

Critical behavior of the two-dimensional Ising model with long-range correlated disorder

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(Received 23 February 2016; revised manuscript received 26 May 2016; published 22 June 2016)

We study critical behavior of the diluted two-dimensional Ising model in the presence of disorder correlations which decay algebraically with distance as $\sim r^{-a}$. Mapping the problem onto two-dimensional Dirac fermions with correlated disorder we calculate the critical properties using renormalization group up to two-loop order. We show that beside the Gaussian fixed point the flow equations have a nontrivial fixed point which is stable for $0.995 < a < 2$ and is characterized by the correlation length exponent $\nu = 2/a + O((2-a)^3)$. Using bosonization, we also calculate the averaged square of the spin-spin correlation function and find the corresponding critical exponent $\eta_2 = 1/2 - (2-a)/4 + O((2-a)^2)$.

DOI: [10.1103/PhysRevB.93.224422](https://doi.org/10.1103/PhysRevB.93.224422)

I. INTRODUCTION

Effects of quenched disorder on critical behavior attracted considerable attention for several decades [1–6]. Among various aspects of this problem influence of disorder correlations is of particular interest. Examples include spin models with correlated random bonds [7–14] and random fields [15–18], quantum transport and localization [19–21], polymers in random media [22], disordered elastic systems [23,24], and percolation [25].

The two-dimensional (2D) Ising model is historically important for studying criticality since its critical behavior deviates from the mean-field picture but still allows for an exact solution [26]. According to the Harris criterion [27] uncorrelated random bond or random site disorder modifies the critical behavior provided that the heat capacity exponent of the pure system is positive, $\alpha_{\text{pure}} > 0$. Although uncorrelated disorder is only marginally irrelevant for the 2D Ising system, since $\alpha_{\text{pure}} = 0$, its effects on the critical behavior were a subject of intensive theoretical and numerical studies [28]. Apart from the purely academic interest, this problem has potential applications; e.g., it was observed that domain formation in membranes with quenched protein obstacles without preferred affinity can be described by a diluted 2D Ising model [29].

The solution of the pure 2D Ising model can be formulated in terms of free 2D Majorana fermions whose mass is proportional to the reduced temperature [30]. The presence of disorder adds a four-fermion interaction with the coupling constant proportional to the concentration of impurities [31–33]. The resulting model has been intensely studied by renormalization group methods. These studies not only confirmed the marginal irrelevance of the disorder but also revealed the presence of logarithmic corrections to the critical behavior of the pure model. In particular, it was found that the specific heat singularity modifies from $C \sim \ln(1/\tau)$ to $C \sim \ln \ln(1/\tau)$ where $\tau = (T_c - T)/T_c$ if the temperature goes sufficiently close to the critical temperature T_c [31,32]. The calculation of the correlation function is a much more difficult task since in the fermionic picture the spin operator is a nonlocal object so that even for the pure case it requires some efforts to recover the well-known result $\eta_{\text{pure}} = \frac{1}{4}$. Initially it was argued [31,33] that disorder modifies the critical exponent

to $\eta = 0$, but later it was realized that the behavior of the N th moment of the spin-spin correlation function averaged over disorder configurations is [34,35]

$$\overline{G(r)^N} \sim \frac{(\ln r)^{N(N-1)/8}}{r^{N/4}}, \quad (1)$$

while in the pure model $G(r)^N \sim r^{-N/4}$.

Real systems may contain extended defects such as linear dislocations or grain boundaries which are either aligned in space or may have random orientation. The presence of extended defects or long-range (LR) correlated disorder modifies the Harris criterion opening a possibility for relevance of disorder in two dimensions. Almost a half century ago, McCoy and Wu proposed the disordered 2D Ising model in which impurities are perfectly correlated in one direction and uncorrelated in the transverse direction [7]. Though it was originally argued that the phase transition in this model is smeared, later it was shown that it is sharp but controlled by an infinite-randomness fixed point [36]. An extension of this model to d dimensions was proposed in Ref. [9], where extended defects are infinitely correlated in ε_d dimensions and randomly distributed in the remaining $\tilde{d} = d - \varepsilon_d$ dimensions. Values $\varepsilon_d = 0, 1, 2$ correspond to uncorrelated point-like, linear, and planar defects, respectively, while noninteger values of ε_d may describe systems containing fractal-like defects [37]. The critical equilibrium and dynamic behavior of these and related models were studied using a double expansion in $\varepsilon = 4 - d$ and ε_d in Refs. [10,11,37–43]. The numerical studies of systems with parallel linear [44] and planar defects [45,46] were also performed.

Weinrib and Halperin proposed an alternative model [8] with LR correlated disorder whose correlations decay with the distance r as a power law, $g(r) \propto r^{-a}$. The critical behavior of this model has been studied to two-loop order using a double $\varepsilon = 4 - d$, $\delta = 4 - a$ expansion [12] and also direct calculations in $d = 3$ [13]. These studies suggest that the phase transition belongs to a universality class different from that for systems with uncorrelated disorder if the correlation length exponent of the pure (undiluted) model satisfies $\nu_{\text{pure}} < 2/a$. The condition holds for $a < d$, while for $a > d$ the usual Harris criterion [27] is recovered and this condition is substituted by $\nu_{\text{pure}} < 2/d$. Although results of Refs. [8,12,13] are in

qualitative agreement and predict an emergence of the new type of critical behavior governed by the so-called LR disorder fixed point, they do not agree on quantitative level. In particular, results of Refs. [8,12] suggest that in the new universality class the correlation length exponent is $\nu = 2/a$ to the second order in $\varepsilon = 4 - d$ and $\delta = 4 - a$ (and even probably to all orders; see [8,47]), whereas calculations performed directly in three dimensions [13] are in favor of a nontrivial value of the exponent, which differs from $\nu = 2/a$ already in the two-loop approximation. In principle the discrepancy can be explained by breaking down the $\varepsilon = 4 - d$ -expansion at large ε . In order to verify this conjecture one needs a controllable method which does not rely on $\varepsilon = 4 - d$ expansion with analytical continuation to $\varepsilon = 2$. Subsequently, these analytic results have been checked by numerical calculations [48–51]. In turn, these have not led so far to common agreement either. Results of computer simulations in Ref. [48,51] support the analytic result $\nu = 2/a$, whereas the critical exponents obtained in numerical studies in Refs. [49,50] deviate from this prediction raising the question about dependence of the critical exponents on the peculiarities of disorder distribution.

In this paper we reconsider this problem using mapping of the 2D Ising model with LR correlated disorder to disordered 2D Dirac fermions, thus approaching the problem from low dimensions. This has been done to one-loop order in Refs. [20,52]. We extend these calculations to two-loop order and also compute the averaged square of the spin-spin correlation function to the lowest order using bosonization. Since the calculations are done directly in two dimensions and are well controlled in the limit of small $\delta = 2 - a$ they provide a test for the possible breaking down of the $\varepsilon = 4 - d$ expansion.

The rest of the paper is organized as follows: Section II introduces fermionic representation of the 2D Ising model with correlated disorder. We give a short description of renormalization of this model in Sec. III. We present two-loop scaling functions in Sec. IV together with their analysis within the framework of δ expansion. Section V is devoted to calculation of the averaged square of the spin-spin correlation function using mapping to the sine-Gordon model. We end the paper with conclusions in Sec. VI. Some technical points are given in the appendices.

II. MODEL

The random bond 2D Ising model can be described by two-dimensional real Majorana fermions whose action reads [53]

$$S_M = \int d\bar{z}dz [\chi \bar{\partial} \chi + \bar{\chi} \partial \bar{\chi} + im(z) \bar{\chi} \chi], \quad (2)$$

where $\bar{\chi}(z)$ and $\chi(z)$ are one-component Grassmann fields, $z = x + iy$, $\partial = \frac{1}{2}(\partial_x - i\partial_y)$, and $m(z) = m_0 + \delta m(z)$ is coupled to the energy operator $\epsilon(z) = i\bar{\chi}(z)\chi(z)$. Here $m_0 = (T_c - T)/T_c$ and $\delta m(z)$ encodes spatial variations in bond strength for a given realization of disorder. Using the two-component spinor notation $\Psi = (\chi, \bar{\chi})^T$ action (2) can be rewritten as

$$S_M = \frac{1}{2} \int d^2r \bar{\Psi}(r) [\not{\partial} + m(r)] \Psi(r), \quad (3)$$

where $\not{\partial} = \gamma_j \partial^j$ with $\gamma_j = \sigma_j$ ($j = 1, 2$) being the Pauli matrices. Note that $\bar{\Psi}$ is not an independent field; it is related to Ψ by $\bar{\Psi} = \Psi^T \gamma_0$ with $\gamma_0 = \sigma_2$. We assume that $\delta m(r)$ is a Gaussian random variable with zero mean and a variance decaying as a power law

$$\overline{\delta m(r) \delta m(0)} = g(r) \sim r^{-a}, \quad r \rightarrow \infty. \quad (4)$$

To simplify calculations we follow [34] and introduce two Majorana fermions Ψ_1 and Ψ_2 which combine to form a complex Dirac fermion $\psi = (\Psi_1 + i\Psi_2)/\sqrt{2}$. The corresponding action reads

$$S_D = \int d^2r \bar{\psi}(r) [\not{\partial} + m(r)] \psi(r). \quad (5)$$

Note that $\bar{\psi}$ and ψ are independent and we may change variable $\bar{\psi} \rightarrow -i\bar{\psi}$. Then the resulting action at criticality, $m_0 = 0$, corresponds to the Dirac fermions in the presence of random imaginary chemical potential $-i\delta m(r)$. Changing variable $\bar{\psi} \rightarrow -\bar{\psi}\sigma_3$ one can see that action (5) also describes the 2D Dirac fermions with random mass disorder [54].

In what follows we are going to use dimensional regularization. To that end we have to generalize the problem to arbitrary d and replace the Pauli matrices by a Clifford algebra represented by the matrices γ_i satisfying the anticommutation relations [55]:

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij} \mathbb{1}, \quad i, j = 1, \dots, d. \quad (6)$$

To average over disorder we use the replica trick introducing n copies of the original system [56]. The resulting replicated action reads

$$S = -i \sum_{\alpha=1}^n \int d^d r \bar{\psi}_\alpha(r) (\not{\partial} + m_0) \psi_\alpha(r) + \frac{1}{2} \sum_{\alpha, \beta=1}^n \int d^d r d^d r' g(r - r') \bar{\psi}_\alpha(r) \psi_\alpha(r) \bar{\psi}_\beta(r') \psi_\beta(r'). \quad (7)$$

The properties of the original system with quenched disorder are then obtained by taking the limit $n \rightarrow 0$. It is convenient to fix the normalization of the disorder distribution (4) in Fourier space. We take disorder potential to be random Gaussian with zero mean and the correlator

$$\overline{\delta m(k) \delta m(k')} = (2\pi)^d \delta^d(k + k') g(k). \quad (8)$$

We choose

$$g(k) = u_0 + v_0 k^{a-d}; \quad (9)$$

here u_0 and v_0 are bare coupling constants. The LR coupling constant v_0 is relevant only for $a < d$. Note that if one neglects the SR term u_0 in Eq. (9) it will be ultimately generated by the RG flow. The bare propagator of the action (7) can be written as

$$\langle \bar{\psi}_\alpha(k) \psi_\beta(-k) \rangle_0 = \delta_{\alpha\beta} \frac{\gamma_j k_j + im_0}{k^2 + m_0^2}. \quad (10)$$

III. RENORMALIZATION OF THE MODEL

Using the bare propagator (10) one can calculate the correlation functions for the action (7) perturbatively in u_0

and v_0 . The integrals entering this perturbation series turn out to be ultraviolet (UV) divergent in $d = 2$. To make the theory finite we are using the dimensional regularization [57] and compute all integrals in $d = 2 - \varepsilon$. Following the works [8,20] we perform a double expansion in $\varepsilon = 2 - d$ and $\delta = 2 - a$ so that all divergences are transformed into the poles in ε and δ while the ratio ε/δ remains finite. In the framework of the minimal subtraction scheme we do not include these finite ratios into the counterterms choosing them to be the pole part only. We are interested in the case $0 < a < 2$, so that $0 < \delta < 2$; however, one has to take with caution the numerical estimations for $\delta > 1$ computed using the results obtained perturbatively in δ . We define the renormalized fields ψ , $\bar{\psi}$, mass m , and dimensionless coupling constants u and v in such a way that all poles can be hidden in the renormalization factors Z_ψ , Z_m , Z_u , and Z_v , leaving finite the correlation functions computed with the renormalized action

$$S_R = \sum_{\alpha=1}^n \int_k \bar{\psi}_\alpha(-k) (Z_\psi \gamma_j k_j - Z_m i m) \psi_\alpha(k) + \frac{1}{2} \sum_{\alpha, \beta=1}^n \int_{k_1, k_2, k_3} [\mu^\varepsilon Z_u u + \mu^\delta Z_v v |k_1 + k_2|^{a-d}] \times \bar{\psi}_\alpha(k_1) \psi_\alpha(k_2) \bar{\psi}_\beta(k_3) \psi_\beta(-k_1 - k_2 - k_3), \quad (11)$$

where $\int_k := \int \frac{d^d k}{(2\pi)^d}$ and we have introduced a renormalization scale μ . Since the renormalized action is obtained from the bare one by the fields rescaling

$$\psi_0 = Z_\psi^{1/2} \psi, \quad \bar{\psi}_0 = Z_\psi^{1/2} \bar{\psi}, \quad (12)$$

the bare and renormalized parameters are related by

$$m_0 = Z_m Z_\psi^{-1} m, \quad (13)$$

$$u_0 = \mu^\varepsilon Z_u Z_\psi^{-2} u, \quad v_0 = \mu^\delta Z_v Z_\psi^{-2} v, \quad (14)$$

where we have included $K_d/2$ in redefinition of u and v . $K_d = 2\pi^{d/2}/[(2\pi)^d \Gamma(d/2)]$ is the surface area of the d -dimensional unit sphere divided by $(2\pi)^d$. The renormalized \mathcal{N} -point vertex function $\Gamma^{(\mathcal{N})}$ is related to the bare $\bar{\Gamma}^{(\mathcal{N})}$ by

$$\bar{\Gamma}^{(\mathcal{N})}(k_i; m_0, u_0, v_0) = Z_\psi^{-\mathcal{N}/2} \Gamma^{(\mathcal{N})}(k_i; m, u, v, \mu). \quad (15)$$

To calculate the renormalization constants it is enough to renormalize the two-point vertex function $\Gamma^{(2)}$ and the four-point vertex function $\Gamma^{(4)}$. We impose that they are finite at $m = \mu$ and find the renormalization constants using minimal subtraction scheme [58]. To that end it is convenient to split the four-point function in the short-range (SR) and long-range (LR) parts:

$$\Gamma^{(4)}(k_1, k_2, k_3, k_4) = \Gamma_u^{(4)}(k_i) + \Gamma_v^{(4)}(k_i) |k_1 + k_2|^{a-d}. \quad (16)$$

The renormalization constants are determined from the condition that $\Gamma_u^{(4)}(0; m = \mu)$ and $\Gamma_v^{(4)}(0; m = \mu)$ are finite.

Since the bare vertex function does not depend on the renormalization scale μ the renormalized vertex function

satisfies the renormalization group equation

$$\left[\mu \frac{\partial}{\partial \mu} - \beta_u(u, v) \frac{\partial}{\partial u} - \beta_v(u, v) \frac{\partial}{\partial v} - \frac{\mathcal{N}}{2} \eta_\psi(u, v) - \gamma(u, v) m \frac{\partial}{\partial m} \right] \Gamma^{(\mathcal{N})}(k_i; m, u, v, \mu) = 0, \quad (17)$$

where we have introduced the scaling functions

$$\beta_u(u, v) = -\mu \left. \frac{\partial u}{\partial \mu} \right|_0, \quad \beta_v(u, v) = -\mu \left. \frac{\partial v}{\partial \mu} \right|_0, \quad (18)$$

$$\eta_\psi(u, v) = -\beta_u(u, v) \frac{\partial \ln Z_\psi}{\partial u} - \beta_v(u, v) \frac{\partial \ln Z_\psi}{\partial v}, \quad (19)$$

$$\eta_m(u, v) = -\beta_u(u, v) \frac{\partial \ln Z_m}{\partial u} - \beta_v(u, v) \frac{\partial \ln Z_m}{\partial v}, \quad (20)$$

$$\gamma(u, v) = \eta_m(u, v) - \eta_\psi(u, v). \quad (21)$$

The subscript ‘‘0’’ stands for derivatives at fixed u_0 , v_0 , and m_0 . The dimensional analysis gives

$$\Gamma^{(\mathcal{N})}(k_i; m, u, v, \mu) = \lambda^{-d + \mathcal{N}(d-1)/2} \Gamma^{(\mathcal{N})}(\lambda k_i; \lambda m, u, v, \lambda \mu), \quad (22)$$

which can be rewritten in an infinitesimal form as

$$\left[\mu \frac{\partial}{\partial \mu} + \sum_i k_i \frac{\partial}{\partial k_i} + m \frac{\partial}{\partial m} - d + \frac{\mathcal{N}(d-1)}{2} \right] \times \Gamma^{(\mathcal{N})}(k_i; m, u, v, \mu) = 0. \quad (23)$$

Subtracting Eq. (17) from Eq. (23) we arrive at

$$\left[\beta_u \frac{\partial}{\partial u} + \beta_v \frac{\partial}{\partial v} + \sum_i k_i \frac{\partial}{\partial k_i} + (1 + \gamma) m \frac{\partial}{\partial m} + \frac{\mathcal{N}}{2} [d - 1 + \eta_\psi] - d \right] \Gamma^{(\mathcal{N})}(k_i; m, u, v) = 0. \quad (24)$$

Equation (24) is a linear first-order partial differential equation which can be solved by the method of characteristics [59]. It reduces this equation to a set of ordinary differential equations which determines a family of curves along which the solution can be integrated from some initial conditions given on a suitable hypersurface. The characteristics lines of Eq. (24) can be found from the flow equations

$$\frac{d\tilde{u}(\xi)}{d \ln \xi} = \beta_u(\tilde{u}(\xi), \tilde{v}(\xi)), \quad (25)$$

$$\frac{d\tilde{v}(\xi)}{d \ln \xi} = \beta_v(\tilde{u}(\xi), \tilde{v}(\xi)), \quad (26)$$

$$\frac{d\tilde{k}_i(\xi)}{d \ln \xi} = \tilde{k}_i(\xi), \quad (27)$$

$$\frac{d\tilde{m}(\xi)}{d \ln \xi} = [1 + \gamma(\tilde{u}(\xi), \tilde{v}(\xi))] \tilde{m}(\xi) \quad (28)$$

with the initial conditions $\tilde{u}(1) = u$, $\tilde{v}(1) = v$, $\tilde{k}_i(1) = k_i$, $\tilde{m}(1) = m$. The solution of Eq. (24) propagates along the characteristics lines (25)–(28) according to

$$\frac{d \ln M_{\mathcal{N}}(\xi)}{d \ln \xi} = d - \frac{\mathcal{N}}{2} [d - 1 + \eta_\psi(\tilde{u}(\xi), \tilde{v}(\xi))], \quad (29)$$

with the initial condition $M_{\mathcal{N}}(1) = 1$, and thus satisfies the scaling relation

$$\Gamma^{(\mathcal{N})}(\tilde{k}_i(\xi); \tilde{m}(\xi), \tilde{u}(\xi), \tilde{v}(\xi)) = M_{\mathcal{N}}(\xi) \Gamma^{(\mathcal{N})}(k_i; m, u, v). \quad (30)$$

$M_{\mathcal{N}}(\xi)$ encodes the anomalous scaling dimension of the fields ψ and $\bar{\psi}$. The critical behavior of the system is expected to be controlled by a stable fixed point (FP) of the renormalization group (RG) flow which is defined as simultaneous zero of β functions (18):

$$\beta_u(u^*, v^*) = 0, \quad \beta_v(u^*, v^*) = 0. \quad (31)$$

Stability of a FP can be determined from the eigenvalues of the stability matrix

$$\mathcal{M} = \begin{pmatrix} \frac{\partial \beta_u(u, v)}{\partial u} & \frac{\partial \beta_u(u, v)}{\partial v} \\ \frac{\partial \beta_v(u, v)}{\partial u} & \frac{\partial \beta_v(u, v)}{\partial v} \end{pmatrix}. \quad (32)$$

The FP is stable provided that both eigenvalues calculated at the FP (31) have negative real parts. We can identify ξ in Eqs. (25)–(30) with the correlation length. Then the solution (30) can be written in the vicinity of the FP (31) as

$$\Gamma^{(\mathcal{N})}(k_i, m) = \xi^{\mathcal{N}d_\psi - d} f_{\mathcal{N}}(k_i \xi, m \xi^{1/v}), \quad (33)$$

where we have identified the correlation length exponent

$$\frac{1}{\nu} = 1 + \gamma(u^*, v^*), \quad (34)$$

and the anomalous scaling dimension of the fields ψ and $\bar{\psi}$

$$d_\psi = \frac{1}{2}[d - 1 + \eta_\psi(u^*, v^*)]. \quad (35)$$

For instance, in the critical point we have

$$\overline{\langle \bar{\psi}(r) \psi(0) \rangle} \sim r^{-2d_\psi}. \quad (36)$$

IV. FIXED POINTS, THEIR STABILITY AND SCALING BEHAVIOR

A. Renormalization to two-loop order

In order to renormalize the theory (11) to two-loop order we need the diagrams contributing to the two- and four-point vertex functions $\Gamma^{(2)}(p)$ and $\Gamma^{(4)}(p_i = 0)$ in the replica limit $n \rightarrow 0$ which are shown in Fig. 1 and Fig. 2. In the one-loop approximation there are two diagrams contributing to the two-point vertex function $\Gamma^{(2)}(p)$ each of which we split into two parts A_1^a and A_2^a . The first part is computed at zero external momentum and the second part is the part which is linear in the external momentum \vec{p} . The same is applied to the two-loop diagrams C_2^{ab} and C_3^{ab} . The diagrams contributing to the four-point function are computed at zero external momenta and expanded in small parameters ε and δ keeping the ratio $\frac{\varepsilon}{\delta}$ finite. Within the minimal subtraction scheme we need only the poles in ε and δ for the two-loop diagrams while for the one-loop diagrams one has to keep also the contributions which are finite in the limit $\varepsilon, \delta \rightarrow 0$. The poles of the diagrams shown in Fig. 1 and Fig. 2 are calculated with the help of the formulas given in Appendix B and collected in Tables I and II, respectively. The vertex functions $\Gamma^{(2)}(p)$, $\Gamma_u^{(4)}(p_i = 0)$, and $\Gamma_v^{(4)}(p_i = 0)$ are computed in Appendix A. Using these

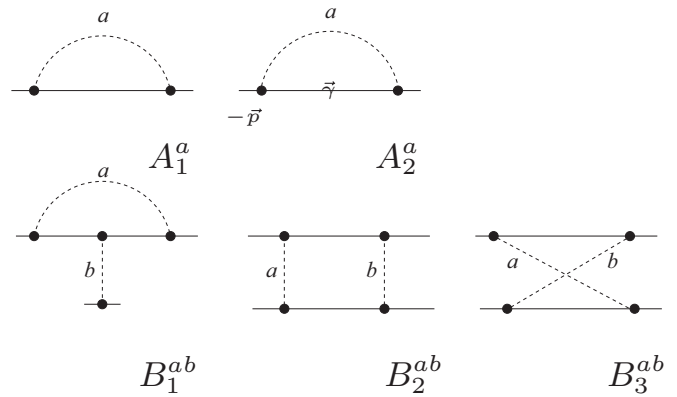


FIG. 1. The one-loop diagrams contributing to the two-point vertex function $\Gamma^{(2)}$ (first row) and to the four-point vertex function $\Gamma^{(4)}$ (second row) in the replica limit $n \rightarrow 0$. Solid lines correspond to the propagator (10) and dashed lines to the disorder vertex (9) which is split in the SR and LR parts. The indices a, b, c take values 0 or 1, depending on whether the dashed line stands for u vertex or v vertex.

functions we find the Z factors:

$$Z_u = 1 + \frac{4u}{\varepsilon} + \frac{4v}{\delta} - u^2 \left(\frac{2}{\varepsilon} - \frac{16}{\varepsilon^2} \right) + \frac{4v^3}{u(\varepsilon - 3\delta)} - uv \left(\frac{28}{\delta + \varepsilon} - \frac{32}{\delta\varepsilon} - \frac{8}{\delta} \right) - v^2 \left(\frac{10}{\delta} - \frac{4\varepsilon}{\delta^2} - \frac{16}{\delta^2} \right), \quad (37)$$

$$Z_v = 1 + \frac{4u}{\varepsilon} + \frac{4v}{\delta} + \frac{16u^2}{\varepsilon^2} - \frac{4v^2}{\delta} \left(1 - \frac{\varepsilon}{\delta} - \frac{4}{\delta} \right) - \frac{8uv}{\delta + \varepsilon} \left(1 - \frac{\varepsilon}{\delta} - \frac{4}{\delta} - \frac{4}{\varepsilon} \right), \quad (38)$$

$$Z_m = 1 + \frac{2u}{\varepsilon} + \frac{2v}{\delta} + \frac{6u^2}{\varepsilon^2} + v^2 \left(\frac{2\varepsilon}{\delta^2} + \frac{6}{\delta^2} - \frac{2}{\delta} \right) + 4uv \left(\frac{3}{\delta\varepsilon} + \frac{1}{\delta} - \frac{2}{\delta + \varepsilon} \right), \quad (39)$$

$$Z_\psi = 1 + \frac{u^2}{\varepsilon} + \frac{4uv\varepsilon}{\delta(\delta + \varepsilon)} + \frac{v^2(2\varepsilon - \delta)}{\delta^2}. \quad (40)$$

For the SR disorder it was argued that the contribution coming from the nonzero mass in the numerator of the bare propagator (10) vanishes, so that one can neglect it from the beginning [60]. We have found that this holds also for the case of the LR disorder at least to the two-loop order; i.e., the contributions in the angular brackets in Tables I and II cancel each other in Eqs. (37)–(40). From Eqs. (37)–(40) using the definitions (18)–(21) we obtain the two-loop expressions for the β functions

$$\beta_u(u, v) = \varepsilon u - 4u(u+v) + 8u(u+v)^2 + 4v(u+v)^2, \quad (41)$$

$$\beta_v(u, v) = \delta v - 4v(u+v) + 4v(u+v)^2, \quad (42)$$

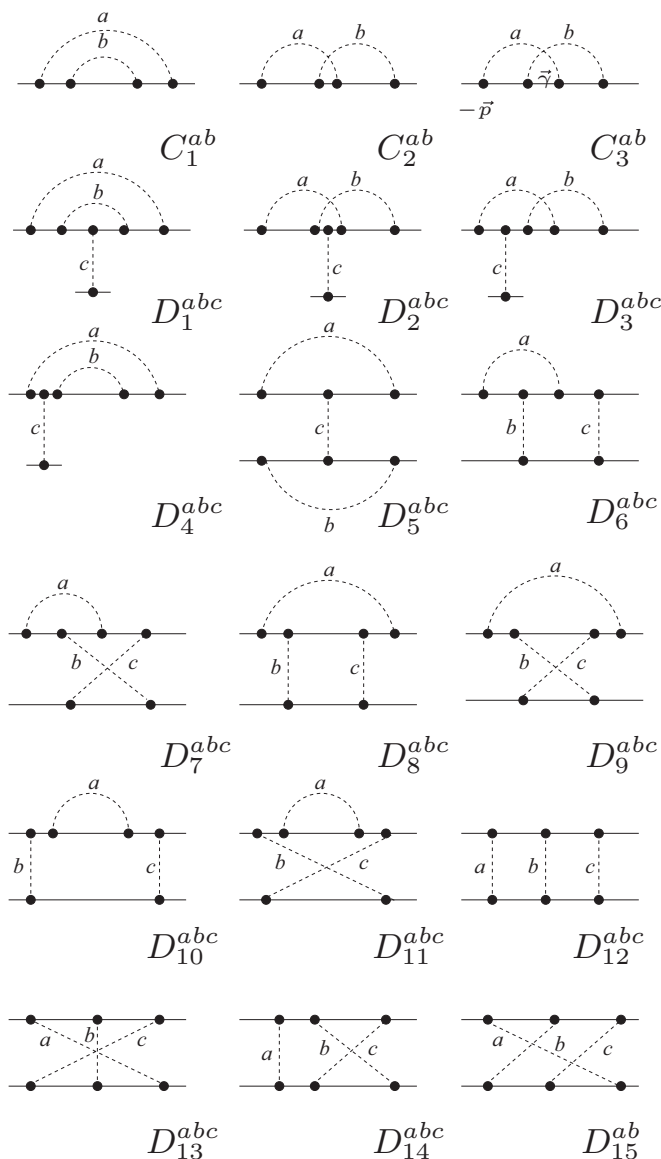


FIG. 2. The two-loop diagrams contributing to the two-point vertex function $\Gamma^{(2)}$ (first row) and to the four-point vertex function $\Gamma^{(4)}$ in the replica limit $n \rightarrow 0$. The indices a, b, c take values 0 or 1, depending on whether the dashed line stands for u vertex or v vertex.

and for the other scaling functions giving the critical exponents:

$$\eta_\psi(u, v) = -2u^2 + 2v^2 - \frac{4\varepsilon}{\delta}uv - \frac{4\varepsilon}{\delta}v^2, \quad (43)$$

$$\eta_m(u, v) = -2u - 2v + 4uv + 4v^2 - \frac{4\varepsilon}{\delta}uv - \frac{4\varepsilon}{\delta}v^2,$$

$$\gamma(u, v) = -2(u + v) + 2(u + v)^2. \quad (44)$$

Note that the ratio $\frac{\varepsilon}{\delta}$ is finite within our regularization scheme. Though it is present in the scaling functions η_m and η_ψ , all these ratios magically cancel each other in the β and γ functions, leaving their coefficients pure integer constants.

B. Expansions in small ε and δ

We now analyze the renormalization group flow using expansion in small ε and δ . The β functions have three FPs: Gaussian, short-range correlated, and long-range correlated disordered FPs.

(i) *Gaussian fixed point*, given by

$$u_G^* = v_G^* = 0, \quad (45)$$

describes the pure 2D Ising model with the correlation length exponent $\nu_{\text{pure}} = 1/[1 + \gamma(u_G^*, v_G^*)] = 1$. Following Ref. [34] one can estimate singularity in the free energy. Using the action (5) one can express the partition function of the Ising model as $Z_{\text{Ising}}^2 = \int D\bar{\psi} D\psi e^{-S\nu} \sim \det[\not{\partial} + m_0]$ with $m_0 \equiv \tau$. Applying the identity $\ln \det = \text{tr} \ln$ we find $F_{\text{sing}} \sim \tau^2 \ln \tau$, so that $C \sim \ln(\tau^{-1})$ and $\alpha_{\text{pure}} = 0$.

(ii) *Short-range correlated disordered fixed point (SR FP)*, given by

$$u_{\text{SR}}^* = \frac{\varepsilon}{4} + \frac{\varepsilon^2}{8}, \quad v_{\text{SR}}^* = 0, \quad (46)$$

merges with the Gaussian FP at $d = 2$. This implies that the SR correlated disorder is marginally irrelevant in two dimensions. As a consequence it results only in logarithmic corrections to the scaling behavior of the pure 2D Ising model. The two-loop logarithmic corrections are calculated in Appendix C. For the correlation length and the specific heat we find

$$\xi \sim \tau^{-1} (\ln \tau^{-1})^{1/2} \left[1 + o\left(\frac{\ln \ln \tau^{-1}}{\ln \tau^{-1}}\right) \right], \quad (47)$$

$$C_{\text{sing}} \sim \ln \ln \tau^{-1} \left[1 + o\left(\frac{1}{\ln \tau^{-1}}\right) \right]; \quad (48)$$

i.e., the subdominant two-loop logarithmic corrections identically vanish.

The Gaussian FP becomes unstable with respect to the LR correlated disorder for $\delta > 0$. This reproduces the extended Harris criterion [8], which states that the critical behavior of the pure system is modified by the LR correlated disorder if $\nu_{\text{pure}} < 2/a$. Indeed, substituting into the last relation $\nu_{\text{pure}} = 1$ one arrives at $a < 2$ which means $\delta > 0$.

(iii) *Long-range correlated disordered fixed point (LR FP)* reads

$$u_{\text{LR}}^* = \frac{\delta^3}{16(\delta - \varepsilon)}, \quad v_{\text{LR}}^* = \frac{\delta}{4} - \frac{\delta^2 \varepsilon}{16(\delta - \varepsilon)}. \quad (49)$$

In two dimensions the LR FP reduces to

$$u_{\text{LR}}^* = \frac{\delta^2}{16} + O(\delta^3), \quad v_{\text{LR}}^* = \frac{\delta}{4} + O(\delta^3). \quad (50)$$

Let us perform the stability analysis of the LR FP. The two eigenvalues of the stability matrix (32) computed at the LR FP (50) at $d = 2$ are shown in Fig. 3 as functions of δ . Both eigenvalues are complex conjugated with the negative real parts for $0 < \delta < \delta_{\text{max}}$, where the LR FP is stable. There are no stable FPs for $\delta > \delta_{\text{max}}$. Expansion of the eigenvalues in small δ gives

$$\lambda_{1,2}^{(\text{LR})} = -\delta + \frac{\delta^2}{2} + O(\delta^3) \pm i\sqrt{\frac{\delta}{2}} \left(\delta + \frac{\delta^2}{4} + O(\delta^3) \right). \quad (51)$$

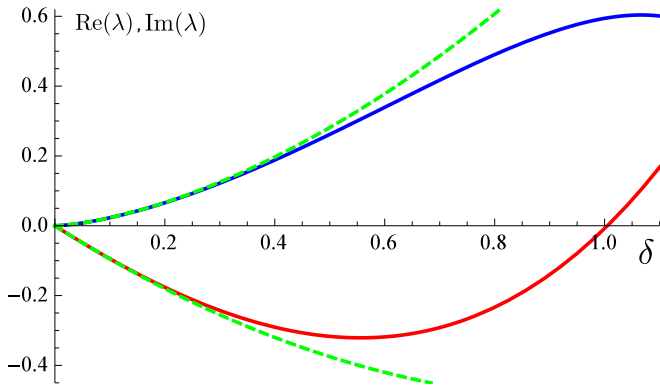


FIG. 3. The eigenvalues of the stability matrix in two dimensions ($d = 2$) as a function of δ . There are two complex conjugated eigenvalues: the red solid curve at the bottom is the real part and the blue curve at the top is the imaginary part. The dashed lines are the series expansions (51).

It is straightforward to see that the value of δ_{\max} that follows from the expansion (51) is $\delta_{\max} = 2$, whereas numerical diagonalization of the stability matrix (32) gives $\delta_{\max} \approx 1.005$ (see Fig. 3 for more details). It is tempting to make more precise the value of δ_{\max} by applying the familiar resummation technique [5] to the two-loop series (41), (42) at fixed ϵ , δ [22,61]. However, at $d = 2$ (i.e., at $\epsilon = 0$) the leading contribution to the first β function (41) vanishes, making the series too short to allow for a reliable resummation.

Thus, while according to the extended Harris criterion the LR FP may be stable for $\delta > 0$ we reveal the existence of the upper bound δ_{\max} for its stability. Indeed, reasonable values of δ lie between 0 and 2, but $\delta = 1$ corresponds to the case of defect lines with random orientation [23]. One can argue that these lines may break the 2D system into disconnected domains: this is the argument that can also be applied to the McCoy and Wu model [7]. Therefore one should take values $\delta > 1$ with caution since strong correlations may destabilize the LR FP and drastically modify the critical behavior. Since we cannot identify any stable and perturbative in disorder FP for $\delta > \delta_{\max}$, two scenarios are possible: (a) smearing of the sharp transition that is manifested in a runaway of the renormalization group flow; (b) a new universality class controlled by a nonperturbative infinite-randomness FP. In the latter case one may expect relevance of rare regions which make a difference between the typical and average correlations: the correlation function between two arbitrary spins separated by a large distance x acquires a broad distribution [62]. Thus, the typical correlation function is very different from the averaged one which is dominated by rare strongly coupled regions of spins with atypical large correlations. As a result, there can be two correlations lengths, typical and averaged, and therefore two critical exponents $\nu_{\text{typ}} \leq \nu_{\text{avr}}$.

Substituting FP (50) into Eqs. (34) and (44) we get the correlation length exponent

$$\frac{1}{\nu} = 1 - \frac{\delta}{2} + O(\delta^3), \quad (52)$$

where the corrections of the second order in δ magically cancel each other. Indeed, comparing Eq. (42) and Eq. (44) one can

observe that at least to two-loop order

$$\beta_v(u, v) = v(\delta + 2\gamma(u, v)). \quad (53)$$

Calculating it at any FP and taking into account Eq. (34) we obtain

$$v^*(\delta + 2(v^{-1} - 1)) = 0, \quad (54)$$

which is in agreement with the conjecture of Refs. [8,47] that the identity

$$v = 2/(2 - \delta) = 2/a \quad (55)$$

is exact at the LR FP with $v^* \neq 0$.

V. SPIN-SPIN CORRELATION AT CRITICALITY: BOSONIZATION

We now focus on the scaling behavior of the two-point correlation function at criticality. Let us denote the correlation function in a given realization of disorder by $G(r)$ and introduce the set of critical exponents

$$\overline{G(r)^N} \sim r^{-\eta_N}. \quad (56)$$

In the absence of multifractality one expects $\eta_N = N\eta_1$ and $\eta_1 \equiv \eta$ is the standard pair correlation function exponent. Since the correspondence between the spin operators in the Ising model and the Majorana fermions is nonlocal, reexpressing the spin-spin correlation function in terms of fermions is complicated and is well defined only in two dimensions. Thus the anomalous dimension calculated in Ref. [52] from the scaling of the two-point fermionic correlation function cannot be directly connected with the critical exponent η . Nevertheless using the Dirac representation allows one to derive a compact formula for the square of the correlation function [34]

$$G(r)^2 = \left\langle \exp \left[i\pi \int_0^r dr' \bar{\psi}(r') \psi(r') \right] \right\rangle, \quad (57)$$

where the averaging is performed with the Dirac action (5). The direct calculation of the spin-spin correlation function from the fermionic representation (57) has been performed only for the pure system and involves a cumbersome algebra [31]. A more simple way to get access to the spin-spin correlation function is to use bosonization. The latter maps the 2D Dirac fermions (5) into the sine-Gordon theory [34,55]

$$S_{\text{SG}} = \int d^2r \left\{ \frac{1}{2} [\nabla\varphi(r)]^2 - \frac{\Lambda m(r)}{\pi} \cos[\sqrt{4\pi}\varphi(r)] \right\}, \quad (58)$$

where Λ is the UV cutoff. The two-point spin correlation function becomes a two-point correlation function of the operator

$$\mathcal{O}(r) = \sin \sqrt{\pi}\varphi(r). \quad (59)$$

Note that we bosonize the Dirac fermions so that this method gives not the two-point function but the square of the two-point function

$$G(r)^2 = \langle \mathcal{O}(r)\mathcal{O}(0) \rangle_{\text{SG}} \quad (60)$$

since the two Majorana fermions, i.e., two copies of the Ising model, have been combined to the Dirac fermions. Averaging

in (60) is performed with action (58). After averaging over disorder we obtain

$$\overline{G(r)^2} = \overline{\langle \mathcal{O}(r)\mathcal{O}(0) \rangle_{\text{SG}}}. \quad (61)$$

To get a perturbative expansion for the correlation functions of the operator (59) one has to compute the correlation functions of exponentials of field $\varphi(r)$:

$$\left\langle \prod_{j=1}^n e^{i\beta_j \varphi(r_j)} \right\rangle_0 = \int D\varphi \exp \left\{ -\frac{1}{2} \int d^2r [\nabla \varphi(r)]^2 + \sum_{j=1}^n i\beta_j \varphi(r_j) \right\}. \quad (62)$$

It can be shown [55] that this correlation function is not vanishing only for $\sum_{j=1}^n \beta_j = 0$ and is given by

$$\left\langle \prod_{j=1}^n e^{i\beta_j \varphi(r_j)} \right\rangle_0 = \prod_{j < k} (\Lambda |r_j - r_k|)^{\beta_j \beta_k / (2\pi)}. \quad (63)$$

For the pure 2D Ising model at criticality, i.e., at $m(r) = 0$, one finds

$$G(r)^2 = \langle \mathcal{O}(r)\mathcal{O}(0) \rangle_0 = \frac{1}{2} \langle e^{i\sqrt{\pi}[\varphi(r) - \varphi(0)]} \rangle = \frac{1}{2} (\Lambda r)^{-1/2}, \quad (64)$$

and thus $\eta_{\text{pure}} = 1/4$ for the pure system. We now calculate the first-order correction in disorder. Applying the replica trick to the action (58) we derive the replicated action

$$S = \frac{1}{2} \sum_{\alpha=1}^n \int d^2r [\nabla \varphi_{\alpha}(r)]^2 - \frac{\Lambda^2}{2\pi^2} \sum_{\alpha, \beta=1}^n \int d^2r d^2r' g(r - r') \cos[\sqrt{4\pi} \varphi_{\alpha}(r)] \cos[\sqrt{4\pi} \varphi_{\beta}(r')]. \quad (65)$$

Here we perform calculations directly in two dimensions to one-loop order that allows us to put $u_0 = 0$. We calculate the averaged squared spin-spin correlation function for one replica $\alpha = 1$ to the first order in u_0 and v_0 in Appendix D and obtain

$$\langle \mathcal{O}_1(r)\mathcal{O}_1(0) \rangle_S = \frac{1}{2} (\Lambda |r|)^{-1/2} \left[1 + \frac{u_0 \ln r \Lambda}{4\pi} + \frac{v_0 |r|^{\delta}}{4\pi \delta} \right]. \quad (66)$$

To renormalize the spin-spin correlation function we introduce the renormalization constant

$$\overset{\circ}{\mathcal{O}} = Z_{\mathcal{O}}^{1/2} \mathcal{O}, \quad (67)$$

which can be found from the relation

$$\overline{G(r)^2} = Z_{\mathcal{O}} \overline{G(r)^2}. \quad (68)$$

Using the dimensional method developed in Ref. [63] we can convert the logarithm in Eq. (66) into a pole as $\ln r \Lambda \rightarrow \frac{|r|^{\epsilon}}{\epsilon}$. Taking into account that to the lowest order $u_0 = 2m^{\epsilon} v / K_d = 4\pi m^{\epsilon} u$ and $v_0 = 2m^{\delta} v / K_d = 4\pi m^{\delta} v$ we obtain

$$Z_{\mathcal{O}} = 1 + \frac{u}{\epsilon} + \frac{v}{\delta} + O(u^2, v^2). \quad (69)$$

The β functions and the FP coordinates can be taken from the results obtained for the Dirac fermions [Eqs. (41), (42) and

(50)]. The resulting scaling function reads

$$\eta_2 = \frac{1}{2} - \beta_u \frac{\partial \ln Z_{\mathcal{O}}}{\partial u} - \beta_v \frac{\partial \ln Z_{\mathcal{O}}}{\partial v} \quad (70)$$

and to one-loop order is given by

$$\eta_2 = \frac{1}{2} - u - v + O(u^2, v^2). \quad (71)$$

Using that to the one-loop order $u^* = 0$ and $v^* = \delta/4$ [see Eq. (50)] we obtain the critical exponent

$$\eta_2 = \frac{1}{2} - \frac{\delta}{4}, \quad (72)$$

which describes algebraic decay of the square of the spin-spin correlation function averaged over disorder:

$$\overline{G(r)^2} = r^{-\eta_2}. \quad (73)$$

Since $\overline{G^2} \geq \overline{G}^2$ and $\eta < \eta_{\text{pure}}$ the exponent η should satisfy the inequality

$$\frac{\eta_2}{2} \approx \frac{1}{4} - \frac{\delta}{8} \leq \eta \leq \frac{1}{4}. \quad (74)$$

To go beyond the one-loop approximation is a nontrivial task which is left for a forthcoming study.

VI. CONCLUSIONS

We have studied the 2D Ising model with LR correlated disorder using the mapping of the model to the 2D Dirac fermions in the presence of LR correlated random mass disorder. Using dimensional regularization with double expansion in $\epsilon = 2 - d$ and $\delta = 2 - a$ we renormalize the corresponding field theory up to the two-loop order. In two dimensions we have found two FPs: Gaussian FP ($u^* = 0, v^* = 0$) and LR FP [$u^* = O(\delta^2), v^* = O(\delta)$]. The Gaussian FP describes the 2D Ising model with SR disorder. The SR disorder is marginally irrelevant in 2D and leads to logarithmic correction to scaling. The SR FP is stable for $\delta < 0$ in accordance with the generalized Harris criterion $a\nu_{\text{pure}} - d > 0$ since $\nu_{\text{pure}} = 1$ in two dimensions.

We have shown that the LR FP is stable for $0 < \delta < \delta_{\text{max}}$ with $\delta_{\text{max}} \approx 1.005$ to two-loop order. The LR FP is characterized by the critical exponent $\nu = 2/a + O(\delta^3)$ in accordance with the prediction $\nu = 2/a$. Using mapping to the sine-Gordon model we have also studied behavior of the averaged square of the spin-spin correlation function at the LR FP which has been found algebraically decaying with the distance as $\overline{G^2(r)} \sim r^{-\eta_2}$. To the lowest order in disorder we have $\eta_2 = \frac{1}{2} - \frac{\delta}{4} + O(\delta^2)$ that gives the bounds for the usual exponent η describing the algebraic decay of the averaged correlation function: $\frac{1}{2}\eta_2 \leq \eta \leq \frac{1}{4}$.

We have not found a stable FP for $\delta > \delta_{\text{max}}$. This runaway can be a sign of either a smeared phase transition or a critical behavior controlled by an infinite randomness FP with different critical exponents. In the last case one can expect difference between the typical and averaged correlation length exponents. The latter is supposed to be due to rare regions with strong correlations so that one can expect $\nu_{\text{avr}} > \nu_{\text{typ}}$. In order to

study the nonperturbative effects for $\delta > \delta_{\max}$ one can try to allow replica symmetry breaking following Refs. [64,65].

Let us compare our finding with the known numerical results. In Ref. [51] it was found that $\eta = 0.2588(14)$ and $\nu = 2.005(5)$ for $a = 1$ ($\delta = 1$). The exponent η satisfies the inequality (74) while the exponent ν is very close to the prediction $\nu = 2/a$. In Ref. [66] it was found that $\eta = 0.204(14)$ and $\nu = 7.14$ for $a = 2/3$ ($\delta = 4/3$). It seems that the exponent η also satisfies the inequality but the exponent ν is much higher than the prediction corresponding to the perturbative LR FP. That was ascribed to hyperscaling violation in the Griffiths phase due to large disorder fluctuations. In the light of our work this is not surprising. Indeed, the runaway of the RG flow for $\delta > \delta_{\max}$ suggests that either the system flows towards an inaccessible within a weak disorder RG infinite randomness FP which controls the transition or the transition is smeared out. The numerical simulations of Ref. [66] are in favor of the first scenario but this still remains an open question.

Another reason for such discrepancy may be due to peculiarities of the spatial distribution of disorder in the model analyzed in [66]. There, the spin configurations of the Ashkin-Teller model at the critical point were used to construct correlated distribution of random couplings. In turn, these displayed large self-similar clusters of strong/weak bonds [67]. Although, by construction, the disorder correlations in Ref. [66] were governed by the power-law decay (4), formal description of their impact might call for the model that differs from the one analyzed in our paper since the bare disorder distribution is strongly non-Gaussian. Note that all above values of the exponent ν satisfy the Chayes-Chayes-Fisher-Spencer inequality for the correlation length exponent of disordered systems, $\nu \geq 2/d$ [68]. This indicates absence of difference between the intrinsic correlation length and the finite-size correlation lengths in this problem.

ACKNOWLEDGMENTS

It is our pleasure to thank Christophe Chatelain and Victor Dotsenko for stimulating discussions. M.D. and V.B. thank the Laboratoire de Physique de l'ENS Lyon for hospitality during preparation of this work. This work was supported in part by the European Union's Research and Innovation funding program FP7 IRSES, Marie Curie Grants No. PIRSES-GA-2011-295302 "Statistical Physics in Diverse Realizations," No. PIRSES-GA-2010-269139 "Dynamics and Cooperative Phenomena in Complex Physical and Biological Environments" (M.D. and V.B.), and No. 612707 "Dynamics of and in Complex Systems" and No. 612669 "Structure and Evolution of Complex Systems with Applications in Physics and Life Sciences" (Yu.H.). A.A.F. acknowledges support from the French Agence Nationale de la Recherche through Grants No. ANR-12-BS04-0007 (SemiTopo), No. ANR-13-JS04-0005-01 (ArtiQ), and No. ANR-14-ACHN-0031 (TopoDyn).

APPENDIX A: VERTEX FUNCTIONS

Here we present the expressions for vertex functions using diagrammatic presentation (see Fig. 1 and Fig. 2). Taking into account the combinatorial factors for the diagrams, the two-

point function is

$$\begin{aligned} \Gamma^{(2)}(p) = \sigma p \{ & 1 + A_2^1 v_0 - C_3^{0,0} u_0^2 - (C_3^{1,0} + C_3^{0,1}) u_0 v_0 - C_3^{1,1} v_0^2 \} \\ & - im_0 \{ 1 - A_1^0 u_0 - A_1^1 v_0 + (C_1^{0,0} + C_2^{0,0}) u_0^2 \\ & + (C_1^{1,0} + C_1^{0,1} + C_2^{1,0} + C_2^{0,1}) u_0 v_0 \\ & + (C_1^{1,1} + C_2^{1,1}) v_0^2 \}. \end{aligned} \quad (\text{A1})$$

The SR part of the full four-point vertex functions reads

$$\begin{aligned} \Gamma_u^{(4)}(0) = & u - 2B_1^{0,0} u^2 - 2B_1^{0,1} uv + u^2 v (\tilde{D}_{12}^{1,0,0} + 2\tilde{D}_{14}^{1,0,0} \\ & + 2D_1^{1,0,0} + 2D_2^{1,0,0} + 4D_3^{1,0,0} + 4D_4^{1,0,0} + D_5^{1,0,0} \\ & + 4\tilde{D}_6^{1,0,0} + \tilde{D}_{12}^{0,1,0} + 2\tilde{D}_{14}^{0,1,0} + 2D_1^{0,1,0} + 2D_2^{0,1,0} \\ & + 4D_3^{0,1,0} + 4D_4^{0,1,0} + D_5^{0,1,0} + 4\tilde{D}_8^{0,1,0} + \tilde{D}_{12}^{0,0,1} \\ & + 2\tilde{D}_{14}^{0,0,1} + 4\tilde{D}_6^{0,0,1}) + uv^2 (\tilde{D}_{12}^{1,1,0} + 2\tilde{D}_{14}^{1,1,0} \\ & + 2D_1^{1,1,0} + 2D_2^{1,1,0} + 4D_3^{1,1,0} + 4D_4^{1,1,0} + D_5^{1,1,0} \\ & + 4\tilde{D}_6^{1,1,0} + \tilde{D}_{12}^{1,0,1} + 2\tilde{D}_{14}^{1,0,1} + 4\tilde{D}_6^{1,0,1} \\ & + \tilde{D}_{12}^{0,1,1} + 2\tilde{D}_{14}^{0,1,1} + 4\tilde{D}_6^{0,1,1}) + v^3 (\tilde{D}_{12}^{1,1,1} \\ & + 2\tilde{D}_{14}^{1,1,1} + 4\tilde{D}_6^{1,1,1}) + u^3 (\tilde{D}_{12}^{0,0,0} + 2\tilde{D}_{14}^{0,0,0} \\ & + 2D_1^{0,0,0} + 2D_2^{0,0,0} + 4D_3^{0,0,0} \\ & + 4D_4^{0,0,0} + D_5^{0,0,0} + 4\tilde{D}_6^{0,0,0}), \end{aligned} \quad (\text{A2})$$

where $\tilde{D}_i^{a,b,c} = D_i^{a,b,c} + D_{i+1}^{a,b,c}$. The LR part of the four-point vertex is given by

$$\begin{aligned} \Gamma_v^{(4)}(0) = & v - 2B_1^{1,1} v^2 - 2B_1^{1,0} uv + v^3 (2D_1^{1,1,1} + 2D_2^{1,1,1} \\ & + 4D_3^{1,1,1} + 4D_4^{1,1,1} + D_5^{1,1,1}) + uv^2 (2D_1^{1,0,1} \\ & + 2D_2^{1,0,1} + 4D_3^{1,0,1} + 4D_4^{1,0,1} + D_5^{1,0,1} + 2D_1^{0,1,1} \\ & + 2D_2^{0,1,1} + 4D_3^{0,1,1} + 4D_4^{0,1,1} + D_5^{0,1,1}) \\ & + u^2 v (2D_1^{0,0,1} + 2D_2^{0,0,1} + 4D_3^{0,0,1} + 4D_4^{0,0,1} \\ & + D_5^{0,0,1}). \end{aligned} \quad (\text{A3})$$

The poles and finite parts of the one-loop diagrams [A_i^a in (A1) and $B_i^{a,b}$ in (A2) and (A3) shown in Fig. 1] are given in Table I together with their combinatorial factors. The poles of

TABLE I. Poles and finite parts of one-loop diagrams in the units of $\frac{K_d}{2}$. C.F. is the combinatorial factor. The angular brackets denote contribution resulting from the mass in the numerator of the bare propagator (10).

Diag.	Value	C.F.
A_1^0	$\langle \frac{2}{\varepsilon} \rangle$	1
A_1^1	$\langle \frac{2}{\delta} \rangle$	1
A_2^0	0	1
A_2^1	$1 - \frac{\varepsilon}{\delta}$	1
$B_1^{0,0} = B_1^{1,0}$	$\frac{2}{\varepsilon} - 1 - \langle 1 \rangle$	2
$B_1^{0,1} = B_1^{1,1}$	$\frac{2}{\delta} - 1 - \langle 1 \rangle$	2
$B_2^{a,b} + B_3^{a,b}$	$-\langle 1 \rangle$	2

TABLE II. Poles of two-loop diagrams in the units of \hat{K} . C.F. is the combinatorial factor. The angular brackets denote contribution resulting from the mass in the numerator of the bare propagator (10).

Diagram	Different vertices $\{a,b,c\}$	Poles	C.F.
$C_1^{a,b}$	$\{0,0\}$	$\frac{4}{\varepsilon^2}(1-\varepsilon)$	1
	$\{1,0\}$	$\frac{4}{\delta\varepsilon}(1-\delta)$	1
	$\{0,1\}$	$\frac{8}{\delta(\varepsilon+\delta)} - \frac{8\varepsilon}{\delta(\varepsilon+\delta)}$	1
	$\{1,1\}$	$\frac{2(3\delta-\varepsilon)}{\delta^2(2\delta-\varepsilon)} - \frac{2(\delta+\varepsilon)}{\delta^2}$	1
$C_2^{a,b}$	$\{0,0\}$	$\frac{2}{\varepsilon^2}$	1
	$\{1,0\}, \{0,1\}$	$\frac{4}{\varepsilon(\delta+\varepsilon)}$	1
	$\{1,1\}$	$\frac{2(3\delta-2\varepsilon)}{\delta^2(2\delta-\varepsilon)}$	1
$C_3^{a,b}$	$\{0,0\}$	$\frac{1}{\varepsilon}$	1
	$\{1,0\}, \{0,1\}$	$\frac{2}{\varepsilon} - \frac{2}{\varepsilon+\delta}$	1
	$\{1,1\}$	$\frac{(3\delta-2\varepsilon)}{\delta^2}$	1
$D_1^{a,b,c}$	$\{0,0,0\}, \{0,0,1\}$	$\frac{4}{\varepsilon^2}(1-\varepsilon) - \langle \frac{4}{\varepsilon} \rangle$	2
	$\{1,0,0\}, \{1,0,1\}$	$\frac{4}{\varepsilon\delta}(1 - \frac{\varepsilon+\delta}{2}) - \langle \frac{2}{\varepsilon} \rangle - \langle \frac{2}{\delta} \rangle$	2
	$\{0,1,0\}, \{0,1,1\}$	$\frac{8}{\delta(\varepsilon+\delta)}(1 - \frac{\varepsilon+\delta}{2}) - \langle \frac{4}{\delta} \rangle$	2
	$\{1,1,0\}, \{1,1,1\}$	$\frac{2(3\delta-\varepsilon)}{\delta^2(2\delta-\varepsilon)} - \frac{2(3\delta-\varepsilon)}{\delta(2\delta-\varepsilon)} - \langle \frac{2(3\delta-\varepsilon)}{\delta(2\delta-\varepsilon)} \rangle$	2
$D_2^{a,b,c}$	$\{0,0,0\}, \{0,0,1\}$	$-\frac{2}{\varepsilon^2} + \frac{2}{\varepsilon} + \langle \frac{2}{\varepsilon} \rangle$	2
	$\{1,0,0\}, \{0,1,0\}, \{1,0,1\}, \{0,1,1\}$	$-\frac{4}{\delta(\varepsilon+\delta)} + \frac{2}{\delta} + \langle \frac{2}{\delta} \rangle$	2
	$\{1,1,0\}, \{1,1,1\}$	$-\frac{2}{\delta(2\delta-\varepsilon)} + \frac{2}{2\delta-\varepsilon} + \langle \frac{2}{2\delta-\varepsilon} \rangle$	2
$D_3^{a,b,c}$	$\{0,0,0\}, \{0,0,1\}$	$\frac{2}{\varepsilon^2} - \frac{2}{\varepsilon} - \langle \frac{2}{\varepsilon} \rangle$	4
	$\{1,0,0\}, \{1,0,1\}$	$\frac{4}{\varepsilon(\varepsilon+\delta)} - \frac{2}{\varepsilon} - \langle \frac{2}{\varepsilon} \rangle$	4
	$\{0,1,0\}, \{0,1,1\}$	$\frac{4}{\delta(\varepsilon+\delta)} - \frac{2}{\delta} - \langle \frac{2}{\delta} \rangle$	4
	$\{1,1,0\}, \{1,1,1\}$	$\frac{2}{\delta^2} - \frac{2}{\delta} - \langle \frac{2}{\delta} \rangle$	4
$D_4^{a,b,c}$	$\{0,0,0\}, \{0,0,1\}, \{1,0,0\}, \{1,0,1\}$	$-\langle \frac{2}{\varepsilon} \rangle$	4
	$\{0,1,0\}, \{0,1,1\}$	$-\langle \frac{2}{\delta} \rangle + \frac{2(\delta-\varepsilon)}{\delta(\varepsilon+\delta)}$	4
	$\{1,1,0\}, \{1,1,1\}$	$-\langle \frac{2}{\delta} \rangle + \frac{(\delta-\varepsilon)}{\delta^2}$	4
$D_5^{a,b,c}$	$\{0,0,0\}, \{0,0,1\}$	$\frac{4}{\varepsilon^2} - \frac{4}{\varepsilon} - \langle \frac{4}{\varepsilon} \rangle$	1
	$\{1,0,0\}, \{0,1,0\}, \{1,0,1\}, \{0,1,1\}$	$\frac{4}{\varepsilon\delta} - \frac{2}{\varepsilon} - \frac{2}{\delta} - \langle (\frac{2}{\varepsilon} + \frac{2}{\delta}) \rangle$	1
	$\{1,1,0\}, \{1,1,1\}$	$\frac{4}{\delta^2} - \frac{4}{\delta} - \langle \frac{4}{\delta} \rangle$	1
$D_6^{a,b,c} + D_7^{a,b,c}$	$\{0,0,0\}, \{0,1,0\}, \{0,0,1\}, \{0,1,1\}$	$-\langle \frac{4}{\varepsilon} \rangle$	4
	$\{1,0,0\}, \{1,1,0\}, \{1,0,1\}, \{1,1,1\}$	$-\langle \frac{4}{\delta} \rangle$	4
$D_8^{a,b,c} + D_9^{a,b,c}$	$\{a,b,c\}$	0	2
$D_{10}^{a,b,c} + D_{11}^{a,b,c}$	$\{a,b,c\}$	0	2
$D_{12}^{a,b,c} + D_{13}^{a,b,c}$	$\{0,0,0\}$	$\frac{4}{\varepsilon^2} - \frac{2}{\varepsilon}$	1
	$\{1,0,0\}, \{0,0,1\}$	$\frac{4}{\delta\varepsilon} - \frac{2}{\varepsilon}$	1
	$\{0,1,0\}$	$\frac{8}{\delta(\delta+\varepsilon)} - \frac{4}{\delta+\varepsilon}$	1
	$\{1,1,0\}, \{0,1,1\}$	$\frac{2(3\delta-\varepsilon)}{\delta^2(2\delta-\varepsilon)} - \frac{3\delta-\varepsilon}{\delta^2}$	1
	$\{1,0,1\}$	$\frac{4}{\delta^2} - \frac{2(2\delta-\varepsilon)}{\delta^2}$	1
	$\{1,1,1\}$	$\frac{8}{(2\delta-\varepsilon)(3\delta-\varepsilon)} - \frac{4(3\delta-2\varepsilon)}{(2\delta-\varepsilon)(3\delta-\varepsilon)}$	1
	$\{0,0,0\}$	$-\frac{2}{\varepsilon^2} + \frac{2}{\varepsilon}$	2
$D_{14}^{a,b,c} + D_{15}^{a,b,c}$	$\{1,0,0\}$	$\frac{4}{\delta(\delta+\varepsilon)} - \frac{4}{\varepsilon\delta} + \frac{2}{\varepsilon}$	2
	$\{0,1,0\}, \{0,0,1\}$	$-\frac{4}{\delta(\delta+\varepsilon)} + \frac{4}{\delta+\varepsilon}$	2
	$\{1,1,0\}, \{1,0,1\}$	$-\frac{2}{\delta^2} + \frac{3}{\delta} - \frac{\varepsilon}{\delta^2}$	2
	$\{0,1,1\}$	$-\frac{2}{\delta(2\delta-\varepsilon)} + \frac{2}{\delta}$	2
	$\{1,1,1\}$	$-\frac{4}{(2\delta-\varepsilon)(3\delta-\varepsilon)} - \frac{2}{2\delta-\varepsilon} + \frac{8}{3\delta-\varepsilon}$	2

the two-loop diagrams [$C_i^{a,b}$ in (A1) and $D_i^{a,b,c}$ in (A2) and (A3) shown in Fig. 2] are summarized in Table II. Some of the two-loop integrals appearing in the calculations of poles are summarized in Appendix B. The angular brackets in Tables I and II denote contributions resulting from the mass in the numerator of the bare propagator (10). These contributions cancel each other in the Z factors (37)–(40) at least to two-loop order as happens in the case of uncorrelated disorder [60].

APPENDIX B: TABLE OF TWO-LOOP INTEGRALS

Here we provide the list of the two-loop integrals, which are helpful in calculation of the two-loop diagrams. To calculate these integrals we used the methods based on the hypergeometric function representation which were developed in Ref. [69] for the φ^4 model with correlated disorder. We introduce the shortcut notations $[1] := q_1^2 + m^2$, $[2] := q_2^2 + m^2$, $[3] := (q_1 + q_2)^2 + m^2$, $\hat{K} := \frac{K_d^2}{4}$ as well as shortcut notation for the integration $\int := \int_{\vec{q}_1} \int_{\vec{q}_2}$. Only the poles are shown so that the omitted terms are of order $O(1)$ unless something else is explicitly stated.

1. $a = b = c = 0$

$$\int \frac{1}{[1][2]} = \int \frac{1}{[1][3]} = \hat{K} m^{-2\varepsilon} \left[\frac{4}{\varepsilon^2} \right], \quad (\text{B1})$$

$$\int \frac{1}{[1][3]^2} = \int \frac{1}{[1][2]^2} = \hat{K} m^{-2\varepsilon-2} \left[\frac{2}{\varepsilon} + O(\varepsilon) \right]. \quad (\text{B2})$$

2. $a \neq 0$

$$\int \frac{q_1^{a-d}}{[1][2]} = \int \frac{q_1^{a-d}}{[1][3]} = \hat{K} m^{-\varepsilon-\delta} \left[\frac{4}{\varepsilon\delta} \right], \quad (\text{B3})$$

$$\int \frac{q_1^{a-d}}{[2][3]} = \hat{K} m^{-\varepsilon-\delta} \left[\frac{8}{\delta(\delta+\varepsilon)} \right], \quad (\text{B4})$$

$$\int \frac{q_1^{a-d}}{[1][2]^2} = \int \frac{q_1^{a-d}}{[1][3]^2} = \hat{K} m^{-\varepsilon-\delta-2} \left[\frac{2}{\delta} \right], \quad (\text{B5})$$

$$\int \frac{q_1^{a-d}}{[2]^2[3]} = \int \frac{q_1^{a-d}}{[2][3]^2} = \hat{K} m^{-\varepsilon-\delta-2} \left[\frac{2}{\delta} \right], \quad (\text{B6})$$

$$\int \frac{q_1^2 q_1^{a-d}}{[2]^2[3]^2} = \hat{K} m^{-\varepsilon-\delta-2} \left[\frac{4}{\delta} \right], \quad (\text{B7})$$

$$\int \frac{q_1^{a-d}}{[1]^2[2]} = \hat{K} m^{-\varepsilon-\delta-2} \left[\frac{2}{\varepsilon} \right], \quad (\text{B8})$$

$$\int \frac{q_1^2 q_1^{a-d}}{[2]^3[3]} = \hat{K} m^{-\varepsilon-\delta-2} [-1 + O(\delta, \varepsilon)], \quad (\text{B9})$$

$$\int \frac{q_1^{a-d}}{[2]^3[3]} = \hat{K} m^{-\varepsilon-\delta-4} \left[\frac{1}{\delta} \right]. \quad (\text{B10})$$

3. $a \neq 0, b \neq 0$

$$\int \frac{q_1^{a-d} q_2^{a-d}}{[1][2]} = \hat{K} m^{-2\delta} \left[\frac{4}{\delta^2} \right], \quad (\text{B11})$$

$$\int \frac{q_1^{a-d} q_2^{a-d}}{[1][3]} = \int \frac{q_1^{a-d} q_2^{a-d}}{[2][3]} = \hat{K} m^{-2\delta} \left[\frac{2(3\delta-\varepsilon)}{\delta^2(2\delta-\varepsilon)} \right], \quad (\text{B12})$$

$$\int \frac{q_1^{a-d} q_2^{a-d}}{[1][3]^2} = \int \frac{q_1^{a-d} q_2^{a-d}}{[2][3]^2} = \hat{K} m^{-2\delta-2} \left[\frac{2}{2\delta-\varepsilon} \right], \quad (\text{B13})$$

$$\int \frac{q_1^{a-d} q_2^{a-d}}{[2]^2[3]} = \int \frac{q_1^{a-d} q_2^{a-d}}{[1][2]^2} = \hat{K} m^{-2\delta-2} \left[\frac{2}{\delta} \right], \quad (\text{B14})$$

$$\int \frac{q_2^2 q_1^{a-d} q_2^{a-d}}{[1]^2[3]} = \hat{K} m^{-2\delta} \left[\frac{2(3\delta-\varepsilon)}{\delta^2(2\delta-\varepsilon)} - \frac{2(3\delta-\varepsilon)}{\delta^2} \right], \quad (\text{B15})$$

$$\int \frac{[2] q_1^{a-d} q_2^{a-d}}{[1]^2[3]^2} = \hat{K} m^{-2\delta-2} \left[\frac{2(3\delta-\varepsilon)}{\delta(2\delta-\varepsilon)} \right], \quad (\text{B16})$$

$$\int \frac{[2] q_1^{a-d} q_2^{a-d}}{[1][3]^2} = \hat{K} m^{-2\delta} \left[\frac{2(3\delta-\varepsilon)}{\delta^2(2\delta-\varepsilon)} - \frac{2}{\delta} \right], \quad (\text{B17})$$

$$\int \frac{[2] q_1^{a-d} q_2^{a-d}}{[1]^2[3]} = \hat{K} m^{-2\delta} \left[\frac{2(3\delta-\varepsilon)}{\delta^2(2\delta-\varepsilon)} - \frac{2(2\delta-\varepsilon)}{\delta^2} \right], \quad (\text{B18})$$

$$\int \frac{q_2^4 q_1^{a-d} q_2^{a-d}}{[1]^2[3]^2} = \hat{K} m^{-2\delta} \left[\frac{8(3\delta-\varepsilon)}{\delta^2(2\delta-\varepsilon)} - \frac{2(8\delta-3\varepsilon)(3\delta-\varepsilon)}{\delta^2(2\delta-\varepsilon)} \right]. \quad (\text{B19})$$

4. $b \neq 0, c \neq 0$

$$\int \frac{q_2^{2(a-d)}}{[1][2]} = \int \frac{q_2^{2(a-d)}}{[2][3]} = \hat{K} m^{-2\delta} \left[\frac{4}{\varepsilon(2\delta-\varepsilon)} \right], \quad (\text{B20})$$

$$\int \frac{q_2^{2(a-d)}}{[1][3]} = \hat{K} m^{-2\delta} \left[\frac{4}{\delta(2\delta-\varepsilon)} \right], \quad (\text{B21})$$

$$\int \frac{q_2^{2(a-d)}}{[1][2]^2} = \int \frac{q_2^{2(a-d)}}{[2]^2[3]} = \hat{K} m^{-2\delta-2} \left[\frac{2}{\varepsilon} \right]. \quad (\text{B22})$$

5. $a \neq 0, b \neq 0, c \neq 0$

$$\int \frac{q_1^{a-d} q_2^{2(a-d)}}{[1][2]} = \hat{K} m^{-3\delta+\varepsilon} \left[\frac{4}{\delta(2\delta-\varepsilon)} \right], \quad (\text{B23})$$

$$\int \frac{q_1^{a-d} q_2^{2(a-d)}}{[1][3]} = \hat{K} m^{-3\delta+\varepsilon} \left[\frac{4(5\delta-3\varepsilon)}{(3\delta-2\varepsilon)(2\delta-\varepsilon)(3\delta-\varepsilon)} \right], \quad (\text{B24})$$

$$\int \frac{q_1^{a-d} q_2^{2(a-d)}}{[2][3]} = \hat{K} m^{-3\delta+\varepsilon} \left[\frac{8(2\delta-\varepsilon)}{\delta(3\delta-2\varepsilon)(3\delta-\varepsilon)} \right], \quad (\text{B25})$$

$$\int \frac{q_1^{a-d} q_2^{2(a-d)}}{[2]^2[3]} = \hat{K} m^{-3\delta+\varepsilon-2} \left[\frac{2}{\delta} \right], \quad (\text{B26})$$

$$\int \frac{q_1^{a-d} q_2^{a-d} |q_1+q_2|^{a-d}}{[1][2]} = \hat{K} m^{-3\delta+\varepsilon} \left[\frac{8}{(2\delta-\varepsilon)(3\delta-\varepsilon)} \right]. \quad (\text{B27})$$

APPENDIX C: LOGARITHMIC CORRECTIONS FOR SR DISORDER

In order to calculate the subdominant logarithmic corrections to scaling behavior in two dimensions due to SR disorder

we have to find the asymptotic flow to the Gaussian FP. Here we do this to two-loop order. The flow equations read

$$\frac{du}{dl} = \beta_u(u, v=0) = -4u^2 + 8u^3 + O(u^4), \quad (\text{C1})$$

$$\frac{d \ln \tau}{dl} = -[1 + \gamma(u, 0)] = -1 + 2u - 2u^2 + O(u^3), \quad (\text{C2})$$

$$\frac{d \ln F}{dl} = \gamma(u, 0) = -2u + 2u^2 + O(u^3), \quad (\text{C3})$$

where $l = \ln \xi$ and F is the vertex function with insertion of the composite operator $\bar{\psi}(0)\psi(0)$ defined in Refs. [70,71]. The asymptotic behavior of the solution of Eq. (C1) in the limit $l \rightarrow \infty$ is

$$u(l) = \frac{1}{4l} + \frac{\ln l}{8l^2} + O\left(\frac{1}{l^2}\right). \quad (\text{C4})$$

Substituting the flow (C4) to Eq. (C2) we obtain

$$\tau^{-1} \sim \xi(\ln \xi)^{-1/2} \left[1 + \frac{\ln \ln \xi}{4 \ln \xi} \right]. \quad (\text{C5})$$

Inverting this equality with logarithmic accuracy we arrive at Eq. (47). The singular part of the specific heat in the asymptotic regime is given by $C_{\text{sing}} = \int dl F^2(l)$ [70]. Solving Eq. (C3) we obtain

$$C_{\text{sing}}(l) = \ln l \left[1 - \frac{1}{2 \ln l} \right]. \quad (\text{C6})$$

Using $l = \ln \xi$ where ξ is given by Eq. (47) we derive Eq. (48).

APPENDIX D: CORRELATION FUNCTION

We now calculate the two-point function (66) for the replica $\alpha = 1$ to the lowest order in disorder. The first-order correction in disorder can be split into the SR and LR parts as follows:

$$\begin{aligned} \langle \mathcal{O}_1(r)\mathcal{O}_1(0) \rangle_S &= \langle \mathcal{O}_1(r)\mathcal{O}_1(0) \rangle_0 + \delta_{\text{SR}}^{(1)} \langle \mathcal{O}_1(r)\mathcal{O}_1(0) \rangle \\ &+ \delta_{\text{LR}}^{(1)} \langle \mathcal{O}_1(r)\mathcal{O}_1(0) \rangle. \end{aligned} \quad (\text{D1})$$

The leading term in Eq. (D1) gives the two-point function of the pure system:

$$\begin{aligned} \langle \mathcal{O}_1(r)\mathcal{O}_1(0) \rangle_0 &= \langle \sin \sqrt{\pi} \phi_1(r) \sin \sqrt{\pi} \phi_1(0) \rangle_0 \\ &= \frac{1}{(2i)^2} \langle (e^{i\sqrt{\pi} \phi_1(r)} - e^{-i\sqrt{\pi} \phi_1(r)}) \\ &\quad \times (e^{i\sqrt{\pi} \phi_1(0)} - e^{-i\sqrt{\pi} \phi_1(0)}) \rangle_0 \\ &= \frac{1}{2} \langle e^{i\sqrt{\pi}(\phi_1(r) - \phi_1(0))} \rangle_0 = \frac{1}{2} (\Lambda r)^{-1/2}, \end{aligned} \quad (\text{D2})$$

where we used Eqs. (62) and (63). The first-order correction in the SR correlated disorder has been calculated in Ref. [34] using bosonization of the 2D massive Thirring model [55]. The latter allows one to eliminate the terms in action (65) which are diagonal in replicas and local in space by means of the

identity

$$\left[\frac{\Lambda}{\pi} \cos \sqrt{4\pi} \varphi(r) \right]^2 = -\frac{1}{2\pi} (\nabla \varphi)^2. \quad (\text{D3})$$

As a result, the kinetic term is rescaled by the factor of $1 + u_0/(2\pi)$. The nondiagonal in replicas terms do not contribute to the one-loop order. Using the rescaling $\varphi = [1 + u_0/(2\pi)]^{-1/2} \varphi'$ we obtain for the SR disorder

$$\begin{aligned} &\langle \mathcal{O}_1(r)\mathcal{O}_1(0) \rangle_0 + \delta_{\text{SR}}^{(1)} \langle \mathcal{O}_1(r)\mathcal{O}_1(0) \rangle \\ &= \frac{1}{2} (\Lambda r)^{-1/(2[1+u_0/(2\pi)])} \approx \frac{1}{2} (\Lambda r)^{-1/2} \left[1 + \frac{u_0 \ln r \Lambda}{4\pi} \right]. \end{aligned} \quad (\text{D4})$$

For the LR disorder we calculate the correction explicitly:

$$\begin{aligned} &\delta_{\text{LR}}^{(1)} \langle \mathcal{O}_1(r)\mathcal{O}_1(0) \rangle \\ &= \frac{\Lambda^2}{2\pi^2} \int d^2 r_1 d^2 r_2 g(r_1 - r_2) \langle \sin \sqrt{\pi} \phi_1(r) \\ &\quad \times \sin \sqrt{\pi} \phi_1(0) \cos \sqrt{4\pi} \phi_1(r_1) \cos \sqrt{4\pi} \phi_1(r_2) \rangle \\ &= -\frac{\Lambda^2}{2\pi^2} \frac{2^2}{2^2 (2i)^2} \int d^2 r_1 d^2 r_2 g(r_1 - r_2) \\ &\quad \times \langle e^{i\sqrt{\pi} \phi_1(r)} e^{-i\sqrt{\pi} \phi_1(0)} e^{i\sqrt{4\pi} \phi_1(r_1)} e^{-i\sqrt{4\pi} \phi_1(r_2)} \rangle \\ &= \frac{\Lambda^2}{8\pi^2} (\Lambda r)^{-1/2} \int d^2 r_1 d^2 r_2 g(r_1 - r_2) (\Lambda |r_1 - r_2|)^{-2} \\ &\quad \times \frac{|r - r_1||r_2|}{|r - r_2||r_1|}. \end{aligned} \quad (\text{D5})$$

Taking $g(r_1 - r_2)$ as the inverse Fourier transform of (9) at $d = 2$,

$$g(r_1 - r_2) = u_0 \delta^{(2)}(r_1 - r_2) + \frac{v_0 \delta}{2\pi} |r_1 - r_2|^{-a}, \quad (\text{D6})$$

where $\delta^{(2)}$ is the two-dimensional δ function, and setting $u_0 = 0$, we find

$$\delta_{\text{LR}}^{(1)} \langle \mathcal{O}_1(r)\mathcal{O}_1(0) \rangle = \frac{(\Lambda r)^{-1/2}}{8\pi^2} \frac{v_0 \delta}{2\pi} |r|^\delta J\left(\frac{1}{2}, \frac{\delta}{4}\right), \quad (\text{D7})$$

where we have introduced the integral

$$J(p, \tau) = \mathcal{FP} \int d^2 r_1 d^2 r_2 |r_1 - r_2|^{4(\tau-1)} \left[\frac{|e - r_1||r_2|}{|e - r_2||r_1|} \right]^{2p}. \quad (\text{D8})$$

Here e is an arbitrary unit vector, and \mathcal{FP} means ‘‘finite part’’ in the sense of dimensional regularization. The method of computing integrals of type (D8) has been developed in Refs. [63,72,73]. It reads

$$\frac{J(p, \tau)}{4\pi^2} = \frac{p^2}{8\tau^2} + O(\tau^{-1}). \quad (\text{D9})$$

Collecting all factors we arrive at Eq. (66).

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