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Classification of topological phases in periodically driven interacting systems

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We consider topological phases in periodically driven (Floquet) systems exhibiting many-body localization, protected by a symmetry G. We argue for a general correspondence between such phases and topological phases of undriven systems protected by symmetry $\mathbb{Z} \rtimes G$ where the additional $\mathbb{Z}$ accounts for the discrete time-translation symmetry. Thus, for example, the bosonic phases in d spatial dimensions without intrinsic topological order [symmetry-protected topological (SPT) phases] are classified by the cohomology group $H^{d+1}[\mathbb{Z} \rtimes G, \mathrm{U}(1)]$. For unitary symmetries, we interpret the additional resulting Floquet phases in terms of the lower-dimensional SPT phases that are pumped to the boundary during one time step. These results also imply the existence of novel symmetry-enriched topological (SET) orders protected solely by the periodicity of the drive.

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Introduction. There are now many known examples of phases of matter which are distinguished not by the symmetries they break spontaneously but through more subtle "topological" orders [1]. Most such phases are not robust to thermal excitations and therefore were thought to exist only at zero temperature [2,3]. However, recently it has been appreciated that, in the presence of strong disorder, it is possible for highly excited eigenstates of a many-body system to be *many-body localized* (MBL) [4–13]. Such MBL states are not thermal and indeed more closely resemble gapped ground states; for example, they obey an area law for the entanglement entropy. This means that they can exhibit topological phases previously thought to be restricted to zero temperature [14–18].

The lifting of the restriction to ground states also allows us to consider more general "Floquet" systems [19–28] in which the Hamiltonian $H(t)$ is allowed to vary in time but with periodicity T. The "eigenstates" of such a system are the eigenstates of the Floquet operator $U = U(T)$ which describes the unitary evolution of the system over one time period. Such eigenstates can also be MBL in the presence of strong disorder [29–34] and hence can exhibit topological phases. However, the classification of topological phases in such "Floquet-MBL" systems is in general richer than in the stationary case.

Recently, progress has begun to be made in understanding the classification of topological phases in Floquet-MBL systems with interactions [35,36]. In particular, Ref. [36] classified phases with a symmetry G and no intrinsic topological order (i.e., *symmetry-protected topological* (SPT) phases [37–55]) in (1+1) dimensions [(1+1)D]. The purpose of this Rapid Communication is to reexpress the classification of Ref. [36] in a concise way, which we feel clarifies the issues involved and streamlines the derivation. We then consider natural extensions, building up to a (conjectured) general correspondence between topological phases in Floquet-MBL systems with symmetry group G and topological phases in stationary systems with symmetry group $\mathbb{Z} \rtimes G$, where the extra $\mathbb{Z}$ accounts for the discrete time-translation symmetry.

Assumptions. We will assume that the Floquet operator U can be expressed as a time evolution of a local time-dependent Hamiltonian $H(t)$ with $H(t+T) = H(t)$. Thus,

$$U = \mathcal{T} \exp\left(-i \int_0^T H(t)dt\right), \quad \mathcal{T} = \text{time ordering}, \quad (1)$$

where we assume that the Hamiltonian $H(t)$ is invariant under a representation $V(g)$ of a symmetry group G, where G can contain antiunitary elements corresponding to a time-reversal symmetry. For antiunitary $g \in G$, what we mean by the Hamiltonian being invariant is that $V(g)H(t)V(g)^{-1} = H(T-t)$. This ensures that, in general,

$$V(g)UV(g)^{-1} = U^{\alpha(g)}, \quad (2)$$

where $\alpha(g) = -1$ if g is antiunitary and $+1$ otherwise.

The SPT classification. The classification of Ref. [36] can be reexpressed in the following way. We define an enlarged symmetry group $\widetilde{G}$ to be the *full* symmetry group of the system, *including* the discrete time-translation symmetry inherent in the Floquet setup. Thus, if all of the symmetries of G are unitary, we have $\widetilde{G} = G \times \mathbb{Z}$. More generally, for antiunitary elements $g \in G$, we have $g\mathbb{T}g^{-1} = \mathbb{T}^{-1}$, where $\mathbb{T}$ is the generator of time translations. Thus, in general $\widetilde{G}$ is a semidirect product $\widetilde{G} = \mathbb{Z} \rtimes G$. Then in the bosonic case, the classification of Ref. [36] can be reformulated as follows (see the Supplemental Material for a proof [56]):

Result 1. The symmetry-protected topological phases in a periodically driven (1+1) bosonic system exhibiting MBL are classified by the second cohomology group $H^2[\widetilde{G}, \mathrm{U}(1)]$.

[Here, and later, we will take it to be implicit that U(1) is to be interpreted as a nontrivial $\widetilde{G}$-module with antiunitary elements of $\widetilde{G}$ acting as inversion as in the original classification of SPT phases with antiunitary symmetries, e.g., see Ref. [46]).

Recall that the bosonic topological phases in a stationary system are classified by $H^2[G, \mathrm{U}(1)]$; to obtain the classification in a driven system one simply replaces G by $\widetilde{G}$. In retrospect, this result should be quite natural. Indeed, the classification of stationary SPT phases in (1+1)D [38,39,42,43], although sometimes expressed in terms of Hamiltonians, is really at its core a classification of *short-range entangled* states (states which are equivalent to a product state by a local unitary) invariant under some local (anti)unitary representation of a symmetry group (see the Appendix for more details). The gapped ground states of a Hamiltonian are examples of such states but so are MBL eigenstates of a Floquet operator. (We could even consider eigenstates of the Floquet operator which are not MBL but are separated from all other eigenstates by a quasienergy gap.) Thus, the standard classification of (1+1)D SPT phases can be applied to any such states. However, there is one difference in the Floquet case: As well as the representation of the symmetry G, a Floquet eigenstate is, by definition, also invariant (up to a phase factor) under the Floquet operator U, which is a local unitary since it is the time evolution of a local Hamiltonian. Therefore, we should really include U in the symmetry group to obtain the full classification. [Equation (2) ensures that we then have a representation of the enlarged symmetry group $\widetilde{G} = \mathbb{Z} \rtimes G$.]

It is true that, when classifying SPT phases, one normally assumes that the action of the symmetry is "on site," that is, that each symmetry operator $V(g)$ is a tensor product of its action on each site of the lattice $V(g) = [v(g)]^{\otimes N}$, which would not be true of the Floquet unitary U. However, all we actually need is that all the symmetry operators (including the Floquet unitary U) can be *restricted* to a region A with boundary while still remaining a representation of $\widetilde{G}$ where by restriction of a local unitary U we mean [52] a unitary U_A acting only on the region A which acts the same as U in the interior of A, well away from the boundary. See the Appendix for the derivation of the classification, given such an assumption.

To see that such a restriction is possible, consider for simplicity the case of unitary symmetries. Then if the Hamiltonian $H(t)$ can be written as a sum $H(t) = \sum_X h_X(t)$ of terms supported on local regions X [each of which commutes with the symmetry $V(g)$], then we can define the restriction of the Floquet operator by simply retaining only the terms which act within A, or in other words,

$$U_A = \mathcal{T} \exp\left(-i \int_0^T dt \sum_{X \subseteq A} h_X(t)\right). \tag{3}$$

Meanwhile, we define the restriction of $V_A(g)$ in the obvious way by only acting with the on-site action on sites contained within A. It is easily seen that $V_A(g)$ is still a representation of G and U_A commutes with $V_A(g)$, so together they form a representation of $\widetilde{G} = \mathbb{Z} \times G$. Similar arguments can be made for antiunitary symmetries.

We emphasize that our derivation of Result 1 is actually more general than that of Ref. [36]. First, in Ref. [36] the result for non-Abelian G was only stated as a conjecture. Our derivation clearly applies to such a G as well. Second, we did not need to assume as did Ref. [36] that *all* the eigenstates of the Floquet operator are MBL; our classification result applies to any of the eigenstates that happen to be MBL or separated from the rest of the quasienergy spectrum by a gap. Finally, since our derivation was based on individual eigenstates, it allows for the possibility of different SPT phases coexisting as eigenstates of a single Floquet operator, separated by an eigenstate transition [14,16].

Higher-dimensional results. When stated in the form given here, the classification result of Ref. [36] has obvious generalizations to higher dimensions. In particular, in Ref. [52] we derived the classification of (2+1)-dimensional [(2+1)D] SPT phases in ground states by considering how the symmetry acts on the boundary. In Ref. [52], we did use the Hamiltonian to argue that the symmetry action on the boundary is well defined; however, the Appendix shows how to formulate this concept for a single short-range entangled state without reference to a Hamiltonian (and without assuming that the symmetry in the bulk is on site). Therefore, we can repeat the analysis of Ref. [52] (but taking care to include the Floquet unitary U in the symmetry group), and one finds that:

Result 2. The symmetry-protected topological phases in a periodically driven (2+1)D bosonic system exhibiting MBL are classified by the third cohomology group $H^3[\widetilde{G},\mathrm{U}(1)]$.

Again, we simply replace $G \to \widetilde{G}$ compared to the usual stationary case. The antiunitary case was not explicitly treated in Ref. [52], but it is a straightforward generalization [57]. One can also prove a similar result for fermionic systems.

General correspondence between stationary and Floquet-MBL topological phases. The above results relied on the method of Ref. [52], which did not consider (at least, not in full generality) SPT phases in higher dimensions or topological phases beyond SPT. Nevertheless, they motivate us to formulate the following conjecture.

Conjecture 1. The topological phases in a (bosonic/fermionic) periodically driven MBL system in d spatial dimensions with on-site symmetry group G are in one-to-one correspondence with the topological phases in a (bosonic/fermionic) *stationary* MBL system in d spatial dimensions with symmetry group $\widetilde{G} = \mathbb{Z} \rtimes G$ (as defined above).

Here by topological phases, we mean both SPT phases and symmetry-*enriched* topological (SET) phases [58–64]. The rationale for this conjecture is as follows. The classification of gapped ground states is known to depend only on the ground states themselves, not on their parent Hamiltonians [43]. Furthermore, since eigenstates in an MBL system look, roughly speaking, like gapped ground states, one expects to obtain the same classification for such eigenstates. However, in a periodically driven system there is an extra local unitary, beyond the symmetries in group G, under which these eigenstates are invariant (up to a phase factor)—namely, the Floquet unitary U. Thus, one should treat U as a symmetry for the purpose of obtaining the classification.

The only way we could envision this conjecture failing would be if the non-on-site nature of the Floquet unitary U turned out to be important in a way that it was not in the case of (1+1)D and (2+1)D SPTs. This seems to us unlikely. In fact, we expect that *any* derivation of the classification of SPT/SET phases—or at least, any derivation which can be formulated in terms of short-range entangled states without reference to Hamiltonians—could probably be applied just as well in the Floquet context, which would prove the conjecture.

We note, however, that probably not all topological phases which can exist at zero temperature can be stabilized in MBL excited states [17]; for this reason, we have been careful to formulate Conjecture 1 in terms of a correspondence with stationary MBL systems, not with zero-temperature states.

Interpretation of the classification in terms of pumping. Results 1 and 2, and Conjecture 1 in higher dimensions imply that the classification of SPT phases in bosonic Floquet-MBL systems in d spatial dimensions is $H^{d+1}[\widetilde{G},\mathrm{U}(1)]$. In the case of a unitary symmetry such that $\widetilde{G}$ is just a direct product $\mathbb{Z} \times G$, we can give a simple physical interpretation of this result. From the Künneth formula for group cohomology [64], one finds that

$$H^{d+1}[\mathbb{Z} \times G,\mathrm{U}(1)] = H^{d+1}[G,\mathrm{U}(1)] \times H^{d}[G,\mathrm{U}(1)]. \quad (4)$$

Thus, the classification is just the usual classification for ground states, plus an extra piece of data given by an element of $H^d[G,\mathrm{U}(1)]$. We expect that this extra piece of data can be interpreted as characterizing the fact that each application of the Floquet unitary U "pumps" an additional $(d-1)$-dimensional SPT phase onto the boundary. This is a generalization of the observation in Ref. [36] that in (1+1)D the extra data is the charge pumped onto each component of the boundary by the Floquet unitary.

A rough physical justification for this interpretation in (2+1)D (which readily generalizes also to higher dimensions) is as follows. For simplicity we assume that G is Abelian. One can then show that the $H^2[G,\mathrm{U}(1)]$ piece of Eq. (4) can be extracted from a 3-cocycle $\omega(\widetilde{g}_1,\widetilde{g}_2,\widetilde{g}_3)$ of the full symmetry group $\widetilde{G}$ by calculating a 2-cocycle of G according to

$$\omega(g_1,g_2) = \frac{\omega(\mathbb{T},g_1,g_2)\omega(g_1,g_2,\mathbb{T})}{\omega(g_1,\mathbb{T},g_2)} \quad (5)$$

(where $\mathbb{T}$ is the generator of discrete time translations). The object Eq. (5) has a familiar interpretation [65]. Indeed, suppose we *gauge* the full symmetry group $\widetilde{G} = \mathbb{Z} \times G$. Then the point excitations in the resulting twisted (2+1)D gauge theory can be classified by the flux $\widetilde{g} \in \widetilde{G}$ they carry. In general, a particle carrying nontrivial flux also carries a *projective* representation of the gauge group. In particular, Eq. (5) describes the projective representation of subgroup G on a particle carrying flux $\mathbb{T}$. Now, in the original ungauged SPT phase, the analog of a flux is a "symmetry twist defect" [63,66–68] which (since fluxes are confined) must occur at the end point of a symmetry twist line. The fact that the end points of such symmetry twist lines carry projective representations of G (which can also be derived directly using the theory of twist defects developed in Ref. [63]) shows that the lines themselves must be in a (1+1)D SPT phase with respect to G. On the other hand, a *closed* symmetry twist line (with no end points) on the boundary ∂A of a region A can be interpreted as the result of applying to the original MBL eigenstate the Floquet unitary U, restricted to region A. The fact that such a state carries a (1+1)D SPT on the boundary ∂A indeed shows that the effect of U is to pump a (1+1)D SPT to the boundary.

On the other hand, we do not expect there to be any similarly simple physical picture in the antiunitary case; in Ref. [36] it was found that the extra data for (1+1)D systems is a somewhat strange "twisted" representation of the symmetry with no obvious physical interpretation.

Floquet topological phases without symmetry. The above considerations allow us to establish the existence of topological phases in driven MBL systems that are distinct in the Floquet context, even in the absence of any additional symmetry, but not in the stationary case. Indeed, imagine we take a Floquet system in (2+1) dimensions or higher with symmetry group $\widetilde{G} = G \times \mathbb{Z}$ and then gauge just symmetry G. In general, gauging a subgroup of the full symmetry group relates SPT phases to symmetry-enriched topological (SET) phases protected by the remaining global symmetry [61–63]; which, in this case, is simply the discrete time-translation symmetry.

Explicit realization. We have already argued above that the invariants which classify Floquet-MBL topological phases with symmetry G should be the same as in the case of stationary topological phases with symmetry $\mathbb{Z} \rtimes G$. However, one might ask whether there might be an obstruction to realizing any of these "potential" Floquet-MBL topological phases in an explicit model. We argue that this is not the case, *provided* that the corresponding stationary topological phase with symmetry $\mathbb{Z} \rtimes G$ can be realized in a stationary MBL system with symmetry $\widetilde{G}_n = \mathbb{Z}_n \rtimes G$ for some sufficiently large n. Such a system, by definition, consists of a Hamiltonian H which commutes with an on-site representation $V(\widetilde{g})$ of $\widetilde{G}_n$. (A faithful on-site representation of $\mathbb{Z}$ does not make sense in a lattice system with finite-dimensional Hilbert space per site, hence why we consider $\mathbb{Z}_n$ instead. A system acted on by $\mathbb{Z}_n$ can always be thought of as being acted on by $\mathbb{Z}$ nonfaithfully.) Then we claim that the Floquet system with Floquet operator $U = e^{iHT}V(\alpha)$ (where α is the generator of $\mathbb{Z}_n$) indeed realizes the desired Floquet-MBL topological phase.

To see this, note that the eigenstates of H are also eigenstates of $V(\alpha)$ [since H commutes with $V(\alpha)$ by assumption] and therefore of U. We can analyze the SPT order of these states by thinking of them either as eigenstates of a stationary system with symmetry $\widetilde{G}$ or as eigenstates of a Floquet system with symmetry G. In fact, the analysis proceeds identically in both cases with only one difference: In the stationary context, the $\mathbb{Z}$ part of the symmetry is taken to be generated by $V(\alpha)$, whereas in the Floquet context, it is generated by U. However, we can make $U = V(\alpha)$ by sending $T \to 0$ continuously. Since the classification of topological phases is *discrete*, we do not expect that this can change the diagnosed phase. This can be checked explicitly in the (2+1)D SPT case.

Conclusion. The perspective on topological phases in Floquet-MBL systems detailed in this Rapid Communication opens up many intriguing questions for future study. Indeed, *every* phenomenon that has been studied in the usual stationary case—for example, symmetry fractionalization on topological excitations in symmetry-enriched topological (SET) phases [55,59,63]—ought to have analogs in the Floquet-MBL case, but in many cases the possibilities will be richer due to the extra $\mathbb{Z}$ symmetry. We leave further exploration of these phases and their physical properties for future work.

Note added. Soon after we posted this work on the arXiv, two more preprints appeared [69,70], whose results overlap with ours.

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Appendix. Here we will briefly recap the argument for the $H^{d+1}[G,\mathrm{U}(1)]$ classification of SPT ground states in $d=1$ and $d=2$ taking care to formulate it in such a way as to make it clear that it can also be applied to give a $H^{d+1}[\mathbb{Z} \rtimes G,\mathrm{U}(1)]$ classification in Floquet systems. Suppose we have some short-range entangled state $|\Psi\rangle$ defined on a system without boundary such that $|\Psi\rangle$ is invariant under the local unitary (or antiunitary) representation $V(g)$ of a symmetry. Now imagine some subregion M of the whole system, and consider the *subspace* $\mathcal{P}_{M,|\Psi\rangle}$ of "boundary states" defined in the Hilbert space of M which *completes* to $|\Psi\rangle$, in the sense that they are identical to $|\Psi\rangle$ away from the boundary of M. The restriction $V_M(g)$ of the symmetry operation $V(g)$ to the region M must preserve this subspace (note that this restriction is still well defined even for antiunitary symmetries since we can take it to act only on the Hilbert space of M). Thus, it is well defined to talk about the action of the symmetry on the boundary states.

Moreover, if we assume that $|\Psi\rangle$ is short-range entangled, this implies that there exists a local unitary $\mathcal{D}$ which transforms $|\Psi\rangle$ into a product state $|\phi\rangle^{\otimes N}$. The restriction $\mathcal{D}_M$ must then transform the states in $\mathcal{P}_{M,|\Psi\rangle}$ into the states which look like a product of $|\psi\rangle$'s away from the boundary. Thus, if we started with a system in d spatial dimensions, we can identify the boundary states with the states of a $(d-1)$-dimensional system. In the case of $d=1$, the boundary is just a set of points, and we classify the SPT order from the *projective* representation of the symmetry on a boundary point [38,39,42,43]. In $d=2$, we can classify the SPT order by considering a symmetry restriction procedure as described in Ref. [52].

[1] X.-G. Wen, Int. J. Mod. Phys. B **04**, 239 (1990).
[2] Z. Nussinov and G. Ortiz, Phys. Rev. B **77**, 064302 (2008).
[3] R. Alicki, M. Fannes, and M. Horodecki, J. Phys. A: Math. Theor. **42**, 065303 (2009).
[4] D. M. Basko, I. L. Aleiner, and B. L. Altshuler, Ann. Phys. (NY) **321**, 1126 (2006).
[5] V. Oganesyan and D. A. Huse, Phys. Rev. B **75**, 155111 (2007).
[6] A. Pal and D. A. Huse, Phys. Rev. B **82**, 174411 (2010).
[7] J. H. Bardarson, F. Pollmann, and J. E. Moore, Phys. Rev. Lett. **109**, 017202 (2012).
[8] M. Serbyn, Z. Papić, and D. A. Abanin, Phys. Rev. Lett. **110**, 260601 (2013).
[9] M. Serbyn, Z. Papić, and D. A. Abanin, Phys. Rev. Lett. **111**, 127201 (2013).
[10] B. Bauer and C. Nayak, J. Stat. Mech. (2013) P09005.
[11] J. Z. Imbrie, J. Stat. Phys. **163**, 998 (2016).
[12] D. A. Huse, R. Nandkishore, and V. Oganesyan, Phys. Rev. B **90**, 174202 (2014).
[13] R. Nandkishore and D. A. Huse, Annu. Rev. Condens. Matter Phys. **6**, 15 (2015).
[14] D. A. Huse, R. Nandkishore, V. Oganesyan, A. Pal, and S. L. Sondhi, Phys. Rev. B **88**, 014206 (2013).
[15] A. Chandran, V. Khemani, C. R. Laumann, and S. L. Sondhi, Phys. Rev. B **89**, 144201 (2014).
[16] K. Slagle, Z. Bi, Y.-Z. You, and C. Xu, arXiv:1505.05147.
[17] A. C. Potter and A. Vishwanath, arXiv:1506.00592.
[18] Y. Bahri, R. Vosk, E. Altman, and A. Vishwanath, Nat. Commun. **6**, 7341 (2015); arXiv:1307.4092.
[19] T. Kitagawa, E. Berg, M. Rudner, and E. Demler, Phys. Rev. B **82**, 235114 (2010).
[20] N. H. Lindner, G. Refael, and V. Galitski, Nat. Phys. **7**, 490 (2011).
[21] L. Jiang, T. Kitagawa, J. Alicea, A. R. Akhmerov, D. Pekker, G. Refael, J. I. Cirac, E. Demler, M. D. Lukin, and P. Zoller, Phys. Rev. Lett. **106**, 220402 (2011).
[22] M. Thakurathi, A. A. Patel, D. Sen, and A. Dutta, Phys. Rev. B **88**, 155133 (2013).
[23] J. K. Asbóth, B. Tarasinski, and P. Delplace, Phys. Rev. B **90**, 125143 (2014).
[24] P. Titum, N. H. Lindner, M. C. Rechtsman, and G. Refael, Phys. Rev. Lett. **114**, 056801 (2015).
[25] P. Titum, E. Berg, M. S. Rudner, G. Refael, and N. H. Lindner, arXiv:1506.00650.
[26] F. Nathan and M. S. Rudner, New J. Phys. **17**, 125014 (2015).
[27] T. Iadecola, L. H. Santos, and C. Chamon, Phys. Rev. B **92**, 125107 (2015).
[28] D. Carpentier, P. Delplace, M. Fruchart, and K. Gawędzki, Phys. Rev. Lett. **114**, 106806 (2015).
[29] L. D'Alessio and A. Polkovnikov, Ann. Phys. **333**, 19 (2013).
[30] P. Ponte, A. Chandran, Z. Papić, and D. A. Abanin, Ann. Phys. **353**, 196 (2015).
[31] A. Lazarides, A. Das, and R. Moessner, Phys. Rev. Lett. **115**, 030402 (2015).
[32] P. Ponte, Z. Papić, F. Huveneers, and D. A. Abanin, Phys. Rev. Lett. **114**, 140401 (2015).
[33] D. A. Abanin, W. D. Roeck, and F. Huveneers, arXiv:1412.4752.
[34] D. A. Abanin, W. De Roeck, and F. Huveneers, Phys. Rev. Lett. **115**, 256803 (2015).
[35] V. Khemani, A. Lazarides, R. Moessner, and S. L. Sondhi, arXiv:1508.03344.
[36] C. W. von Keyserlingk and S. L. Sondhi, arXiv:1602.02157.
[37] Z.-C. Gu and X.-G. Wen, Phys. Rev. B **80**, 155131 (2009).
[38] F. Pollmann, A. M. Turner, E. Berg, and M. Oshikawa, Phys. Rev. B **81**, 064439 (2010).
[39] F. Pollmann, E. Berg, A. M. Turner, and M. Oshikawa, Phys. Rev. B **85**, 075125 (2012).
[40] L. Fidkowski and A. Kitaev, Phys. Rev. B **81**, 134509 (2010).
[41] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B **82**, 155138 (2010).
[42] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B **83**, 035107 (2011).
[43] N. Schuch, D. Pérez-García, and I. Cirac, Phys. Rev. B **84**, 165139 (2011).
[44] L. Fidkowski and A. Kitaev, Phys. Rev. B **83**, 075103 (2011).
[45] X. Chen, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B **84**, 235141 (2011).
[46] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B **87**, 155114 (2013).
[47] M. Levin and Z.-C. Gu, Phys. Rev. B **86**, 115109 (2012).
[48] A. Vishwanath and T. Senthil, Phys. Rev. X **3**, 011016 (2013).
[49] C. Wang, A. C. Potter, and T. Senthil, Science **343**, 629 (2014).
[50] A. Kapustin, arXiv:1403.1467.
[51] Z.-C. Gu and X.-G. Wen, Phys. Rev. B **90**, 115141 (2014).
[52] D. V. Else and C. Nayak, Phys. Rev. B **90**, 235137 (2014).
[53] F. J. Burnell, X. Chen, L. Fidkowski, and A. Vishwanath, Phys. Rev. B **90**, 245122 (2014).

[54] J. C. Wang, Z.-C. Gu, and X.-G. Wen, Phys. Rev. Lett. **114**, 031601 (2015).
[55] M. Cheng, Z. Bi, Y.-Z. You, and Z.-C. Gu, arXiv:1501.01313.
[56] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.93.201103 for a proof.
[57] D. V. Else and C. Nayak (unpublished).
[58] J. Maciejko, X.-L. Qi, A. Karch, and S.-C. Zhang, Phys. Rev. Lett. **105**, 246809 (2010).
[59] A. M. Essin and M. Hermele, Phys. Rev. B **87**, 104406 (2013).
[60] Y.-M. Lu and A. Vishwanath, Phys. Rev. B **93**, 155121 (2016).
[61] A. Mesaros and Y. Ran, Phys. Rev. B **87**, 155115 (2013).
[62] L.-Y. Hung and X.-G. Wen, Phys. Rev. B **87**, 165107 (2013).
[63] M. Barkeshli, P. Bonderson, M. Cheng, and Z. Wang, arXiv:1410.4540.
[64] M. Cheng, M. Zaletel, M. Barkeshli, A. Vishwanath, and P. Bonderson, arXiv:1511.02263.
[65] F. A. Bais, P. van Driel, and M. de Wild Propitius, Nucl. Phys. B **393**, 547 (1993).
[66] H. Bombin, Phys. Rev. Lett. **105**, 030403 (2010).
[67] M. Barkeshli, C.-M. Jian, and X.-L. Qi, Phys. Rev. B **87**, 045130 (2013).
[68] X. Chen, F. J. Burnell, A. Vishwanath, and L. Fidkowski, Phys. Rev. X **5**, 041013 (2015).
[69] A. C. Potter, T. Morimoto, and A. Vishwanath, arXiv:1602.05194.
[70] R. Roy and F. Harper, arXiv:1602.08089.