rm LS}$ is the coupling constant. Third, $\hat{H}_{\rm int}$ represents the onsite multiorbital Hubbard interaction terms [40]

$$\begin{aligned}\hat{H}_{\rm int} = {} & U \sum_{\boldsymbol{j}} \sum_{a} \hat{n}_{\boldsymbol{j}a\uparrow} \hat{n}_{\boldsymbol{j}a\downarrow} + U' \sum_{\boldsymbol{j}} \sum_{a} \sum_{b<a} \hat{n}_{\boldsymbol{j}a} \hat{n}_{\boldsymbol{j}b} \\ & - J_{\rm H} \sum_{\boldsymbol{j}} \sum_{a} \sum_{b<a} \left(2\hat{\boldsymbol{s}}_{\boldsymbol{j}a} \cdot \hat{\boldsymbol{s}}_{\boldsymbol{j}b} + \frac{1}{2} \hat{n}_{\boldsymbol{j}a} \hat{n}_{\boldsymbol{j}b} \right) \\ & + J' \sum_{\boldsymbol{j}} \sum_{a} \sum_{b\neq a} \hat{c}^\dagger_{\boldsymbol{j}a\uparrow} \hat{c}^\dagger_{\boldsymbol{j}a\downarrow} \hat{c}_{\boldsymbol{j}b\downarrow} \hat{c}_{\boldsymbol{j}b\uparrow},\end{aligned} \tag{4}$$

where $\hat{n}_{\boldsymbol{j}as}$ is $\hat{n}_{\boldsymbol{j}as} = \hat{c}^\dagger_{\boldsymbol{j}as} \hat{c}_{\boldsymbol{j}as}$, $\hat{n}_{\boldsymbol{j}a}$ is $\hat{n}_{\boldsymbol{j}a} = \sum_s \hat{n}_{\boldsymbol{j}as}$, $\hat{\boldsymbol{s}}_{\boldsymbol{j}a}$ is $\hat{\boldsymbol{s}}_{\boldsymbol{j}a} = \frac{1}{2} \sum_{s,s'} \hat{c}^\dagger_{\boldsymbol{j}as} \boldsymbol{\sigma}_{s,s'} \hat{c}_{\boldsymbol{j}as'}$ with the Pauli matrices $\boldsymbol{\sigma}_{s,s'}$, U is the intraorbital Coulomb interaction, U' is the interorbital Coulomb interaction, $J_{\rm H}$ is the Hund's rule coupling, and J' is the pair hopping term. Fourth, $\hat{H}_{\rm imp}$ represents the onsite scattering potential of dilute nonmagnetic impurities [19]

$$\hat{H}_{\rm imp} = I_{\rm imp} \sum_{\boldsymbol{j}} \sum_{a} \sum_{s} \hat{c}^\dagger_{\boldsymbol{j}as} \hat{c}_{\boldsymbol{j}as}, \tag{5}$$

where $I_{\rm imp}$ is the potential amplitude.

This model is sufficient for a realistic analysis of the intrinsic AHE and SHE of an interacting metal because of the following four reasons. First, we can choose any form of $\epsilon_{ab}(\boldsymbol{k})$ if $\epsilon_{ab}(\boldsymbol{k})$ contains the interorbital hopping whose mirror symmetries for a xz and a yz plane are odd [e.g., the next-nearest-neighbor hopping between the d_{yz} and d_{xz} orbitals in Fig. 4(a) of Appendix A]; as we will see in Sec. III A 2, such interorbital hopping is necessary to obtain finite $\sigma^{\rm C}_{xy}$ or $\sigma^{\rm S}_{xy}$ [17,25]. Second, among several possibilities of the SOCs, only the onsite SOC is sufficient because its effect is leading in a solid and because we can analyze the intrinsic AHE or SHE of a metal even without the inversion symmetry at an ab plane by not using the nonlocal SOC [21,26]; the effect of that inversion-symmetry breaking can be included in $\epsilon_{ab}(\boldsymbol{k})$ [41]. Third, we may sufficiently describe the screened short-ranged electron-electron interaction in an interacting multiorbital metal by the onsite multiorbital Hubbard interactions because those interactions have not only the intraorbital term but also the interorbital terms; our formalism can be easily extended if the interactions are short-ranged. Fourth, $\hat{H}_{\rm imp}$ may be sufficient to include effects of dilute nonmagnetic impurities, which exist in a realistic situation, because the effects can be roughly described by their weak onsite scattering potential [42].

B. Treatment of each Hamiltonian

To analyze the intrinsic AHE or SHE of an interacting multiorbital metal, we use $\hat{H}_0 + \hat{H}_{\rm LS}$ as the nonperturbative Hamiltonian

$$\hat{H}_0 + \hat{H}_{\rm LS} = \sum_{\boldsymbol{k}} \sum_{a,b} \sum_{s,s'} \bar{\epsilon}^{ss'}_{ab}(\boldsymbol{k}) \hat{c}^\dagger_{\boldsymbol{k}as} \hat{c}_{\boldsymbol{k}bs'}, \tag{6}$$

and $\hat{H}_{\rm int} + \hat{H}_{\rm imp}$ as the perturbative Hamiltonian. In particular, for a simple treatment of $\hat{H}_{\rm imp}$, we assume both that nonmagnetic impurities are randomly distributed and that $I_{\rm imp}$ is smaller than the bandwidth so as to satisfy $k_{\rm F} l \gg 1$ (i.e., case away from the Mott-Ioffe-Regel limit), where $k_{\rm F}$ is of the order of magnitude the Fermi momentum and l is the mean-free path. The first assumption is standard [17,19,42], and the second is reasonable in several transition metals or transition-metal oxides. Then, because of the first assumption, we can use the averaging over each impurity position [42]; because of the second, we can neglect the combination terms of $\hat{H}_{\rm int}$ and $\hat{H}_{\rm imp}$ in the self-energy and sufficiently treat $\hat{H}_{\rm imp}$ in the Born approximation [43].

In this treatment, we can use simple treatments about several quantities. First, the retarded self-energy is given by

$$\Sigma^{ss'}_{ab}(\tilde{k}) = \Sigma^{ss'}_{\text{el-el};ab}(\tilde{k}) + \frac{n_{\rm imp} I^2_{\rm imp}}{N} \sum_{\boldsymbol{k}'} G^{ss'}_{ab}(\boldsymbol{k}', i\epsilon_m), \tag{7}$$

where $\tilde{k}$ is $\tilde{k} \equiv (\boldsymbol{k}, i\epsilon_m)$, $\Sigma^{ss'}_{\text{el-el};ab}(\tilde{k})$ is the self-energy arising from $\hat{H}_{\rm int}$ in the perturbation theory, and the second term is the self-energy arising from $\hat{H}_{\rm imp}$ in the Born approximation [19,42] with N the number of lattice sites. Correspondingly, we obtain the QP damping arising from $\hat{H}_{\rm int}$ and $\hat{H}_{\rm imp}$ because the QP damping is defined as

$$\gamma^*_\alpha(\boldsymbol{k}) = -z_\alpha(\boldsymbol{k}) {\rm Im} \Sigma^{\rm (R)}_\alpha(\boldsymbol{k}, \xi^*_\alpha(\boldsymbol{k})), \tag{8}$$

where α is the band index of a QP, $\xi^*_\alpha(\boldsymbol{k})$ is the QP energy determined by the solution of $\det|\epsilon \delta_{a,b} \delta_{s,s'} - \bar{\epsilon}^{ss'}_{ab}(\boldsymbol{k}) - {\rm Re}\Sigma^{{\rm (R)}ss'}_{ab}(k)| = 0$, $\Sigma^{\rm (R)}_\alpha(k)$ with $k \equiv (\boldsymbol{k}, \epsilon)$ is the retarded self-energy of the QP band α, and $z_\alpha(\boldsymbol{k})$ is the QP weight

$$z_\alpha(\boldsymbol{k}) = \left[1 - \left. \frac{\partial\, {\rm Re}\Sigma^{\rm (R)}_\alpha(\boldsymbol{k}, \epsilon)}{\partial \epsilon} \right|_{\epsilon \to \xi^*_\alpha(\boldsymbol{k})} \right]^{-1}. \tag{9}$$

In general, $\gamma^*_\alpha(\boldsymbol{k})$ depends on temperature because of the temperature dependence of $\Sigma^{{\rm (R)}ss'}_{\text{el-el};ab}(\boldsymbol{k}, \epsilon)$, e.g., the T^2 dependence of $\gamma^*_\alpha(\boldsymbol{k})$ near $\boldsymbol{k} = \boldsymbol{k}_{\rm F}$ in the Fermi liquid [42]. Then, the irreducible four-point vertex function in Matsubara-frequency representation $\Gamma^{(1)\{s_1\}}_{\{a\}}(\tilde{k}, \tilde{k}'; \tilde{q}) \equiv \Gamma^{(1)s_1 s_2 s_3 s_4}_{abcd}(\boldsymbol{k}+\boldsymbol{q}, i\epsilon_{m+n}, \boldsymbol{k}, i\epsilon_m, \boldsymbol{k}'+\boldsymbol{q}, i\epsilon_{m'+n}, \boldsymbol{k}', i\epsilon_{m'})$, is given

by

$$\Gamma^{(1)\{s_1\}}_{\{a\}}(\tilde{k},\tilde{k}';\tilde{q}) = \frac{\delta\Sigma^{ss'}_{\text{el-el};ab}(\tilde{k})}{\delta G^{s''s'''}_{cd}(\tilde{k}')} + n_{\text{imp}} I^2_{\text{imp}} \delta_{a,c}\delta_{b,d}\delta_{s,s''}\delta_{s',s'''}, \quad (10)$$

where the first term is the irreducible four-point vertex function arising from $\hat{H}_{\text{int}}$, and the second term is the irreducible four-point vertex function arising from $\hat{H}_{\text{imp}}$ in the Born approximation [43]. Because of this decomposition, $\hat{H}_{\text{imp}}$ causes no correction to the charge and the spin current for even-parity systems, resulting in the disappearance of the extrinsic terms of the dc anomalous Hall or spin Hall conductivity in the similar way in noninteracting case [17,19] [see the sentences below Eq. (37)].

C. Linear response theory

To formulate σ^{C}_{xy} and σ^{S}_{xy} as general as possible, we use the linear response theory [39]. This is because the linear response theory provides exact expressions of σ^{C}_{xy} and σ^{S}_{xy} within the linear response of an external electric field and because that theory with appropriate approximations has several advantages compared with the phenomenological theory.

We can derive an exact expression of σ^{C}_{xy} within the linear response of an external electric field from the Kubo formula [39] for the charge current perpendicular to it without an external magnetic field:

$$\sigma^{\text{C}}_{xy} = \lim_{\omega\to 0}\lim_{\boldsymbol{q}\to\boldsymbol{0}} \frac{\tilde{K}^{\text{C(R)}}_{xy}(\boldsymbol{q},\omega) - \tilde{K}^{\text{C(R)}}_{xy}(\boldsymbol{q},0)}{i\omega}, \quad (11)$$

where $\tilde{K}^{\text{C(R)}}_{xy}(\omega) \equiv \tilde{K}^{\text{C(R)}}_{xy}(\boldsymbol{0},\omega)$ is obtained by the analytic continuation of $\tilde{K}^{\text{C}}_{xy}(i\Omega_n)$ with bosonic Matsubara frequency $\Omega_n = 2\pi T n$:

$$\tilde{K}^{\text{C(R)}}_{xy}(\omega) = \tilde{K}^{\text{C}}_{xy}(i\Omega_n)|_{i\Omega_n\to\omega+i0+}, \quad (12)$$

with

$$\begin{aligned}\tilde{K}^{\text{C}}_{xy}(i\Omega_n) &= \frac{1}{N}\lim_{\boldsymbol{q}\to\boldsymbol{0}}\int_0^{T^{-1}} d\tau\, e^{i\Omega_n\tau}\langle \text{T}_\tau \hat{J}^{\text{C}}_{\boldsymbol{q}x}(\tau)\hat{J}^{\text{C}}_{-\boldsymbol{q}y}(0)\rangle \\ &= \frac{1}{N}\sum_{\boldsymbol{k},\boldsymbol{k}'}\sum_{\{a\}}\sum_{\{s\}}\int_0^{T^{-1}} d\tau\, e^{i\Omega_n\tau}(-e)\delta_{s',s}(v_{\boldsymbol{k}x})^{ss}_{ba} \\ &\quad\times(-e)\delta_{s'',s'''}(v_{\boldsymbol{k}'y})^{s''s''}_{cd} \\ &\quad\times\langle \text{T}_\tau \hat{c}^\dagger_{\boldsymbol{k}bs'}(\tau)\hat{c}_{\boldsymbol{k}as}(\tau)\hat{c}^\dagger_{\boldsymbol{k}'cs''}\hat{c}_{\boldsymbol{k}'ds'''}\rangle \\ &= \frac{(-e)^2}{N}\sum_{\boldsymbol{k},\boldsymbol{k}'}\sum_{\{a\}}\sum_{\{s\}}\delta_{s',s}(v_{\boldsymbol{k}x})^{ss}_{ba}\delta_{s'',s'''}(v_{\boldsymbol{k}'y})^{s''s''}_{cd} \\ &\quad\times K^{ss''}_{abcd}(\boldsymbol{k},\boldsymbol{k}';i\Omega_n). \end{aligned} \quad (13)$$

(Note that we should carry out the integration about τ before carrying out $i\Omega_n \to \omega + i0+$ [44].) In Eq. (13), T_τ is the time-ordering operator [42], $\sum_{\{a\}}$ is $\sum_{\{a\}} \equiv \sum_{a,b,c,d}$, $\sum_{\{s\}}$ is $\sum_{\{s\}} \equiv \sum_{s,s',s'',s'''}$, the charge current operator is

$$\hat{J}^{\text{C}}_{\boldsymbol{q}\nu} = (-e)\sum_{\boldsymbol{k}}\sum_{a,b}\sum_{s,s'}\delta_{s',s}(v_{\boldsymbol{k}\nu})^{ss}_{ba}\hat{c}^\dagger_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}bs'}\hat{c}_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}as}, \quad (14)$$

and the noninteracting group velocity is

$$(v_{\boldsymbol{k}\nu})^{ss}_{ab} = \frac{\partial\epsilon_{ab}(\boldsymbol{k})}{\partial k_\nu}. \quad (15)$$

The noninteracting group velocity is not affected by the onsite SOC because that is independent of momentum [19].

Also, we can exactly derive σ^{S}_{xy} within the linear response in the similar way for σ^{C}_{xy} if we define the spin-current operator. Let us use a standard definition [13,45]

$$\hat{J}^{\text{S}}_{\boldsymbol{q}\nu} = \frac{1}{2}\sum_{\boldsymbol{k}}\sum_{a,b}\sum_{s,s'}\text{sgn}(s)\delta_{s',s}(v_{\boldsymbol{k}\nu})^{ss}_{ba}\hat{c}^\dagger_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}bs'}\hat{c}_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}as}, \quad (16)$$

with $\text{sgn}(\uparrow) = +1$ or $\text{sgn}(\downarrow) = -1$. In this definition, the spin current is the difference between the spin-up and -down components of the charge current [19]:

$$\hat{J}^{\text{S}}_{\boldsymbol{q}\nu} = \frac{1}{2(-e)}\left[\left(\hat{J}^{\text{C}}_{\boldsymbol{q}\nu}\right)_{\uparrow\uparrow} - \left(\hat{J}^{\text{C}}_{\boldsymbol{q}\nu}\right)_{\downarrow\downarrow}\right], \quad (17)$$

where $(\hat{J}^{\text{C}}_{\boldsymbol{q}\nu})_{ss}$ is defined by $\hat{J}^{\text{C}}_{\boldsymbol{q}\nu} = \sum_s(\hat{J}^{\text{C}}_{\boldsymbol{q}\nu})_{ss}$. Even if we use a different but single-body definition, we can carry out the general formulation in the similar way. By adopting this definition (16) to the Kubo formula for σ^{S}_{xy}, its exact expression is obtained:

$$\sigma^{\text{S}}_{xy} = \lim_{\omega\to 0}\lim_{\boldsymbol{q}\to\boldsymbol{0}} \frac{\tilde{K}^{\text{S(R)}}_{xy}(\boldsymbol{q},\omega) - \tilde{K}^{\text{S(R)}}_{xy}(\boldsymbol{q},0)}{i\omega}, \quad (18)$$

with

$$\tilde{K}^{\text{S(R)}}_{xy}(\omega) \equiv \tilde{K}^{\text{S(R)}}_{xy}(\boldsymbol{0},\omega) = \tilde{K}^{\text{S}}_{xy}(i\Omega_n)|_{i\Omega_n\to\omega+i0+} \quad (19)$$

and

$$\begin{aligned}\tilde{K}^{\text{S}}_{xy}(i\Omega_n) &= \frac{1}{N}\lim_{\boldsymbol{q}\to\boldsymbol{0}}\int_0^{T^{-1}} d\tau\, e^{i\Omega_n\tau}\langle \text{T}_\tau \hat{J}^{\text{S}}_{\boldsymbol{q}x}(\tau)\hat{J}^{\text{C}}_{-\boldsymbol{q}y}(0)\rangle \\ &= \frac{1}{N}\sum_{\boldsymbol{k},\boldsymbol{k}'}\sum_{\{a\}}\sum_{\{s\}}\int_0^{T^{-1}} d\tau\, e^{i\Omega_n\tau}\frac{1}{2}\text{sgn}(s)\delta_{s',s}(v_{\boldsymbol{k}x})^{ss}_{ba} \\ &\quad\times(-e)\delta_{s'',s'''}(v_{\boldsymbol{k}'y})^{s''s''}_{cd} \\ &\quad\times\langle \text{T}_\tau \hat{c}^\dagger_{\boldsymbol{k}bs'}(\tau)\hat{c}_{\boldsymbol{k}as}(\tau)\hat{c}^\dagger_{\boldsymbol{k}'cs''}\hat{c}_{\boldsymbol{k}'ds'''}\rangle \\ &= \frac{-e}{2N}\sum_{\boldsymbol{k},\boldsymbol{k}'}\sum_{\{a\}}\sum_{\{s\}}\text{sgn}(s)\delta_{s',s}(v_{\boldsymbol{k}x})^{ss}_{ba} \\ &\quad\times\delta_{s'',s'''}(v_{\boldsymbol{k}'y})^{s''s''}_{cd}K^{ss''}_{abcd}(\boldsymbol{k},\boldsymbol{k}';i\Omega_n). \end{aligned} \quad (20)$$

Then, the linear response theory [39] has several advantages compared with the phenomenological theory such as the Boltzmann theory in the relaxation-time approximation [14]. The most important advantage is about the treatment of the dominant excitations. The linear response theory does not assume whether the dominant excitations are either Fermi surface or Fermi sea type; instead, the dominant excitations are naturally determined as a result of the treatment of the perturbation terms. On the other hand, the Boltzmann theory assumes the dominant excitations as a result of assuming the distribution function [e.g., $f_{\boldsymbol{k}} = f^0_{\boldsymbol{k}} - \Phi_{\boldsymbol{k}}\frac{\partial f^0_{\boldsymbol{k}}}{\partial\epsilon_{\boldsymbol{k}}}$ in Eq. (7.7.1) of Ref. [36]]. Thus, the linear response theory is suitable to analyze the roles of the Fermi surface and the Fermi sea terms. Then, in the linear response theory we can analyze

the interaction effects with keeping momentum conservation in combination with Baym-Kadanoff's conserving approximation [43,46,47]; in the relaxation-time approximation [14], momentum conservation is violated because of the introduction of the momentum- and frequency-independent relaxation time [43]. This is one of the advantages because the appropriate treatment of momentum conservation is vital to analyze transport phenomena. Actually, only if we use the appropriate treatment, we can obtain the disappearance of the resistivity without the lattice and impurities [37]; in the relaxation-time approximation, the resistivity remains finite. In addition, the linear response theory is useful to study a variety of material dependence because the material dependence arises from the differences in the electronic structure and because we can naturally include those differences in the linear response theory; in the relaxation-time approximation, it is difficult to include the differences in the interaction effects.

In the remaining part of this section, we explain the derivation only for $\sigma^{\rm C}_{xy}$. This is because the difference between $\sigma^{\rm C}_{xy}$ and $\sigma^{\rm S}_{xy}$ comes from the difference between $\hat{J}^{\rm C}_{\boldsymbol{q}x}(\tau)$ and $\hat{J}^{\rm S}_{\boldsymbol{q}x}(\tau)$ and because we obtain $\sigma^{\rm S}_{xy}$ by replacing $(-e)\delta_{s',s}(v_{\boldsymbol{k}x})^{ss}_{ba}$ in $\sigma^{\rm C}_{xy}$ by $\frac{1}{2}{\rm sgn}(s)\delta_{s',s}(v_{\boldsymbol{k}x})^{ss}_{ba}$ [compare Eqs. (13) and (20)].

Before the details of the derivation, we remark on the importance of the order of taking the limits. To obtain the finite observable current, we should take $\boldsymbol{q}\to\boldsymbol{0}$ before taking $\omega\to 0$ [46]. Then, for dc conductivities, only after taking $\omega\to 0$, we can take $\gamma^*_\alpha(\boldsymbol{k})\to 0$ because we should hold $\omega\tau_{\rm trans}\ll 1$ [46,48,49], where $\tau_{\rm trans}$ is the transport relaxation time which is of the order of magnitude the inverse of the QP damping. Namely, to discuss the dc conductivities in the clean and zero-temperature limit, we should take $\lim_{\gamma^*_\alpha(\boldsymbol{k})\to 0}\lim_{\omega\to 0}\lim_{\boldsymbol{q}\to\boldsymbol{0}}$ in this order. If we take $\gamma^*_\alpha(\boldsymbol{k})\to 0$ before taking $\omega\to 0$, the results become unphysical. In particular, the order of those limits is crucial for interacting systems because the important difference between cases in $\omega\tau_{\rm trans}\ll 1$ and $\omega\tau_{\rm trans}\gg 1$ is known as the difference between the first and the zero sound in the Fermi liquid [50]. However, in noninteracting systems only in the clean and zero-temperature limit, the unphysical limit [31] leads to the same $\sigma^{\rm C}_{xy}$ or $\sigma^{\rm S}_{xy}$ as that in the physical limit [17,19]. Since we cannot expect such accidental agreement in interacting systems, we should care about the order of taking the limits.

The derivation for $\sigma^{\rm C}_{xy}$ consists of three steps.

The first step is to express $K^{ss''}_{abcd}(\boldsymbol{k},\boldsymbol{k}';i\Omega_n)$ in terms of the single-particle Green's functions and the reducible four-point vertex function [51]; the latter describes the multiple electron-hole scattering. We can carry out that procedure by the perturbative expansion using the Bloch–De Dominicis theorem [49]:

$$K^{ss''}_{abcd}(\boldsymbol{k},\boldsymbol{k}';i\Omega_n)=-\delta_{\boldsymbol{k},\boldsymbol{k}'}T\sum_m G^{ss''}_{ac}(\tilde{k_+})G^{s''s}_{db}(\tilde{k})-\frac{T^2}{N}\sum_{m,m'}\sum_{\{A\}}\sum_{\{s_1\}}G^{ss_1}_{aA}(\tilde{k_+})G^{s''s_4}_{dD}(\tilde{k'})\times G^{s_3s''}_{Cc}(\tilde{k'_+})G^{s_2s}_{Bb}(\tilde{k})\times\Gamma^{\{s_1\}}_{\{A\}}(\tilde{k},\tilde{k'};\boldsymbol{0},i\Omega_n),\tag{21}$$

with $\tilde{k_+}\equiv(\boldsymbol{k},i\epsilon_m+i\Omega_n)$, fermionic Matsubara frequency $\epsilon_m=2\pi T(m+\frac{1}{2})$, and the reducible four-point vertex function in Matsubara-frequency representation $\Gamma^{\{s_1\}}_{\{A\}}(\tilde{k},\tilde{k'};\boldsymbol{q},i\Omega_n)\equiv\Gamma^{s_1s_2s_3s_4}_{ABCD}(\boldsymbol{k}+\boldsymbol{q},i\epsilon_{m+n},\boldsymbol{k},i\epsilon_m,\boldsymbol{k}'+\boldsymbol{q},i\epsilon_{m'+n},\boldsymbol{k}',i\epsilon_{m'})$. In principle, we can calculate $G^{ss'}_{ab}(\tilde{k})$ from Dyson's equation using the self-energy

$$G^{ss'}_{ab}(\tilde{k})=G^{(0)ss'}_{ab}(\tilde{k})+\sum_{c,d}\sum_{s'',s'''}G^{(0)ss''}_{ac}(\tilde{k})\Sigma^{(0)s''s'''}_{cd}(\tilde{k})G^{s'''s'}_{db}(\tilde{k}),\tag{22}$$

with the noninteracting single-particle Green's function $G^{(0)ss'}_{ab}(\tilde{k})$ and $\Gamma^{\{s_1\}}_{\{a\}}(\tilde{k},\tilde{k'};\boldsymbol{q},i\Omega_n)$ from the Bethe-Salpeter equation using the irreducible four-point vertex function [51,52]

$$\Gamma^{\{s_1\}}_{\{a\}}(\tilde{k},\tilde{k'};\boldsymbol{q},i\Omega_n)=\Gamma^{(1)\{s_1\}}_{\{a\}}(\tilde{k},\tilde{k'};\boldsymbol{q},i\Omega_n)+\frac{T}{N}\sum_{\boldsymbol{k}''}\sum_{m''}\sum_{\{A\}}\times\sum_{\{s'_1\}}\Gamma^{(1)s_1s_2s'_3s'_4}_{abCD}(\tilde{k},\tilde{k''};\boldsymbol{q},i\Omega_n)G^{s'_3s'_1}_{CA}(\tilde{k''}+\tilde{q})G^{s'_2s'_4}_{BD}(\tilde{k''})\times\Gamma^{s'_1s'_2s_3s_4}_{ABcd}(\tilde{k''},\tilde{k'};\boldsymbol{q},i\Omega_n).\tag{23}$$

The second step is to carry out the analytic continuation of $\tilde{K}^{\rm C}_{xy}(i\Omega_n)$. This procedure is the same for $\sigma^{\rm C}_{xx}$ [52] with $\hat{H}_0$ and $\hat{H}_{\rm int}$ without $\hat{H}_{\rm LS}$ and $\hat{H}_{\rm imp}$ because the relevant parameters in this procedure are only frequencies [49] and spin indices are irrelevant. In this procedure, we use the analytic properties [49] of the single-particle Green's function and reducible four-point vertex function and rewrite the sum of the Matsubara frequency by the corresponding contour integral; $G^{ss'}_{ab}(\boldsymbol{k},\epsilon)$ is singular on the horizontal line ${\rm Im}\epsilon=0$; $\Gamma^{\{s_1\}}_{\{a\}}(\boldsymbol{k},\epsilon,\boldsymbol{k}',\epsilon';\boldsymbol{0},\omega)$ is singular on the horizontal lines ${\rm Im}\epsilon=0$, ${\rm Im}(\epsilon+\omega)=0$, ${\rm Im}\epsilon'=0$, ${\rm Im}(\epsilon'+\omega)=0$, ${\rm Im}(\epsilon+\epsilon'+\omega)=0$, and ${\rm Im}(\epsilon-\epsilon')=0$, where the horizontal line ${\rm Im}\omega=0$ is excluded because we consider ${\rm Im}\omega>0$ [see Eq. (12)]. As derived in Appendix B, we obtain

$$\tilde{K}^{\rm C(R)}_{xy}(\omega)=-\frac{(-e)^2}{2i}\sum_k\sum_{\boldsymbol{k}'}\sum_{\{a\}}\sum_{\{s\}}\delta_{s',s}(v_{\boldsymbol{k}x})^{ss}_{ba}\times\delta_{s'',s'''}(v_{\boldsymbol{k}'y})^{s'''s''}_{cd}\sum_{l=1}^{3}T_l(\epsilon,\omega)K^{ss''}_{l;abcd}(\boldsymbol{k},\boldsymbol{k}';\epsilon;\omega),\tag{24}$$

where

$$\sum_k\equiv\frac{1}{N}\sum_{\boldsymbol{k}}\int_{-\infty}^{\infty}\frac{d\epsilon}{2\pi},\tag{25}$$

$$T_1(\epsilon,\omega)=\tanh\frac{\epsilon}{2T},\tag{26}$$

$$T_2(\epsilon,\omega)=\tanh\frac{\epsilon+\omega}{2T}-\tanh\frac{\epsilon}{2T},\tag{27}$$

$$T_3(\epsilon,\omega)=-\tanh\frac{\epsilon+\omega}{2T},\tag{28}$$

and

$$K^{ss''}_{l;abcd}(\boldsymbol{k},\boldsymbol{k}';\epsilon;\omega) = \delta_{\boldsymbol{k},\boldsymbol{k}'} g^{ss''s''s}_{l;acdb}(k;\omega) + \frac{1}{N}\int_{-\infty}^{\infty}\frac{d\epsilon'}{4\pi i}\sum_{\{A\}}\sum_{\{s_1\}} g^{ss_1s_2s}_{l;aABb}(k;\omega) \times \sum_{l'=1}^{3}\mathcal{J}^{\{s_1\}}_{ll';\{A\}}(k,k';\omega) g^{s_3s''s''s_4}_{l';CcdD}(k';\omega), \tag{29}$$

with

$$g^{ss''s'''s'}_{1;acdb}(k;\omega) = G^{(\mathrm{R})ss''}_{ac}(\boldsymbol{k},\epsilon+\omega) G^{(\mathrm{R})s'''s'}_{db}(\boldsymbol{k},\epsilon), \tag{30}$$

$$g^{ss''s'''s'}_{2;acdb}(k;\omega) = G^{(\mathrm{R})ss''}_{ac}(\boldsymbol{k},\epsilon+\omega) G^{(\mathrm{A})s'''s'}_{db}(\boldsymbol{k},\epsilon), \tag{31}$$

$$g^{ss''s'''s'}_{3;acdb}(k;\omega) = G^{(\mathrm{A})ss''}_{ac}(\boldsymbol{k},\epsilon+\omega) G^{(\mathrm{A})s'''s'}_{db}(\boldsymbol{k},\epsilon), \tag{32}$$

and $\mathcal{J}^{\{s_1\}}_{ll';\{a\}}(k,k';\omega)$, connected with $\mathcal{J}^{(1)\{s_1\}}_{ll';\{a\}}(k,k';\omega)$ by the Bethe-Salpeter equation

$$\mathcal{J}^{\{s_1\}}_{ll';\{a\}}(k,k';\omega) = \mathcal{J}^{(1)\{s_1\}}_{ll';\{a\}}(k,k';\omega) + \frac{1}{2i}\sum_{k''}\sum_{\{A\}}\sum_{\{s'_1\}}\sum_{l''=1}^{3} \times \mathcal{J}^{(1)s_1s_2s'_3s'_4}_{ll'';abCD}(k,k'';\omega) g^{s'_3s'_1s'_2s'_4}_{l'';CABD}(k'';\omega) \times \mathcal{J}^{s'_1s'_2s'_3s'_4}_{l''l';ABcd}(k'',k';\omega). \tag{33}$$

Here, $\mathcal{J}^{\{s_1\}}_{ll';\{a\}}(k,k';\omega)$ is connected with $\Gamma^{\{s_1\}}_{ll';\{a\}}(k,k';\omega)$, the reducible four-point vertex function in real-frequency representation, as shown in Eqs. (B3)–(B11); for the connections between $\mathcal{J}^{(1)\{s_1\}}_{ll';\{a\}}(k,k';\omega)$ and $\Gamma^{(1)\{s_1\}}_{ll';\{a\}}(k,k';\omega)$, we should add the superscript (1) to $\mathcal{J}^{\{s_1\}}_{ll';\{a\}}(k,k';\omega)$ and $\Gamma^{\{s_1\}}_{ll';\{a\}}(k,k';\omega)$ in those equations.

The third step is to rewrite $\tilde{K}^{\mathrm{C(R)}}_{xy}(\omega)$ in a more compact form by using the vertex function of the charge current. The vertex function of the charge current in Matsubara-frequency representation $\Lambda^{\mathrm{C};s's''}_{\nu;AB}(\tilde{k};\tilde{q}) \equiv \Lambda^{\mathrm{C};s's''}_{\nu;AB}(\boldsymbol{k}+\boldsymbol{q},i\omega_{m+n},\boldsymbol{k},i\omega_m)$ ($\nu = x,y$) is defined as follows [53]:

$$\sum_{A,B,s',s''} G^{ss'}_{aA}(\tilde{k}+\tilde{q})\Lambda^{\mathrm{C};s's''}_{\nu;AB}(\tilde{k};\tilde{q})G^{s''s}_{Bb}(\tilde{k}) = \int_0^{T^{-1}} d\tau\, e^{i\omega_{m+n}\tau}\int_0^{T^{-1}} d\tau' e^{-i\Omega_n\tau'}\langle \mathrm{T}_\tau \hat{c}_{\boldsymbol{k}+\boldsymbol{q}as}(\tau)\hat{J}^{\mathrm{C}}_{-\boldsymbol{q}\nu}(\tau')\hat{c}^\dagger_{kbs}\rangle. \tag{34}$$

Thus, $\Lambda^{\mathrm{C};s's''}_{\nu;AB}(\tilde{k};\tilde{q})$ is connected with $\Gamma^{s's''s_3s_4}_{ABC'D'}(\tilde{k},\tilde{k}';\boldsymbol{q},i\Omega_n)$ through the Bethe-Salpeter equation

$$\Lambda^{\mathrm{C};s's''}_{\nu;AB}(\tilde{k};\tilde{q}) = \delta_{s',s''}(v_{\boldsymbol{k}\nu})^{s's'}_{AB} + \frac{T}{N}\sum_{\boldsymbol{k}'}\sum_{m'}\sum_{\{A'\}}\sum_{\{s_1\}}\Gamma^{s's''s_3s_4}_{ABC'D'}(\tilde{k},\tilde{k}';\boldsymbol{q},i\Omega_n) \times G^{s_3s_1}_{C'A'}(\tilde{k}'+\tilde{q})G^{s_2s_4}_{B'D'}(\tilde{k}')\delta_{s_1,s_2}(v_{\boldsymbol{k}'\nu})^{s_1s_2}_{A'B'}. \tag{35}$$

Then, to convert this relation into the relation in real-frequency representation, we should carry out the analytic continuation of $\Lambda^{\mathrm{C};s's''}_{\nu;AB}(\tilde{k};\tilde{q})$. Since this procedure is similar for the second term of Eq. (21), we can carry out this procedure in the similar way. As a result, we obtain that connection

$$\Lambda^{\mathrm{C};s''s'''}_{\nu;l;cd}(k;\omega) = \delta_{s'',s'''}(v_{\boldsymbol{k}y})^{s''s'''}_{cd} + \sum_{k'}\sum_{A,B,s_1,s_2}\alpha^{s''s'''s_1s_2}_{l;cdAB}(k,k';\omega)\delta_{s_1,s_2}(v_{\boldsymbol{k}'y})^{s_1s_1}_{AB}, \tag{36}$$

with $\Lambda^{\mathrm{C};s''s'''}_{y;l;cd}(k;\omega) \equiv \Lambda^{\mathrm{C};s''s'''}_{y;l;cd}(\boldsymbol{k},\epsilon+\omega,\boldsymbol{k},\epsilon)$ and

$$\alpha^{s''s'''s_1s_2}_{l;cdAB}(k,k';\omega) = \frac{1}{2i}\sum_{C,D,s_3,s_4}\sum_{l'=1}^{3}\mathcal{J}^{s''s'''s_3s_4}_{ll';cdCD}(k,k';\omega) \times g^{s_3s_1s_2s_4}_{l';CABD}(k';\omega). \tag{37}$$

These equations with Eq. (33) show that the correction terms to the noninteracting charge current come from the multiple electron-hole scattering, described by the reducible four-point vertex function [51]. Furthermore, we can show that the correction term arising from $\hat{H}_{\mathrm{imp}}$ disappears for even-parity systems because we can rewrite Eq. (36) as

$$\Lambda^{\mathrm{C};s''s'''}_{\nu;l;cd}(k;\omega) = \delta_{s'',s'''}(v_{\boldsymbol{k}y})^{s''s'''}_{cd} + \frac{1}{2i}\sum_{k'}\sum_{\{A\}}\sum_{\{s_1\}}\sum_{l'=1}^{3}\mathcal{J}^{(1)s''s'''s_3s_4}_{ll';cdCD}(k,k';\omega) \times g^{s_3s_1s_2s_4}_{l';CABD}(k';\omega)\Lambda^{\mathrm{C};s_1s_2}_{\nu;l';AB}(k';\omega), \tag{38}$$

and because part of the above second term arising from $\hat{H}_{\mathrm{imp}}$ exactly vanishes in even-parity systems due to the combination of the momentum-independent irreducible four-point vertex function in the Born approximation [17,19] [see Eq. (10)], the even-parity symmetry of the single-particle Green's functions, and the odd-parity symmetry of the noninteracting group velocity, which results in the odd-parity symmetry of the vertex function of the charge current. Namely, for even-parity systems with the weak onsite scattering potential of the impurities, the correction terms in $\Lambda^{\mathrm{C};s''s'''}_{\nu;l;cd}(k;\omega)$ arise from only $\hat{H}_{\mathrm{int}}$. Rewriting part of Eq. (24) by using the relation

$$\begin{aligned}\sum_{\boldsymbol{k}'}\sum_{c,d,s'',s'''}K^{ss''}_{l;abcd}(\boldsymbol{k},\boldsymbol{k}';\epsilon;\omega)\delta_{s'',s'''}(v_{\boldsymbol{k}'y})^{s''s''}_{cd} &= \sum_{c,d,s'',s'''}g^{ss''s''s}_{l;acdb}(k;\omega)\delta_{s'',s'''}(v_{\boldsymbol{k}y})^{s''s''}_{cd} \\ &+ \sum_{c,d,s'',s'''}\sum_{A,B,s_1,s_2}g^{ss_1s_2s}_{l;aABb}(k;\omega)\frac{1}{2i}\sum_{k'}\sum_{C,D,s_3,s_4}\sum_{l'=1}^{3} \\ &\times \mathcal{J}^{\{s_1\}}_{ll';\{A\}}(k,k';\omega)g^{s_3s''s'''s_4}_{l';CcdD}(k';\omega)\delta_{s'',s'''}(v_{\boldsymbol{k}'y})^{s''s''}_{cd} \\ &= \sum_{c,d,s'',s'''}g^{ss''s'''s}_{l;acdb}(k;\omega)\Lambda^{\mathrm{C};s''s'''}_{y;l;cd}(k;\omega),\end{aligned} \tag{39}$$

we can rewrite $\tilde{K}^{\mathrm{C(R)}}_{xy}(\omega)$ as follows:

$$\tilde{K}^{\mathrm{C(R)}}_{xy}(\omega) = -\frac{(-e)^2}{2i}\sum_{k}\sum_{\{a\}}\sum_{\{s\}}\delta_{s',s}(v_{\boldsymbol{k}x})^{ss}_{ba}\sum_{l=1}^{3}T_l(\epsilon,\omega) \times g^{ss''s'''s'}_{l;acdb}(k;\omega)\Lambda^{\mathrm{C};s''s'''}_{y;l;cd}(k;\omega). \quad (40)$$

A set of Eqs. (11), (26)–(32), (36), and (37) provides a framework to obtain an exact expression of σ^{C}_{xy} within the linear response of an external electric field.

We also obtain an exact framework for σ^{S}_{xy} by replacing $(-e)\delta_{s',s}(v_{\boldsymbol{k}x})^{ss}_{ba}$ in σ^{C}_{xy} by $\frac{1}{2}\mathrm{sgn}(s)\delta_{s',s}(v_{\boldsymbol{k}x})^{ss}_{ba}$. Namely, instead of Eqs. (11) and (40), we use Eq. (18) and

$$\tilde{K}^{\mathrm{S(R)}}_{xy}(\omega) = -\frac{(-e)}{2^2 i}\sum_{k}\sum_{\{a\}}\sum_{\{s\}}\delta_{s',s}\mathrm{sgn}(s)(v_{\boldsymbol{k}x})^{ss}_{ba}\sum_{l=1}^{3} \times T_l(\epsilon,\omega)g^{ss''s'''s'}_{l;acdb}(k;\omega)\Lambda^{\mathrm{C};s''s'''}_{y;l;cd}(k;\omega). \quad (41)$$

III. RESULTS

Since it is difficult to solve the exact expressions of σ^{C}_{xy} and σ^{S}_{xy} in the linear response theory, we adopt the approximation appropriate for an interacting metal to these expressions, and analyze the interaction effects on σ^{C}_{xy} and σ^{S}_{xy}. First, we derive the approximate expressions of σ^{C}_{xy} and σ^{S}_{xy} in Éliashberg's approximation [49]; in part of this derivation, we use Appendix C. This approximation is usually used to derive transport coefficients of an interacting metal microscopically [43,48,49,52,54], and its result [49] can reproduce the phenomenological transport equation in the Fermi liquid [42,51,55]; thus, this approximation may be appropriate if the terms included remain considerable. Comparing the derived σ^{C}_{xy} or σ^{S}_{xy} with the corresponding noninteracting result, we analyze the interaction effects on the derived σ^{C}_{xy} or σ^{S}_{xy}. Second, we address the applicability of this approximation, and show its limit in clean and low-T case. The correct understanding of this applicability is important to understand the difference between σ^{C}_{xy} and σ^{C}_{xx}. Third, we introduce an approximation beyond Éliashberg's approximation in order to describe the outside of the applicable region of Éliashberg's approximation, and derive the approximate expressions of σ^{C}_{xy} and σ^{S}_{xy}. We also analyze how the additional terms are affected by the electron-electron interaction.

A. Éliashberg's approximation

After reviewing the singular property of a retarded-advanced product of two single-particle Green's functions in the presence of the Fermi surface with several long-lived QPs, we derive the approximate expressions of σ^{C}_{xy} and σ^{S}_{xy} by utilizing this property. Then, let us argue the interaction effects due to the modifications from the noninteracting result.

1. Formulation

We begin with the singular properties [42,51] of a retarded-advanced product of two single-particle Green's functions such as $G^{(\mathrm{R})s_1s_3}_{ac}(k+\frac{q}{2})G^{(\mathrm{A})s_4s_2}_{db}(k-\frac{q}{2})$ in the limits $q\to 0$ and $\gamma^*_\alpha(\boldsymbol{k}_{\mathrm{F}})/T\to 0$ in the presence of the Fermi surface. In the presence of the Fermi surface, we can well define QPs with the long-lived lifetime for at least several Fermi momenta [56–58]. These QPs are well described by the coherent part of the single-particle Green's function [42,46,51], given by

$$G^{(\mathrm{R})ss'}_{\mathrm{coh};ab}(k) = \sum_{\alpha}(U_{\boldsymbol{k}})^{s}_{a\alpha}\frac{z_\alpha(\boldsymbol{k})}{\epsilon-\xi^*_\alpha(\boldsymbol{k})+i\gamma^*_\alpha(\boldsymbol{k})}(U^\dagger_{\boldsymbol{k}})^{s'}_{\alpha b}, \quad (42)$$

where $(U_{\boldsymbol{k}})^{s}_{a\alpha}$ is the unitary matrix used to obtain $\xi^*_\alpha(\boldsymbol{k})$. Then, for analyses of the limiting properties of the products of two single-particle Green's functions, it is sufficient to consider only the coherent parts [51]. This is because in the limits under consideration, the incoherent part [i.e., $G^{(\mathrm{R})ss'}_{ab}(k)-G^{(\mathrm{R})ss'}_{\mathrm{coh};ab}(k)$] is well defined and only the product of the coherent parts can be singular due to the merging of their poles [51]. Such singular behavior is obtained only for a retarded-advanced product because the poles of the coherent parts merge only if one of the poles crosses over the Fermi surface and because such crossing occurs only for a retarded-advanced product [51]. As a result, a retarded-advanced product gives the leading dependence on external momentum and frequency and the QP damping, and the dependence of a retarded-retarded or an advanced-advanced product is approximately negligible [49]. This treatment remains reasonable even for finite $\gamma^*_\alpha(\boldsymbol{k}_{\mathrm{F}})/T$ if $\gamma^*_\alpha(\boldsymbol{k}_{\mathrm{F}})/T$ satisfies $\gamma^*_\alpha(\boldsymbol{k}_{\mathrm{F}})/T<1$ because this treatment is regarded as a lowest-order expansion in terms of $\gamma^*_\alpha(\boldsymbol{k}_{\mathrm{F}})/T$.

Utilizing the singular property of a retarded-advanced product of two single-particle Green's functions, we derive approximate expressions of σ^{C}_{xy} and σ^{S}_{xy} in Éliashberg's approximation. (Because of the same reason for the exact formulation in the linear response theory, we explain the derivation for σ^{C}_{xy} in detail.) To utilize the singular property, we introduce two quantities $\mathcal{J}^{(0)\{s_1\}}_{ll';\{a\}}(k,k';\omega)$ and $\Lambda^{\mathrm{C}(0)ss'}_{v;l;ab}(k;\omega)$:

$$\mathcal{J}^{(0)\{s_1\}}_{ll';\{a\}}(k,k';\omega) = \mathcal{J}^{(1)\{s_1\}}_{ll';\{a\}}(k,k';\omega) + \frac{1}{2i}\sum_{k''}\sum_{\{A\}}\sum_{\{s'_1\}}\sum_{l'=1,3}\mathcal{J}^{(1)s_1s_2s'_3s'_4}_{ll'';abCD}(k,k'';\omega) \times g^{s'_3s'_1s'_2s'_4}_{l'';CABD}(k'';\omega)\mathcal{J}^{(0)s'_1s'_2s_3s_4}_{l''l';ABcd}(k'',k';\omega) \quad (43)$$

and

$$\Lambda^{\mathrm{C}(0)ss'}_{v;l;ab}(k;\omega) = \delta_{s,s'}(v_{\boldsymbol{k}v})^{ss}_{ab} + \sum_{k'}\sum_{A,B,s_1,s_2}\alpha^{(0)ss's_1s_2}_{l;abAB}(k,k';\omega)\delta_{s_1,s_2}(v_{\boldsymbol{k}'v})^{s_1s_1}_{AB}, \quad (44)$$

where $\alpha^{(0)ss's_1s_2}_{l;abAB}(k,k';\omega)$ is

$$\alpha^{(0)ss's_1s_2}_{l;abAB}(k,k';\omega) = \frac{1}{2i}\sum_{C,D,s_3,s_4}\sum_{l'=1,3}\mathcal{J}^{(0)ss's_3s_4}_{ll';abCD}(k,k';\omega) \times g^{s_3s_1s_2s_4}_{l';CABD}(k';\omega). \quad (45)$$

Equations (43) and (44) show that $\mathcal{J}^{(0)\{s_1\}}_{ll';\{a\}}(k,k';\omega)$ and $\Lambda^{\mathrm{C}(0)ss'}_{v;l;ab}(k;\omega)$ do not include a retarded-advanced product of two single-particle Green's functions. Thus, those quantities can be used to exclude the terms including at least a

retarded-advanced product from the terms of $\tilde{K}_{xy}^{\mathrm{C(R)}}(\omega)$ in Eq. (40). Among those terms, we need to decompose the terms for $l=1$ and 3 in $\tilde{K}_{xy}^{\mathrm{C(R)}}(\omega)$ into the terms without and with the retarded-advanced product. This is because $\Lambda_{\nu;l;ab}^{\mathrm{C}(0)ss'}(k;\omega)$ is connected with $\Lambda_{\nu;l;ab}^{\mathrm{C};ss'}(k;\omega)$ through the Bethe-Salpeter equation

$$\Lambda_{\nu;l;ab}^{\mathrm{C};ss'}(k;\omega)=\Lambda_{\nu;l;ab}^{\mathrm{C}(0)ss'}(k;\omega)+\frac{1}{2i}\sum_{k'}\sum_{\{A\}}\sum_{\{s_1\}}\mathcal{J}_{l2;abCD}^{(0)ss's_3s_4}(k,k';\omega)\times g_{2;CABD}^{s_3s_1s_2s_4}(k';\omega)\Lambda_{\nu;2;AB}^{\mathrm{C};s_1s_2}(k';\omega). \quad (46)$$

After the decomposition of the terms for $l=1$ and 3 in $\tilde{K}_{xy}^{\mathrm{C(R)}}(\omega)$ in Eq. (40), explained in Appendix C, we obtain

$$\begin{aligned}\tilde{K}_{xy}^{\mathrm{C(R)}}(\omega)=&-\frac{(-e)^2}{2i}\sum_{k}\sum_{\{a\}}\sum_{\{s\}}\delta_{s',s}(v_{\boldsymbol{k}x})_{ba}^{ss}\sum_{l=1,3}\\&\times\mathrm{sgn}(2-l)T_l(\epsilon,\omega)g_{l;acdb}^{ss''s'''s'}(k;\omega)\Lambda_{y;l;cd}^{\mathrm{C}(0)s''s'''}(k;\omega)\\&-\frac{(-e)^2}{2i}\sum_{k}\sum_{\{a\}}\sum_{\{s\}}\Lambda_{x;2;ba}^{\mathrm{C}(0)s's}(\boldsymbol{k},\epsilon,\boldsymbol{k},\epsilon+\omega)\\&\times T_2(\epsilon,\omega)g_{2;acdb}^{ss''s'''s'}(k;\omega)\Lambda_{y;2;cd}^{\mathrm{C};s''s'''}(k;\omega). \quad (47)\end{aligned}$$

This equation shows that only the second term includes a retarded-advanced product of two single-particle Green's functions. Since we assume in Éliashberg's approximation [49] that the leading terms of $\tilde{K}_{xy}^{\mathrm{C(R)}}(\omega)$ come from the most divergent terms in $q\to 0$ and $\gamma_\alpha^*(\boldsymbol{k}_{\mathrm{F}})/T\to 0$, we obtain σ_{xy}^{C} in this approximation $\sigma_{xy}^{\mathrm{C}}=\sigma_{xy}^{\mathrm{C(I)}}$,

$$\begin{aligned}\sigma_{xy}^{\mathrm{C(I)}}=&(-e)^2\sum_{k}\sum_{\{a\}}\sum_{\{s\}}\left(-\frac{\partial f(\epsilon)}{\partial\epsilon}\right)\Lambda_{x;2;ba}^{\mathrm{C}(0)s's}(k;0)\\&\times G_{ac}^{(\mathrm{R})ss''}(k)G_{db}^{(\mathrm{A})s'''s'}(k)\Lambda_{y;2;cd}^{\mathrm{C};s''s'''}(k;0), \quad (48)\end{aligned}$$

where $f(\epsilon)$ is the Fermi distribution function. We also obtain σ_{xy}^{S} in this approximation $\sigma_{xy}^{\mathrm{S}}=\sigma_{xy}^{\mathrm{S(I)}}$,

$$\begin{aligned}\sigma_{xy}^{\mathrm{S(I)}}=&\frac{(-e)}{2}\sum_{k}\sum_{\{a\}}\sum_{\{s\}}\left(-\frac{\partial f(\epsilon)}{\partial\epsilon}\right)\Lambda_{x;2;ba}^{\mathrm{S}(0)s's}(k;0)\\&\times G_{ac}^{(\mathrm{R})ss''}(k)G_{db}^{(\mathrm{A})s'''s'}(k)\Lambda_{y;2;cd}^{\mathrm{C};s''s'''}(k;0), \quad (49)\end{aligned}$$

by adopting the same argument to Eqs. (18) and (41) and introducing the vertex function of the spin current,

$$\begin{aligned}&\Lambda_{\nu;l;ab}^{\mathrm{S}(0)ss'}(k;\omega)\\&=\delta_{s,s'}\mathrm{sgn}(s)(v_{\boldsymbol{k}\nu})_{ab}^{ss}+\sum_{k'}\sum_{A,B,s_1,s_2}\alpha_{l;abAB}^{(0)ss's_1s_2}(k,k';\omega)\\&\times\delta_{s_1,s_2}\mathrm{sgn}(s_1)(v_{\boldsymbol{k}'\nu})_{AB}^{s_1s_1}. \quad (50)\end{aligned}$$

By using that vertex function, $\tilde{K}_{xy}^{\mathrm{S(R)}}(\omega)$ can be exactly rewritten as follows:

$$\begin{aligned}\tilde{K}_{xy}^{\mathrm{S(R)}}(\omega)=&-\frac{(-e)}{2^2i}\sum_{k}\sum_{\{a\}}\sum_{\{s\}}\delta_{s',s}\mathrm{sgn}(s)(v_{\boldsymbol{k}x})_{ba}^{ss}\\&\times\sum_{l=1,3}\mathrm{sgn}(2-l)T_l(\epsilon,\omega)g_{l;acdb}^{ss''s'''s'}(k;\omega)\\&\times\Lambda_{y;l;cd}^{\mathrm{C}(0)s''s'''}(k;\omega)\\&-\frac{(-e)}{2^2i}\sum_{k}\sum_{\{a\}}\sum_{\{s\}}\Lambda_{x;2;ba}^{\mathrm{S}(0)s's}(\boldsymbol{k},\epsilon,\boldsymbol{k},\epsilon+\omega)\\&\times T_2(\epsilon,\omega)g_{2;acdb}^{ss''s'''s'}(k;\omega)\Lambda_{y;2;cd}^{\mathrm{C};s''s'''}(k;\omega). \quad (51)\end{aligned}$$

Because of the same reason for $\Lambda_{y;2;cd}^{\mathrm{C};s''s'''}(k;\omega)$, $\Lambda_{x;2;ba}^{\mathrm{C}(0)s's}(k;\omega)$ and $\Lambda_{x;2;ba}^{\mathrm{S}(0)s's}(k;\omega)$ include the corrections to the noninteracting charge and spin currents, respectively, due to the multiple electron-hole scattering arising from $\hat{H}_{\mathrm{int}}$ and such corrections arising from $\hat{H}_{\mathrm{imp}}$ completely vanish in even-parity systems for the weak onsite scattering potential of dilute nonmagnetic impurities. Note that in the similar way for $\Lambda_{y;2;cd}^{\mathrm{C};s''s'''}(k;\omega)$, we can rewrite Eqs. (44) and (50) using Eq. (43) as follows:

$$\begin{aligned}\Lambda_{\nu;l;ab}^{\mathrm{C}(0)ss'}(k;\omega)=&\delta_{s,s'}(v_{\boldsymbol{k}\nu})_{ab}^{ss}\\&+\frac{1}{2i}\sum_{k'}\sum_{\{A\}}\sum_{\{s_1\}}\sum_{l'=1,3}\mathcal{J}_{ll';abCD}^{(1)ss's_3s_4}(k,k';\omega)\\&\times g_{l';CABD}^{s_3s_1s_2s_4}(k';\omega)\Lambda_{\nu;l';AB}^{\mathrm{C}(0)s_1s_2}(k';\omega) \quad (52)\end{aligned}$$

and

$$\begin{aligned}\Lambda_{\nu;l;ab}^{\mathrm{S}(0)ss'}(k;\omega)=&\delta_{s,s'}\mathrm{sgn}(s)(v_{\boldsymbol{k}\nu})_{ab}^{ss}\\&+\frac{1}{2i}\sum_{k'}\sum_{\{A\}}\sum_{\{s_1\}}\sum_{l'=1,3}\mathcal{J}_{ll';abCD}^{(1)ss's_3s_4}(k,k';\omega)\\&\times g_{l';CABD}^{s_3s_1s_2s_4}(k';\omega)\Lambda_{\nu;l';AB}^{\mathrm{S}(0)s_1s_2}(k';\omega). \quad (53)\end{aligned}$$

2. *Interaction effects*

Since the comparison between the derived Fermi surface term and the noninteracting Fermi surface term is useful to deduce the interaction effects on the Fermi surface term, we show the noninteracting Fermi surface terms [17,19] of the intrinsic AHE and SHE, $\sigma_{xy}^{\mathrm{C(0;I)}}$ and $\sigma_{xy}^{\mathrm{S(0;I)}}$:

$$\begin{aligned}\sigma_{xy}^{\mathrm{C(0;I)}}=&(-e)^2\sum_{k}\sum_{\{a\}}\sum_{\{s\}}\left(-\frac{\partial f(\epsilon)}{\partial\epsilon}\right)\delta_{s',s}(v_{\boldsymbol{k}x})_{ba}^{ss}\\&\times G_{ac}^{(0;\mathrm{R})ss''}(k)G_{db}^{(0;\mathrm{A})s'''s'}(k)\delta_{s'',s'''}(v_{\boldsymbol{k}'y})_{cd}^{s''s''} \quad (54)\end{aligned}$$

and

$$\begin{aligned}\sigma_{xy}^{\mathrm{S(0;I)}}=&\frac{(-e)}{2}\sum_{k}\sum_{\{a\}}\sum_{\{s\}}\left(-\frac{\partial f(\epsilon)}{\partial\epsilon}\right)\delta_{s',s}(v_{\boldsymbol{k}x})_{ba}^{ss}\\&\times G_{ac}^{(0;\mathrm{R})ss''}(k)G_{db}^{(0;\mathrm{A})s'''s'}(k)\delta_{s'',s'''}(v_{\boldsymbol{k}'y})_{cd}^{s''s''}. \quad (55)\end{aligned}$$

Comparing Eq. (48) or (49) with Eq. (54) or (55), respectively, we see the electron-electron interaction causes three modifications. First, the x component of the noninteracting

charge or spin current becomes, respectively, $(-e)\Lambda^{C(0)s's}_{x;2;ba}(k;0)$ in $\sigma^{C(I)}_{xy}$ or $\frac{1}{2}\Lambda^{S(0)s's}_{x;2;ba}(k;0)$ in $\sigma^{S(I)}_{xy}$. Second, the two single-particle Green's functions change from noninteracting to interacting. Third, the y component of the noninteracting charge current becomes $(-e)\Lambda^{C;s''s'''}_{y;2;cd}(k;0)$.

Let us begin with the interaction effect due to the replacement of the single-particle Green's functions. Since the interaction effects on the single-particle Green's function arise from the self-energy [Eq. (22)] and the self-energy causes the QP damping [Eq. (8)], we analyze the damping dependence of $\sigma^{C(I)}_{xy}$ or $\sigma^{S(I)}_{xy}$. For that purpose, we need to analyze the damping dependence of $g^{ss''s'''s'}_{2;acdb}(k;0)$ in $\sigma^{C(I)}_{xy}$ or $\sigma^{S(I)}_{xy}$ because the others are $O(\gamma^0)$, where γ is of the order of magnitude the QP damping [49]. As explained, $g^{ss''s'''s'}_{2;acdb}(k;0)$ has the leading damping dependence among several products of two single-particle Green's functions because of the limiting property of the product of the coherent parts of the retarded and the advanced single-particle Green's function. That leading damping dependence is given by [46]

$$g^{ss''s'''s'}_{2;acdb}(k;0) \approx i2\pi \sum_{\alpha,\beta} u^{ss''s'''s'}_{a\alpha c;d\beta b}(\boldsymbol{k}) z_\alpha(\boldsymbol{k}) z_\beta(\boldsymbol{k}) \times \frac{\delta[\epsilon - \xi^*_\alpha(\boldsymbol{k})]}{\Delta\xi^*_{\beta\alpha}(\boldsymbol{k}) + i[\gamma^*_\alpha(\boldsymbol{k}) + \gamma^*_\beta(\boldsymbol{k})]}, \quad (56)$$

with $\Delta\xi^*_{\beta\alpha}(\boldsymbol{k}) \equiv \xi^*_\beta(\boldsymbol{k}) - \xi^*_\alpha(\boldsymbol{k})$ and $u^{ss''s'''s'}_{a\alpha c;d\beta b}(\boldsymbol{k}) \equiv (U_{\boldsymbol{k}})^s_{a\alpha}(U^\dagger_{\boldsymbol{k}})^{s'''}_{\alpha c}(U_{\boldsymbol{k}})^{s'''}_{d\beta}(U^\dagger_{\boldsymbol{k}})^{s'}_{\beta b}$. In deriving Eq. (56), we have used Eq. (42) and the identity

$$\lim_{\delta\to 0+}\left[\frac{1}{z - X + i\delta} - \frac{1}{z - X - i\delta}\right] = -2\pi i\delta(z - X). \quad (57)$$

Equation (56) can be also rewritten as

$$g^{ss''s'''s'}_{2;acdb}(k;0) \approx i\pi \sum_{\alpha,\beta} u^{ss''s'''s'}_{a\alpha c;d\beta b}(\boldsymbol{k}) z_\alpha(\boldsymbol{k}) z_\beta(\boldsymbol{k}) \times [\delta(\epsilon - \xi^*_\alpha(\boldsymbol{k})) + \delta(\epsilon - \xi^*_\beta(\boldsymbol{k}))] \times \frac{\Delta\xi^*_{\beta\alpha}(\boldsymbol{k}) - i[\gamma^*_\alpha(\boldsymbol{k}) + \gamma^*_\beta(\boldsymbol{k})]}{\Delta\xi^*_{\beta\alpha}(\boldsymbol{k})^2 + [\gamma^*_\alpha(\boldsymbol{k}) + \gamma^*_\beta(\boldsymbol{k})]^2} \quad (58)$$

by using two equalities

$$G^{(R)ss'}_{ab}(k) = G^{(A)s's}_{ba}(k)^* \quad (59)$$

and

$$u^{ss''s'''s'}_{a\alpha c;d\beta b}(\boldsymbol{k}) = u^{s's'''s''s}_{b\beta d;c\alpha a}(\boldsymbol{k})^*. \quad (60)$$

To see the finite components of $g^{ss''s'''s'}_{2;acdb}(k;0)$ in $\sigma^{C(I)}_{xy}$ or $\sigma^{S(I)}_{xy}$, we should detect the terms odd with respect to k_x and k_y. This is because $\Lambda^{C(0)s's}_{x;2;ba}(k;0)$ and $\Lambda^{S(0)s's}_{x;2;ba}(k;0)$ are odd with respect to k_x due to the k_x derivative in $(v_{\boldsymbol{k}x})^{ss}_{ba}$ [see Eqs. (44) and (50)] and $\Lambda^{C;s''s'''}_{y;2;cd}(k;0)$ is odd with respect to k_y due to the k_y derivative in $(v_{\boldsymbol{k}y})^{ss}_{ba}$ [see Eq. (36)], i.e., the terms other than $\Lambda^{C(0)s's}_{x;2;ba}(k;0)$ [$\Lambda^{S(0)s's}_{x;2;ba}(k;0)$] and $\Lambda^{C;s''s'''}_{y;2;cd}(k;0)$ in $\sigma^{C(I)}_{xy}$ ($\sigma^{S(I)}_{xy}$) should be odd with respect to k_x and k_y to obtain finite terms after taking the $\boldsymbol{k}$ summation. Note that an integrand of the $\boldsymbol{k}$ summation should be even about each k_η to obtain the finite value. Since such odd terms arise from the terms proportional to $u^{ss''s'''s'}_{a\alpha c;d\beta b}(\boldsymbol{k})\Delta\xi^*_{\beta\alpha}(\boldsymbol{k})$ ($\alpha \neq \beta$) in Eq. (56), the dominant multiband excitations for $\sigma^{C(I)}_{xy}$ or $\sigma^{S(I)}_{xy}$ are interband; to obtain finite odd terms arising from those terms, the hopping integral with the odd mirror symmetry is necessary. For further argument, let us consider a simple but sufficient situation: the finite terms of σ^{C}_{xy} or σ^{S}_{xy} come from the interband excitations only at $\boldsymbol{k} = \boldsymbol{k}_0$. In this situation, the leading terms of $g^{ss''s'''s'}_{2;acdb}(k;0)$ in $\sigma^{C(I)}_{xy}$ or $\sigma^{S(I)}_{xy}$ become $O(\gamma^{-2})$ in $|\Delta\xi^*_{\beta\alpha}(\boldsymbol{k}_0)| \ll [\gamma^*_\alpha(\boldsymbol{k}_0) + \gamma^*_\beta(\boldsymbol{k}_0)]$, and $O(\gamma^0)$ in $|\Delta\xi^*_{\beta\alpha}(\boldsymbol{k}_0)| \gg [\gamma^*_\alpha(\boldsymbol{k}_0) + \gamma^*_\beta(\boldsymbol{k}_0)]$. As a result, σ^{C}_{xy} or σ^{S}_{xy} becomes $O(\gamma^{-2})$ in the former limit and $O(\gamma^0)$ in the latter limit. More precisely, the leading terms of $\sigma^{C(I)}_{xy}$ and $\sigma^{S(I)}_{xy}$ in $|\Delta\xi^*_{\beta\alpha}(\boldsymbol{k}_0)| \ll [\gamma^*_\alpha(\boldsymbol{k}_0) + \gamma^*_\beta(\boldsymbol{k}_0)]$ are given by

$$\sigma^{C(I)}_{xy} \approx \frac{-(-e)^2}{2N}\sum_\alpha\sum_{\beta\neq\alpha}\frac{\Delta\xi^*_{\beta\alpha}(\boldsymbol{k}_0)}{[\gamma^*_\alpha(\boldsymbol{k}_0) + \gamma^*_\beta(\boldsymbol{k}_0)]^2} \times \left\{\mathrm{Im}\left[\tilde\Lambda^{C(0)}_{x;2;\beta\alpha}(\boldsymbol{k}_0,\xi^*_\alpha(\boldsymbol{k}_0))\tilde\Lambda^{C}_{y;2;\alpha\beta}(\boldsymbol{k}_0,\xi^*_\alpha(\boldsymbol{k}_0))\right] + \mathrm{Im}\left[\tilde\Lambda^{C(0)}_{x;2;\beta\alpha}(\boldsymbol{k}_0,\xi^*_\beta(\boldsymbol{k}_0))\tilde\Lambda^{C}_{y;2;\alpha\beta}(\boldsymbol{k}_0,\xi^*_\beta(\boldsymbol{k}_0))\right]\right\} \quad (61)$$

and

$$\sigma^{S(I)}_{xy} \approx \frac{-(-e)}{2^2N}\sum_\alpha\sum_{\beta\neq\alpha}\frac{\Delta\xi^*_{\beta\alpha}(\boldsymbol{k}_0)}{[\gamma^*_\alpha(\boldsymbol{k}_0) + \gamma^*_\beta(\boldsymbol{k}_0)]^2} \times \left\{\mathrm{Im}\left[\tilde\Lambda^{S(0)}_{x;2;\beta\alpha}(\boldsymbol{k}_0,\xi^*_\alpha(\boldsymbol{k}_0))\tilde\Lambda^{C}_{y;2;\alpha\beta}(\boldsymbol{k}_0,\xi^*_\alpha(\boldsymbol{k}_0))\right] + \mathrm{Im}\left[\tilde\Lambda^{S(0)}_{x;2;\beta\alpha}(\boldsymbol{k}_0,\xi^*_\beta(\boldsymbol{k}_0))\tilde\Lambda^{C}_{y;2;\alpha\beta}(\boldsymbol{k}_0,\xi^*_\beta(\boldsymbol{k}_0))\right]\right\}, \quad (62)$$

respectively; in $|\Delta\xi^*_{\beta\alpha}(\boldsymbol{k}_0)| \gg [\gamma^*_\alpha(\boldsymbol{k}_0) + \gamma^*_\beta(\boldsymbol{k}_0)]$, $\sigma^{C(I)}_{xy}$ and $\sigma^{S(I)}_{xy}$ are given by

$$\sigma^{C(I)}_{xy} \approx \frac{-(-e)^2}{2N}\sum_\alpha\sum_{\beta\neq\alpha}\frac{1}{\Delta\xi^*_{\beta\alpha}(\boldsymbol{k}_0)} \times \left\{\mathrm{Im}\left[\tilde\Lambda^{C(0)}_{x;2;\beta\alpha}(\boldsymbol{k}_0,\xi^*_\alpha(\boldsymbol{k}_0))\tilde\Lambda^{C}_{y;2;\alpha\beta}(\boldsymbol{k}_0,\xi^*_\alpha(\boldsymbol{k}_0))\right] + \mathrm{Im}\left[\tilde\Lambda^{C(0)}_{x;2;\beta\alpha}(\boldsymbol{k}_0,\xi^*_\beta(\boldsymbol{k}_0))\tilde\Lambda^{C}_{y;2;\alpha\beta}(\boldsymbol{k}_0,\xi^*_\beta(\boldsymbol{k}_0))\right]\right\} \quad (63)$$

and

$$\sigma^{S(I)}_{xy} \approx \frac{-(-e)}{2^2N}\sum_\alpha\sum_{\beta\neq\alpha}\frac{1}{\Delta\xi^*_{\beta\alpha}(\boldsymbol{k}_0)} \times \left\{\mathrm{Im}\left[\tilde\Lambda^{S(0)}_{x;2;\beta\alpha}(\boldsymbol{k}_0,\xi^*_\alpha(\boldsymbol{k}_0))\tilde\Lambda^{C}_{y;2;\alpha\beta}(\boldsymbol{k}_0,\xi^*_\alpha(\boldsymbol{k}_0))\right] + \mathrm{Im}\left[\tilde\Lambda^{S(0)}_{x;2;\beta\alpha}(\boldsymbol{k}_0,\xi^*_\beta(\boldsymbol{k}_0))\tilde\Lambda^{C}_{y;2;\alpha\beta}(\boldsymbol{k}_0,\xi^*_\beta(\boldsymbol{k}_0))\right]\right\}, \quad (64)$$

respectively. In those equations, we have introduced three quantities:

$$\tilde\Lambda^{C(0)}_{\nu;2;\beta\alpha}(\boldsymbol{k},\epsilon) = \sqrt{z_\beta(\boldsymbol{k})z_\alpha(\boldsymbol{k})} \times \sum_{a,b,s,s'}(U^\dagger_{\boldsymbol{k}})^{s'}_{\beta b}\Lambda^{C(0)s's}_{\nu;2;ba}(k;0)(U_{\boldsymbol{k}})^s_{a\alpha}, \quad (65)$$

$$\tilde\Lambda^{C}_{\nu;2;\alpha\beta}(\boldsymbol{k},\epsilon) = \sqrt{z_\alpha(\boldsymbol{k})z_\beta(\boldsymbol{k})} \times \sum_{c,d,s'',s'''}(U^\dagger_{\boldsymbol{k}})^{s''}_{\alpha c}\Lambda^{C;s''s'''}_{\nu;2;cd}(k;0)(U_{\boldsymbol{k}})^{s'''}_{d\beta}, \quad (66)$$

and

$$\tilde{\Lambda}^{\mathrm{S}(0)}_{\nu;2;\beta\alpha}(\boldsymbol{k},\epsilon) = \sqrt{z_\beta(\boldsymbol{k})z_\alpha(\boldsymbol{k})} \times \sum_{a,b,s,s'} (U^\dagger_{\boldsymbol{k}})^{s'}_{\beta b}\Lambda^{\mathrm{S}(0)s's}_{\nu;2;ba}(k;0)(U_{\boldsymbol{k}})^{s}_{a\alpha}. \quad (67)$$

For more complex situations with the interband excitations at $\boldsymbol{k} = \boldsymbol{k}_0, \boldsymbol{k}_1, \ldots, \boldsymbol{k}_{K-1}$, we need to apply the above argument for the simple situation to each term of the interband excitations at $\boldsymbol{k}_j$ and combine each other's damping dependence: if at least one of the interband excitations satisfies $|\Delta\xi^*_{\beta\alpha}(\boldsymbol{k}_j)| \ll [\gamma^*_\alpha(\boldsymbol{k}_j) + \gamma^*_\beta(\boldsymbol{k}_j)]$, $\sigma^{\mathrm{C(I)}}_{xy}$ or $\sigma^{\mathrm{S(I)}}_{xy}$ becomes damping dependent; on the other hand, if all the interband excitations satisfy $|\Delta\xi^*_{\beta\alpha}(\boldsymbol{k}_j)| \gg [\gamma^*_\alpha(\boldsymbol{k}_j) + \gamma^*_\beta(\boldsymbol{k}_j)]$, $\sigma^{\mathrm{C(I)}}_{xy}$ or $\sigma^{\mathrm{S(I)}}_{xy}$ is damping independent. Thus, the electron-electron interaction causes the finite damping dependence of σ^{C}_{xy} and σ^{S}_{xy} at high temperatures even without impurities. Furthermore, since the interaction-induced QP damping decreases with decreasing temperature [42], the electron-electron interaction causes the emergence of the temperature dependence of σ^{C}_{xy} and σ^{S}_{xy} and a crossover from damping dependent to damping independent σ^{C}_{xy} or σ^{S}_{xy} even without impurities (see Fig. 2).

Then, we see the interaction effect due to the replacement of the spin current for $\sigma^{\mathrm{S(I)}}_{xy}$. This is related to the effects of the SCD because the difference between $\sigma^{\mathrm{C(I)}}_{xy}$ and $\sigma^{\mathrm{S(I)}}_{xy}$ comes from the difference between $(-e)\Lambda^{\mathrm{C}(0)s's}_{x;2;ba}(k;0)$ and $\frac{1}{2}\Lambda^{\mathrm{S}(0)s's}_{x;2;ba}(k;0)$ [see Eqs. (48) and (49)]. Actually, rewriting $\Lambda^{\mathrm{S}(0)s's}_{2;x;ba}(k;0)$ by using $\Lambda^{\mathrm{C}(0)s's}_{2;x;ba}(k;0)$ as

$$\Lambda^{\mathrm{S}(0)s's}_{2;x;ba}(k;0) = \mathrm{sgn}(s)\Lambda^{\mathrm{C}(0)s's}_{2;x;ba}(k;0) - 2\,\mathrm{sgn}(s) \times \sum_{k'}\sum_{A,B} \alpha^{(0)s's-s-s}_{2;baAB}(k,k';0)(v_{k'x})^{-s-s}_{AB}, \quad (68)$$

and substituting Eq. (68) into (49), we can show that the second term of Eq. (68) leads to a SCD-induced correction of $\sigma^{\mathrm{S(I)}}_{xy}$,

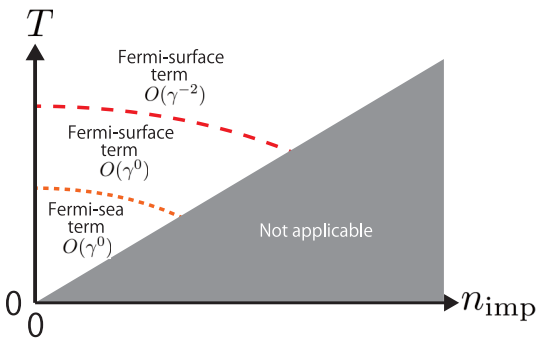

FIG. 2. Schematic diagram about the dominant term and damping dependence of σ^{C}_{xy} or σ^{S}_{xy}. Our formalism is applicable outside the gray triangle region because that region satisfies $\gamma^*_\alpha(\boldsymbol{k}_{\mathrm{F}})/T \geqslant 1$ due to the impurity-induced QP damping. The crossovers occur at the red and the orange dotted lines. The form of the red dotted line depends strongly on the temperature dependence of the QP damping.

$\Delta\sigma^{(\mathrm{SCD})}_{xy}$:

$$\begin{aligned}\sigma^{\mathrm{S(I)}}_{xy} &= \frac{1}{2(-e)}\sum_{\{s\}}\mathrm{sgn}(s)(-e)^2\sum_k\sum_{\{a\}}\left(-\frac{\partial f(\epsilon)}{\partial\epsilon}\right) \times \Lambda^{\mathrm{C}(0)s's}_{x;2;ba}(k;0)g^{ss''s'''s'}_{2;acdb}(k;0)\Lambda^{\mathrm{C};s''s'''}_{y;2;cd}(k;0) \\ &\quad - \frac{(-e)}{N}\sum_{\{s\}}\mathrm{sgn}(s)\sum_k\sum_{\{a\}}\left(-\frac{\partial f(\epsilon)}{\partial\epsilon}\right) \times \sum_{k'}\sum_{A,B}\alpha^{(0)s's-s-s}_{2;baAB}(k,k';0)(v_{k'x})^{-s-s}_{AB} \\ &\quad \times g^{ss''s'''s'}_{2;acdb}(k;0)\Lambda^{\mathrm{C};s''s'''}_{y;2;cd}(k;0) \\ &= \frac{1}{2(-e)}\sum_{\{s\}}\mathrm{sgn}(s)\left(\sigma^{\mathrm{C(I)}}_{xy}\right)^{s'ss''s'''} + \Delta\sigma^{(\mathrm{SCD})}_{xy}, \end{aligned} \quad (69)$$

with spin-decomposed component of σ^{C}_{xy}, $(\sigma^{\mathrm{C}}_{xy})^{s'ss''s'''}$, defined as $\sigma^{\mathrm{C(I)}}_{xy} = \sum_{\{s\}}(\sigma^{\mathrm{C(I)}}_{xy})^{s'ss''s'''}$.

I believe this interpretation is appropriate because of the following arguments. Since the SCD [33,34] affects only spin transports, it is reasonable to suppose that the difference between $\sigma^{\mathrm{S(I)}}_{xy}$ and $\sigma^{\mathrm{C(I)}}_{xy}$ is related to the effects of the SCD on the Fermi surface term. In addition, it is consistent with the general property of the SCD in metals to suppose that the second term of Eq. (68) causes the correction due to the SCD because the second term represents the correction of the spin current due to the multiple scattering of the electron-electron interaction between different spins (see the second term for $s' = s$). Here, the general property is that only such multiple scattering causes the SCD in metals because for the onsite bare electron-electron interactions such as the Hubbard interactions, the multiple scattering is necessary to obtain the finite momentum transfer. Note that this general property of metals indicates the importance of the momentum dependence of the self-energy due to the electron-electron interaction in discussing the SCD in metals because that momentum dependence is necessary to obtain finite second term of Eq. (68).

Finally, let us see the interaction effects due to the other modifications, i.e., the replacement of the x component of the charge current in $\sigma^{\mathrm{C(I)}}_{xy}$ and the replacement of the y component of the charge current in $\sigma^{\mathrm{C(I)}}_{xy}$ or $\sigma^{\mathrm{S(I)}}_{xy}$. First, the former replacement causes a magnitude decrease of $\sigma^{\mathrm{C(I)}}_{xy}$ from a noninteracting value. This is because the correction term in $\Lambda^{\mathrm{C}(0)s's}_{x;2;ba}(k;0)$, the second term of Eq. (52) for $\omega = 0$, is related to the k_x derivative of the real part of the self-energy due to a Ward identity [51,59] and because its effect on the charge current, the renormalization of the group velocity, reduces a magnitude of the charge current [49]. Then, the latter replacement maybe changes not only the magnitude of $\sigma^{\mathrm{C(I)}}_{xy}$ or $\sigma^{\mathrm{S(I)}}_{xy}$, but also its sign in some cases near an antiferromagnetic quantum-critical point due to the similar mechanism for the weak-field usual Hall effect [43]. For the weak-field usual Hall effect without the onsite SOC, the angle change of the charge current can be induced near the antiferromagnetic quantum-critical point due to the momentum dependence of the irreducible four-point vertex function, and that angle change causes the sign change of the

usual Hall conductivity [43]. To check this possibility for the intrinsic AHE or SHE, we need a numerical calculation for $\sigma_{xy}^{\mathrm{C(I)}}$ or $\sigma_{xy}^{\mathrm{S(I)}}$ by applying an approximation appropriate near the antiferromagnetic quantum-critical point to the particular band structure. Since that is a next step, it remains an important issue to clarify the interaction effects of that replacement on $\sigma_{xy}^{\mathrm{C(I)}}$ or $\sigma_{xy}^{\mathrm{S(I)}}$.

B. Applicability of Éliashberg's approximation

We turn to applicability of Éliashberg's approximation [46,49,52] for σ_{xy}^{C} or σ_{xy}^{S}. First, we should restrict arguments to cases for $\gamma_\alpha^*(\boldsymbol{k}_\mathrm{F})/T < 1$ because Éliashberg's approximation is reasonable only for $\gamma_\alpha^*(\boldsymbol{k}_\mathrm{F})/T < 1$ (see Sec. III A). Thus, the gray triangle region in Fig. 2 is the outside of the applicable region. Then, there are two key factors to argue whether $\sigma_{xy}^{\mathrm{C(I)}}$ or $\sigma_{xy}^{\mathrm{S(I)}}$ become finite or not, i.e., the broadening of the QP spectra due to the QP damping and the broadening of $(-\frac{\partial f(\epsilon)}{\partial \epsilon})$ due to temperature. This is because of the following three facts: $\sigma_{xy}^{\mathrm{C(I)}}$ or $\sigma_{xy}^{\mathrm{S(I)}}$ consists of the integral of $(-\frac{\partial f(\epsilon)}{\partial \epsilon})$ for interband excitations; such integral becomes finite only for the finite overlap between the QP spectra of the two bands; that overlap arises from the above two key factors. Thus, in the range of $\gamma_\alpha^*(\boldsymbol{k}_\mathrm{F})/T < 1$, we have three distinct cases, i.e., high-T case, intermediate-T case, and low-T case: in the high-T case, both of the two factors lead to a finite overlap between the QP spectra of the two bands for at least an interband excitation; in the intermediate-T case, the finite overlap arises only from the broadening of $(-\frac{\partial f(\epsilon)}{\partial \epsilon})$; in the low-$T$ case, the overlap becomes negligible. For example, those three cases for the interband excitation at $\boldsymbol{k} = \boldsymbol{k}_j$ are shown schematically in Figs. 3(a), 3(b), and 3(c). As we see from Figs. 3(a) and 3(b), Éliashberg's approximation gives finite $\sigma_{xy}^{\mathrm{C(I)}}$ or $\sigma_{xy}^{\mathrm{S(I)}}$ in the high-T case and the intermediate-T case. However, in the low-T case, corresponding to Fig. 3(c), $\sigma_{xy}^{\mathrm{C(I)}}$ or $\sigma_{xy}^{\mathrm{S(I)}}$ become very small; thus, Éliashberg's approximation becomes insufficient. Since the high-T case, the intermediate-T case, and the low-T case correspond to, respectively, the upper, the middle, and the lower regions of the left triangle of Fig. 2, Éliashberg's approximation is sufficient to analyze the intrinsic AHE and SHE in the upper and the middle regions; for the analysis in the lower region, we need to take Éliashberg's approximation a step further.

C. Approximation beyond Éliashberg's approximation

Starting to explain the points missing in Éliashberg's approximation and being important in the low-T case, we construct an approximation beyond Éliashberg's approximation and derive the approximate expressions of σ_{xy}^{C} and σ_{xy}^{S} in this approximation. Then, we see the damping dependence and the effects of the SCD in this approximation. Furthermore, by comparison with the noninteracting Fermi sea term, we deduce how the electron-electron interaction affects the Fermi sea term of σ_{xy}^{C} or σ_{xy}^{S}.

1. Formulation

As we see in Sec. III B, in the low-T case, where temperature is low and the QP damping is small, the term of σ_{xy}^{C} or σ_{xy}^{S} considered in Éliashberg's approximation becomes very small. For analyses in such case, we need to use an appropriate approximation beyond Éliashberg's approximation. In particular, we should take account of the terms of the interband excitations including $f(\epsilon)$ because those terms remain finite even in clean and low-T case. Since Éliashberg's approximation has succeeded in getting reasonable descriptions of several transports of interacting metals (e.g., the resistivity [37,49] and the weak-field usual Hall effect [48,54]), I suppose that Éliashberg's approximation is not so bad even for the description of the intrinsic AHE or SHE, and that an approximation appropriate for analyses in the low-T case can be obtained by extending Éliashberg's approximation.

On the basis of those suppositions, we construct an approximation beyond Éliashberg's approximation by going back to the exact expression of $\tilde{K}_{xy}^{\mathrm{C(R)}}(\omega)$ or $\tilde{K}_{xy}^{\mathrm{S(R)}}(\omega)$ [Eq. (47) or (51)] and taking account of not only the terms considered in Éliashberg's approximation, but also the terms leading among the terms of the Fermi sea integral. Such leading terms come from the terms proportional to the ω-linear term of $g_{l;acdb}^{ss''s'''s'}(k;\omega)$ or $\Lambda_{y;l;cd}^{\mathrm{C}(0)s''s'''}(k;\omega)$ $(l = 1,3)$ in the first term of Eq. (47) or (51) because we need to take the leading ω-linear

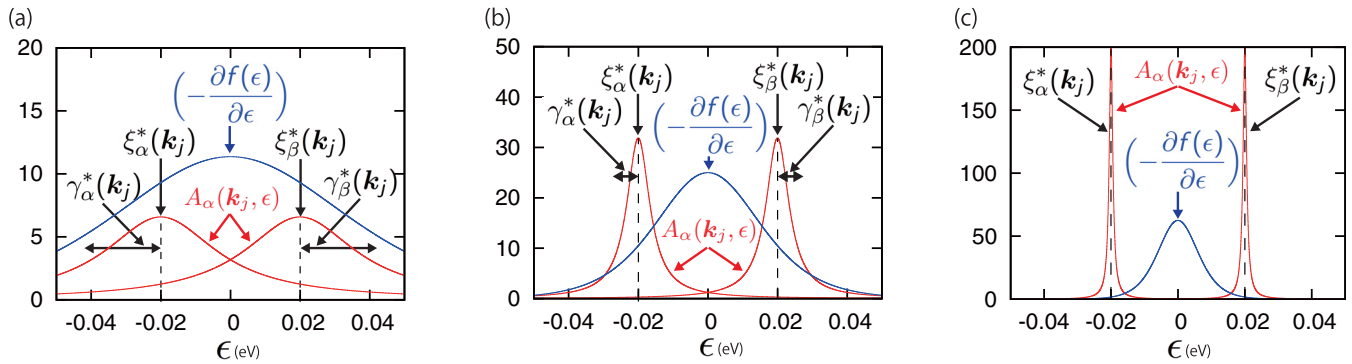

FIG. 3. QP spectral function $A_\alpha(\boldsymbol{k},\epsilon)$ and $(-\frac{\partial f(\epsilon)}{\partial \epsilon})$ for the interband excitation at $\boldsymbol{k} = \boldsymbol{k}_j$ in (a) the high-T case, (b) the intermediate case, and (c) the low-T case. In those panels, the QP spectral function is given by $A_\alpha(\boldsymbol{k}_j,\epsilon) = \frac{z_\alpha(\boldsymbol{k})}{\pi}\frac{\gamma_\alpha^*(\boldsymbol{k})}{[\epsilon-\xi_\alpha^*(\boldsymbol{k})]^2+\gamma_\alpha^*(\boldsymbol{k})^2}$; the parameters are chosen as $z_\alpha(\boldsymbol{k}_j) = z_\beta(\boldsymbol{k}_j) = 0.4$, $\xi_\alpha^*(\boldsymbol{k}_j) = -\xi_\beta^*(\boldsymbol{k}_j) = 0.02$ eV, and $\gamma_\alpha^*(\boldsymbol{k}_j) = \gamma_\beta^*(\boldsymbol{k}_j) = 40T^2$; T in panels (a), (b), and (c) are 0.004, 0.01, and 0.022 (eV), respectively.

terms to obtain σ_{xy}^{C} or σ_{xy}^{S} [see Eq. (11) or (18)]. Thus, the terms leading among the terms of the Fermi sea integral lead to additional terms of σ_{xy}^{C} and σ_{xy}^{S}.

As a result, σ_{xy}^{C} and σ_{xy}^{S} in this approximation become $\sigma_{xy}^{\mathrm{C}} = \sigma_{xy}^{\mathrm{C(I)}} + \sigma_{xy}^{\mathrm{C(II)}}$ and $\sigma_{xy}^{\mathrm{S}} = \sigma_{xy}^{\mathrm{S(I)}} + \sigma_{xy}^{\mathrm{S(II)}}$, respectively, where $\sigma_{xy}^{\mathrm{C(II)}}$ is

$$\sigma_{xy}^{\mathrm{C(II)}} = (-e)^2 \sum_k \sum_{\{a\}} \sum_{\{s\}} f(\epsilon)\delta_{s',s}(v_{\boldsymbol{k}x})_{ba}^{ss} \sum_{l=1,3} \times \mathrm{sgn}(l-2) \lim_{\omega\to 0} \frac{\partial}{\partial\omega}\left[g_{l;acdb}^{ss''s'''s'}(k;\omega)\Lambda_{y;l;cd}^{\mathrm{C}(0)s''s'''}(k;\omega)\right], \tag{70}$$

and $\sigma_{xy}^{\mathrm{S(II)}}$ is

$$\sigma_{xy}^{\mathrm{S(II)}} = \frac{(-e)}{2} \sum_k \sum_{\{a\}} \sum_{\{s\}} f(\epsilon)\mathrm{sgn}(s)\delta_{s',s}(v_{\boldsymbol{k}x})_{ba}^{ss} \sum_{l=1,3} \times \mathrm{sgn}(l-2) \lim_{\omega\to 0} \frac{\partial}{\partial\omega}\left[g_{l;acdb}^{ss''s'''s'}(k;\omega)\Lambda_{y;l;cd}^{\mathrm{C}(0)s''s'''}(k;\omega)\right]. \tag{71}$$

For the direct comparison with the noninteracting Fermi sea term, we rewrite part of the terms proportional to the ω derivative of $g_{l;acdb}^{ss''s'''s'}(k;\omega)(l=1,3)$ in Eqs. (70) and (71) as follows:

$$\begin{aligned}\sum_{l=1,3} \mathrm{sgn}(l-2) \lim_{\omega\to 0} \frac{\partial g_{l;acdb}^{ss''s'''s'}(k;\omega)}{\partial\omega} \Lambda_{y;l;cd}^{\mathrm{C}(0)s''s'''}(k;0) &= -\frac{\partial G_{ac}^{(\mathrm{R})ss''}(k)}{\partial\epsilon} G_{db}^{(\mathrm{R})s'''s'}(k) \Lambda_{y;1;cd}^{\mathrm{C}(0)s''s'''}(k;0) \\ &+ \frac{\partial G_{ac}^{(\mathrm{A})ss''}(k)}{\partial\epsilon} G_{db}^{(\mathrm{A})s'''s'}(k) \Lambda_{y;3;cd}^{\mathrm{C}(0)s''s'''}(k;0),\end{aligned} \tag{72}$$

where we use the identity

$$\lim_{\omega\to 0} \frac{\partial F(\epsilon+\omega)}{\partial\omega} = \frac{\partial F(\epsilon)}{\partial\epsilon}. \tag{73}$$

Namely, Eqs. (70) and (71) become

$$\begin{aligned}\sigma_{xy}^{\mathrm{C(II)}} = -(-e)^2 \sum_k \sum_{\{a\}} \sum_{\{s\}} f(\epsilon)\delta_{s',s}(v_{\boldsymbol{k}x})_{ba}^{ss} &\times \Bigg[\frac{\partial G_{ac}^{(\mathrm{R})ss''}(k)}{\partial\epsilon} G_{db}^{(\mathrm{R})s'''s'}(k) \Lambda_{y;1;cd}^{\mathrm{C}(0)s''s'''}(k;0) \\ &- \frac{\partial G_{ac}^{(\mathrm{A})ss''}(k)}{\partial\epsilon} G_{db}^{(\mathrm{A})s'''s'}(k) \Lambda_{y;3;cd}^{\mathrm{C}(0)s''s'''}(k;0) \\ &+ G_{ac}^{(\mathrm{R})ss''}(k) G_{db}^{(\mathrm{R})s'''s'}(k) \lim_{\omega\to 0} \frac{\partial \Lambda_{y;1;cd}^{\mathrm{C}(0)s''s'''}(k;\omega)}{\partial\omega} \\ &- G_{ac}^{(\mathrm{A})ss''}(k) G_{db}^{(\mathrm{A})s'''s'}(k) \lim_{\omega\to 0} \frac{\partial \Lambda_{y;3;cd}^{\mathrm{C}(0)s''s'''}(k;\omega)}{\partial\omega} \Bigg]\end{aligned} \tag{74}$$

and

$$\begin{aligned}\sigma_{xy}^{\mathrm{S(II)}} = -\frac{(-e)}{2} \sum_k \sum_{\{a\}} \sum_{\{s\}} f(\epsilon)\mathrm{sgn}(s)\delta_{s',s}(v_{\boldsymbol{k}x})_{ba}^{ss} &\times \Bigg[\frac{\partial G_{ac}^{(\mathrm{R})ss''}(k)}{\partial\epsilon} G_{db}^{(\mathrm{R})s'''s'}(k) \Lambda_{y;1;cd}^{\mathrm{C}(0)s''s'''}(k;0) \\ &- \frac{\partial G_{ac}^{(\mathrm{A})ss''}(k)}{\partial\epsilon} G_{db}^{(\mathrm{A})s'''s'}(k) \Lambda_{y;3;cd}^{\mathrm{C}(0)s''s'''}(k;0) \\ &+ G_{ac}^{(\mathrm{R})ss''}(k) G_{db}^{(\mathrm{R})s'''s'}(k) \lim_{\omega\to 0} \frac{\partial \Lambda_{y;1;cd}^{\mathrm{C}(0)s''s'''}(k;\omega)}{\partial\omega} \\ &- G_{ac}^{(\mathrm{A})ss''}(k) G_{db}^{(\mathrm{A})s'''s'}(k) \lim_{\omega\to 0} \frac{\partial \Lambda_{y;3;cd}^{\mathrm{C}(0)s''s'''}(k;\omega)}{\partial\omega} \Bigg],\end{aligned} \tag{75}$$

respectively.

2. *Interaction effects*

Before comparing the derived Fermi sea term with the noninteracting Fermi sea term, we analyze the damping dependence of $\sigma_{xy}^{\mathrm{C(II)}}$ or $\sigma_{xy}^{\mathrm{S(II)}}$ and the effects of the SCD on $\sigma_{xy}^{\mathrm{S(II)}}$ in order to clarify how the two important properties obtained in Éliashberg's approximation modify in the low-T case. Those properties are the crossover from the damping dependent to the damping independent σ_{xy}^{C} or σ_{xy}^{S} with decreasing temperature and the correction term of σ_{xy}^{S}, as shown in Sec. III A 2.

First, $\sigma_{xy}^{\mathrm{C(II)}}$ and $\sigma_{xy}^{\mathrm{S(II)}}$ become $O(\gamma^0)$ in $T\to 0$ and $\gamma_\alpha^*(\boldsymbol{k}_{\mathrm{F}})/T \to 0$ because we can neglect the damping dependence of $g_{1;acdb}^{ss''s'''s'}(k;0)$ or $g_{3;acdb}^{ss''s'''s'}(k;0)$ [49]. Since that result remains qualitatively the same in the low-T case, σ_{xy}^{C} and σ_{xy}^{S} become $\sigma_{xy}^{\mathrm{C}} \approx \sigma_{xy}^{\mathrm{C(II)}} = O(\gamma^0)$ and $\sigma_{xy}^{\mathrm{S}} \approx \sigma_{xy}^{\mathrm{S(II)}} = O(\gamma^0)$, respectively. In addition, another crossover occurs at the orange line in Fig. 2 because the dominant term changes from the Fermi surface term to the Fermi sea term with decreasing temperature. It should be noted that $\sigma_{xy}^{\mathrm{C(II)}}$ or $\sigma_{xy}^{\mathrm{S(II)}}$ becomes negligible compared with $\sigma_{xy}^{\mathrm{C(I)}}$ or $\sigma_{xy}^{\mathrm{S(I)}}$, respectively, if the QP damping is larger than the energy of the interband excitation which gives the finite contribution to $\sigma_{xy}^{\mathrm{C(II)}}$ or $\sigma_{xy}^{\mathrm{S(II)}}$. This is because in $\sigma_{xy}^{\mathrm{C(II)}}$ or $\sigma_{xy}^{\mathrm{S(II)}}$ we neglect the dependence on the QP damping as a result of the leading-term expansion of the products of the two single-particle Green's functions in terms of $\gamma_\alpha^*(\boldsymbol{k}_{\mathrm{F}})/T \to 0$, while we consider the dependence on the energy of the interband excitation. Thus, when the QP damping is larger, $\sigma_{xy}^{\mathrm{C(II)}}$ or $\sigma_{xy}^{\mathrm{S(II)}}$ becomes less dominant than $\sigma_{xy}^{\mathrm{C(I)}}$ or $\sigma_{xy}^{\mathrm{S(I)}}$, respectively, because only $\sigma_{xy}^{\mathrm{C(I)}}$ or $\sigma_{xy}^{\mathrm{S(I)}}$ has the leading dependence on the QP damping.

Next, since the spin current in $\sigma_{xy}^{\mathrm{S(II)}}$ is the same as the noninteracting one, $\sigma_{xy}^{\mathrm{S(II)}}$ is not affected by the SCD. Thus, the SCD affects σ_{xy}^{S} except at low temperatures. Then, to understand how the electron-electron interaction affects the Fermi sea term, we compare Eqs. (74) and (75) with the noninteracting Fermi sea terms [17,19] of σ_{xy}^{C} and σ_{xy}^{S}, respectively, $\sigma_{xy}^{\mathrm{C(0;II)}}$ and $\sigma_{xy}^{\mathrm{S(0;II)}}$, and deduce the interaction effects on the Fermi sea

terms. $\sigma_{xy}^{\mathrm{C(0;II)}}$ and $\sigma_{xy}^{\mathrm{S(0;II)}}$ are given [17,19] by

$$\sigma_{xy}^{\mathrm{C(0;II)}} = -(-e)^2 \sum_k \sum_{\{a\}} \sum_{\{s\}} f(\epsilon)\delta_{s',s}(v_{\boldsymbol{k}x})_{ba}^{ss} \times \left[\frac{\partial G_{ac}^{(0;\mathrm{R})ss''}(k)}{\partial\epsilon} G_{db}^{(0;\mathrm{R})s'''s'}(k) - \frac{\partial G_{ac}^{(0;\mathrm{A})ss''}(k)}{\partial\epsilon} G_{db}^{(0;\mathrm{A})s'''s'}(k) \right] \delta_{s'',s'''}(v_{\boldsymbol{k}y})_{cd}^{s''s''} \quad (76)$$

and

$$\sigma_{xy}^{\mathrm{S(0;II)}} = -\frac{(-e)}{2} \sum_k \sum_{\{a\}} \sum_{\{s\}} f(\epsilon)\mathrm{sgn}(s)\delta_{s',s}(v_{\boldsymbol{k}x})_{ba}^{ss} \times \left[\frac{\partial G_{ac}^{(0;\mathrm{R})ss''}(k)}{\partial\epsilon} G_{db}^{(0;\mathrm{R})s'''s'}(k) - \frac{\partial G_{ac}^{(0;\mathrm{A})ss''}(k)}{\partial\epsilon} G_{db}^{(0;\mathrm{A})s'''s'}(k) \right] \delta_{s'',s'''}(v_{\boldsymbol{k}y})_{cd}^{s''s''}, \quad (77)$$

respectively. After carrying out the ϵ integral in Eqs. (76) and (77), $\sigma_{xy}^{\mathrm{C(0;II)}}$ or $\sigma_{xy}^{\mathrm{S(0;II)}}$ is decomposed into the Berry-curvature term and the other part of the Fermi sea term [17,19]. Comparing Eqs. (74) and (75) with Eqs. (76) and (77), respectively, we find that each of $\sigma_{xy}^{\mathrm{C(II)}}$ and $\sigma_{xy}^{\mathrm{S(II)}}$ has three modifications due to the electron-electron interaction. Those modifications are the replacement of the single-particle Green's functions by the interacting ones, the replacement of the y component of the charge current by its vertex function, and the appearance of the ω derivative term of the y component of the vertex functions of the charge current.

Each of those modifications affects $\sigma_{xy}^{\mathrm{C(II)}}$ and $\sigma_{xy}^{\mathrm{S(II)}}$ as follows. First, the replacement of the single-particle Green's functions will little affect $\sigma_{xy}^{\mathrm{C(II)}}$ and $\sigma_{xy}^{\mathrm{S(II)}}$ because the QP damping of the retarded-retarded or advanced-advanced product is negligible [49] and because the effects of $z_\alpha(\boldsymbol{k})$ in the numerator and the denominator of the coherent parts of that product for finite ϵ nearly cancel out each other when the band dependence of $z_\alpha(\boldsymbol{k})$ is not strong. Second, the effects of the replacement of the y component of the charge current on $\sigma_{xy}^{\mathrm{C(II)}}$ or $\sigma_{xy}^{\mathrm{S(II)}}$ may be also not large because, as described in Sec. III A 2, the difference between $\Lambda_{y;l;cd}^{\mathrm{C(0)}s''s'''}(\boldsymbol{k},\epsilon;0)$ and $\delta_{s'',s'''}(v_{\boldsymbol{k}y})_{cd}^{s''s''}$ just causes the renormalization of the group velocity. Third, the modification about the appearance of the ω derivative term of the charge current may lead to the finite correction term if the dynamical effects of the electron-electron interaction are considerable. If the effects of the electron-electron interaction can be either neglected or treated in a mean-field approximation, the ω derivative is exactly zero. Actual estimations of those three interaction effects by numerical calculations are remaining issues for a future study.

IV. DISCUSSION

In this section, we discuss the origin of the differences between σ_{xy}^{C} and σ_{xx}^{C}, the differences between the present formalism and Haldane's formalism, and the correspondences between our results and experiments.

Before discussing the origin of the differences between σ_{xy}^{C} and σ_{xx}^{C}, we show σ_{xx}^{C} in Éliashberg's approximation for $\hat{H}$, and see its properties about the dominant multiband excitations, the damping dependence, and applicability of Éliashberg's approximation. Since we obtain the exact expression of σ_{xx}^{C} in the linear response theory by replacing $\hat{J}_{-\boldsymbol{q}y}^{\mathrm{C}}(0)$ in $\tilde{K}_{xy}^{\mathrm{C}}(i\Omega_n)$ in Eq. (11) by $\hat{J}_{-\boldsymbol{q}x}^{\mathrm{C}}(0)$, we can derive σ_{xx}^{C} in Éliashberg's approximation in the similar way for σ_{xy}^{C}. Thus, σ_{xx}^{C} in this approximation becomes $\sigma_{xx}^{\mathrm{C}} = \sigma_{xx}^{\mathrm{C(I)}}$ with

$$\sigma_{xx}^{\mathrm{C(I)}} = (-e)^2 \sum_k \sum_{\{a\}} \sum_{\{s\}} \left(-\frac{\partial f(\epsilon)}{\partial\epsilon} \right) \Lambda_{x;2;ba}^{\mathrm{C(0)}s's}(k;0) \times G_{ac}^{(0;\mathrm{R})ss''}(k;0) G_{db}^{(0;\mathrm{A})s'''s'}(k;0) \Lambda_{x;2;cd}^{\mathrm{C};s''s'''}(k;0). \quad (78)$$

The difference between $\sigma_{xx}^{\mathrm{C(I)}}$ and $\sigma_{xy}^{\mathrm{C(I)}}$ is the difference between $\Lambda_{x;2;cd}^{\mathrm{C};s''s'''}(k;0)$ and $\Lambda_{y;2;cd}^{\mathrm{C};s''s'''}(k;0)$. In addition, from the similar argument for $\sigma_{xy}^{\mathrm{C(I)}}$, we can deduce several properties of σ_{xx}^{C}. First, because of the same reason for $\sigma_{xy}^{\mathrm{C(I)}}$, we can determine the dominant multiband excitations and damping dependence of $\sigma_{xx}^{\mathrm{C(I)}}$ by analyzing the leading terms of $g_{2;acdb}^{ss''s'''s'}(k;0)$ which give the finite terms of $\sigma_{xx}^{\mathrm{C(I)}}$. Since $\sigma_{xx}^{\mathrm{C(I)}}$ includes two k_x derivatives arising from the k_x derivatives of the noninteracting group velocity in $\Lambda_{x;2;ba}^{\mathrm{C(0)}s's}(k;0)$ and $\Lambda_{x;2;cd}^{\mathrm{C};s''s'''}(k;0)$ [see Eqs. (44) and (36)], the terms in $\sigma_{xx}^{\mathrm{C(I)}}$ other than those should be even with respect to k_x and k_y. Due to this property, the terms proportional to $-i[\gamma_\alpha^*(\boldsymbol{k}) + \gamma_\beta^*(\boldsymbol{k})]$ in the leading terms of $g_{2;acdb}^{ss''s'''s'}(k;0)$ give the finite terms of $\sigma_{xx}^{\mathrm{C(I)}}$ [see Eq. (58)]. In addition, since the denominator of the leading terms of $g_{2;acdb}^{ss''s'''s'}(k;0)$ includes $\Delta\xi_{\beta\alpha}^*(\boldsymbol{k})^2 (\geqslant 0)$ [see Eq. (58)], the dominant multiband excitations become intraband (i.e., $\beta = \alpha$). Furthermore, due to that property, $\sigma_{xx}^{\mathrm{C(I)}}$ is always $O(\gamma^{-1})$ because we can approximate $\sigma_{xx}^{\mathrm{C(I)}}$ as

$$\sigma_{xx}^{\mathrm{C}} \approx \frac{(-e)^2}{N} \sum_{\boldsymbol{k}} \sum_\alpha \frac{1}{2\gamma_\alpha^*(\boldsymbol{k})} \times \mathrm{Re}\left[\tilde{\Lambda}_{x;2;\alpha\alpha}^{\mathrm{C(0)}}(\boldsymbol{k},\xi_\alpha^*(\boldsymbol{k}))\tilde{\Lambda}_{x;2;\alpha\alpha}^{\mathrm{C}}(\boldsymbol{k},\xi_\alpha^*(\boldsymbol{k}))\right] \quad (79)$$

with Eqs. (65) and (66). Then, the dominance of the intraband excitations for $\sigma_{xx}^{\mathrm{C(I)}}$ indicates that Éliashberg's approximation is always applicable in the left triangle region of Fig. 2 because for the intraband excitations the overlap between the QP spectra is unimportant.

Combining the above properties of σ_{xx}^{C} with the corresponding properties of σ_{xy}^{C}, we can clarify the origin of the differences between σ_{xx}^{C} and σ_{xy}^{C}. Namely, the origin is the difference in the dominant multiband excitations.

In addition, we can deduce the general principles in formulating transport coefficients of an interacting multiorbital metal. If the dominant multiband excitations are intraband, we can sufficiently treat the electron-electron interaction in Éliashberg's approximation. If the interband excitations are dominant, we need to use, instead of Éliashberg's approximation, an approximation beyond it only in the low-T case.

Then, we argue the differences between the present formalism and Haldane's formalism [32]. Assuming that σ_{xy}^{C} is given

only by the Berry-curvature term, Haldane proposed that the term of the Berry-curvature term after partial integral about ϵ could describe the excitations near the Fermi level [32]. However, the exact σ_{xy}^{C} includes the Fermi surface term which qualitatively differs from the Berry-curvature term [see Eq. (11) with Eq. (47)]; the difference arises from the effects of a retarded-advanced product of two single-particle Green's functions, which are important in the resistivity [37,49] and the weak-field usual Hall conductivity [48,54] in the Fermi liquid. Thus, if the Fermi surface term is dominant, Haldane's formalism is inapplicable. Since we find the dominance of the Fermi surface term in the high-T and the intermediate-T regions of Fig. 2 even without impurities, the present formalism reveals the remarkable interaction effects arising from the non-Berry-curvature term outside the applicable region of Haldane's formalism [32].

Finally, we discuss the correspondences between our results and experiments. First, we can check the interaction-driven mechanism of the damping dependence of σ_{xy}^{C} or σ_{xy}^{S} and crossover between damping dependent to damping independent σ_{xy}^{C} or σ_{xy}^{S} by measuring its temperature dependence in a clean system. This is because that temperature dependence is induced by the temperature dependence of the interaction-induced QP damping, as explained in Sec. III A 2. Also, we may observe the difference of the form of the red dotted line in Fig. 2 between weakly interacting and strongly interacting metals because the Fermi liquid and the nearly antiferromagnetic or nearly ferromagnetic metal show the different temperature dependence of the QP damping [42,60]. Moreover, although it is difficult to detect the crossover between the damping-independent Fermi surface and Fermi sea terms only by experiments, we can check its existence by combination of experiments and first-principle calculations if we find the material in which the sign of σ_{xy}^{C} or σ_{xy}^{S} changes at the crossover line: to find such material, we need to systematically analyze the intrinsic AHE or SHE on the basis of a realistic band structure in the presence of the electron-electron interaction by using the first-principle calculation; after the finding, we need to experimentally analyze the sign of σ_{xy}^{C} or σ_{xy}^{S} as a function of temperature around the crossover temperature. Then, the results about the SCD indicate, first, that in a measurement of the SHE in the low-T case, σ_{xy}^{S} behaves as if the nonconservation of the spin current is not important; second, that we may observe the effects of the SCD on the intrinsic SHE at high or slightly low temperatures where the Fermi surface term is dominant. However, it remains a challenging issue to clarify how large its effects are among several transition metals and transition-metal oxides.

V. SUMMARY

In summary, we have constructed the general formalism for the intrinsic AHE and SHE of the interacting multiorbital metal by using the linear response theory with the appropriate approximations, and have clarified the roles of the Fermi surface term and Fermi sea term of the dc conductivity and the effects of the SCD on these terms. In the high-T and the intermediate-T regions of Fig. 2, we have used Éliashberg's approximation, and in the low-T region, we have constructed the approximation beyond Éliashberg's approximation. Most importantly, we highlight the important roles of the Fermi surface term, a non-Berry-curvature term, even without impurities in the high-T and the intermediate-T regions. Actually, this Fermi surface term leads to the interaction-driven temperature dependence of σ_{xy}^{C} or σ_{xy}^{S} in the high-T region and the SCD-induced correction of σ_{xy}^{S}. These results considerably develop our understanding of the intrinsic AHE and SHE. In addition to those achievements, we have found that the differences between σ_{xy}^{C} and σ_{xx}^{C} arise from the difference in the dominant multiband excitations. Namely, due to the dominance of the interband excitations in σ_{xy}^{C}, the Fermi sea term such as the Berry-curvature term becomes dominant in clean and low-T case, while due to the dominance of the intraband excitations in σ_{xx}^{C}, the Fermi surface term is always dominant. This answers how to construct the FL theory for the intrinsic AHE or SHE. Moreover, we have shown the principles to construct general formalism of transport coefficients including the interaction effects and the multiband effects. This may be useful for further research of charge, spin, and heat transports for an interacting multiorbital metal.

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APPENDIX A: UNDERSTANDING OF THE INTRINSIC AHE OR SHE AS ORBITAL AHARANOV-BOHM EFFECT

In this appendix, we see that the origin of finite terms of σ_{xy}^{C} or σ_{xy}^{S} can be understood by analyzing the corresponding motion of an electron or a QP in real space, and that the origin of the intrinsic AHE or SHE in several metals is the orbital Aharanov-Bohm (AB) effect [25].

First, we can obtain the intuitive insight of the origin of finite σ_{xy}^{C} or σ_{xy}^{S} by expressing its finite term as the corresponding motion of an electron or a QP in real space [21,25]. For simplicity of arguments, let us argue noninteracting case of σ_{xy}^{C} because that argument for an electron is similarly applicable for σ_{xy}^{S} and because the similar argument holds even for a QP in interacting case. In the linear response theory, σ_{xy}^{C} has four matrix elements, the x and the y components of the charge current and two single-particle Green's functions [17,30] [see Eqs. (54) and (76)]; each term is the matrix element of the corresponding operator. Then, the charge current operator is single-body [see Eq. (14)], and the operator of the retarded or advanced noninteracting single-particle Green's function is given by the inverse matrix of $(\omega\hat{1} - \hat{H}_0 - \hat{H}_{\mathrm{LS}} + i\delta\hat{1})$ for $\delta = +0$ or -0, respectively. Since we can express it in terms of the series of $\frac{\hat{H}_0+\hat{H}_{\mathrm{LS}}}{\omega}$, a single-body operator, we can decompose the terms of σ_{xy}^{C} into the corresponding motion of an electron in real space [21,25]. That motion helps understand which terms in the Hamiltonian are essential to obtain finite σ_{xy}^{C}. Thus, the analysis of that motion helps understand the origin of the finite terms of σ_{xy}^{C} [21,25].

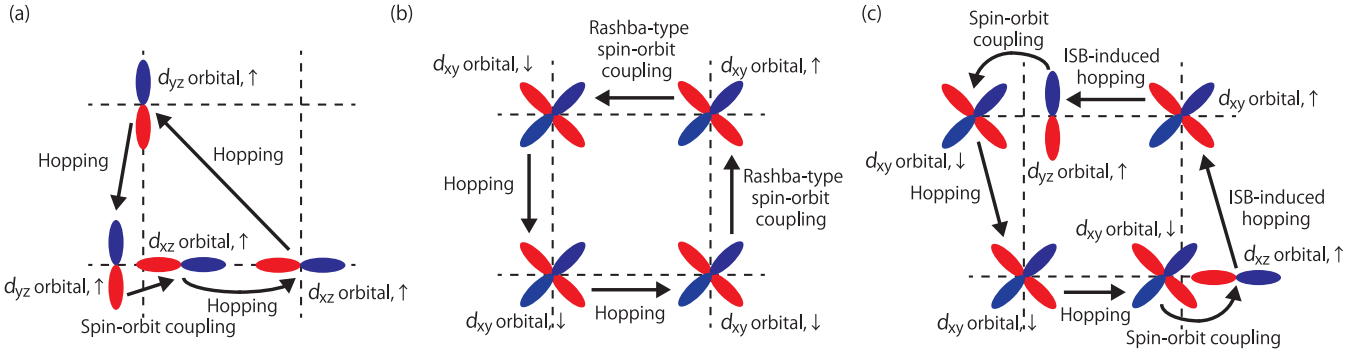

FIG. 4. Schematic pictures about the motion of an electron that gives a finite term of σ^{C}_{xy} or σ^{S}_{xy} for (a) $SrRuO_3$ [17] or Sr_2RuO_4 [25], and (b) an effective single-orbital metal [9] without the inversion symmetry at an ab plane, and (c) the schematic picture for the t_{2g}-orbital metal [21] whose single-orbital limit corresponds to case of (b). In those panels, ↑ and ↓ denote spin up and spin down, respectively, each black arrow denotes each motion due to the single-body operator, and the color difference of an orbital denotes the sign difference of its wave function. ISB-induced hopping in panel (c) denotes the hopping integral induced by the inversion-symmetry breaking at an ab plane [41]. Several similarities in panels (c) and (a) should be noted: the two ISB-induced hoppings in panel (c) play a similar role for the next-nearest-neighbor hopping between the d_{yz} and d_{xz} orbitals in panel (a); a sequence of the SOCs between the spin-down d_{xy} and the spin-up d_{xz} orbital and between the spin-up d_{yz} and the spin-down d_{xy} orbital in panel (c) play a similar role for the SOC between the spin-up d_{xz} and the spin-up d_{yz} orbital in panel (a).

Then, we consider three examples, and see the finite terms of σ^{C}_{xy} or σ^{S}_{xy} arise from the acquisition of the AB-type phase factor of an electron because of the onsite SOC and several hopping integrals. The following arguments are applicable to other cases of the intrinsic AHE or SHE of a metal.

The first example is case of a t_{2g}-orbital metal on a square lattice, corresponding to the AHE [17,18] in $SrRuO_3$ and the SHE [25] in Sr_2RuO_4. By the analysis of the motions for the finite terms of σ^{C}_{xy} or σ^{S}_{xy}, we find that one of the finite terms in this case arises from the motion shown in Fig. 4(a) [25]. This figure shows that the SOC from the spin-up d_{yz} orbital to the spin-up d_{xz} orbital causes $-\frac{\pi}{2}$ rotation, resulting in a complex phase factor of the wave function of an electron $\exp\frac{i\pi l_z}{2} = i$ [25]. This phase factor is similar to the AB phase factor [24] in the presence of an external magnetic field. Thus, we can regard the acquisition of such phase factor using orbital degrees of freedom as the orbital AB effect [25]. Namely, the orbital AB effect causes the intrinsic AHE or SHE in this case. In addition to the onsite SOC, the direct hopping integral between the d_{yz} and d_{xz} orbitals is important to obtain finite σ^{C}_{xy}.

Second, we can apply the similar mechanism to case of the intrinsic SHE in Pt [19,20]. In this case, we can acquire the AB-type phase factor by using several hopping integrals and the onsite SOC; e.g., the onsite SOC from the spin-up d_{xy} orbital to the spin-up $d_{x^2-y^2}$ orbital leads to $-\frac{\pi}{4}$ rotation, resulting in a complex phase factor $\exp\frac{i\pi l_z}{4} = i$ [19].

Third, we can similarly understand the intrinsic AHE or SHE in an effective single-orbital metal [9] without the inversion symmetry at an ab plane. For the explicit argument, let us consider the situation of the d_{xy}-orbital system on a square lattice. (The following argument is applicable even for other single-orbital systems without the inversion symmetry.) Since the electronic structure in this situation may be described by the single-orbital Rashba model [61], we can determine the motion which gives the finite term of σ^{C}_{xy} or σ^{S}_{xy} in the Rashba model [9] [see Fig. 4(b)]. Although that motion seems to be not categorized as the orbital AB effect, that motion can also be understood as the orbital AB effect [21]. This is because a t_{2g}-orbital model with the onsite SOC without the inversion symmetry at an ab plane becomes an effective single-orbital Rashba model for a large difference of the single-body energy level between the d_{xy} and $d_{xz/yz}$ orbitals [41]: the microscopic origin of the Rashba-type SOC is the combination of the transverse components of the onsite SOC and the hopping integral induced by the inversion-symmetry breaking in the presence of the large single-body energy difference between the d_{xy} and $d_{xz/yz}$ orbitals [see Fig. 4(c)]. Note that except the case for the large single-body energy difference the t_{2g}-orbital model qualitatively differs from the effective single-orbital Rashba model [21,41], and that the differences play important roles in obtaining the intrinsic term, which defeats the extrinsic term in the Born approximation [26].

APPENDIX B: DERIVATION OF EQ. (24)

In this appendix, we derive Eq. (24) by carrying out the analytic continuations of the first and the second terms of Eq. (21). First, we can carry out the analytic continuation of the first term of Eq. (21) as follows:

$$-\delta_{\boldsymbol{k},\boldsymbol{k}'}T\sum_{m} G^{ss''}_{ac}(\tilde{k_+})G^{s''s}_{db}(\tilde{k})$$
$$= -\delta_{\boldsymbol{k},\boldsymbol{k}'}\int_{C}\frac{d\epsilon}{4\pi i}\tanh\frac{\epsilon}{2T}G^{ss''}_{ac}(\tilde{k_+})G^{s''s}_{db}(\tilde{k}) \to -\delta_{\boldsymbol{k},\boldsymbol{k}'}\int_{-\infty}^{\infty}\frac{d\epsilon}{4\pi i}\tanh\frac{\epsilon}{2T}G^{(\mathrm{R})ss''}_{ac}(\boldsymbol{k},\epsilon+\omega)$$

$$\times\left[G^{(\mathrm{R})s''s}_{db}(\boldsymbol{k},\epsilon)-G^{(\mathrm{A})s''s}_{db}(\boldsymbol{k},\epsilon)\right]-\delta_{\boldsymbol{k},\boldsymbol{k}'}\int_{-\infty}^{\infty}\frac{d\epsilon}{4\pi i}\tanh\frac{\epsilon+\omega}{2T}\left[G^{(\mathrm{R})ss''}_{ac}(\boldsymbol{k},\epsilon+\omega)-G^{(\mathrm{A})ss''}_{ac}(\boldsymbol{k},\epsilon+\omega)\right]G^{(\mathrm{A})s''s}_{db}(\boldsymbol{k},\epsilon)$$

$$=-\delta_{\boldsymbol{k},\boldsymbol{k}'}\int_{-\infty}^{\infty}\frac{d\epsilon}{4\pi i}\sum_{l=1}^{3}T_l(\epsilon,\omega)g^{ss''s'''s'}_{l;acdb}(k;\omega),\tag{B1}$$

where the contour C consists of the three parts into which the circle is divided by inserting two horizontal lines $\mathrm{Im}(\epsilon+i\Omega_n)=0$ and $\mathrm{Im}\epsilon=0$, $\to$ represents taking $i\Omega_n\to\omega+i0+$, $T_l(\epsilon,\omega)$ are given by Eqs. (26)–(28), and $g^{ss''s'''s'}_{l;acdb}(k;\omega)$ are given by Eqs. (30)–(32).

Second, we can similarly carry out the analytic continuation of the second term of Eq. (21):

$$-\frac{T^2}{N}\sum_{m,m'}\sum_{\{A\}}\sum_{\{s_1\}}G^{ss_1}_{aA}(\tilde{k_+})G^{s''s_4}_{dD}(\tilde{k'})G^{s_3s''}_{Cc}(\tilde{k_+'})G^{s_2s}_{Bb}(\tilde{k})\Gamma^{\{s_1\}}_{\{A\}}(\tilde{k},\tilde{k'},i\epsilon_{m'};\boldsymbol{0},i\Omega_n)$$

$$=-\frac{1}{N}\int_{\mathrm{C}}\frac{d\epsilon}{4\pi i}\tanh\frac{\epsilon}{2T}\sum_{\{A\}}\sum_{\{s_1\}}G^{ss_1}_{aA}(\boldsymbol{k},\epsilon+i\Omega_n)G^{s_2s}_{Bb}(\boldsymbol{k},\epsilon)$$

$$\times\left[\int_{\mathrm{C}'}\frac{d\epsilon'}{4\pi i}\tanh\frac{\epsilon'}{2T}G^{s''s_4}_{dD}(\boldsymbol{k}',\epsilon')G^{s_3s''}_{Cc}(\boldsymbol{k}',\epsilon'+i\Omega_n)\Gamma^{\{s_1\}}_{\{A\}}(\boldsymbol{k},\epsilon,\boldsymbol{k}',\epsilon';\boldsymbol{0},i\Omega_n)\right.$$

$$+T\,G^{s''s_4}_{dD}(\boldsymbol{k}',\epsilon)G^{s_3s''}_{Cc}(\boldsymbol{k}',\epsilon+i\Omega_n)\Gamma^{\{s_1\}}_{\{A\}}(\boldsymbol{k},\epsilon,\boldsymbol{k}',\epsilon;\boldsymbol{0},i\Omega_n)$$

$$\left.+T\,G^{s''s_4}_{dD}(\boldsymbol{k}',-\epsilon-i\Omega_n)G^{s_3s''}_{Cc}(\boldsymbol{k}',-\epsilon)\Gamma^{\{s_1\}}_{\{A\}}(\boldsymbol{k},\epsilon,\boldsymbol{k}',-\epsilon-i\Omega_n;\boldsymbol{0},i\Omega_n)\right]$$

$$\to-\frac{1}{N}\sum_{\{A\}}\sum_{\{s_1\}}\int_{-\infty}^{\infty}\frac{d\epsilon}{4\pi i}P\int_{-\infty}^{\infty}\frac{d\epsilon'}{4\pi i}\coth\frac{\epsilon'-\epsilon}{2T}\left\{\left(\tanh\frac{\epsilon+\omega}{2T}-\tanh\frac{\epsilon}{2T}\right)g^{ss_1s_2s}_{2;aABb}(k;\omega)g^{s_3s''s''s_4}_{2;CcdD}(k';\omega)\right.$$

$$\times\left[\Gamma^{\{s_1\}}_{22\text{-II};\{A\}}(k,k';\boldsymbol{0},\omega)-\Gamma^{\{s_1\}}_{22\text{-III};\{A\}}(k,k';\boldsymbol{0},\omega)\right]-\tanh\frac{\epsilon+\omega}{2T}g^{ss_1s_2s}_{3;aABb}(k;\omega)g^{s_3s''s''s_4}_{3;CcdD}(k';\omega)\left[\Gamma^{\{s_1\}}_{33\text{-I};\{A\}}(k,k';\boldsymbol{0},\omega)\right.$$

$$\left.\left.-\Gamma^{\{s_1\}}_{33\text{-II};\{A\}}(k,k';\boldsymbol{0},\omega)\right]+\tanh\frac{\epsilon}{2T}g^{ss_1s_2s}_{1;aABb}(k;\omega)g^{s_3s''s''s_4}_{1;CcdD}(k';\omega)\left[\Gamma^{\{s_1\}}_{11\text{-II};\{A\}}(k,k';\boldsymbol{0},\omega)-\Gamma^{\{s_1\}}_{11\text{-I};\{A\}}(k,k';\boldsymbol{0},\omega)\right]\right\}$$

$$-\frac{1}{N}\sum_{\{A\}}\sum_{\{s_1\}}\int_{-\infty}^{\infty}\frac{d\epsilon}{4\pi i}\int_{-\infty}^{\infty}\frac{d\epsilon'}{4\pi i}\tanh\frac{\epsilon'}{2T}$$

$$\times\left\{\left(\tanh\frac{\epsilon+\omega}{2T}-\tanh\frac{\epsilon}{2T}\right)g^{ss_1s_2s}_{2;aABb}(k;\omega)\left[\Gamma^{\{s_1\}}_{21;\{A\}}(k,k';\boldsymbol{0},\omega)g^{s_3s''s''s_4}_{1;CcdD}(k';\omega)-\Gamma^{\{s_1\}}_{22\text{-II};\{A\}}(k,k';\boldsymbol{0},\omega)g^{s_3s''s''s_4}_{2;CcdD}(k';\omega)\right]\right.$$

$$-\tanh\frac{\epsilon+\omega}{2T}g^{ss_1s_2s}_{3;aABb}(k;\omega)\left[\Gamma^{\{s_1\}}_{31\text{-I};\{A\}}(k,k';\boldsymbol{0},\omega)g^{s_3s''s''s_4}_{1;CcdD}(k';\omega)-\Gamma^{\{s_1\}}_{32;\{A\}}(k,k';\boldsymbol{0},\omega)g^{s_3s''s''s_4}_{2;CcdD}(k';\omega)\right]$$

$$\left.+\tanh\frac{\epsilon}{2T}g^{ss_1s_2s}_{1;aABb}(k;\omega)\left[\Gamma^{\{s_1\}}_{11\text{-I};\{A\}}(k,k'\boldsymbol{0},\omega)g^{s_3s''s''s_4}_{1;CcdD}(k';\omega)-\Gamma^{\{s_1\}}_{12;\{A\}}(k,k';\boldsymbol{0},\omega)g^{s_3s''s''s_4}_{2;CcdD}(k';\omega)\right]\right\}$$

$$-\frac{1}{N}\sum_{\{A\}}\sum_{\{s_1\}}\int_{-\infty}^{\infty}\frac{d\epsilon}{4\pi i}\int_{-\infty}^{\infty}\frac{d\epsilon'}{4\pi i}\tanh\frac{\epsilon'+\omega}{2T}$$

$$\times\left\{\left(\tanh\frac{\epsilon+\omega}{2T}-\tanh\frac{\epsilon}{2T}\right)g^{ss_1s_2s}_{2;aABb}(k;\omega)\left[\Gamma^{\{s_1\}}_{22\text{-IV};\{A\}}(k,k';\boldsymbol{0},\omega)g^{s_3s''s''s_4}_{2;CcdD}(k';\omega)-\Gamma^{\{s_1\}}_{23;\{A\}}(k,k';\boldsymbol{0},\omega)g^{s_3s''s''s_4}_{3;CcdD}(k';\omega)\right]\right.$$

$$-\tanh\frac{\epsilon+\omega}{2T}g^{ss_1s_2s}_{3;aABb}(k;\omega)\left[\Gamma^{\{s_1\}}_{32;\{A\}}(k,k';\boldsymbol{0},\omega)g^{s_3s''s''s_4}_{2;CcdD}(k';\omega)-\Gamma^{\{s_1\}}_{33\text{-I};\{A\}}(k,k';\boldsymbol{0},\omega)g^{s_3s''s''s_4}_{3;CcdD}(k';\omega)\right]$$

$$\left.+\tanh\frac{\epsilon}{2T}g^{ss_1s_2s}_{1;aABb}(k;\omega)\left[\Gamma^{\{s_1\}}_{12;\{A\}}(k,k';\boldsymbol{0},\omega)g^{s_3s''s''s_4}_{2;CcdD}(k';\omega)-\Gamma^{\{s_1\}}_{13\text{-I};\{A\}}(k,k';\boldsymbol{0},\omega)g^{s_3s''s''s_4}_{3;CcdD}(k';\omega)\right]\right\}$$

$$-\frac{1}{N}\sum_{\{A\}}\sum_{\{s_1\}}\int_{-\infty}^{\infty}\frac{d\epsilon}{4\pi i}P\int_{-\infty}^{\infty}\frac{d\epsilon'}{4\pi i}\coth\frac{\epsilon'+\epsilon+\omega}{2T}$$

$$\times\left\{\left(\tanh\frac{\epsilon+\omega}{2T}-\tanh\frac{\epsilon}{2T}\right)g^{ss_1s_2s}_{2;aABb}(k;\omega)g^{s_3s''s''s_4}_{2;CcdD}(k';\omega)\left[\Gamma^{\{s_1\}}_{22\text{-III};\{A\}}(k,k';\boldsymbol{0},\omega)-\Gamma^{\{s_1\}}_{22\text{-IV};\{A\}}(k,k';\boldsymbol{0},\omega)\right]\right.$$

$$\left.-\tanh\frac{\epsilon+\omega}{2T}g^{ss_1s_2s}_{3;aABb}(k;\omega)g^{ss_1s_2s}_{1;CcdD}(k';\omega)\left[\Gamma^{\{s_1\}}_{31\text{-II};\{A\}}(k,k';\boldsymbol{0},\omega)-\Gamma^{\{s_1\}}_{31\text{-I};\{A\}}(k,k';\boldsymbol{0},\omega)\right]\right.$$

$$+\tanh\frac{\epsilon}{2T}g^{ss_1s_2s}_{1;aABb}(k;\omega)g^{s_3s''s''s_4}_{3;CcdD}(k';\omega)\Big[\Gamma^{\{s_1\}}_{13\text{-I};\{A\}}(k,k';\mathbf{0},\omega)-\Gamma^{\{s_1\}}_{13\text{-II};\{A\}}(k,k';\mathbf{0},\omega)\Big]\Bigg\}$$

$$=-\frac{1}{N}\int_{-\infty}^{\infty}\frac{d\epsilon}{4\pi i}\int_{-\infty}^{\infty}\frac{d\epsilon'}{4\pi i}\sum_{l=1}^{3}T_l(\epsilon,\omega)\sum_{\{A\}}\sum_{\{s_1\}}g^{ss_1s_2s}_{l;aABb}(k;\omega)\sum_{l'=1}^{3}\mathcal{J}^{\{s_1\}}_{ll';\{A\}}(k,k';\omega)g^{s_3s''s''s_4}_{l';CcdD}(k';\omega). \tag{B2}$$

In Eq. (B2), the contour C is the same as that used for the first term of Eq. (21), the contour C' consists of the five parts into which the circle is divided by inserting four horizontal lines $\mathrm{Im}(\epsilon'-i\epsilon_m)=0$, $\mathrm{Im}\epsilon'=0$, $\mathrm{Im}(\epsilon'+i\Omega_n)=0$, and $\mathrm{Im}(\epsilon'+i\epsilon_m+i\Omega_n)=0$, without the terms corresponding to $\mathrm{Im}\epsilon'=\epsilon_m$ and $\mathrm{Im}\epsilon'=-\epsilon_m-\Omega_n$, P represents taking the Cauchy principal value of the integral, and $\mathcal{J}^{\{s_1\}}_{ll';\{A\}}(k,k';\omega)$ is connected with the reducible four-point vertex functions in real-frequency representation as follows:

$$\mathcal{J}^{\{s_1\}}_{11;\{a\}}(k,k';\omega)=\tanh\frac{\epsilon'}{2T}\Gamma^{\{s_1\}}_{11\text{-I};\{a\}}(k,k';\mathbf{0},\omega)+\coth\frac{\epsilon'-\epsilon}{2T}\Big[\Gamma^{\{s_1\}}_{11\text{-II};\{a\}}(k,k';\mathbf{0},\omega)-\Gamma^{\{s_1\}}_{11\text{-I};\{a\}}(k,k';\mathbf{0},\omega)\Big], \tag{B3}$$

$$\mathcal{J}^{\{s_1\}}_{12;\{a\}}(k,k';\omega)=\left(\tanh\frac{\epsilon'+\omega}{2T}-\tanh\frac{\epsilon'}{2T}\right)\Gamma^{\{s_1\}}_{12;\{a\}}(k,k';\mathbf{0},\omega), \tag{B4}$$

$$\mathcal{J}^{\{s_1\}}_{13;\{a\}}(k,k';\omega)=-\tanh\frac{\epsilon'+\omega}{2T}\Gamma^{\{s_1\}}_{13\text{-I};\{a\}}(k,k';\mathbf{0},\omega)-\coth\frac{\epsilon+\epsilon'+\omega}{2T}\Big[\Gamma^{\{s_1\}}_{13\text{-II};\{a\}}(k,k';\mathbf{0},\omega)-\Gamma^{\{s_1\}}_{13\text{-I};\{a\}}(k,k';\mathbf{0},\omega)\Big], \tag{B5}$$

$$\mathcal{J}^{\{s_1\}}_{21;\{a\}}(k,k';\omega)=\tanh\frac{\epsilon'}{2T}\Gamma^{\{s_1\}}_{21;\{a\}}(k,k';\mathbf{0},\omega), \tag{B6}$$

$$\mathcal{J}^{\{s_1\}}_{22;\{a\}}(k,k';\omega)=\coth\frac{\epsilon'-\epsilon}{2T}\Big[\Gamma^{\{s_1\}}_{22\text{-II};\{a\}}(k,k';\mathbf{0},\omega)-\Gamma^{\{s_1\}}_{22\text{-III};\{a\}}(k,k';\mathbf{0},\omega)\Big]-\tanh\frac{\epsilon'}{2T}\Gamma^{\{s_1\}}_{22\text{-II};\{a\}}(k,k';\mathbf{0},\omega)$$
$$+\coth\frac{\epsilon'+\epsilon+\omega}{2T}\Big[\Gamma^{\{s_1\}}_{22\text{-III};\{a\}}(k,k';\mathbf{0},\omega)-\Gamma^{\{s_1\}}_{22\text{-IV};\{a\}}(k,k';\mathbf{0},\omega)\Big]+\tanh\frac{\epsilon'+\omega}{2T}\Gamma^{\{s_1\}}_{22\text{-IV};\{a\}}(k,k';\mathbf{0},\omega), \tag{B7}$$

$$\mathcal{J}^{\{s_1\}}_{23;\{a\}}(k,k';\omega)=-\tanh\frac{\epsilon'+\omega}{2T}\Gamma^{\{s_1\}}_{23;\{a\}}(k,k';\mathbf{0},\omega), \tag{B8}$$

$$\mathcal{J}^{\{s_1\}}_{31;\{a\}}(k,k';\omega)=\tanh\frac{\epsilon'}{2T}\Gamma^{\{s_1\}}_{31\text{-I};\{a\}}(k,k';\mathbf{0},\omega)+\coth\frac{\epsilon+\epsilon'+\omega}{2T}\Big[\Gamma^{\{s_1\}}_{31\text{-II};\{a\}}(k,k';\mathbf{0},\omega)-\Gamma^{\{s_1\}}_{31\text{-I};\{a\}}(k,k';\mathbf{0},\omega)\Big], \tag{B9}$$

$$\mathcal{J}^{\{s_1\}}_{32;\{a\}}(k,k';\omega)=\left(\tanh\frac{\epsilon'+\omega}{2T}-\tanh\frac{\epsilon'}{2T}\right)\Gamma^{\{s_1\}}_{32;\{a\}}(k,k';\mathbf{0},\omega), \tag{B10}$$

and

$$\mathcal{J}^{\{s_1\}}_{33;\{a\}}(k,k';\omega)=-\tanh\tfrac{\epsilon'+\omega}{2T}\Gamma^{\{s_1\}}_{33\text{-I};\{a\}}(k,k';\omega)-\coth\tfrac{\epsilon'-\epsilon}{2T}\Big[\Gamma^{\{s_1\}}_{33\text{-II};\{a\}}(k,k';\omega)-\Gamma^{\{s_1\}}_{33\text{-I};\{a\}}(k,k';\omega)\Big], \tag{B11}$$

where the subscript X in $\Gamma^{\{s_1\}}_{X;\{a\}}(k,k';\mathbf{0},\omega)$ represents the inequalities about ϵ, ϵ', and ω of the reducible four-point vertex functions in real-frequency representation: $\Gamma^{\{s_1\}}_{X;\{a\}}(k,k';\mathbf{0},\omega)$ for $X=$ 11-I, 11-II, 21, 31-II, 31-I, 32, 33-I, 33-II, 23, 13-II, 13-I, 12, 22-III, 22-II, 22-I, and 22-IV satisfy, respectively,

$$\mathrm{Im}\epsilon>0,\ \mathrm{Im}\epsilon+\mathrm{Im}\omega>0,\ \mathrm{Im}\epsilon'>0,\ \mathrm{Im}\epsilon'+\mathrm{Im}\omega>0,\ \mathrm{Im}\epsilon+\mathrm{Im}\epsilon'+\mathrm{Im}\omega>0,\ \mathrm{Im}\epsilon-\mathrm{Im}\epsilon'>0, \tag{B12}$$

$$\mathrm{Im}\epsilon>0,\ \mathrm{Im}\epsilon+\mathrm{Im}\omega>0,\ \mathrm{Im}\epsilon'>0,\ \mathrm{Im}\epsilon'+\mathrm{Im}\omega>0,\ \mathrm{Im}\epsilon+\mathrm{Im}\epsilon'+\mathrm{Im}\omega>0,\ \mathrm{Im}\epsilon-\mathrm{Im}\epsilon'<0, \tag{B13}$$

$$\mathrm{Im}\epsilon<0,\ \mathrm{Im}\epsilon+\mathrm{Im}\omega>0,\ \mathrm{Im}\epsilon'>0,\ \mathrm{Im}\epsilon'+\mathrm{Im}\omega>0,\ \mathrm{Im}\epsilon+\mathrm{Im}\epsilon'+\mathrm{Im}\omega>0,\ \mathrm{Im}\epsilon-\mathrm{Im}\epsilon'<0, \tag{B14}$$

$$\mathrm{Im}\epsilon<0,\ \mathrm{Im}\epsilon+\mathrm{Im}\omega<0,\ \mathrm{Im}\epsilon'>0,\ \mathrm{Im}\epsilon'+\mathrm{Im}\omega>0,\ \mathrm{Im}\epsilon+\mathrm{Im}\epsilon'+\mathrm{Im}\omega>0,\ \mathrm{Im}\epsilon-\mathrm{Im}\epsilon'<0, \tag{B15}$$

$$\mathrm{Im}\epsilon<0,\ \mathrm{Im}\epsilon+\mathrm{Im}\omega<0,\ \mathrm{Im}\epsilon'>0,\ \mathrm{Im}\epsilon'+\mathrm{Im}\omega>0,\ \mathrm{Im}\epsilon+\mathrm{Im}\epsilon'+\mathrm{Im}\omega<0,\ \mathrm{Im}\epsilon-\mathrm{Im}\epsilon'<0, \tag{B16}$$

$$\mathrm{Im}\epsilon<0,\ \mathrm{Im}\epsilon+\mathrm{Im}\omega<0,\ \mathrm{Im}\epsilon'<0,\ \mathrm{Im}\epsilon'+\mathrm{Im}\omega>0,\ \mathrm{Im}\epsilon+\mathrm{Im}\epsilon'+\mathrm{Im}\omega<0,\ \mathrm{Im}\epsilon-\mathrm{Im}\epsilon'<0, \tag{B17}$$

$$\mathrm{Im}\epsilon<0,\ \mathrm{Im}\epsilon+\mathrm{Im}\omega<0,\ \mathrm{Im}\epsilon'<0,\ \mathrm{Im}\epsilon'+\mathrm{Im}\omega<0,\ \mathrm{Im}\epsilon+\mathrm{Im}\epsilon'+\mathrm{Im}\omega<0,\ \mathrm{Im}\epsilon-\mathrm{Im}\epsilon'<0, \tag{B18}$$

$$\mathrm{Im}\epsilon<0,\ \mathrm{Im}\epsilon+\mathrm{Im}\omega<0,\ \mathrm{Im}\epsilon'<0,\ \mathrm{Im}\epsilon'+\mathrm{Im}\omega<0,\ \mathrm{Im}\epsilon+\mathrm{Im}\epsilon'+\mathrm{Im}\omega<0,\ \mathrm{Im}\epsilon-\mathrm{Im}\epsilon'>0, \tag{B19}$$

$$\mathrm{Im}\epsilon<0,\ \mathrm{Im}\epsilon+\mathrm{Im}\omega>0,\ \mathrm{Im}\epsilon'<0,\ \mathrm{Im}\epsilon'+\mathrm{Im}\omega<0,\ \mathrm{Im}\epsilon+\mathrm{Im}\epsilon'+\mathrm{Im}\omega<0,\ \mathrm{Im}\epsilon-\mathrm{Im}\epsilon'>0, \tag{B20}$$

$$\mathrm{Im}\epsilon>0,\ \mathrm{Im}\epsilon+\mathrm{Im}\omega>0,\ \mathrm{Im}\epsilon'<0,\ \mathrm{Im}\epsilon'+\mathrm{Im}\omega<0,\ \mathrm{Im}\epsilon+\mathrm{Im}\epsilon'+\mathrm{Im}\omega<0,\ \mathrm{Im}\epsilon-\mathrm{Im}\epsilon'>0, \tag{B21}$$

$$\mathrm{Im}\epsilon > 0,\ \mathrm{Im}\epsilon + \mathrm{Im}\omega > 0,\ \mathrm{Im}\epsilon' < 0,\ \mathrm{Im}\epsilon' + \mathrm{Im}\omega < 0,\ \mathrm{Im}\epsilon + \mathrm{Im}\epsilon' + \mathrm{Im}\omega > 0,\ \mathrm{Im}\epsilon - \mathrm{Im}\epsilon' > 0, \tag{B22}$$

$$\mathrm{Im}\epsilon > 0,\ \mathrm{Im}\epsilon + \mathrm{Im}\omega > 0,\ \mathrm{Im}\epsilon' < 0,\ \mathrm{Im}\epsilon' + \mathrm{Im}\omega > 0,\ \mathrm{Im}\epsilon + \mathrm{Im}\epsilon' + \mathrm{Im}\omega > 0,\ \mathrm{Im}\epsilon - \mathrm{Im}\epsilon' > 0, \tag{B23}$$

$$\mathrm{Im}\epsilon < 0,\ \mathrm{Im}\epsilon + \mathrm{Im}\omega > 0,\ \mathrm{Im}\epsilon' < 0,\ \mathrm{Im}\epsilon' + \mathrm{Im}\omega > 0,\ \mathrm{Im}\epsilon + \mathrm{Im}\epsilon' + \mathrm{Im}\omega > 0,\ \mathrm{Im}\epsilon - \mathrm{Im}\epsilon' > 0, \tag{B24}$$

$$\mathrm{Im}\epsilon < 0,\ \mathrm{Im}\epsilon + \mathrm{Im}\omega > 0,\ \mathrm{Im}\epsilon' < 0,\ \mathrm{Im}\epsilon' + \mathrm{Im}\omega > 0,\ \mathrm{Im}\epsilon + \mathrm{Im}\epsilon' + \mathrm{Im}\omega > 0,\ \mathrm{Im}\epsilon - \mathrm{Im}\epsilon' < 0, \tag{B25}$$

$$\mathrm{Im}\epsilon < 0,\ \mathrm{Im}\epsilon + \mathrm{Im}\omega > 0,\ \mathrm{Im}\epsilon' < 0,\ \mathrm{Im}\epsilon' + \mathrm{Im}\omega > 0,\ \mathrm{Im}\epsilon + \mathrm{Im}\epsilon' + \mathrm{Im}\omega < 0,\ \mathrm{Im}\epsilon - \mathrm{Im}\epsilon' < 0, \tag{B26}$$

and

$$\mathrm{Im}\epsilon < 0,\ \mathrm{Im}\epsilon + \mathrm{Im}\omega > 0,\ \mathrm{Im}\epsilon' < 0,\ \mathrm{Im}\epsilon' + \mathrm{Im}\omega > 0,\ \mathrm{Im}\epsilon + \mathrm{Im}\epsilon' + \mathrm{Im}\omega < 0,\ \mathrm{Im}\epsilon - \mathrm{Im}\epsilon' > 0. \tag{B27}$$

We also obtain the connections between $\mathcal{J}^{(1)\{s_1\}}_{ll';\{a\}}(k,k';\omega)$ and $\Gamma^{(1)\{s_1\}}_{X;\{a\}}(k,k';\mathbf{0},\omega)$, the irreducible four-point vertex functions in real-frequency representation, by adding the superscript (1) in both $\mathcal{J}^{\{s_1\}}_{ll';\{a\}}(k,k';\omega)$ and $\Gamma^{\{s_1\}}_{X;\{a\}}(k,k';\mathbf{0},\omega)$ in Eqs. (B3)–(B11).

Combining Eqs. (B1) and (B2) with Eqs. (12) and (21), we obtain Eq. (24).

APPENDIX C: DERIVATION OF EQ. (47)

In this appendix, we derive Eq. (47). The derivation is in the following way: First, by using Eq. (46), we can rewrite the terms for $l = 1$ and 3 in Eq. (40) as

$$\begin{aligned}
&-\frac{(-e)^2}{2i}\sum_{k}\sum_{\{a\}}\sum_{\{s\}}\delta_{s',s}(v_{\boldsymbol{k}x})^{ss}_{ba}\sum_{l=1,3}T_l(\epsilon,\omega)g^{ss''s'''s'}_{l;acdb}(k;\omega)\Lambda^{\mathrm{C};s''s'''}_{y;l;cd}(k;\omega)\\
&=-\frac{(-e)^2}{2i}\sum_{k}\sum_{\{a\}}\sum_{\{s\}}\delta_{s',s}(v_{\boldsymbol{k}x})^{ss}_{ba}\tanh\frac{\epsilon}{2T}g^{ss''s'''s'}_{1;acdb}(k;\omega)\Lambda^{\mathrm{C}(0)s''s'''}_{y;1;cd}(k;\omega)\\
&-\frac{(-e)^2}{(2i)^2}\sum_{k}\sum_{\{a\}}\sum_{\{s\}}\delta_{s',s}(v_{\boldsymbol{k}x})^{ss}_{ba}\tanh\frac{\epsilon}{2T}g^{ss''s'''s'}_{1;acdb}(k;\omega)\sum_{k'}\sum_{\{A\}}\sum_{\{s_1\}}\mathcal{J}^{(0)s''s'''s_3s_4}_{12;cdCD}(k,k';\omega)g^{s_3s_1s_2s_4}_{2;CABD}(k';\omega)\Lambda^{\mathrm{C};s_1s_2}_{y;2;AB}(k';\omega)\\
&+\frac{(-e)^2}{2i}\sum_{k}\sum_{\{a\}}\sum_{\{s\}}\delta_{s',s}(v_{\boldsymbol{k}x})^{ss}_{ba}\tanh\frac{\epsilon+\omega}{2T}g^{ss''s'''s'}_{3;acdb}(k;\omega)\Lambda^{\mathrm{C}(0)s''s'''}_{y;3;cd}(k;\omega)\\
&+\frac{(-e)^2}{(2i)^2}\sum_{k}\sum_{\{a\}}\sum_{\{s\}}\delta_{s',s}(v_{\boldsymbol{k}x})^{ss}_{ba}\tanh\frac{\epsilon+\omega}{2T}g^{ss''s'''s'}_{3;acdb}(k;\omega)\\
&\times\sum_{k'}\sum_{\{A\}}\sum_{\{s_1\}}\mathcal{J}^{(0)s''s'''s_3s_4}_{32;cdCD}(k,k';\omega)g^{s_3s_1s_2s_4}_{2;CABD}(k';\omega)\Lambda^{\mathrm{C};s_1s_2}_{y;2;AB}(k';\omega).
\end{aligned}\tag{C1}$$

Furthermore, the second and the fourth terms in Eq. (C1) can be rewritten as, respectively,

$$\begin{aligned}
&\frac{(-e)^2}{4}\sum_{k}\sum_{k'}\sum_{\{a\}}\sum_{\{A\}}\sum_{\{s\}}\sum_{\{s_1\}}\delta_{s',s}(v_{\boldsymbol{k}x})^{ss}_{ba}\tanh\frac{\epsilon}{2T}g^{ss''s'''s'}_{1;acdb}(k;\omega)\mathcal{J}^{(0)s''s'''s_3s_4}_{12;cdCD}(k,k';\omega)g^{s_3s_1s_2s_4}_{2;CABD}(k';\omega)\Lambda^{\mathrm{C};s_1s_2}_{y;2;AB}(k';\omega)\\
&=\frac{(-e)^2}{4}\sum_{k'}\sum_{k}\sum_{\{a\}}\sum_{\{A\}}\sum_{\{s\}}\sum_{\{s_1\}}\delta_{s',s}(v_{\boldsymbol{k}'x})^{ss}_{ba}\tanh\frac{\epsilon'}{2T}g^{ss''s'''s'}_{1;acdb}(k';\omega)\left(\tanh\frac{\epsilon+\omega}{2T}-\tanh\frac{\epsilon}{2T}\right)\Gamma^{(0)s''s'''s_3s_4}_{12;cdCD}(k',k;\mathbf{0},\omega)\\
&\times g^{s_3s_1s_2s_4}_{2;CABD}(k;\omega)\Lambda^{\mathrm{C};s_1s_2}_{y;2;AB}(k;\omega)\\
&=\frac{(-e)^2}{4}\sum_{k}\sum_{k'}\sum_{\{a\}}\sum_{\{A\}}\sum_{\{s\}}\sum_{\{s_1\}}\mathcal{J}^{(0)s's's_3s_4}_{21;baCD}(\boldsymbol{k},\epsilon,\boldsymbol{k},\epsilon+\omega,\boldsymbol{k}',\epsilon',\boldsymbol{k}',\epsilon'+\omega)g^{s_3s_1s_2s_4}_{1;CABD}(\boldsymbol{k}',\epsilon',\boldsymbol{k}',\epsilon'+\omega)\delta_{s_1,s_2}(v_{\boldsymbol{k}'x})^{s_1s_1}_{AB}\\
&\times\left(\tanh\frac{\epsilon+\omega}{2T}-\tanh\frac{\epsilon}{2T}\right)g^{ss''s'''s'}_{2;acdb}(k;\omega)\Lambda^{\mathrm{C};s''s'''}_{y;2;cd}(k;\omega)
\end{aligned}\tag{C2}$$

and

$$
\begin{aligned}
&-\frac{(-e)^2}{4}\sum_{k}\sum_{k'}\sum_{\{a\}}\sum_{\{A\}}\sum_{\{s\}}\sum_{\{s_1\}}\delta_{s',s}(v_{\boldsymbol{k}x})^{ss}_{ba}\tanh\frac{\epsilon+\omega}{2T}g^{ss''s'''s'}_{3;acdb}(k;\omega)\mathcal{J}^{(0)s''s'''s_3s_4}_{32;cdCD}(k,k';\omega)g^{s_3s_1s_2s_4}_{2;CABD}(k';\omega)\Lambda^{\mathrm{C};s_1s_2}_{y;2;AB}(k';\omega)\\
&=-\frac{(-e)^2}{4}\sum_{k'}\sum_{k}\sum_{\{a\}}\sum_{\{A\}}\sum_{\{s\}}\sum_{\{s_1\}}\delta_{s',s}(v_{\boldsymbol{k}'x})^{ss}_{ba}\tanh\frac{\epsilon'+\omega}{2T}g^{ss''s'''s'}_{3;acdb}(k';\omega)\left(\tanh\frac{\epsilon+\omega}{2T}-\tanh\frac{\epsilon}{2T}\right)\Gamma^{(0)s''s'''s_3s_4}_{32;cdCD}(k',k;\mathbf{0},\omega)\\
&\times g^{s_3s_1s_2s_4}_{2;CABD}(k;\omega)\Lambda^{\mathrm{C};s_1s_2}_{y;2;AB}(k;\omega)\\
&=\frac{(-e)^2}{4}\sum_{k}\sum_{k'}\sum_{\{a\}}\sum_{\{A\}}\sum_{\{s\}}\sum_{\{s_1\}}\mathcal{J}^{(0)s's''s_3s_4}_{23;baCD}(\boldsymbol{k},\epsilon,\boldsymbol{k},\epsilon+\omega,\boldsymbol{k}',\epsilon',\boldsymbol{k}',\epsilon'+\omega)g^{s_3s_1s_2s_4}_{3;CABD}(\boldsymbol{k}',\epsilon',\boldsymbol{k}',\epsilon'+\omega)\delta_{s_1,s_2}(v_{\boldsymbol{k}'x})^{s_1s_1}_{AB}\\
&\times\left(\tanh\frac{\epsilon+\omega}{2T}-\tanh\frac{\epsilon}{2T}\right)g^{ss''s'''s'}_{2;acdb}(k;\omega)\Lambda^{\mathrm{C};s''s'''}_{y;2;cd}(k;\omega).
\end{aligned}
\tag{C3}
$$

In deriving Eq. (C2), we have used Eqs. (B4) and (B6) and an identity [49,52]

$$
\Gamma^{(0)ss's''s'''}_{12;abcd}(\boldsymbol{k}',\epsilon'+\omega,\boldsymbol{k}',\epsilon',\boldsymbol{k},\epsilon+\omega,\boldsymbol{k},\epsilon)=\Gamma^{(0)s'''s''s's}_{21;dcba}(\boldsymbol{k},\epsilon,\boldsymbol{k},\epsilon+\omega,\boldsymbol{k}',\epsilon',\boldsymbol{k}',\epsilon'+\omega).
\tag{C4}
$$

In addition, to derive Eq. (C3), we have used Eqs. (B8) and (B10) and another identity [49,52]

$$
\Gamma^{(0)ss's''s'''}_{32;abcd}(\boldsymbol{k}',\epsilon'+\omega,\boldsymbol{k}',\epsilon',\boldsymbol{k},\epsilon+\omega,\boldsymbol{k},\epsilon)=\Gamma^{(0)s'''s''s's}_{23;dcba}(\boldsymbol{k},\epsilon,\boldsymbol{k},\epsilon+\omega,\boldsymbol{k}',\epsilon',\boldsymbol{k}',\epsilon'+\omega).
\tag{C5}
$$

Returning Eqs. (C2) and (C3) to Eq. (C1), we obtain

$$
\begin{aligned}
&-\frac{(-e)^2}{2i}\sum_{k}\sum_{\{a\}}\sum_{\{s\}}\delta_{s',s}(v_{\boldsymbol{k}x})^{ss}_{ba}\sum_{l=1,3}T_l(\epsilon,\omega)g^{ss''s'''s'}_{l;acdb}(k;\omega)\Lambda^{\mathrm{C};s''s'''}_{y;l;cd}(k';\omega)\\
&=-\frac{(-e)^2}{2i}\sum_{k}\sum_{\{a\}}\sum_{\{s\}}\delta_{s',s}(v_{\boldsymbol{k}x})^{ss}_{ba}\sum_{l=1,3}T_l(\epsilon,\omega)g^{ss''s'''s'}_{l;acdb}(k;\omega)\Lambda^{\mathrm{C}(0)s''s'''}_{y;l;cd}(k';\omega)\\
&-\frac{(-e)^2}{2i}\sum_{k}\sum_{\{a\}}\sum_{\{s\}}\left[\frac{1}{2i}\sum_{k'}\sum_{\{A\}}\sum_{\{s_1\}}\sum_{l'=1,3}\mathcal{J}^{(0)s's''s_3s_4}_{2l';baCD}(\boldsymbol{k},\epsilon,\boldsymbol{k},\epsilon+\omega,\boldsymbol{k}',\epsilon',\boldsymbol{k}',\epsilon'+\omega)g^{s_3s_1s_2s_4}_{l';CABD}(\boldsymbol{k}',\epsilon',\boldsymbol{k}',\epsilon'+\omega)\right.\\
&\left.\times\,\delta_{s_1,s_2}(v_{\boldsymbol{k}'x})^{s_1s_1}_{AB}\right]T_2(\epsilon,\omega)g^{ss''s'''s'}_{2;acdb}(k;\omega)\Lambda^{\mathrm{C};s''s'''}_{y;2;cd}(k;\omega)\\
&=-\frac{(-e)^2}{2i}\sum_{k}\sum_{\{a\}}\sum_{\{s\}}\delta_{s',s}(v_{\boldsymbol{k}x})^{ss}_{ba}\sum_{l=1,3}T_l(\epsilon,\omega)g^{ss''s'''s'}_{l;acdb}(k;\omega)\Lambda^{\mathrm{C}(0)s''s'''}_{y;l;cd}(k';\omega)\\
&-\frac{(-e)^2}{2i}\sum_{k}\sum_{\{a\}}\sum_{\{s\}}\left[\Lambda^{\mathrm{C}(0)s's}_{x;2;ba}(\boldsymbol{k},\epsilon,\boldsymbol{k},\epsilon+\omega)-\delta_{s',s}(v_{\boldsymbol{k}x})^{ss}_{ba}\right]T_2(\epsilon,\omega)g^{ss''s'''s'}_{2;acdb}(k;\omega)\Lambda^{\mathrm{C};s''s'''}_{y;2;cd}(k;\omega).
\end{aligned}
\tag{C6}
$$

Then, combining Eq. (C6) with Eq. (40), we obtain Eq. (47). This is another exact expression of $\tilde{K}^{\mathrm{C(R)}}_{xy}(\omega)$.

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