Classification and properties of symmetry-enriched topological phases: Chern-Simons approach with applications to Z_2 spin liquids

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We study (2+1)-dimensional phases with topological order, such as fractional quantum Hall states and gapped spin liquids, in the presence of global symmetries. Phases that share the same topological order can then differ depending on the action of symmetry, leading to symmetry-enriched topological (SET) phases. Here, we present a K-matrix Chern-Simons approach to identify distinct phases with Abelian topological order, in the presence of unitary or antiunitary global symmetries. A key step is the identification of a smooth edge sewing condition that is used to check if two putative phases are indeed distinct. We illustrate this method by classifying Z_2 topological order (Z_2 spin liquids) in the presence of an internal Z_2 global symmetry for which we find six distinct phases. These include two phases with an unconventional action of symmetry that permutes anyons leading to symmetry-protected Majorana edge modes. Other routes to realizing protected edge states in SET phases are identified. Symmetry-enriched Laughlin states and double-semion theories are also discussed. Somewhat surprisingly, we observe that (i) gauging the global symmetry of distinct SET phases leads to topological orders with the same total quantum dimension, and (ii) a pair of distinct SET phases can yield the same topological order on gauging the symmetry.

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I. INTRODUCTION

It was long believed that phases of matter arose from different patterns of symmetry breaking [1,2]. The discovery of integer [3] and fractional [4] quantum Hall (FQH) effects demonstrated, however, that there exist many different phases of matter which lie outside this paradigm. In particular, the FQH states differ in their internal quasiparticle structure as well as their boundary excitations while preserving all symmetries of the system. This phenomenon is robust against any perturbation and is called [5,6] "topological order," which implies ground-state degeneracy (GSD) on a closed manifold (a Riemann surface of genus g), and emergent anyon excitations [7] which obey neither bosonic nor fermionic statistics. Another class of topologically ordered phases are gapped quantum spin liquids [6]. Recently, several examples of gapped spin liquids have appeared in numerical studies of fairly natural spin-$\frac{1}{2}$ Heisenberg models, on the kagome [8] and square lattice (with nearest- and next-neighbor exchange) [9,10]. Calculations of topological entanglement entropy [11,12] point to Z_2 topological order [13,14]. However, the precise identification of these phases requires understanding the interplay between topological order and symmetry in these systems. The symmetries include both onsite global spin rotation and time-reversal symmetries, as well as the space-group symmetries of the lattice. Kagome lattice antiferromagnets, such as herbertsmithite, may provide experimental realization of this physics, although experimental challenges arising from disorder and residual interactions continue to be actively studied. This motivates the study of distinct topologically ordered phases that may arise in the presence of symmetry [15–21].

In the presence of symmetry, the structure of topological order is even richer. The microscopic degrees of freedom in the system are either bosons or fermions, and they must form a linear representation of the symmetry group [22] G_s. The emergent anyons, however, do not need to form a linear representation of G_s. Instead, they could transform projectively under symmetry operation, i.e., each of them can carry a fractional quantum number of symmetry. For example, in Laughlin FQH states [5] at filling fraction $\nu = 1/m$, each elementary quasiparticle carries a fraction $(1/m)$ of the electron charge. This phenomenon is widely known as fractionalization, although a more appropriate name is perhaps *symmetry fractionalization* [18,19,23]. The associated symmetry in Laughlin states is the U(1) charge conservation of electrons. While the emergent quasiparticles transform projectively (instead of linearly), the microscopic degrees of freedom always transform linearly under symmetry, simply because each microscopic degrees of freedom can be regarded as a conglomerate of multiple emergent quasiparticles.

Even in the absence of topological order, when symmetry G_s is preserved, different symmetry-protected topological (SPT) phases [24] emerge which are separated from each other through phase transitions. These SPT phases feature symmetry-protected boundary states which will be gapless, unless symmetry G_s is (spontaneously or explicitly) broken on the boundary. Well-known examples of SPT phases are topological insulators [25,26] and superconductors [27]. In (2+1) dimensions [(2+1)D] all SPT phases have symmetry-protected nonchiral edge modes [28–30].

The existence of SPT phases further enriches the structure of symmetric topological orders. In other words, the topologically ordered phase is not fully determined by how its (anyon) quasiparticles transform (projectively or not) under symmetry: its microscopic degrees of freedom could form a SPT state in parallel with the topological order [19]. The formation of SPT state can, e.g., bring in new structures to the edge states of the

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topologically ordered system, and lead to a distinct symmetry-enriched topological (SET) order. Therefore, two different SET phases sharing the same topological order can differ by the symmetry transformation on their anyon quasiparticles, or by their distinct boundary excitations. Clearly, we have a question here: Given two states sharing the same topological order while preserving symmetry G_s, can they be continuously connected to each other without a phase transition, if symmetry G_s is preserved?

We address this issue for (2+1)D (Abelian) topological orders. Focusing on onsite (global, instead of spatial) symmetries, we present a universal criterion (Criterion I in Sec. II E) related to the edge states of these (2+1)D SET phases, which works for both unitary and antiunitary onsite symmetries. The physical picture behind this criterion is demonstrated in Fig. 1. Two SET phases No. 1 and No. 2 living on the two cylinders are considered the same if they can be smoothly connected together via tunneling of microscopic degrees of freedom between the two edges. Distinct SET phases, on the other hand, present an obstruction to such a smooth sewing.

The above criterion allows us to clarify the structure of symmetry-enriched topological (SET) orders in (2+1)D. The method we follow is the Chern-Simons approach, which provides a unified description for low-energy bulk and edge properties of a generic Abelian topological order [31–33] in (2+1)D. In particular, the bulk-edge correspondence [34,35] in Chern-Simons approach enables us to identify all edge excitations with their bulk counterparts, such as the microscopic degrees of freedom (bosons/fermions) and anyons. Therefore, the above criterion of smooth sewing boundary conditions for two different SET phases can be made precise within the Chern-Simons approach (see Sec. II E).

More concretely, a (2+1)D Abelian topological phase is fully characterized by a symmetric integer matrix $\mathbf{K}$ in the

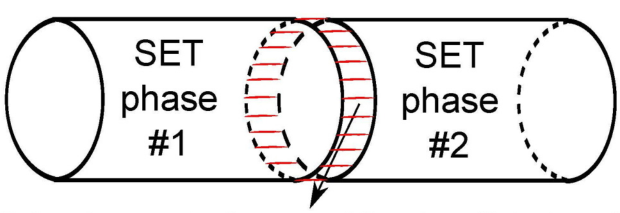

FIG. 1. Edge sewing criterion to distinguish symmetry-enriched topological (SET) phases. Only the microscopic degrees of freedom, i.e., "electrons" (and not gauge charged objects such as anyons/fractionalized quasiparticles), can tunnel between the two edges of a pair of semi-infinite cylinders. If two SET phases can be continuously tuned into one another without a phase transition (while preserving symmetry), there is a "smooth" sewing between the two cylinders of SET phases No. 1 and No. 2. This implies that all edge excitations are gapped by a few symmetry-allowed terms that tunnel "electrons" between the two edges. In the thermodynamic limit these tunneling terms lead to M degenerate ground states, corresponding exactly to the M-fold torus degeneracy of the topological order. On the other hand, if the two SET phases are different, there is no such "smooth" boundary condition to sew the two edges. A precise version of this statement is formulated in Criterion I in Sec. II E.

TABLE I. Two different Z_2 spin liquids with (antiunitary) time-reversal symmetry Z_2^T classified by Abelian Chern-Simons theory. In SET phase No. 1 all quasiparticles in (26) transform linearly under Z_2^T symmetry, while in SET phase No. 2 quasiparticle m transforms projectively under Z_2^T symmetry. The data set in the second line completely characterize these SET phases. "Proj. sym." is short for "projective realization of symmetry" in the table.

$\mathbf{K} \simeq \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ with symmetry $G_s = Z_2^T = \{\boldsymbol{g},\boldsymbol{e} = \boldsymbol{g}^2\}$		
Data set in (19): $\left[\mathbf{K} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \{\eta^g = -1, \mathbf{W}^g, \delta\vec{\phi}^g\}\right]$		
Label	No. 1	No. 2
$\mathbf{W}^g$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$\delta\vec{\phi}^g$	$(0,n\pi)^T$	$(0,\pi/2 + n\pi)^T$
Proj. sym. $[m = (0,1)^T]$	No	Yes
Gapless edges	No	No

Chern-Simons approach. When symmetry G_s is preserved in the system, the anyons could carry a fractional symmetry quantum number (or transform projectively under the symmetry [15]), while the microscopic degrees of freedom (bosons/fermions) must form linear representations of the symmetry group $G_s = \{\boldsymbol{g}\}$. The relation between microscopic degrees of freedom and fractionalized anyon excitations is especially clear in the Chern-Simons approach.

Based on the mentioned Criterion I to differentiate distinct SET phases, we can classify all different SET phases with the same topological order $\{K\}$ and symmetry G_s. In this work, we studied various examples: Z_2 spin liquids [36], double-semion theory [37,38], and bosonic/fermionic Laughlin states [5] at filling fraction $\nu = 1/m$. We consider both antiunitary time-reversal symmetry $G_s = Z_2^T$, and unitary $G_s = Z_2$ or $Z_2 \times Z_2$. These are the analogs of the spin rotation symmetry of Heisenberg magnets. Classification of these SET phases with symmetry G_s is summarized in Tables I–V. In the case of antiunitary time-reversal symmetry ($G_s = Z_2^T$), our classification based on Chern-Simons approach is unable to capture one extra SET phase, in which distinct anyons are permuted by time-reversal operation.

In particular, we highlight the classification of the simplest class of SETs, Z_2 (toric code) topological order with a $G_s = Z_2$ onsite symmetry. We find a total of six phases with our method. Of these, only two nontrivial phases are understood in terms of distinct fractional charges. Other phases include combination with SPTs, or unconventional symmetry action that permutes anyons and leads to protected Majorana edge states.

Unconventional SET phases. We divide SET phases into two types: conventional and unconventional. In "conventional" SET phases, all (anyon) quasiparticles merely obtain a U(1) phase under any symmetry operation. In contrast, in the more exotic "unconventional" SET phases, certain symmetry operations *exchange* two inequivalent anyons, instead of just acquiring U(1) phases. For example, under onsite unitary Z_2 symmetry operation, the two anyons, i.e., the electric charge e and magnetic vortex m of a Z_2 spin liquid are exchanged (see Table III). Previously, such a transformation law was considered in the Wen "plaquette" model [39–41] for translation symmetry, in contrast to the internal $G_s = Z_2$ symmetry considered here. These "unconventional" SET

TABLE II. Classification of "conventional" Z_2 spin liquids enriched by onsite (unitary) $G_s = Z_2$ symmetry. There are four different "conventional" SET phases, where under Z_2 symmetry all quasiparticles (e,m,f) merely obtain a U(1) phase factor. The data set in the second line completely characterizes these SET phases. $\mathbf{K}_g$ denotes the topological order, which is obtained by gauging the unitary $G_s = Z_2$ symmetry in the Z_2 spin liquid. Some of these SET phases have Z_2 symmetry-protected edge states, which will be gapless unless Z_2 symmetry is spontaneously broken. On gauging the Z_2 symmetry (blue entries), new quasiparticle excitations (coined "Z_2 symmetry fluxes") $\{q_g\}$ are obtained, as described in Appendix B. Their statistics (B3)–(B5) are also summarized in the table: its self-statistics $\theta_{q_g} = 2\pi h_{q_g}$ has a one-to-one correspondence with its topological spin $\Theta_{q_g} = \exp(i\theta_{q_g}) = \exp(2\pi i h_{q_g})$.

$\mathbf{K} \simeq \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ with unitary symmetry $G_s = Z_2 = \{\boldsymbol{g}, \boldsymbol{e} = \boldsymbol{g}^2\}$ Data set in (19): $\left[\mathbf{K} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \{\eta^g = +1, \mathbf{W}^g = 1_{4\times4}, \delta\vec{\phi}^g\}\right]$				
Label	No. 1	No. 2	No. 3	No. 4
$\delta\vec{\phi}^g$	$\begin{pmatrix} 0 \\ 0 \\ \pi \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ \pi \\ \pi \end{pmatrix}$	$\begin{pmatrix} \pi/2 \\ 0 \\ \pi \\ 0 \end{pmatrix} \simeq \begin{pmatrix} \pi/2 \\ 0 \\ \pi \\ \pi \end{pmatrix}$	$\begin{pmatrix} \pi/2 \\ \pi/2 \\ \pi \\ 0 \end{pmatrix} \simeq \begin{pmatrix} \pi/2 \\ \pi/2 \\ \pi \\ \pi \end{pmatrix}$
Proj. sym. $[e \simeq (1,0,0,0)^T]$	No	No	Yes	Yes
Proj. sym. $[m \simeq (0,1,0,0)^T]$	No	No	No	Yes
Proj. sym. $[f \simeq (1,1,0,0)^T]$	No	No	Yes	No
Symmetry-protected edge states	No	Yes	No	Yes
Central charge c of edge states	0	1	0	1
After gauging symmetry $\boldsymbol{g}$:				
$\mathbf{K}_g \simeq$	$\begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$	$\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$	$\begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix}$
$\theta_{q_g}/2\pi \equiv h_{q_g} \mod 1$	$0,1/2$	$\pm 1/4$	$0, \pm 1/4, 1/2$	$\pm 1/8, \pm 3/8$
$\tilde{\theta}_{q_g,e}/2\pi \mod 1$	$0,1/2$	$0,1/2$	$\pm 1/4$	$\pm 1/4$
$\tilde{\theta}_{q_g,m}/2\pi \mod 1$	$0,1/2$	$0,1/2$	$0,1/2$	$\pm 1/4$
Comparison to [21]	(000)	(100)	(010),(110),$m_1 = 0$	$m_1 = 2$

phases have some striking properties. First, the edge features gapless Majorana edge modes that are protected by symmetry. Next, if Z_2 symmetry is broken at the edge, then a Majorana fermion is trapped at the edge domain wall. Finally, as illustrated in Fig. 2, when a pair of electric charge (e) is created at opposite sides of a sphere, we can divide the system into two subsystems A and B, so that there is one electric charge e localized in each subsystem. Now, if we perform the Z_2 symmetry operation only in subsystem A (flip all the spins), the electric charge e therein will become a magnetic vortex m. Since an electric charge e and a magnetic vortex m differ by a fermion f ($e \times f = m$ or $m \times f = e$) in the Z_2 spin liquid, this means a fermion mode f must simultaneously appear at the boundary separating subsystems A and B, as the Ising symmetry is acted on A. This is discussed in Secs. III B 2 and III B 3.

Symmetry-protected edge states. In general, the nonchiral topological orders, like Z_2 topological order and double-

TABLE III. Classification of "unconventional" Z_2 spin liquids enriched by onsite (unitary) $G_s = Z_2$ symmetry. There are two different "unconventional" SET phases, where under Z_2 symmetry quasiparticles e and m will exchange. The data set in the second line completely characterizes these SET phases. Both SET phases have Z_2 symmetry-protected edge states, which will be gapless unless Z_2 symmetry is broken. The central charges of these gapless edge states are half integers, in contrast to "conventional" Z_2 spin liquids where $c \in \mathbb{Z}$ (see Table II). As described in Appendix E, gauging this "unconventional" Z_2 symmetry leads to new non-Abelian quasiparticles (blue entries), which has quantum dimension $d_{q_g} = \sqrt{2}$ and topological spin $\Theta_{q_g} = \exp(2\pi i h_{q_g})$. These new quasiparticle excitations are Z_2 symmetry fluxes, labeled by $\{q_g\}$. All quasiparticle contents of the non-Abelian topological orders obtained by gauging Z_2 symmetry are summarized in Table VII. The "gauged" non-Abelian topological orders for both SET phases have ninefold GSD on a torus, corresponding to nine different superselection sectors.

$\mathbf{K} \simeq \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ with unitary symmetry $G_s = Z_2 = \{\boldsymbol{g}, \boldsymbol{e} = \boldsymbol{g}^2\}$ Data set in (19): $\left[\mathbf{K} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \{\eta^g = +1, \mathbf{W}^g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 1_{2\times2}, \delta\vec{\phi}^g\}\right]$		
Label	No. 5	No. 6
$\delta\vec{\phi}^g$	$(0,0,\pi,0)^T \simeq (\pi/2,\pi/2,\pi,0)^T$	$(0,0,\pi,\pi)^T \simeq (\pi/2,\pi/2,\pi,\pi)^T$
Proj. sym. $[f \simeq (1,1,0,0)^T]$	No	No
Symmetry-protected edge states	Yes	Yes
Central charge c of gapless edge states	1/2	3/2
$h_{q_g} \mod 1$	$\pm\frac{1}{16}, \pm\frac{9}{16}$	$\pm\frac{3}{16}, \pm\frac{5}{16}$
Relation to Kitaev's 16-fold way [44]	$(\nu = 1) \otimes (\nu = 15) \simeq (\nu = 7) \otimes (\nu = 9)$	$(\nu = 5) \otimes (\nu = 11) \simeq (\nu = 3) \otimes (\nu = 13)$

TABLE IV. Classification of "conventional" $\nu = \frac{1}{2k}$, $k =$ odd bosonic Laughlin states (or chiral spin liquids with $2k$-fold GSD on torus) enriched by onsite (unitary) $G_s = Z_2$ symmetry. There are four different conventional SET phases, where under Z_2 symmetry all quasiparticles merely obtain a U(1) phase factor. The data set in the second line completely characterizes these SET phases. $\mathbf{K}_g$ denotes the topological order, which is obtained by gauging the unitary $G_s = Z_2$ symmetry in the Z_2 spin liquid. On gauging the Z_2 symmetry (blue entries), new quasiparticle excitations $\{q_g\}$ (Z_2 symmetry fluxes) are obtained. Their statistics (42) and (43) are also summarized in the table.

$\mathbf{K} \simeq 2k$ ($k =$ odd) with unitary symmetry $G_s = Z_2 = \{\boldsymbol{g}, \boldsymbol{e} = \boldsymbol{g}^2\}$ Data set in (19): $\left[\mathbf{K} = (2k) \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \{\eta^g = +1, \mathbf{W}^g = 1_{3\times 3}, \delta\vec{\phi}^g\}\right]$				
Label	No. 1	No. 2	No. 3	No. 4
$\delta\vec{\phi}^g$	$(0,\pi,0)^T$	$(0,\pi,\pi)^T$	$(\pi/2k,\pi,0)^T$	$(\pi/2k,\pi,\pi)^T$
After gauging symmetry:				
$\mathbf{K}_g \simeq$	$\begin{pmatrix} 2k & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 2k & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$	$\begin{pmatrix} 8k & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \simeq 8k$	$\begin{pmatrix} 2k & -1 & 0 \\ -1 & -2 & 2 \\ 0 & 2 & 0 \end{pmatrix}$
$h_{q_g} = \theta_{q_g}/2\pi \mod 1$	$\frac{n^2}{4k}, \frac{n^2}{4k} + \frac{1}{2}$ ($n \in \mathbb{Z}$)	$\frac{n^2}{4k} \pm \frac{1}{4}$ ($n \in \mathbb{Z}$)	$\frac{4k+1+4n(n+1)}{16k} \pm \frac{1}{4}$ ($n \in \mathbb{Z}$)	$\frac{1+4n(n+1)}{16k} \pm \frac{1}{4}$ ($n \in \mathbb{Z}$)
$\tilde{\theta}_{q_g,p}/2\pi \mod 1$	$\frac{n}{2k}, \frac{n}{2k} + \frac{1}{2}$ ($n \in \mathbb{Z}$)	$\frac{n}{2k}, \frac{n}{2k} + \frac{1}{2}$ ($n \in \mathbb{Z}$)	$\frac{2n+1}{4k}, \frac{2n+1}{4k} + \frac{1}{2}$ ($n \in \mathbb{Z}$)	$\frac{2n+1}{4k}, \frac{2n+1}{4k} + \frac{1}{2}$ ($n \in \mathbb{Z}$)

semion models, do not have gapless excitations at the edge. However, these may appear with additional symmetry. Indeed, the unconventional Z_2 SET phases have Majorana edge states. Since they are protected by an onsite Z_2 symmetry, they are stable even in the presence of disorder that breaks translation symmetry along the edge. Two further mechanisms for gapless edge modes in "conventional" SET phases may be identified. The first is the trivial observation that adding an SPT phase could lead to a corresponding protected edge state. The second mechanism operates when both the electric and magnetic particles of the Z_2 gauge theory transforms projectively under symmetry. Then, one cannot condense either of them at the edge, implying a protected edge. Details and a sufficient condition for protected edge states will appear in Sec. III F.

Gauging symmetry. A powerful tool in studying the effect of an onsite unitary symmetry G_s is the consequence of gauging it [21,29]. This means the global G_s symmetry is promoted to a local "gauge symmetry," which leads to new topological orders. Distinct topological orders can help distinguish different actions of the symmetry in the ungauged theory. By this procedure in (2+1)D, nonlinear sigma models with topological terms, which describe SPT phases [24], can be mapped to gauge theories with a topological term [21,29], discussed by Dijkgraaf and Witten [42]. In this work, we systematically study the consequences of gauging unitary onsite symmetry in Abelian SET phases. For many "conventional" SET phases with Abelian symmetries, the new topological order obtained by gauging symmetry is Abelian, and Chern-Simons theory is a natural framework to derive it. There are some cases of "conventional" SET phases that will lead to non-Abelian topological orders by gauging symmetry, though, an example being Z_2 spin liquids with $Z_2 \times Z_2$ spin rotational symmetry (see Sec. III E).

In the examples studied in this work (Z_2 spin liquids, double-semion theories, and $\nu = 1/2k$ bosonic Laughlin states with onsite $G_s = Z_2$ symmetry), different SET phases seem to lead to distinct topological orders (with different anyon contents) by gauging the unitary symmetry $G_s = Z_2$. However, this does not always happen for a general symmetry group G_s. Remarkably for all different G_s symmetry-enriched topological phases with the same topological orders (same GSD and anyon statistics), once we gauge the unitary symmetry

TABLE V. Classification of conventional $\nu = \frac{1}{2k}$, $k =$ even, bosonic Laughlin states (or chiral spin liquids with $2k$-fold GSD on torus) enriched by onsite (unitary) $G_s = Z_2$ symmetry. There are three different conventional SET phases, where under Z_2 symmetry all quasiparticles merely obtain a U(1) phase factor. This is in contrast to four distinct SET phases when $k =$ odd (see Table IV). The data set in the second line completely characterizes these SET phases. $\mathbf{K}_g$ denotes the topological order, which is obtained by gauging the unitary $G_s = Z_2$ symmetry in the Z_2 spin liquid. On gauging the Z_2 symmetry (blue entries) new quasiparticle excitations $\{q_g\}$ (Z_2 symmetry fluxes) are obtained. Their statistics (42) and (43) are also summarized in the table: its self-statistics $\theta_{q_g} = 2\pi h_{q_g}$ has a one-to-one correspondence with its topological spin $\Theta_{q_g} = \exp(2\pi i h_{q_g})$.

$\mathbf{K} \simeq 2k$ ($k =$ even) with unitary symmetry $G_s = Z_2 = \{\boldsymbol{g}, \boldsymbol{e} = \boldsymbol{g}^2\}$ Data set in (19): $\left[\mathbf{K} = (2k) \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \{\eta^g = +1, \mathbf{W}^g = 1_{3\times 3}, \delta\vec{\phi}^g\}\right]$			
Label	No. 1	No. 2	No. 3
$\delta\vec{\phi}^g$	$(0,\pi,0)^T$	$(0,\pi,\pi)^T$	$(\pi/2k,\pi,0)^T \simeq (\pi/2k,\pi,\pi)^T$
After gauging symmetry:			
$\mathbf{K}_g \simeq$	$\begin{pmatrix} 2k & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 2k & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$	$\begin{pmatrix} 8k & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \simeq \begin{pmatrix} 2k & -1 & 0 \\ -1 & -2 & 2 \\ 0 & 2 & 0 \end{pmatrix}$
$h_{q_g} = \theta_{q_g}/2\pi \mod 1$	$\frac{n^2}{4k}, \frac{n^2}{4k} + \frac{1}{2}$ ($n \in \mathbb{Z}$)	$\frac{n^2}{4k} \pm \frac{1}{4}$ ($n \in \mathbb{Z}$)	$\frac{1+4n(n+1)}{16k} \pm \frac{1}{4}$ ($n \in \mathbb{Z}$)
$\tilde{\theta}_{q_g,p}/2\pi \mod 1$	$\frac{n}{2k}, \frac{n}{2k} + \frac{1}{2}$ ($n \in \mathbb{Z}$)	$\frac{n}{2k}, \frac{n}{2k} + \frac{1}{2}$ ($n \in \mathbb{Z}$)	$\frac{2n+1}{4k}, \frac{2n+1}{4k} + \frac{1}{2}$ ($n \in \mathbb{Z}$)

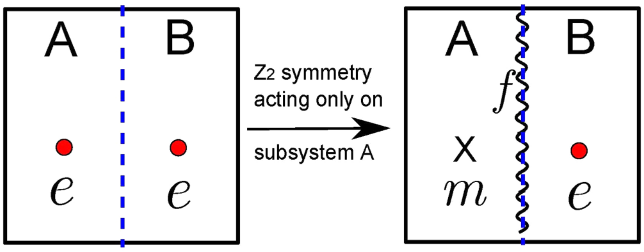

FIG. 2. A fermion mode (f) localized at the boundary between two subsystems A and B which from a bipartition of the on a sphere, where the unconventional Ising symmetry-enriched Z_2 spin liquid resides. Under the Ising symmetry operation, an electric charge e will transform into a magnetic vortex m. Consider one electric charge is created in each subsystem. If we perform Ising (Z_2) symmetry only on subsystem A, a fermion mode will emerge on the boundary, as the electric e charge turns into a magnetic vortex m in A.

G_s, they lead to distinct intrinsic topological orders with the *same* total quantum dimension [43] $\mathcal{D} = \sqrt{\sum_\alpha d_\alpha^2}$. Therefore, these distinct topological orders obtained by gauging unitary symmetry G_s also share the same topological entanglement entropy [11,12] $\gamma = \ln \mathcal{D}$. Although this is an observation from the examples studied in this paper, we conjecture that it generically holds for all SET phases with a finite unitary symmetry group G_s.

Somewhat surprisingly, this furnishes examples where two gauge theories with distinct Dijkgraaf-Witten [42] topological terms correspond to the same topological order. Here, the topological terms arising for the gauge group $Z_2 \times Z_2$ are obtained by gauging SPT phases and correspond to elements of $H^3[Z_2 \times Z_2, \mathrm{U}(1)]$. Theories for distinct elements are shown to be equivalent on relabeling quasiparticles [a $GL(4,Z)$ transformation]. Therefore, the distinction between these theories requires additional information such as specification of electric versus magnetic vortices (Appendix D).

For "unconventional" SET phases, however, gauging the symmetry always leads to non-Abelian topological orders. A general argument for this conclusion is provided in Sec. III B 4. For example, the unconventional Ising symmetry-enriched Z_2 spin liquids, after gauging the Ising ($G_s = Z_2$) symmetry, lead to non-Abelian topological orders with ninefold GSD on a torus. Interestingly, they can be naturally embedded within Kitaev's 16-fold way classification [44] of (2+1)D Z_2 gauge theories (see Tables III and VII). Notably as mentioned earlier, these non-Abelian topological orders also have total quantum dimension $\mathcal{D} = 16$, the same as that of Abelian $Z_2 \times Z_2$ (or Z_4) gauge theories which are obtained by gauging Z_2 symmetry in "conventional" SET phases. In this case, a vertex algebra approach [45] can be introduced to extract all information of the non-Abelian topological order (Appendix E). In particular, after gauging the onsite Ising (Z_2) symmetry, new quasiparticles $\{q_g\}$ (coined Z_2 symmetry fluxes) emerge as deconfined excitations. It is a non-Abelian anyon in the unconventional SET case, which corresponds to the edge domain-wall bound state in Fig. 4.

Spin-$\frac{1}{2}$ from K-matrix CS *theory*: We demonstrate how an emergent "spin-$\frac{1}{2}$" excitation can be realized in the Chern-Simons formalism, by studying Z_2 gauge theories with $Z_2 \times Z_2$ symmetry. The latter has a projective representation that can protect a twofold degenerate state, analogous to spin $\frac{1}{2}$. This is accomplished by expanding the 2×2 K matrix of a Z_2 gauge theory to a 4×4 matrix by adding a trivial insulator layer (a 2×2 "trivial" block). Symmetry transformations implemented in this expanded space have the desired properties (in Sec. III E).

Connection to other work. A symmetry-based approach was used to classify Z_2 spin liquids in Ref. [18]. An advantage of that approach is that it treated both internal and space-group symmetries. However, topological distinctions and the appearance of edge states are not captured. Also, the "unconventional" symmetry realizations were not discussed. Finally, as mentioned in Ref. [18], the symmetry-based approach produces forbidden SETs, that cannot be realized in (2+1)D, but only as the surface state of a (3+1)D SPT phase [46]. Our Chern-Simons approach does not produce such states. A different classification scheme in Ref. [19] produces a subset of our "conventional" phases although explicit lattice realizations are given for them. Finally, Ref. [21] gave a classification based on gauging the symmetry, which misses distinctions between phases as discussed previously. In this work, we show that plausibly different phases given in Refs. [19,21] (belonging to distinct Dijkgraaf-Witten topological terms) actually correspond to the same SET phase. Our approach is perhaps closest to that adopted in Ref. [17], which, however, was restricted to time-reversal-symmetric topological states. Thus, the results in this paper go beyond previous classifications of Z_2 symmetry-enriched Z_2 gauge theories (including Z_2 spin liquid and double-semion theory), and a detailed comparison is given in Appendix D.

This paper is organized as follows. In Sec. II, we introduce the Chern-Simons K-matrix approach to (Abelian) symmetry-enriched topological (SET) phases in (2+1)D. Rules for implementing onsite symmetry in a topologically ordered phase are discussed in Sec. II D, with criteria to differentiate distinct SET phases in Sec. II E. Next, in Sec. III, we demonstrate our approach by classifying SET phases in a few examples. They include (i) Z_2 spin liquid with (antiunitary) time-reversal symmetry ($G_s = Z_2^T$) symmetry (Sec. III A, Table I), (ii) Z_2 spin liquid with unitary Ising ($G_s = Z_2$) symmetry (Sec. III B, Tables II and III), (iii) double-semion theory with unitary Ising ($G_s = Z_2$) symmetry (Appendix C, Table VI), and (iv) even-denominator bosonic Laughlin state with unitary Ising ($G_s = Z_2$) symmetry (Sec. III C, Tables IV and V). Appendixes B and E provide detailed instructions on how to gauge unitary symmetries in (2+1)D SET phase.

II. CHERN-SIMONS APPROACH TO SYMMETRY-ENRICHED ABELIAN TOPOLOGICAL ORDERS IN (2+1)D

A. Chern-Simons theory description of (2+1)D Abelian topological orders

In two spatial dimensions, a generic gapped phase of matter is believed to be described by a low-energy effective Chern-Simons theory in the long-wavelength limit [31–33,42,47]. Both the bulk anyon excitations and the gapless

TABLE VI. Classification of double-semion theory (C1) enriched by onsite (unitary) $G_s = Z_2$ symmetry (see Appendix C for details). There are six different "conventional" SET phases, where under Z_2 symmetry all quasiparticles $(s,\bar{s},b)$ merely obtain a U(1) phase factor. The data set in the second line completely characterize these SET phases. $\mathbf{K}_g$ denotes the topological order, which is obtained by gauging the unitary $G_s = Z_2$ symmetry in the double-semion theory. Some of these SET phases have Z_2 symmetry-protected edge states, which will be gapless unless Z_2 symmetry is spontaneously broken. On gauging the Z_2 symmetry (blue entries), new quasiparticles $\{q_{\boldsymbol{g}}\}$ (coined "$\boldsymbol{g}$ symmetry fluxes") are obtained, as described in Appendix B. Its statistics (C5) and (C6) are also summarized in the table: its self-statistics $\theta_{q_{\boldsymbol{g}}} = 2\pi h_{q_{\boldsymbol{g}}}$ has a one-to-one correspondence with its topological spin $\Theta_{q_{\boldsymbol{g}}} \equiv \exp(2\pi i h_{q_{\boldsymbol{g}}})$. Note, there are no "unconventional" symmetry realizations of onsite Z_2 symmetry for this topological order.

$\mathbf{K} \simeq \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ with unitary symmetry $G_s = Z_2 = \{\boldsymbol{g},\boldsymbol{e} = \boldsymbol{g}^2\}$						
Data set in (19): $\left[\mathbf{K} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \{\eta^{\boldsymbol{g}} = +1, \mathbf{W}^{\boldsymbol{g}} = 1_{4\times4}, \delta\vec{\phi}^{\boldsymbol{g}}\}\right]$						
Label	No. 1	No. 2	No. 3	No. 4	No. 5	No. 6
$\delta\vec{\phi}^{\boldsymbol{g}}$	$\begin{pmatrix} 0 \\ 0 \\ \pi \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ \pi \\ \pi \end{pmatrix}$	$\begin{pmatrix} \pi/2 \\ 0 \\ \pi \\ 0 \end{pmatrix} \simeq \begin{pmatrix} \pi/2 \\ 0 \\ \pi \\ \pi \end{pmatrix}$	$\begin{pmatrix} 0 \\ \pi/2 \\ \pi \\ 0 \end{pmatrix} \simeq \begin{pmatrix} 0 \\ \pi/2 \\ \pi \\ \pi \end{pmatrix}$	$\begin{pmatrix} \pi/2 \\ \pi/2 \\ \pi \\ 0 \end{pmatrix}$	$\begin{pmatrix} \pi/2 \\ \pi/2 \\ \pi \\ \pi \end{pmatrix}$
Proj. sym. (s)	No	No	Yes	No	Yes	Yes
Proj. sym. $(\bar{s})$	No	No	No	Yes	Yes	Yes
Proj. sym. (b)	No	No	Yes	Yes	No	No
Symmetry-protected edge	No	No	Yes	Yes	No	Yes
Central charge c	0	0	1	1	0	1
After gauging symmetry $\boldsymbol{g}$:						
$\mathbf{K}_g \simeq$	$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$	$\begin{pmatrix} 8 & 0 \\ 0 & -2 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 \\ 0 & -8 \end{pmatrix}$	$\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}$
$\theta_{q_{\boldsymbol{g}}}/2\pi \equiv h_{q_{\boldsymbol{g}}} \mod 1$	$0, \pm\frac{1}{4}, \frac{1}{2}$	$0, \pm\frac{1}{4}, \frac{1}{2}$	$\frac{1}{16}, -\frac{3}{16}, \frac{5}{16}, -\frac{7}{16}$	$-\frac{1}{16}, \frac{3}{16}, -\frac{5}{16}, \frac{7}{16}$	$0, \frac{1}{2}$	$\pm\frac{1}{4}$
$\tilde{\theta}_{q_{\boldsymbol{g}},s}/2\pi \mod 1$	0,1/2	0,1/2	±1/4	0,1/2	±1/4	±1/4
$\tilde{\theta}_{q_{\boldsymbol{g}},\bar{s}}/2\pi \mod 1$	0,1/2	0,1/2	0,1/2	±1/4	±1/4	±1/4
Notation in Ref. [21]	(001)	(101)	$m_1 = 3$	$m_1 = 1$	(011)	(111)

edge states are captured by the effective theory [35]. Examples include integer and fractional quantum Hall states [6], gapped quantum spin liquids [40,48,49], and topological insulators/superconductors. When we restrict ourselves to the case of gapped Abelian phases where all the elementary excitations in the bulk obey Abelian statistics [7], a complete description is given in terms of Abelian U(1)^N Chern-Simons theory [31–33,35]. To be specific, the low-energy effective Lagrangian of U(1)^N Chern-Simons theory has the following generic form:

$$\mathcal{L}_{\text{CS}} = \frac{\epsilon_{\mu\nu\lambda}}{4\pi} \sum_{I,J=1}^{N} a_\mu^I \mathbf{K}_{I,J} \partial_\nu a_\lambda^J - \sum_{I=1}^{N} a_\mu^I j_I^\mu + \dots, \tag{1}$$

where $\mu,\nu,\lambda = 0,1,2$ in (2+1)D and summation over repeated indices are always assumed. Here, $\dots$ represents higher-order terms, such as Maxwell terms $\sim(\epsilon_{\mu\nu}\partial_\mu a_\nu^I)^2$. $\mathbf{K}$ is a symmetric $N \times N$ matrix with integer entries. Notice that the U(1) gauge fields a_μ^I are all *compact* in the sense that they are coupled to *quantized* gauge charges with currents j_I^μ. In the first quantized language, the quantized quasiparticle currents j_I^μ are written as

$$\forall\ I = 1,\dots,N:\quad j_I^0(\mathbf{r}) = \sum_n l_I^{(n)} \delta(\mathbf{r} - \mathbf{r}^{(n)}),$$

$$j_I^\alpha(\mathbf{r}) = \sum_n l_I^{(n)} \dot{r}_\alpha^{(n)} \delta(\mathbf{r} - \mathbf{r}^{(n)}),\ \ \alpha = 1,2$$

where $\mathbf{r}^{(n)} = (r_1^{(n)}, r_2^{(n)})$ denotes the position of the nth quasiparticle, and gauge charges $l_I^{(n)}$ are all quantized as integers. We can simply label the nth quasiparticle by its gauge charge vector $\mathbf{l}^{(n)} = (l_1^{(n)},\dots,l_N^{(n)})^T$. The self-(exchange) statistics of a quasiparticle $\mathbf{l}$ is given by its statistical angle

$$\theta_{\mathbf{l}} = \pi \mathbf{l}^T \mathbf{K}^{-1} \mathbf{l},\ \ \mathbf{l} \in \mathbb{Z}^N \tag{2}$$

while the mutual (braiding) statistics of a quasiparticle $\mathbf{l}$ and $\mathbf{l}'$ is characterized by

$$\tilde{\theta}_{\mathbf{l},\mathbf{l}'} = 2\pi \mathbf{l}^T \mathbf{K}^{-1} \mathbf{l}',\ \ \mathbf{l},\mathbf{l}' \in \mathbb{Z}^N. \tag{3}$$

The above statistics comes from the nonlocal Hopf Lagrangian [50] of currents j_I^μ, obtained by integrating out the gauge fields a_μ^I in (1). A simple observation from (3) is that for a quasiparticle excitation with gauge charge

$$\tilde{\mathbf{l}} = \mathbf{K}\mathbf{l},\ \ \mathbf{l} \in \mathbb{Z}^N \tag{4}$$

its mutual statistics with any other quasiparticle $\mathbf{l}'$ is a multiple of 2π. In other words, the quasiparticles $\tilde{\mathbf{l}} = \mathbf{K}\mathbf{l}$ are local [51] with respect to any other quasiparticles $\mathbf{l}'$. Therefore, they are interpreted as the "gauge-invariant" microscopic degrees of freedom in the physical system, such as electrons [32] in a fractional quantum Hall state, and spin-1 magnons in a spin-$\frac{1}{2}$ Z_2 spin liquid [15]. Another direct observation is that when all diagonal elements of matrix $\mathbf{K}$ are *even* integers, the microscopic degrees of freedom have bosonic statistics $\theta = 0 \mod 2\pi$, and (1) describes a bosonic system. When at least one diagonal element of $\mathbf{K}$ is an *odd* integer, there are fermionic microscopic degrees of freedom in the system.

The ground-state degeneracy (GSD), as an important character for the topologically ordered phase described by

TABLE VII. Quasiparticle (q.p.) contents of non-Abelian topological orders obtained by gauging the Z_2 symmetry in unconventional SET phases as summarized in Table III. They are related to Z_2 orbifold CFT compactified at radius $R = 2$. The fusion rules of non-Abelian quasiparticles have a one-to-one correspondence to the two copies of Ising CFT (i.e., Ising^2 theory). Each quasiparticle q_a corresponds to a vertex operator (a primary field) in the vertex algebra (which are CFTs) defined through operator product expansion (OPE), and its (conformal) scaling dimension $h_a(\mod 1)$ physically relates to the topological spin $\exp(2\pi i h_a)$ of the quasiparticle. The modular $\mathcal{S}$ matrix of such non-Abelian topological orders is also determined by the OPEs between vertex operators. Allowed quasiparticles must be local w.r.t. any electron operators ~ 1. Any two quasiparticles differing by an electron operator are considered as the same. The scaling dimensions in the Ising^2 CFT (or Z_2 orbifold model) are $h_1 = 0$, $h_j = 1$, $h_{f^1} = h_{f^2} = \frac{1}{2}$, $h_{V_1} = \frac{1}{8}$, $h_{\sigma^1} = h_{\sigma^2} = \frac{1}{16}$, and $h_{\tau^1} = h_{\tau^2} = \frac{9}{16}$. We label these nine different quasiparticles (or superselection sectors) as q_a, $0 \leqslant a \leqslant 8$, which is shown in the $(a+2)$th row of this table. All non-Abelian topological orders in this table have ninefold GSD on a torus, corresponding to nine different superselection sectors.

Unconventional SET phases			No. 5		No. 6		No. 5		No. 6	
Z_2 orbifold fields	Quantum dimension	Ising^2 fields	$\delta\vec{\phi}^g = (0,0,\pi,0)^T$ q.p. q_a	h_a	$\delta\vec{\phi}^g = (0,0,\pi,\pi)^T$ q.p. q_a	h_a	$\delta\vec{\phi}^g = (\pi/2,\pi/2,\pi,0)^T$ q.p. q_a	h_a	$\delta\vec{\phi}^g = (\pi/2,\pi/2,\pi,\pi)^T$ q.p. q_a	h_a
$1 \sim e^{2i\varphi_1}$	1	$1 \otimes 1$	1	0	1	0	1	0	1	0
$j_1 = i\partial\varphi_1$	1	$\psi \otimes \psi$	$\bar{j}_1$	1	$\bar{j}_1$	1	$\bar{j}_1$	1	$\bar{j}_1$	1
$f^1 \sim \cos\varphi_1$	1	$\psi \otimes 1$	$\bar{f}^1$	$\frac{1}{2}$	$\bar{f}^1$	$\frac{1}{2}$	$\bar{f}^1$	$\frac{1}{2}$	$\bar{f}^1$	$\frac{1}{2}$
$f^2 \sim \sin\varphi_1$	1	$1 \otimes \psi$	$\bar{f}^2$	$\frac{1}{2}$	$\bar{f}^2$	$\frac{1}{2}$	$\bar{f}^2$	$\frac{1}{2}$	$\bar{f}^2$	$\frac{1}{2}$
$V_1 \sim \cos\frac{\varphi_1}{2}$	2	$\sigma \otimes \sigma$	$\bar{V}_1 e^{i\varphi_1/2}$	0	$\bar{V}_1 e^{i\varphi_1/2}$	0	$\bar{V}_1 e^{i\varphi_1/2}$	0	$\bar{V}_1 e^{i\varphi_1/2}$	0
σ^1	$\sqrt{2}$	$\sigma \otimes 1$	$\bar{\sigma}^1 e^{i\frac{\varphi_1}{2}} \cdot e^{i\frac{\varphi_2-\bar{\varphi}_2}{2\sqrt{2}}}$	$\frac{1}{16}$	$\bar{\sigma}^1 e^{i(\frac{\varphi_1}{2}+\frac{\varphi_2}{\sqrt{2}})}$	$\frac{5}{16}$	$\bar{\sigma}^1 e^{i\frac{\varphi_2-\bar{\varphi}_2}{2\sqrt{2}}}$	$\frac{-1}{16}$	$\bar{\sigma}^1 e^{i\frac{\varphi_2}{\sqrt{2}}}$	$\frac{3}{16}$
σ^2	$\sqrt{2}$	$1 \otimes \sigma$	$\bar{\sigma}^2 e^{i\frac{\varphi_2-\bar{\varphi}_2}{2\sqrt{2}}}$	$\frac{-1}{16}$	$\bar{\sigma}^2 e^{i\frac{\varphi_2}{\sqrt{2}}}$	$\frac{3}{16}$	$\bar{\sigma}^2 e^{i\frac{\varphi_1}{2}} \cdot e^{i\frac{\varphi_2-\bar{\varphi}_2}{2\sqrt{2}}}$	$\frac{1}{16}$	$\bar{\sigma}^2 e^{i(\frac{\varphi_1}{2}+\frac{\varphi_2}{\sqrt{2}})}$	$\frac{5}{16}$
τ^1	$\sqrt{2}$	$\sigma \otimes \psi$	$\bar{\tau}^1 e^{i\frac{\varphi_1}{2}} \cdot e^{i\frac{\varphi_2-\bar{\varphi}_2}{2\sqrt{2}}}$	$\frac{9}{16}$	$\bar{\tau}^1 e^{i(\frac{\varphi_1}{2}+\frac{\varphi_2}{\sqrt{2}})}$	$\frac{-3}{16}$	$\bar{\tau}^1 e^{i\frac{\varphi_2-\bar{\varphi}_2}{2\sqrt{2}}}$	$\frac{-9}{16}$	$\bar{\tau}^1 e^{i\frac{\varphi_2}{\sqrt{2}}}$	$\frac{-5}{16}$
τ^2	$\sqrt{2}$	$\psi \otimes \sigma$	$\bar{\tau}^2 e^{i\frac{\varphi_2-\bar{\varphi}_2}{2\sqrt{2}}}$	$\frac{-9}{16}$	$\bar{\tau}^2 e^{i\frac{\varphi_2}{\sqrt{2}}}$	$\frac{-5}{16}$	$\bar{\tau}^2 e^{i\frac{\varphi_1}{2}} \cdot e^{i\frac{\varphi_2-\bar{\varphi}_2}{2\sqrt{2}}}$	$\frac{9}{16}$	$\bar{\tau}^2 e^{i(\frac{\varphi_1}{2}+\frac{\varphi_2}{\sqrt{2}})}$	$\frac{-3}{16}$
Electron operator : $1 \sim e^{2i\varphi_1} \sim e^{\sqrt{2}i(\varphi_2 \pm \bar{\varphi}_2)} \sim$			$\bar{f}^1 e^{i\varphi_1} \sim e^{i\frac{\varphi_2-\bar{\varphi}_2}{\sqrt{2}}} \sim \bar{j}_1 e^{i\frac{\varphi_2+\bar{\varphi}_2}{\sqrt{2}}}$		$\bar{f}^1 e^{i\varphi_1} \sim e^{i\sqrt{2}\bar{\varphi}_2} \sim \bar{j}_1 e^{i\frac{\varphi_2+\bar{\varphi}_2}{\sqrt{2}}}$		$\bar{f}^2 e^{i\varphi_1} \sim e^{i\frac{\varphi_2-\bar{\varphi}_2}{\sqrt{2}}} \sim \bar{j}_1 e^{i\frac{\varphi_2+\bar{\varphi}_2}{\sqrt{2}}}$		$\bar{f}^2 e^{i\varphi_1} \sim e^{i\sqrt{2}\bar{\varphi}_2} \sim \bar{j}_1 e^{i\frac{\varphi_2+\bar{\varphi}_2}{\sqrt{2}}}$	
Relation to Kitaev's 16-fold way [44]			$(\nu = 1) \otimes (\nu = 15)$		$(\nu = 5) \otimes (\nu = 11)$		$(\nu = 7) \otimes (\nu = 9)$		$(\nu = 3) \otimes (\nu = 13)$	

effective theory (1), is [35]

$$\mathrm{GSD} = |\det \mathbf{K}|^g$$

on a Riemann surface of genus g. On the torus with $g = 1$, the corresponding $\mathrm{GSD} = |\det \mathbf{K}|$ also equals the numbers of different anyon types (or the number of distinct superselection sectors [44,51]) in the (2+1)D topological ordered system. A simple picture is the following: two anyons differing by a (local) microscopic excitation are the same (or, more precisely, belong to the same superselection sector) in the sense that they share the same braiding properties:

$$\mathbf{l}' \simeq \mathbf{l}'' \Longleftrightarrow \mathbf{l}' - \mathbf{l}'' = \mathbf{K}\mathbf{l}, \quad \mathbf{l},\mathbf{l}',\mathbf{l}'' \in \mathbb{Z}^N.$$

Therefore, different quasiparticle types correspond to inequivalent integer vectors $\mathbf{l} \in \mathbb{Z}^N$ in an N-dimensional lattice, where the Bravais lattice primitive vectors are nothing but the N column vectors of matrix $\mathbf{K}$. As a result $|\det \mathbf{K}|$, the volume of the primitive cell in $\mathbf{l}$ space, counts the number of different quasiparticle types (or superselection sectors) in a topologically ordered system described by effective theory (1).

B. Edge excitations of an Abelian topological order

There is a bulk-edge correspondence [35,52] for effective theory (1). When put on an open manifold $\mathcal{M}$ with a boundary $\partial\mathcal{M}$, the gauge invariance of effective Lagrangian (1) implies the existence of edge states on the boundary $\partial\mathcal{M}$. The N chiral boson fields $\{\phi_I \simeq \phi_I + 2\pi | 1 \leqslant I \leqslant N\}$ capture the edge excitations. To be specific, assuming the manifold $\mathcal{M}$ covers the lower half-plane $r_2 < 0$, then edge excitations localized on the boundary $\partial\mathcal{M} = \{(r_1,r_2)|r_2 = 0\}$ have the following effective Lagrangian:

$$\mathcal{L}_{rE} = \frac{1}{4\pi}\sum_{I,J}(\mathbf{K}_{I,J}\partial_0\phi_I\partial_1\phi_J - \mathbf{V}_{I,J}\partial_1\phi_I\partial_1\phi_J), \tag{5}$$

where rE stands for the *right edge*. On the other hand, if the manifold $\mathcal{M}$ instead covers the upper half-plane $r_2 > 0$, the corresponding edge theory becomes

$$\mathcal{L}_{lE} = -\frac{1}{4\pi}\sum_{I,J}(\mathbf{K}_{I,J}\partial_0\phi_I\partial_1\phi_J + \mathbf{V}_{I,J}\partial_1\phi_I\partial_1\phi_J), \tag{6}$$

where lE means *left edge* here. $\mathbf{V}$ is a positive-definite real symmetric $N \times N$ matrix, determined by microscopic details of the system. The edge effective theories (5) and (6) imply the following Kac-Moody algebra [35] of chiral boson fields:

$$[\phi_I(x),\partial_y\phi_J(y)] = \pm 2\pi \mathbf{K}^{-1}_{I,J} i\delta(x-y), \tag{7}$$

where + (−) sign corresponds to the right (left) edge. The signature (n_+,n_-) of matrix $\mathbf{K}$ now has a clear physical meaning from (5) and (6): each positive (negative) eigenvalue of $\mathbf{K}$ corresponds to a right mover (left mover) on the right edge (5) and a left mover (right mover) on the left edge (6).

Similar to the quasiparticle excitations in the bulk labeled by their gauge charge $\mathbf{l}$, associated quasiparticles on the edge $V_{\mathbf{l}} = \exp(i\sum_I l_I\phi_I)$ are also labeled by an integer vector $\mathbf{l} = (l_1,\ldots,l_N)^T$. This identification between bulk quasiparticle $\mathbf{l}$ and edge excitations $\hat{V}_{\mathbf{l}}$ indicates that each (local) microscopic degree of freedom (4) in the bulk also has a correspondent local excitation on the edge: $\hat{V}_{\hat{\mathbf{l}}} = \hat{V}_{\mathbf{Kl}}$. For a $N \times N$ matrix $\mathbf{K}$, all these local excitations on the edge are composed of the following N-independent local excitations (microscopic degrees of freedom on the edge):

$$e^{i\sum_J \mathbf{K}_{I,J}\phi_J(x,t)}, \quad 1 \leqslant I \leqslant N.$$

In the context of fractional quantum Hall states, these local operators on the edge are called [35,53] "electron operators."

Here, let us go over the simplest case with no symmetry, when symmetry group $G_s = \{\boldsymbol{e}\}$ and $\boldsymbol{e}$ denotes the identity element of a group. In this case, all the (local) microscopic boson degrees of freedom can condense in the bulk, and accordingly on the edge the following Higgs terms can be added to Lagrangians (5) and (6):

$$\mathcal{L}_{\mathrm{Higgs}} = \sum_I C_I(e^{i\chi_I}\hat{M}_I + \mathrm{H.c.}) = \sum_{I=1}^N C_I \cos\left(p_I\sum_J \mathbf{K}_{I,J}\phi_J(x,t) + \chi_I\right), \tag{8}$$

where C_I and χ_I are all real parameters. Notice that constant factor

$$p_I \equiv (3 - (-1)^{\mathbf{K}_{I,I}})/2, \quad \forall\, 1 \leqslant I \leqslant N$$

guarantees the self-statistics (2) of local quasiparticle

$$\hat{M}_I(x,t) \equiv e^{ip_I\sum_J \mathbf{K}_{I,J}\phi_J(x,t)}, \; 1 \leqslant I \leqslant N \tag{9}$$

is bosonic since if $\hat{M}_I$ is fermionic the Higgs term (8) will violate locality. The Abelian topological order (featured by GSD on genus-g Riemann surfaces and anyon statistics) will not be affected by these Higgs terms [30,54,55], since all anyon excitations are local with respect to the microscopic boson degrees of freedom. As a result, the condensation of local bosonic degrees of freedom $\{\hat{M}_I\}$ will not trigger a phase transition, when there is no symmetry in the Abelian topological order. Hence, in a general ground these Higgs terms (8) should be included in the low-energy effective theories (1), (5), and (6) of an Abelian topological order, in the absence of any symmetry. Although naively effective theories (1) and (5) and (6) seem to have N-conserved U(1) currents, this $\mathrm{U}(1)^N$ symmetry will disappear when these Higgs terms are considered.

Since Higgs terms (8) are generally present in the edge effective theory (when there is no symmetry), they will introduce backscattering processes on the edge. A natural question is the stability of gapless edge excitations [56]. When $n_+ \neq n_-$ for the p signature (n_+,n_-) of matrix $\mathbf{K}$, there is a net chirality for edge states (5) and (6) and they cannot be fully gapped out by the Higgs terms (8). A physical consequence is a nonzero thermal Hall conductance in the system [57]. If $n_+ = n_-$, on the other hand, there is no net chirality on the edge. But, this does not mean the edge states can be gapped out by Higgs term (8): the simplest counterexample is $\mathbf{K} = \begin{pmatrix} 3 & 0 \\ 0 & -5 \end{pmatrix}$, whose edge cannot be gapped out even in the absence of any symmetry [58]. When the system preserves symmetry G_s, the structure of edge states is richer. Typically, some Higgs terms in (8) will be forbidden by symmetry, and there will be symmetry-protected edge excitations [30,41,59] in the Abelian topological order. In other words, certain branches of edge excitations will either remain gapless when symmetry G_s is preserved, or become gapped out when symmetry G_s is spontaneously broken on the edge. For a general discussion on the stability of edge modes in an Abelian topological order, we refer the readers to Sec. III of Ref. [30]. For the SET phases studied in this work, their edge stabilities are briefly discussed in Sec. III F.

C. Different Chern-Simons theories can describe the same topological order

For symmetric unimodular $\mathbf{K}$ matrix with $\det \mathbf{K} = \pm 1$, the ground state of system (1) is unique on any closed manifold. Consistent with the nondegenerate ground state on torus, any quasiparticle $\mathbf{l}$ is either bosonic or fermionic with trivial mutual statistics with each other. Hence, there is no topological order in the system [17,30] when $\det \mathbf{K} = \pm 1$. However, the corresponding gapped phase can still have gapless chiral edge modes on its boundary, which are stable against any perturbations. Well-known examples are the integer quantum Hall effects where $\mathbf{K}$ is an $N \times N$ identity matrix. On the other hand, if $\mathbf{K}$ matrix satisfies the following "trivial" condition

$$\text{Trivial phase : for } N = \dim\mathbf{K} = \text{even},$$
$$\det \mathbf{K} = (-1)^{N/2},\ (n_+,n_-) = (N/2,N/2), \qquad (10)$$

the edge excitations will be nonchiral (the same number of right and left movers) and are generally gapped in the absence of symmetry [30]. In these cases, we call the corresponding phase a *trivial phase* in (2+1)D since it is featureless both in the bulk and on the edge and can be continuously connected to a trivial product state without any phase transition [24].

One key point we want to emphasize is that the Chern-Simons theory description for a certain topologically ordered phase is *not unique*. In other words, two different $\mathbf{K}$ matrices for effective theory (1) can correspond to the same topological phase, with the same set of quasiparticle (anyon) excitations. The two features described below are crucial for the classification of symmetry-enriched topological orders.

First of all, the following $GL(N,\mathbb{Z})$ transformation on the $\mathbf{K}$ matrix yields an equivalent description for the same phase

$$\mathbf{K} \simeq \tilde{\mathbf{K}} = \mathbf{X}^T\mathbf{K}\mathbf{X},\ \ \mathbf{X} \in GL(N,\mathbb{Z}) \qquad (11)$$

where $GL(N,\mathbb{Z})$ represents the group of $N \times N$ unimodular matrices. This $GL(N,\mathbb{Z})$ transformation $\mathbf{X}$ merely relabels the quasiparticle (anyon) excitations so that $\mathbf{l} \to \tilde{\mathbf{l}} = \mathbf{X}^{-1}\mathbf{l}$. It is straightforward to see that all the topological properties, such as quasiparticle statistics and GSD, are invariant under such a $GL(N,\mathbb{Z})$ transformation. A brief introduction to $GL(N,\mathbb{Z})$ group is given in Appendix A.

Second, notice that a trivial phase satisfying (10) can always be added to a topologically ordered phase without changing any topological properties (such as quasiparticle statistics, GSD, and chiral central charge of edge excitations [44]). One just needs to enlarge the Hilbert space to include some new microscopic degrees of freedom, which form a trivial phase. Mathematical addition of a topologically ordered phase with matrix $\mathbf{K}$ and a trivial phase with matrix $\mathbf{K}_t$ satisfying (10) is carried out by the matrix direct sum [30]

$$\mathbf{K} \simeq \mathbf{K}' = \mathbf{K} \oplus \mathbf{K}_t,$$
$$\det \mathbf{K}_t = (-1)^{N_t/2},\ \ N_t = \dim\mathbf{K}_t = \text{even}. \qquad (12)$$

Therefore, two $\mathbf{K}$ matrices of different dimensions can describe the same topologically ordered phase. Typically in a bosonic system (where the microscopic degrees of freedom are all bosons), the generic trivial phase is represented by [17,30]

$$\mathbf{K}_t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \qquad (13)$$

Meanwhile, in a fermionic system, both (13) and

$$\mathbf{K}_t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (14)$$

together represent a generic trivial phase.

D. Implementing symmetries in Abelian topological orders

Our discussions in the previous section did not assume any symmetry [60] in the topologically ordered phase. Without any symmetry, an Abelian topological order is fully characterized by its $\mathbf{K}$ matrix. In the presence of symmetry, however, $\mathbf{K}$ matrix alone is not enough to describe a symmetry-enriched topological (SET) phase: e.g., distinct SET phases that are separated from each other by phase transitions can share the same $\mathbf{K}$ matrix. The missing information is how the bulk quasiparticles (with currents j_I^μ) in effective theory (1) transform under the symmetry. The corresponding information in the edge states (5) and (6) is how the chiral boson fields $\{\phi_I,\ 1 \leqslant I \leqslant N\}$ transform under symmetry.

We will restrict to unitary and antiunitary *onsite* (or global) symmetries in this work. By onsite symmetries we mean the local Hilbert space is mapped to itself [28] under the symmetry transformation, so that the symmetries act in an "onsite" fashion. In this case, studying the symmetry transformations of bulk quasiparticles (with currents j_I^μ) is equivalent to [30] studying the symmetry transformations of edge chiral bosons $\{\phi_I,\ 1 \leqslant I \leqslant N\}$. Henceforth, we will focus on the chiral boson variables on the edge to study their transformation rules under symmetry operations, in the presence of a symmetry group G_s.

Most generally, under the operation of symmetry group element $\boldsymbol{g} \in G_s$, the chiral boson fields $\{\phi_I\}$ transform in the following way [30]:

$$\phi_I(x,t) \to \sum_J \eta^g \mathbf{W}^g_{I,J}\phi_J(x,t) + \delta\phi^g_I,$$
$$\eta^g \mathbf{K} = (\mathbf{W}^g)^T\mathbf{K}\mathbf{W}^g,\ \ \mathbf{W}^g \in GL(N,\mathbb{Z}), \qquad (15)$$

where $\eta^g = +1\ (-1)$ for a unitary (antiunitary) onsite symmetry. This is simply because under an antiunitary symmetry operation (such as time reversal $t \to -t$) the Chern-Simons term $\epsilon^{\mu\nu\lambda}a^I_\mu\partial_\nu a^J_\lambda$ changes sign, and in order to keep the Lagrangian (1) in the bulk or (5) and (6) on the edge invariant, $\mathbf{K}$ must change sign under the $GL(N,\mathbb{Z})$ rotation $\mathbf{W}^g$.

Notice that the above symmetry transformations $\{\mathbf{W}^g,\delta\phi^g|\boldsymbol{g} \in G_s\}$ must be compatible with group structure of symmetry group G_s. This provides a strong constraint on the allowed choices of $GL(N,\mathbb{Z})$ rotations $\{\mathbf{W}^g\}$ and U(1) phase shifts $\{\delta\phi^g_I \simeq \delta\phi^g_I + 2\pi\}$. To be precise, the consistent conditions for symmetry transformations $\{\mathbf{W}^g,\delta\phi^g|\boldsymbol{g} \in G_s\}$ on an Abelian topological order characterized by matrix $\mathbf{K}$ are summarized in the following statement: The *(nonlocal)* *quasiparticle* excitations $\{\hat{Q}_I(x,t) \equiv e^{i\phi_I(x,t)}\}$ transform *projectively* under symmetry group G_s, while the *(local) microscopic boson degrees of freedom* $\{\hat{M}_I(x,t) \equiv e^{ip_I \sum_J \mathbf{K}_{I,J}\phi_J(x,t)}\}$ in (9) must form a *linear representation* of symmetry group G_s. Here, constant factor $p_I = 1$ if $\mathbf{K}_{I,I}$ = even, or $p_I = 2$ if $\mathbf{K}_{I,I}$ = odd.

In the following, we will discuss why (local) microscopic boson degrees of freedom must form a linear representation of symmetry group G_s. Imagine an Abelian topolgical ordered phase preserves symmetry G_s. For simplicity, let us consider $G_s = Z_2 = \{\boldsymbol{g},\boldsymbol{e}\}$ for an illustration. We denote the generator of the Z_2 group as $\boldsymbol{g}$. It satisfies the following Z_2 multiplication rule:

$$\boldsymbol{g} \cdot \boldsymbol{g} \equiv \boldsymbol{g}^2 = \boldsymbol{e}. \tag{16}$$

And under this Z_2 symmetry operation $\boldsymbol{g}$ the edge chiral bosons transform as (15). Consider we weakly break the Z_2 symmetry without closing the bulk energy gap (no phase transition). Now, Z_2 operation $\boldsymbol{g}$ is not a symmetry anymore and there is no symmetry in the system. Therefore, all the local bosonic degrees of freedom $\{\hat{M}_I(x,t) \equiv e^{ip_I \sum_J \mathbf{K}_{I,J}\phi_J(x,t)} | 1 \leqslant I \leqslant N\}$ can be condensed and Higgs term (8) should be allowed. At the same time, notice that $\boldsymbol{g}^2 = \boldsymbol{e}$ is still a "symmetry" of the system. When symmetry operation $\boldsymbol{g}$ acts twice, its transformations on chiral bosons $\vec{\phi}(x,t) \equiv (\phi_1(x,t),\dots,\phi_N(x,t))^T$ become

$$\begin{aligned}\vec{\phi}(x,t) &\xrightarrow{\boldsymbol{g}} \eta^{\boldsymbol{g}}\mathbf{W}^{\boldsymbol{g}}\vec{\phi}(x,t) + \delta\vec{\phi}^{\boldsymbol{g}} \\ &\xrightarrow{\boldsymbol{g}} (\mathbf{W}^{\boldsymbol{g}})^2\vec{\phi}(x,t) + (1_{N\times N} + \eta^{\boldsymbol{g}}\mathbf{W}^{\boldsymbol{g}})\delta\vec{\phi}^{\boldsymbol{g}},\end{aligned} \tag{17}$$

where $1_{N\times N}$ denotes an $N \times N$ identity matrix. And we must require all Higgs terms (8) with arbitrary parameters $\{C_I,\chi_I\}$ are allowed by "symmetry" $\boldsymbol{g}^2 = \boldsymbol{e}$ in (17). In other words, all the Higgs terms in (8) should remain invariant when Z_2 operation $\boldsymbol{g}$ acts twice as in (17)! This means the argument of any cosine (Higgs) terms in (8) must be invariant up to a 2π phase, leading to the following relation:

$$\mathbf{PK}(\mathbf{W}^{\boldsymbol{g}})^2 = \mathbf{PK}, \;\; \mathbf{PK}(1_{N\times N} + \eta^{\boldsymbol{g}}\mathbf{W}^{\boldsymbol{g}})\delta\vec{\phi}^{\boldsymbol{g}} = 2\pi\mathbf{n},$$

where we defined $N \times N$ diagonal matrix $\mathbf{P}_{I,J} = p_I\delta_{I,J}$ and $\mathbf{n} = (n_1,\dots,n_N)^T \in \mathbb{Z}^N$ is an integer vector. The above relation can be rewritten as

$$\begin{aligned}(\mathbf{W}^{\boldsymbol{g}})^2 = 1_{N\times N},\; \eta^{\boldsymbol{g}}\mathbf{K} = (\mathbf{W}^{\boldsymbol{g}})^T\mathbf{K}\mathbf{W}^{\boldsymbol{g}}, \\ (1_{N\times N} + \eta^{\boldsymbol{g}}\mathbf{W}^{\boldsymbol{g}})\delta\vec{\phi}^{\boldsymbol{g}} = 2\pi(\mathbf{PK})^{-1}\mathbf{n},\;\; \mathbf{n} \in \mathbb{Z}^N.\end{aligned} \tag{18}$$

These are the *group compatibility conditions* on the symmetry transformation (15) for a Z_2 symmetry group $G_s = \{\boldsymbol{g},\boldsymbol{e} = \boldsymbol{g}^2\}$. These conditions will be applied in the examples later.

In a generic case, symmetry group G_s (and its multiplication table) is fully determined by a set of algebraic relations

$$\mathcal{A}_{m_1,\dots,m_{N_g}} \equiv \boldsymbol{g}_1^{m_1} \cdot \boldsymbol{g}_2^{m_2} \dots \boldsymbol{g}_{N_g}^{m_{N_g}} = \boldsymbol{e},$$

where $\{\boldsymbol{g}_1,\dots,\boldsymbol{g}_{N_g}\}$ is a set of generators in group G_s. Each algebraic relation $\mathcal{A}_{m_1,\dots,m_{N_g}}$ gives rise to a consistent condition with an integer vector $\mathbf{n}_{m_1,\dots,m_{N_g}}$, just like (18) in the $G_s = Z_2$ case. When all these group compatibility conditions are satisfied, any local bosonic degrees of freedom

$$\hat{B}_{\mathbf{l}}(x,t) \equiv \exp(i\mathbf{l}^T\mathbf{PK}\vec{\phi}(x,t)),\;\; \mathbf{l} \in \mathbb{Z}^N$$

are invariant under symmetry operation $\mathcal{A}_{m_1,\dots,m_{N_g}}$. By definition, they form a linear representation of the symmetry group G_s. On the other hand, a generic quasiparticle excitation

$$\hat{V}_{\mathbf{l}}(x,t) \equiv \exp(i\mathbf{l}^T\vec{\phi}(x,t)),\;\; \mathbf{l} \in \mathbb{Z}^N$$

could still transform nontrivially under consecutive symmetry operation $\mathcal{A}_{m_1,\dots,m_{N_g}}$ (which equals identity $\boldsymbol{e}$ in symmetry group G_s). Therefore, these (fermionic or anyonic) excitations transform projectively [15,30] under symmetry group G_s.

E. Criteria for different symmetry-enriched topological orders

In the previous section, we discussed the consistent conditions on the symmetry transformations on the quasiparticle excitations in an Abelian topological order. In terms of chiral boson fields $\vec{\phi}(x,t)$ which capture the quasiparticle contents in an Abelian topological order, under symmetry transformations (15) (labeled by $\{\mathbf{W}^{\boldsymbol{g}},\delta\vec{\phi}^{\boldsymbol{g}} | \boldsymbol{g} \in G_s\}$ for symmetry group G_s), the (local) bosonic degrees of freedom (9) transform linearly while (nonlocal) anyonic degrees of freedom $\{e^{i\phi_I}\}$ can transform projectively. Together with matrix $\mathbf{K}$ which contains all the topological properties, the following set of data

$$[\mathbf{K},\{\eta^{\boldsymbol{g}},\mathbf{W}^{\boldsymbol{g}},\delta\vec{\phi}^{\boldsymbol{g}} | \boldsymbol{g} \in G_s\}] \tag{19}$$

fully characterize a symmetry-enriched topological (SET) phase in the presence of symmetry group G_s.

A natural question is as follows: Is such data a unique fingerprint for a SET phase? Can two different sets of data describe the same SET phase? Not surprisingly, the answer is yes. A trivial example is discussed earlier when symmetry group is trivial $G_s = \{\boldsymbol{e}\}$ (i.e., no symmetry) and two different $\mathbf{K}$ matrices correspond to the same Abelian topological order. So, how can we tell whether two sets of data (19) describe the same SET phase or not? In the following, we will propose a few criteria, which thoroughly address this issue.

The first criterion comes from the physical picture that there is no "smooth" boundary condition under which we can sew two different SET phases with the same topological order and symmetry group G_s. This is rooted in the fact that two different SET phases cannot be continuously (no phase transitions in between) connected to each other without breaking the symmetry. On the other hand, if two symmetric states belong to the same SET phase, there must exist a "transparent" smooth boundary between these two states that preserves symmetry. In the following, we will establish the above physical picture in a more precise mathematical setup.

Consider two symmetric Abelian states described by K matrices $\mathbf{K}^L$ and $\mathbf{K}^R$. First, we require $\mathbf{K}^L \simeq \mathbf{K}^R$ describe the same Abelian topological order in the absence of symmetry, i.e., they have the same topological properties such as GSD ($|\det\mathbf{K}^L| = |\det\mathbf{K}^R|$) and quasiparticle statistics. This is because two SET phases are certainly different if they correspond to different topological orders when symmetry is broken.

Consider a left edge (6) of SET phase $[\mathbf{K}^L,\{\eta^{\boldsymbol{g}},\mathbf{W}^{\boldsymbol{g}}_L,\delta\vec{\phi}^{\boldsymbol{g}}_L | \boldsymbol{g} \in G_s\}]$ and the right edge (5) of SET phase $[\mathbf{K}^R,\{\eta^{\boldsymbol{g}},\mathbf{W}^{\boldsymbol{g}}_R,\delta\vec{\phi}^{\boldsymbol{g}}_R | \boldsymbol{g} \in G_s\}]$ are sewed together by introducing tunneling terms between the two edges (see Fig. 1). We denote the chiral boson fields as $\{\phi^L_I\}$ on the left edge and $\{\phi^R_J\}$ on the right edge. Notice that only microscopic degrees of freedom (4) whose mutual statistics (3) with any quasiparticle are multiples of 2π can appear in the tunneling term between the right and left edges [53] as shown in Fig. 1. Therefore, a general tunneling term has the following Lagrangian density:

$$\mathcal{H}_{\text{tunnel}} = \sum_{\alpha} T_\alpha \cos\left(\left(\mathbf{l}^L_\alpha\right)^T \mathbf{K}^L \vec{\phi}^L - \left(\mathbf{l}^R_\alpha\right)^T \mathbf{K}^R \vec{\phi}^R + \varphi_\alpha\right),$$

$$\mathbf{l}^L_\alpha,\mathbf{l}^R_\alpha \neq 0, \quad \left(\mathbf{l}^L_\alpha\right)^T \mathbf{K}^L \mathbf{l}^L_\alpha - \left(\mathbf{l}^R_\alpha\right)^T \mathbf{K}^R \mathbf{l}^R_\alpha = 0, \forall\, \alpha \tag{20}$$

where T_α,φ_α are real parameters. According to Kac-Moody algebra (7) for the chiral bosons, the condition on the second line means the variables in each cosine term of (20) commute with itself and can be localized at a classical value $\langle (\mathbf{l}^L_\alpha)^T \mathbf{K}^L \vec{\phi}^L - (\mathbf{l}^R_\alpha)^T \mathbf{K}^R \vec{\phi}^R \rangle$. Of course, every tunneling term in (20) must be allowed by symmetry, i.e., they remain invariant under symmetry transformation (15). The edge states are fully gapped [30] if each chiral boson field $\phi^{L/R}_I$ is either pinned at a classical value or does not commute with at least one variable of the cosine terms in (20). Notice that each cosine term in (20) must contain local operators from both edges.

If the set of symmetric tunneling terms (20) cannot fully gap out the boundary between the two states, the two symmetric states clearly cannot belong to the same SET phase. On the contrary, even if the boundary can be fully gapped out by (20), the two states on both sides of the boundary may still correspond to different SET phases, as elaborated below.

For an Abelian topological order characterized by $N \times N$ matrix $\mathbf{K}$, its Abelian anyons $e^{i\vec{\mu}\cdot\vec{\phi}}$ are labeled by vectors $\vec{\mu} \in \mathbb{Z}^N$ which form an N-dimensional integer lattice [32,33,61]. The primitive vectors $\{\mathbf{b}^J\}$ are given by rows of $\mathbf{K}$ matrix, i.e., $\mathbf{b}^J_I = \mathbf{K}_{I,J}$, which forms a Bravais lattice. This defines an equivalence relation between anyon vectors $\vec{l}$: two anyon vectors differing by a Bravais lattice vector correspond to the same sector of Abelian anyon. Distinct anyons in an Abelian topological order thus correspond to different "sublattices" of the anyon lattice. Fusion rule of distinct anyon sectors corresponds to addition of the anyon vectors in the anyon lattice $\Lambda_{\mathbf{K}}$.

The effective K matrix for the boundary between $\mathbf{K}^L$ and $\mathbf{K}^R$ is $\mathbf{K} = \mathbf{K}^L \oplus (-\mathbf{K}^R)$. If a set of tunneling terms (20) fully gap out the boundary, tunneling an anyon $\vec{\mu}_R$ from the right side through the boundary would inject a different anyon $\vec{\mu}_L$ into the left side in a way compatible with the boundary terms (20). Mathematically, we can define the vector product between two anyons $\vec{\mu}$ and $\vec{\mu}'$ in terms of bilinear form $(\vec{\mu},\vec{\mu}') \equiv \vec{\mu}^T \mathbf{K}^{-1} \vec{\mu}'$, and the anyons tunneled through the boundary $(\vec{\mu}^T_L, -\vec{\mu}^T_R)^T$ are compatible with tunneling terms if and only if they are orthogonal to all tunneling vectors $((\mathbf{l}^L)^T \mathbf{K}^L, -(\mathbf{l}^R)^T \mathbf{K}^R)^T$, i.e.,

$$\vec{\mu}^T_L \mathbf{l}^L_\alpha - \vec{\mu}^T_R \mathbf{l}^R_\alpha = 0,\ \forall\, \alpha. \tag{21}$$

This establishes a mapping between anyons $\vec{\mu}_L$ in the left state $\mathbf{K}^L$ and anyons $\vec{\mu}_R$ in the right state $\mathbf{K}^R$, once tunneling terms (20) fully gap out the boundary.

Definition of smooth edge sewing condition. Symmetric tunneling terms (20) provide a smoothing sewing between the left and right sides, if and only if (i) they fully gap out the boundary without breaking symmetry, (ii) the mapping (21) from anyons $\vec{\mu}_L \in \Lambda_L$ on the left side to anyons $\vec{\mu}_R \in \Lambda_R$ on the right side realizes an isomorphism between their anyon lattices Λ_L and Λ_R.

Since an isomorphism between the two anyon lattices preserves all the fusion rules (vector addition) and anyon statistics (vector product) of all anyon sectors. Therefore, the smooth sewed boundary would be transparent for the two sides: tunneling an anyon $\vec{\mu}_R$ from the right side through the boundary would inject an anyon $\vec{\mu}_L$ with the same fractional statistics and fusion rules into the left side, as if the anyon $\vec{\mu}_R$ passes through the boundary. This provides a one-to-one correspondence between anyons on both sides in a way compatible with the symmetry. This motivates our criterion for different Abelian SET phases.

Criterion I. Two sets of data $[\mathbf{K}^L,\{\eta^{\boldsymbol{g}},\mathbf{W}^{\boldsymbol{g}}_L,\delta\vec{\phi}^{\boldsymbol{g}}_L|\boldsymbol{g} \in G_s\}]$ (for the left edge) and $[\mathbf{K}^R,\{\eta^{\boldsymbol{g}},\mathbf{W}^{\boldsymbol{g}}_R,\delta\vec{\phi}^{\boldsymbol{g}}_R|\boldsymbol{g} \in G_s\}]$ (for the right edge) belong to the same SET phase if and only if there exists a set of tunneling terms (20) connecting the two edges, which satisfy the smooth sewing condition defined above.

We have argued the sufficiency of this criterion; following, we show that it is also a necessary condition. Assume that smooth sewing condition cannot be achieved for the boundary between two states. Since the distinct anyon sectors are the same for $\mathbf{K}^L$ and $\mathbf{K}^R$, we only need to consider the case that the mapping from Λ_L to Λ_R is not injective without loss of generality [62]. This means tunneling a nontrivial anyon $\vec{\mu}_{L,0} \neq 0$ from the left side can simply inject a local boson (vacuum sector) $\vec{\mu}_{R,0} \sim 0$ into the right side, via the symmetric tunneling terms (20). This will lead to significant observable effects. Consider a hybrid sphere where the "equator belt" region is our symmetric state $\mathbf{K}^L$, while the rest region (near north and south poles) is occupied by symmetric state $\mathbf{K}^R$. We can immediately show the hybrid system actually have ground-state degeneracy [63,64] on a sphere since we can create quasiparticles $e^{i(\vec{\mu}^T_{L,0}\cdot\vec{\phi}_L - \vec{\mu}^T_{R,0}\cdot\vec{\phi}_R)}$ on the pair of boundaries, and drag the pair of nontrivial anyon $e^{i(\vec{\mu}^T_{L,0}\cdot\vec{\phi}_L)}$ towards the equator before annihilating them on the equator. Such a string operator gives rise to a degenerate ground state. Such degenerate gapped ground states are impossible on a sphere (no noncontractible loops) if the two regions host the same SET phase.

Let us take a look at the simplest case, when the two SET states share exactly the same set of data (19). In this case, the tunneling term (20) essentially sews the left and right edges of the same SET phase. The smooth sewing between left and right edges basically tunnels the same local microscopic degrees of freedom $\hat{V}^L_{\mathbf{Kl}}$ of one edge with its counterpart $\hat{V}^R_{\mathbf{Kl}}$ on the other edge. The following tunneling term

$$\mathcal{H}^0_{\text{tunnel}} = \sum_{I=1}^{N} T_I \cos\left(\sum_J \mathbf{K}_{I,J}(\phi^L_J - \phi^R_J) + \varphi_I\right) \tag{22}$$

is allowed by symmetry G_s and will gap out the edge states. Notice that all cosine terms commute with each other, so they can be minimized simultaneously. One important feature of the above tunneling terms is that there are $|\det K|$ inequivalent classical minima [53] for the $\{\phi^L_J - \phi^R_J\}$ variables of the cosine terms. In other words, the chiral bosons will be pinned at one of the $|\det K|$ classical values by the above tunneling terms. In this case, the one-to-one correspondence between left and right anyons is simply $\vec{\mu}_L = \vec{\mu}_R$.

This criterion applies universally to both unitary and antiunitary symmetries (such as time-reversal symmetry). When $\det \mathbf{K}^{L/R} = \pm 1$ it automatically reduces to the criterion for different symmetry-protected topological (SPT) phases in the Chern-Simons approach [30]. A direct consequence of Criterion I are the following two corollaries.

Corollary I. If the two sets of data share the same matrix $\mathbf{K}^L = \mathbf{K}^R$, and all their local microscopic degrees of freedom (4) transform in the same way under symmetry G_s, then they belong to the same SET phase since their edges can be sewed together smoothly by term (22).

Next, notice that a $GL(N,\mathbb{Z})$ transformation (11) can always be performed on a $\mathbf{K}$ matrix without changing the topological order. It simply relabels different quasiparticles. Besides, U(1) gauge transformations can always be performed on gauge fields a_μ^I and chiral bosons $\{\phi_I\}$. The most general gauge transformations that relabel quasiparticles have the form

$$\phi_I(x,t) \to \sum_J \mathbf{X}_{I,J}\phi_J(x,t) + \Delta\phi_I, \;\; \mathbf{X} \in GL(N,\mathbb{Z}) \quad (23)$$

where $\Delta\phi_I \in [0,2\pi)$ are constants. We denote such a gauge transformation as $\{\mathbf{X},\Delta\vec{\phi}\}$. Under such a gauge transformation, the set of data (19) changes as

$$\begin{aligned} &\mathbf{K} \overset{\{\mathbf{X},\Delta\vec{\phi}\}}{\longrightarrow} \mathbf{X}^T\mathbf{K}\mathbf{X}, \\ \forall\, \boldsymbol{g} \in G_s,\;\; &\mathbf{W}^g \overset{\{\mathbf{X},\Delta\vec{\phi}\}}{\longrightarrow} \mathbf{X}^{-1}\mathbf{W}^g\mathbf{X}, \\ &\delta\vec{\phi}^g \overset{\{\mathbf{X},\Delta\vec{\phi}\}}{\longrightarrow} \mathbf{X}^{-1}(\delta\vec{\phi}^g + (\eta^g\mathbf{W}^g - 1_{N\times N})\Delta\vec{\phi}) \end{aligned} \quad (24)$$

and η^g remains invariant. Here comes the second corollary.

Corollary II. Any two sets of data (19) that can be related to each other by a gauge transformation (24) correspond to the same SET phase.

Last but not least, an important lesson from studying SPT phases is that there is a duality [29,42] between SPT phases and gauge theories (or intrinsic topological orders). This duality is established by gauging the (unitary) symmetry G_s in the SPT phase, i.e., coupling the physical degrees of freedom (which transform under symmetry G_s) to a gauge field [29] (with gauge group G_s). One conjecture is that different SPT phases with G_s symmetry always lead to distinct G_s gauge theories. Naively, one can ask the same question for SET phases [20], i.e., will two different SET phases (with G_s symmetry) always lead to distinct intrinsic topological orders, when the symmetry G_s is gauged? In the examples studied in this work, by gauging unitary symmetry $G_s = Z_2$ different SET phases do lead to distinct topological orders, with different quasiparticle statistics. Besides, these distinct topological orders all share the same total quantum dimension [43] $\mathcal{D}$ (and hence the same topological entanglement entropy [11,12] $\gamma = \ln\mathcal{D}$). However, one can show that for a general symmetry group this not true: i.e., different SET phases can result in the same topological order by gauging the symmetry. For a counterexample, let us consider two different SET phases with $G_s = Z_2 \times Z_2$ symmetry. They are constructed by stacking a topologically ordered layer, which does not transform under symmetry at all, with two different $Z_2 \times Z_2$ SPT layers, respectively [65]. Clearly, they cannot be smoothly connected to each other without phase transitions while preserving $Z_2 \times Z_2$ symmetry, thus are distinct SET phases. However, after gauging the global $Z_2 \times Z_2$ symmetry, they can lead to the same intrinsic topological order, e.g.,

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \oplus \mathbf{K}(010) \simeq \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \oplus \mathbf{K}(110),$$

where $\mathbf{K}(n_1n_2n_3)$ are intrinsic topological orders obtained by gauging $Z_2 \times Z_2$ SPT phases in 2D, as defined in (D1).

Although the above statement is not always true, the converse (or inverse) statement is necessarily true:

Criterion II. After gauging the unitary symmetry G_s, if two SET phases (with symmetry G_s) lead to two different topological orders, they must belong to two distinct SET phases.

Criterion II only applies to unitary symmetries. As will become clear in the examples, in certain cases (including those which we call "unconventional" SET phases), gauging an Abelian symmetry in an Abelian topological order will lead to non-Abelian topological orders [66].

In the following sections, we will demonstrate these criteria by classifying different SET phases with (anti)unitary Z_2 symmetries. The Abelian topological orders that will be studied include Z_2 spin liquid [40,49] with $\mathbf{K} \simeq \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$, double-semion theory [37,38] with $\mathbf{K} \simeq \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$, and Laughlin states [5] ($\mathbf{K} \simeq m$) at different filling fractions $\nu = 1/m$. Among them, Z_2 spin liquid and double-semion theory are nonchiral Abelian phases, in the sense that their edge excitations have no net chirality. And in the absence of symmetry their edge excitations will generically be gapped. On the other hand, Laughlin states are chiral Abelian phases with quantized thermal Hall conductance.

III. EXAMPLES

In this section, we will apply the Chern-Simons approach discussed in previous sections to various Abelian topological orders. We start by classifying Z_2 spin liquids with time-reversal symmetry $G_s = Z_2^T$ and with a unitary Z_2 symmetry $G_s = Z_2$. Usually by Z_2 spin liquids people refer to gapped many-spin ground states supporting fractionalized spin-carrying quasiparticles, coined "spinons," and other fractionalized quasiparticles carrying no spin quantum numbers, coined "visons." The mutual (braiding) statistics of a spinon and a vison is semionic ($\theta_{s,v} = \pi$), while the self-statistics of a spinon/vison is bosonic. A Z_2 spin liquid has fourfold GSD on a torus. All these topological properties are captured by the Chern-Simons theory (1) with

$$\mathbf{K} \simeq \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}. \quad (25)$$

In the context of this work, we do not assume spin rotational symmetry and hence visons/spinons generally cannot be distinguished from their spin quantum numbers. Despite this fact we still use the name "Z_2 spin liquid" to label this Abelian topological order. The four degenerate ground states on a torus correspond to the four superselection sectors, which are associated with the four inequivalent quasiparticles

$$1 \simeq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \simeq \begin{pmatrix} 2 \\ 0 \end{pmatrix} \simeq \begin{pmatrix} 0 \\ 2 \end{pmatrix},$$
$$e \simeq \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ m \simeq \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ f \simeq \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{26}$$

where both e and m have bosonic (self-)statistics and they correspond to electric charge and magnetic vortex in a Z_2 gauge theory [36], respectively. f is the bound state of an electric charge and a magnetic vortex, with fermionic statistic. 0 corresponds to any local excitations (4) with no fractional statistics, belonging to the vacuum sector. In the folklore of Z_2 spin liquid, a vison is e (or m) and accordingly a bosonic spinon is m (or e).

A. Classifying Z_2 spin liquids with time-reversal symmetry

As a warmup we consider Abelian topological order (25) with symmetry group $Z_2^T = \{\boldsymbol{g},\boldsymbol{e} = \boldsymbol{g}^2\}$ with algebra (16). Notice that the generator of Z_2^T group $\boldsymbol{g}$ is an antiunitary operation with $\eta^{\boldsymbol{g}} = -1$ in (15). In this case, we rely on Criterion I and its corollaries to differentiate various Z_2^T SET phases.

The associated group compatibility condition (18) for $G_s = Z_2^T$ in Abelian topological order (25) is

$$(\mathbf{W}^{\boldsymbol{g}})^2 = 1_{2\times 2},\ -\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = (\mathbf{W}^{\boldsymbol{g}})^T \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \mathbf{W}^{\boldsymbol{g}},$$
$$(1_{2\times 2} - \mathbf{W}^{\boldsymbol{g}})\delta\vec{\phi}^{\boldsymbol{g}} = \pi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{n},\ \ \mathbf{n} \in \mathbb{Z}^2. \tag{27}$$

The $\mathbf{W}^{\boldsymbol{g}} \in GL(2,\mathbb{Z})$ solution to the above conditions is $\mathbf{W}^{\boldsymbol{g}} = \pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. However, notice that the following $GL(2,\mathbb{Z})$ gauge transformations (24) keep the $\mathbf{K}$ matrix (25) invariant:

$$\mathbf{X} = \pm 1_{2\times 2},\ \ \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Therefore, $\mathbf{W}^{\boldsymbol{g}} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\mathbf{W}^{\boldsymbol{g}} = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ are equivalent, related by gauge transformation $\mathbf{X} = \pm\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ in (24). We can fix the gauge by choosing

$$\mathbf{W}^{\boldsymbol{g}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for time-reversal operation $\boldsymbol{g}$. Then, solving the second line of conditions (27) we obtain

$$n_2 = 0,\ \ \delta\vec{\phi} = \begin{pmatrix} \delta\phi_1 \\ \frac{n_1}{2}\pi \end{pmatrix} \mod 2\pi.$$

We can always choose a gauge transformation $\{\mathbf{X} = 1_{2\times 2},\Delta\vec{\phi}\}$ in (24) so that $\delta\phi_1 = 0$. Meanwhile, since $\mathbf{l} = (2,0)^T$ and $\mathbf{l} = (0,2)^T$ are the local excitations in the system, according to Corollary II, $n_1 =$ even all correspond to the same SET phase. Meanwhile, $n_1 =$ odd leads to another SET phase, which is distinct from the $n_1 =$ even SET phase. This is because the magnetic vortex $m \simeq (0,1)^T$ transforms projectively in $n_1 =$ odd phase, but transforms linearly in the $n_2 =$ even phase under time-reversal symmetry. It is straightforward to check that there is no way to smoothly sew the two edges of $n_1 =$ even and $n_1 =$ odd SET phases by a time-reversal-invariant tunneling term (20), which has fourfold-degenerate classical minima. Therefore, according to Criterion I they belong to two different SET phases. Hence, Chern-Simons approach produces two different classes of Z_2^T symmetry-enriched Z_2 spin liquids, as summarized in Table I.

Since the previous calculations are based on 2×2 $\mathbf{K}$ matrix (25), it is natural to ask the following: What if we enlarge the dimension of $\mathbf{K}$ matrix (12) by introducing the trivial part with (13)? In this new representation of the same Abelian topological order, will we get more SET phases or not? Notice that the trivial part (13) is nothing but the $\mathbf{K}$ matrix for a bosonic SPT phase [17,30] in (2+1)D. For antiunitary Z_2^T symmetry, there is no nontrivial bosonic SPT phase [24,30] in (2+1)D. This means the edge chiral bosons for the trivial parts can always be gapped out by introducing symmetry-allowed backscattering cosine (Higgs) terms, whose classical minima is pinned at a unique classical value since $|\det \mathbf{K}_t| = 1$. According to Criterion I, in the presence of Z_2^T symmetry, when the dimension of $\mathbf{K}$ is enlarged by adding the trivial parts, it will not introduce any new SET phases.

At the end we discuss the stability of edge excitations in the two SET phases. Notice that the chiral bosons $\{\phi_{1,2}\}$ transform as

$$\begin{pmatrix} \phi_1(x,t) \\ \phi_2(x,t) \end{pmatrix} \xrightarrow{\boldsymbol{g}} \begin{pmatrix} -\phi_1(x,t) \\ \phi_2(x,t) + \frac{n_1}{2}\pi \end{pmatrix}$$

under time-reversal operation $\boldsymbol{g}$. As a result, the edges can be completely gapped by introducing Higgs terms

$$\mathcal{H}_{\text{Higgs}} = C\cos[2\phi_1(x,t)],$$

which pins chiral boson field $\phi_1(x,t)$ to a classical value $\langle\phi_1(x,t)\rangle = 0$ or π, without breaking the time-reversal symmetry. Therefore, in general there are no gapless edge states for the two SET phases with $G_s = Z_2^T$.

Potentially, one could conceive of a phase where both electric and magnetic vortices transform projectively under time-reversal symmetry. However, such a phase is only possible as the surface state of a (3+1)D SPT phase with time-reversal symmetry [46]. The $\mathbf{K}$-matrix classification correctly reproduces the fact that this phase is forbidden.

Meanwhile, in the presence of time-reversal symmetry only, there exists one Z_2 spin liquid phase which cannot be described by Abelian Chern-Simons theory with a $\mathbf{K}$ matrix. This SET phase can be constructed, e.g., in terms of Abrikosov-fermion [67–69] representation of a spin-$\frac{1}{2}$ system, where the spin-$\frac{1}{2}$ fermionic spinons form a topological superconductor in class DIII [70,71]. Roughly speaking, such a topological superconductor consists of a $p+ip$ chiral superconductor of spin-$\uparrow$ fermions, and a $p-ip$ chiral superconductor of spin-$\downarrow$ fermions so that time-reversal symmetry is preserved. In contrast, SET phase No. 2 in Table I corresponds to a nontopological s-wave singlet superconductor of fermionic spinons. On the other hand, phase No. 1 in Table I can be realized in a spin-1 system, where fermionic spinons (carrying integer spins) again form a trivial s-wave superconductor in the fermion representation [72] of spin-1. These three time-reversal-enriched Z_2 spin liquids cannot be adiabatically tuned into each other without a phase transition. A full classification of Z_2 spin liquids with time-reversal symmetry is constituted of these three distinct phases.

B. Classifying Z_2 spin liquids with onsite Z_2 symmetry

As discussed earlier, the reason why 2×2 matrix $\mathbf{K}=\begin{pmatrix}0&2\\2&0\end{pmatrix}$ is enough to describe Z_2^T-symmetric Z_2 spin liquids is that there are no nontrivial Z_2^T SPT phases of bosons in (2+1)D. In other words, the possible trivial part (13) that can be added to $\mathbf{K}$ in (12) does not bring in new structure to SET phases. However, for a unitary $G_s=Z_2$ symmetry, as will become clear later, there is a nontrivial bosonic SPT phase [24,28,29] whose edge cannot be gapped without breaking the Z_2 symmetry. This Z_2 SPT phase can be understood in Chern-Simons approach with $\mathbf{K}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. Therefore, we need to consider the 4×4 matrix

$$\mathbf{K}=\begin{pmatrix}0&2\\2&0\end{pmatrix}\oplus\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}0&2&0&0\\2&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$$

to represent a generic Z_2 spin liquid enriched by a unitary Z_2 symmetry. The group compatibility condition (18) for the unitary Z_2 symmetry ($\eta^g=1$) transformation (15) becomes

$$\begin{gathered}(\mathbf{W}^g)^2=1_{4\times4},\\ \begin{pmatrix}0&2\\2&0\end{pmatrix}\oplus\begin{pmatrix}0&1\\1&0\end{pmatrix}=(\mathbf{W}^g)^T\begin{pmatrix}0&2\\2&0\end{pmatrix}\oplus\begin{pmatrix}0&1\\1&0\end{pmatrix}\mathbf{W}^g,\\ (1_{4\times4}+\mathbf{W}^g)\delta\vec{\phi}^g=\pi\begin{pmatrix}0&1\\1&0\end{pmatrix}\oplus\begin{pmatrix}0&2\\2&0\end{pmatrix}\mathbf{n},\quad \mathbf{n}\in\mathbb{Z}^4.\end{gathered}\tag{28}$$

The gauge-inequivalent solutions to $\mathbf{W}^g$ are the following [73]:

$$\mathbf{W}^g=1_{4\times4},\ \begin{pmatrix}0&1\\1&0\end{pmatrix}\oplus 1_{2\times2}.$$

We will discuss these two cases separately in the following. In particular, we will call the first case ($\mathbf{W}^g=1_{4\times4}$) "conventional" Z_2 spin liquids. In contrast, we will call the second case $[\mathbf{W}^g=\begin{pmatrix}0&1\\1&0\end{pmatrix}\oplus 1_{2\times2}]$ "unconventional" Z_2 spin liquids, in the sense that distinct quasiparticles (e and m) are exchanged under Z_2 symmetry operation $\boldsymbol{g}$.

1. "Conventional" Z_2 symmetry-enriched Z_2 spin liquids

First, we discuss the solution $\mathbf{W}^g=1_{4\times4}$. In this case, each anyon quasiparticle (26) in the Z_2 spin liquids merely obtains a U(1) phase factor under the Z_2 symmetry operation $\boldsymbol{g}$, and we call them "conventional" SET phases. Due to the gauge transformations $\mathbf{X}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\oplus 1_{2\times2}, 1_{2\times2}\oplus\begin{pmatrix}0&1\\1&0\end{pmatrix}$ which leave the $\mathbf{K}$ matrix invariant, we know that the integer vector $\mathbf{n}$ in (28) has the following equivalency relation:

$$\mathbf{n}=\begin{pmatrix}n_1\\n_2\\n_3\\n_4\end{pmatrix}\simeq\begin{pmatrix}n_2\\n_1\\n_3\\n_4\end{pmatrix}\simeq\begin{pmatrix}n_1\\n_2\\n_4\\n_3\end{pmatrix}.$$

Moreover, Corollary II tells us

$$\mathbf{n}\simeq\begin{pmatrix}n_1+2\\n_2\\n_3\\n_4\end{pmatrix}\simeq\begin{pmatrix}n_1\\n_2+2\\n_3\\n_4\end{pmatrix}\simeq\begin{pmatrix}n_1\\n_2\\n_3+2\\n_4\end{pmatrix}\simeq\begin{pmatrix}n_1\\n_2\\n_3\\n_4+2\end{pmatrix}$$

and

$$\begin{pmatrix}n_1\\n_2\\0\\0\end{pmatrix}\simeq\begin{pmatrix}n_1\\n_2\\1\\0\end{pmatrix}\simeq\begin{pmatrix}n_1\\n_2\\0\\1\end{pmatrix}.$$

Naively, there are six inequivalent solutions with $\mathbf{W}^g=1_{4\times4}$ under Z_2 symmetry:

$$\mathbf{n}=\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\Longleftrightarrow$$

$$\delta\vec{\phi}^g=\begin{pmatrix}0\\0\\\pi\\0\end{pmatrix},\begin{pmatrix}0\\0\\\pi\\\pi\end{pmatrix},\begin{pmatrix}\pi/2\\0\\\pi\\0\end{pmatrix},\begin{pmatrix}\pi/2\\0\\\pi\\\pi\end{pmatrix},\begin{pmatrix}\pi/2\\\pi/2\\\pi\\0\end{pmatrix},\begin{pmatrix}\pi/2\\\pi/2\\\pi\\\pi\end{pmatrix}.$$

However, these six SET phases are not all different. According to Criterion I, one can show that

$$\delta\vec{\phi}^g=\begin{pmatrix}\pi/2\\0\\\pi\\0\end{pmatrix}\simeq\begin{pmatrix}\pi/2\\0\\\pi\\\pi\end{pmatrix}\simeq\begin{pmatrix}\pi/2\\-\pi\\\pi\\\pi\end{pmatrix}=\mathbf{X}^{-1}\delta\vec{\phi}^g$$

by a $GL(4,\mathbb{Z})$ gauge transformation (24) with

$$\mathbf{X}=\begin{pmatrix}1&0&0&0\\0&1&1&0\\0&0&1&0\\-2&0&0&1\end{pmatrix},\quad \mathbf{X}^T\mathbf{K}\mathbf{X}=\mathbf{K}.$$

Similarly, another two states belong to the same SET phase by the gauge transformation $\mathbf{X}$:

$$\delta\vec{\phi}^g=\begin{pmatrix}\pi/2\\\pi/2\\\pi\\0\end{pmatrix}\simeq\begin{pmatrix}\pi/2\\\pi/2\\\pi\\\pi\end{pmatrix}\simeq\begin{pmatrix}\pi/2\\-\pi/2\\\pi\\\pi\end{pmatrix}=\mathbf{X}^{-1}\delta\vec{\phi}^g.$$

As a result, we obtain four inequivalent SET phases with $\mathbf{W}^g=1_{4\times4}$ under Z_2 symmetry, with their symmetry transformations $\delta\vec{\phi}^g$ summarized in Table II. We require $\delta\vec{\phi}\neq 0$ so that the local excitations (4) form a faithful representation [30] of symmetry group G_s. In the following, we briefly discuss the consequence of gauging the unitary Z_2 symmetry. A detailed prescription of gauging a unitary symmetry in the Chern-Simons approach is given in Appendix B, where we have shown that gauging Z_2 symmetry $\{\mathbf{W}^g=1_{4\times4},\delta\vec{\phi}^g=\pi(i_1/2,i_2/2,1,i_4)^T\}$ yields an Abelian topological order described by matrix $\mathbf{K}_g$ in (B2). Take No. 4 for an example, after gauging Z_2 symmetry we have

$$\mathbf{K}_g=\begin{pmatrix}0&0&0&2\\0&0&2&-1\\0&2&0&-1\\2&-1&-1&-2\end{pmatrix}\simeq\begin{pmatrix}4&0&0&0\\0&-4&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$$

$$\simeq \begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix},$$

where the first equivalency $\simeq$ is realized by gauge transformation (24) with

$$\mathbf{X} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 3 & 1 & 1 \\ 2 & -2 & 0 & -1 \end{pmatrix}, \quad \det \mathbf{X} = 1.$$

From Table II, one can see that different SET phases lead to distinct (intrinsic) topological orders by gauging their Z_2 symmetry. New quasiparticles $\{q_g\}$ emerge when we gauge the symmetry, whose topological spins $\Theta_{q_g} \equiv \exp(2\pi i h_{q_g})$ and mutual statistics $\tilde{\theta}_{q_g,e}, \tilde{\theta}_{q_g,m}$ with original anyons (e and m) serves as important characters of the SET phase (see Table II).

The stability of edge excitations is also summarized in Table II. For SET phases No. 2 the gapless edge excitations come from the trivial $(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$ part (lower 2×2 part) of the $\mathbf{K}$ matrix, which corresponds to the symmetry-protected edge modes of bosonic Z_2 SPT phases. In fact, No. 2 phase corresponds to nothing but stacking a layer of bosonic Z_2 SPT phase with a layer of Z_2 spin liquid which does not transform under Z_2 symmetry operation. On the other hand, for No. 4, the topologically ordered $(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix})$ part (upper 2×2) of the $\mathbf{K}$ matrix contributes to $c = 1$ gapless edge states. In other words, in a Z_2 spin liquid, if both e and m transform projectively under Z_2 symmetry, the edge excitations are protected to be gapless unless symmetry is broken. The edge chiral bosons $\{\phi_{1,2}\}$ for $\mathbf{K} = (\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix})$ can be refermionized as right-moving branch $\psi_R \sim \exp[i(\phi_1 + \phi_2)]$ and left-moving branch $\psi_L \sim \exp[i(\phi_1 - \phi_2)]$. The edge effective theory (5) can be rewritten as

$$\mathcal{L}_{rE} = i\psi_R^\dagger(\partial_t - v_+\partial_x)\psi_R - i\psi_L^\dagger(\partial_t - v_-\partial_x)\psi_L, \quad (29)$$

where $v_\pm = (\mathbf{V}_{1,2} \pm \sqrt{\mathbf{V}_{1,2}^2 + \mathbf{V}_{1,1}\mathbf{V}_{2,2}})/2$ are the velocities of edge modes. It is easy to see that under Z_2 symmetry $\boldsymbol{g}$ the chiral fermions transform as $\psi_R \to -\psi_R$ and $\psi_L \to \psi_L$ for SET phase No. 4, and hence backscattering term $\psi_L\psi_R +$ H.c. and $\psi_L^\dagger\psi_R +$ H.c. are forbidden by Z_2 symmetry. As a result, there are gapless edge states in No. 4 SET phase, protected by Z_2 symmetry.

2. "Unconventional" Z_2 symmetry-enriched Z_2 spin liquids

Now we turn to solutions to (28) with $\mathbf{W}^g = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}) \oplus 1_{2\times 2}$. Notice that one can always choose a proper gauge $\Delta\vec{\phi}$ in (24) so that $\delta\phi_1^g = \delta\phi_2^g$, and hence $n_1 = n_2$ in (28). Naively, there are four different solutions of this type to (28): they are $\delta\vec{\phi}^g = \pi(n_1/2, n_1/2, 1, n_3)$ with $n_{1,3} = 0,1$. However, one can show that these four solutions are related by a gauge transformation $\Delta\vec{\phi} = (\pi/2,0,0,0)$ and $\mathbf{X} = 1_{4\times 4}$ in (24):

$$\delta\vec{\phi}^g = \begin{pmatrix} \pi/2 \\ \pi/2 \\ \pi \\ n_3\pi \end{pmatrix} \simeq \begin{pmatrix} 0 \\ 0 \\ \pi \\ n_3\pi \end{pmatrix} \simeq \begin{pmatrix} 0 \\ \pi \\ \pi \\ n_3\pi \end{pmatrix}$$
$$= \mathbf{X}^{-1}(\delta\vec{\phi}^g + (\mathbf{W}^g - 1_{4\times 4})\Delta\vec{\phi}),$$

where the second equivalency is due to Criterion I. Consequently, there are only two different "unconventional" Z_2 spin liquids enriched by unitary onsite Z_2 symmetry, as summarized in Table III. We can see that two distinct superselection sectors, i.e., the electric charge e and magnetic vortex m exchange under onsite Z_2 symmetry $\boldsymbol{g}$, hence, we call them "unconventional" SET phases. Notice that unlike "conventional" SET phases in Table II, we cannot determine whether e or m transform projectively under unitary Z_2 symmetry $\boldsymbol{g}$ or not since they transform into each other under $\boldsymbol{g}$. Both SET phases in Table III host symmetry-protected edge excitations on the boundary, but the central charge of the gapless edge states is $c = \frac{1}{2}$, different from $c = 1$ in "conventional" SET phases (see Table II).

For SET phase No. 5, the edge chiral bosons $\phi_{3,4}$ for the trivial $(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$ part of $\mathbf{K}$ matrix can be fully gapped by a $\cos\phi_4$ term. However, the chiral bosons $\phi_{1,2}$ for topologically ordered $(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix})$ part of $\mathbf{K}$ are protected by Z_2 symmetry. To be precise, in the refermionized description (29) for edge states, the chiral fermions transform as

$$\psi_R \xrightarrow{g} (-1)^{n_1}\psi_R, \quad \psi_L \xrightarrow{g} \psi_L^\dagger \quad (30)$$

for $\delta\vec{\phi}^g = \pi(n_1/2, n_1/2, 1, n_3)$ with $n_{1,3} = 0,1$ in Table III. We can rewrite each chiral fermion in terms of two Majorana fermions [40] $\xi_{R/L}$ and $\eta_{R/L}$:

$$\psi_{R/L} \equiv \xi_{R/L} + i\eta_{R/L}.$$

When $n_1 = 0$, the following backscattering term is allowed by Z_2 symmetry

$$\mathcal{H}_{bs} \propto \psi_R(\psi_L + \psi_L^\dagger) + \text{H.c.} = 4\xi_R\xi_L.$$

Therefore, the $\xi_{R/L}$ branch of Majorana fermions is gapped, while the $\eta_{R/L}$ branch is protected by Z_2 symmetry. When $n_1 = 1$, similarly the following backscattering term

$$\mathcal{H}_{bs} \propto \psi_R(\psi_L - \psi_L^\dagger) + \text{H.c.} = 4\eta_R\eta_L$$

is allowed by Z_2 symmetry. It will gap out $\eta_{R/L}$ modes and leave Majorana modes $\xi_{R/L}$ gapless. As a consequence, a $c = \frac{1}{2}$ branch of Majorana fermions will remain gapless, unless Z_2 symmetry is broken on the edge. Together with the $c = 1$ Z_2 symmetry-protected chiral boson edge [30] from $(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$ part for SET phase No. 6, we obtain the total central charge c for both 'unconventional SET phases as summarized in Table III.

Aside from gapless edge states, another important feature for these unconventional SET phases is that they lead to non-Abelian topological orders once the unitary Z_2 symmetry is gauged. For these unconventional SET phases, a vertex algebra approach is introduced in Appendix E to gauge the unitary symmetry. The quasiparticle contents of the "gauged" non-Abelian topological orders for both SET phases are summarized in Table VII. The "gauged" topological orders are related to the unconventional Z_2 gauge theories describing fermions with *odd* Chern number ν coupled to a Z_2 gauge field, as Kitaev described in his 16-fold way classification of (2+1)D Z_2 gauge theories [44]. More specifically, these non-Abelian topological orders are $Z_2 \times Z_2$ gauge theories, the direct product of $\nu =$ odd Z_2 gauge theory and its time-reversal counterpart $\bar{\nu} = 16 - \nu$, as summarized in Table III. Hence, these "gauged" topological orders both have nonchiral edge states (chiral central charge $c_- = 0$), which will generally be gapped in the absence of extra symmetry.

After gauging the symmetry, new quasiparticles with quantum dimension $d_{q_g} = \sqrt{2}$ emerge as deconfined excitations, called Z_2 symmetry fluxes $\{q_g\}$. When any quasiparticle q in the original SET phase is moved adiabatically around a Z_2 symmetry flux, it becomes its image $\hat{\boldsymbol{g}}q$ under Z_2 symmetry operation. These Z_2 symmetry fluxes are similar to a Majorana bound state in the vortex core of a spinless $p + ip$ superconductor [74] in (2+1)D. However, they have different topological spin than those in $p + ip$ superconductors. To be specific, there are four inequivalent Z_2 symmetry fluxes with topological spin $\exp(\pm\frac{\nu}{16}2\pi i)$ and $\exp(\pm\frac{8+\nu}{16}2\pi i)$, as shown in Table III. All these non-Abelian topological orders have ninefold GSD on a torus, corresponding to nine different superselection sectors shown in Table VII. It is not hard to see that SET phases No. 5 and No. 6 do lead to different non-Abelian topological orders after Z_2 symmetry is gauged. In particular, their 9×9 modular $\mathcal{S}$ and $\mathcal{T}$ matrices in the basis of Table VII are shown in the end of Appendix E. Combining the edge states and "gauged" topological orders summarized in Table III, indeed there are two distinct unconventional Z_2 symmetry-enriched Z_2 spin liquids.

Therefore, a full classification of Z_2 spin liquids (or Z_2 toric codes) enriched by unitary onsite Z_2 symmetry includes six different SET phases. Two of these six SET phases are unconventional (Table III), in the sense that distinct superselction sectors are exchanged under onsite Z_2 symmetry, while the other four SET phases are conventional (Table II). Here, many conventional SET phases lead to Abelian topological orders by gauging the unitary symmetry, while unconventional SET phases always lead to non-Abelian topological orders. In spite of these diversities, a general rule seems to apply to all SET phases:

Conjecture I. All different SET phases (with the same "ungauged" topological order and onsite unitary symmetry G_s) always lead to topological orders with the same total quantum dimension [43,44] $\mathcal{D}$ (and hence the same topological entanglement entropy [11,12] $\gamma = \ln \mathcal{D}$), after finite unitary symmetry G_s is gauged.

One can easily see this conjecture holds for all examples studied in this paper, such as Z_2 spin liquids in Tables II and III, double-semion theories in Table VI, and $\nu = 1/2k$ Laughlin states in Tables IV and V. For example, all "gauged" topological orders from Z_2 symmetry-enriched Z_2 spin liquids have $\mathcal{D} = 16$, no matter Abelian (conventional) or non-Abelian (unconventional). Similarly, $\nu = 1/2k$ Laughlin states enriched by Z_2 symmetry has $\mathcal{D} = 8k$.

This conclusion, however, does not apply to continuous (unitary) symmetries. For a simplest example, let us consider bosonic SPT phases protected by $G_s = \mathrm{U}(1)$ symmetry [30] as a special case of $G_s = \mathrm{U}(1)$ SET phases. They are featured by *even integer* Hall conductance $\sigma_{xy} = 2q$, $q \in \mathbb{Z}$ in units of e^2/h where e is the fundamental charge of bosons. Gauging the unitary $G_s = \mathrm{U}(1)$ symmetry leads to Abelian $\mathrm{U}(1)_{2q}$ Chern-Simons theories, whose total quantum dimension $\mathcal{D} = |2q|$ clearly differs for different U(1) SPT phases.

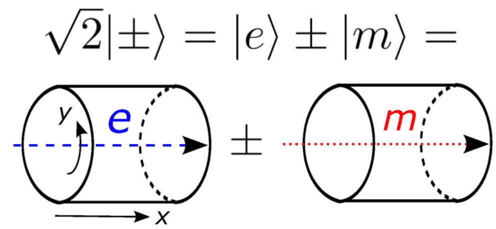

FIG. 3. The Ising symmetry eigenstates are linear combinations of minimal entropy states (MESs) for an unconventional Ising symmetry-enriched Z_2 spin liquid since one MES $|e\rangle$ transforms into another MES $|m\rangle$ under Ising symmetry operation.

3. *Measurable effects of unconventional Z_2 symmetry realizations*

Suppose there is a Z_2 spin liquid which preserves Z_2 spin rotational symmetry (for integer spins), are there measurable effects for these SET phases? More specifically, what are the distinctive measurable features of the unconventional Z_2 SET phases? In this section, we will try to answer these questions in two aspects, i.e., measurements in the bulk and on the edge. We will focus on unconventional SET phases in this section.

First of all, an important ingredient of SET phases is how their quasiparticles transform under symmetry G_s. This gives us one way to measure an SET phase: to create quasiparticle excitations and apply symmetry operation on them. For example, for a Z_2 spin liquid on a closed manifold (a sphere, say), a pair of electric charges $e \simeq (1,0,0,0)^T$ [or magnetic vortices $m \simeq (0,1,0,0)^T$] can be created on top of the ground state. For an onsite unitary symmetry (such as Z_2 spin-flip symmetry $\boldsymbol{g}$), one can choose to perform the symmetry operation only on a part of the whole system. For example, Fig. 2 shows such a striking measurable effect on the unconventional Z_2 symmetry-enriched Z_2 spin liquids. Assume in the SET phase a pair of electric charges e are created, one in subsystem A and the other in subsystem B. The whole system $A \bigcup B$ lives on a closed manifold, say a sphere on which the ground state is unique. If we only perform the unconventional Z_2 symmetry operation $\boldsymbol{g}$ in subsystem A (but not in B), the electric charge $e \simeq (1,0,0,0)^T$ in subsystem A will become a magnetic vortex $m \simeq (0,1,0,0)^T$. However, one electric charge and one magnetic vortex cannot be created simultaneously ($e \times m = f \neq 1$) on top of the ground state: the conservation of "topological charge" requires the existence of an extra fermion f in the system! Such a fermion mode f lives on the boundary (dashed line in Fig. 2) between subsystems A and B, as shown by the wavy line in Fig. 2. This effect happens in all four SET phases in Table III.

Second, there are degenerate ground states once we put the SET phases on a closed manifold with nontrivial topology (with nonzero genus). For example, they have fourfold GSD on a torus (or infinite cylinder). How these ground states transform under symmetry is a reflection of how anyon quasiparticles transform under symmetry. Specifically, one can always choose a set of bases called the minimal entropy states (MESs) [75]. As the name implies, each MES is a superposition of degenerate ground states which minimizes the bipartite entanglement entropy [76] once a certain entanglement cut is chosen, e.g., along the y direction in the middle of the infinite cylinder as shown in Fig. 3. The MESs are flux eigenstates [77], which keep maximum knowledge of the states after the entanglement cut. Specifically for a Z_2 spin liquid, we can label the four MESs as $|1\rangle,|e\rangle,|m\rangle,|f\rangle$ on an infinite cylinder. Remarkably under the unconventional Ising symmetry operation, two MESs ($|e\rangle$ and $|m\rangle$ exchanges) and their linear combinations $|\pm\rangle = (|e\rangle \pm |m\rangle)/\sqrt{2}$ are the Ising symmetry eigenstates. Therefore, the MESs necessarily break Ising symmetry in such an unconventional SET phase! These phenomena can be measured in numerical studies [13].

Third, the edge state structure encodes much information of a SET phase, when it supports symmetry-protected edge modes. For unconventional SET phases, there are always gapless edge excitations protected by symmetry, unless the symmetry is spontaneously broken on the boundary. In the specific case of unconventional Z_2 symmetry-enriched Z_2 spin liquids summarized in Table III, one important feature is that SET phases No. 1 and No. 2 support gapless (nonchiral) Majorana fermion excitations on the boundary, with central charge $c = \frac{1}{2} \mod 1$. However, this is not universal for all unconventional SET phases. A more interesting effect comes from the bound state localized at a Z_2 domain wall on the edge. Take SET phase No. 1 for example, a perturbation on the edge

$$\mathcal{H}_1 = A_1 \cos(2\phi_1) + A_2 \cos(2\phi_2) + A_4 \cos\phi_4 \quad (31)$$

can fully gap out the edge excitations in (5) and (6) with $\mathbf{K} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, if $A_{1,4} \neq 0$ (or $A_{2,4} \neq 0$). On one side of the Z_2 domain wall on the edge of SET phase No. 1, we break Z_2 symmetry with $A_2 = 0$ and $A_{1,4} \neq 0$. On the other side of the Z_2 domain wall, Z_2 symmetry is broken in the opposite way so that $A_1 = 0$ and $A_{2,4} \neq 0$. Physically, the electric charges are condensed on one side of the domain wall, while magnetic vortices condense on the other side. At the domain wall a non-Abelian (Majorana) bound state [39,78] is localized, which has quantum dimension $\sqrt{2}$, as illustrated in Fig. 4. Such a domain-wall bound state is similar to those localized at the (ferromagnetism/superconductivity) mass domain walls of a quantum spin Hall insulator [79]. In the vertex algebra context, these domain-wall bound states correspond to quasiparticle q_6 (in the eighth role) in Table VII. In SET phase No. 5 in Table III, e.g., it has topological spin $\exp(-i\pi/8)$. In the bulk-edge correspondence of SET phases, such a bound state on the edge is related to the Z_2 symmetry flux q_g in the bulk.

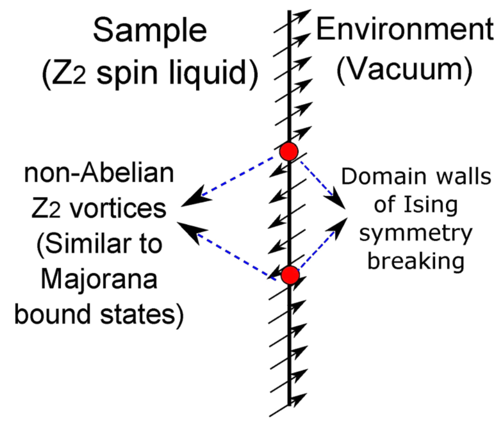

FIG. 4. Domain-wall bound state on the edge of unconventional Ising symmetry-enriched Z_2 spin liquids (see Table III). In these SET phases, under Z_2 symmetry operation, one electric charge will transform into a magnetic vortex and vice versa. The onsite unitary Z_2 (Ising) symmetry can be, e.g., a spin-flip symmetry. On the two sides of the Ising symmetry domain wall, two different backscattering "mass" terms related by spin-flip Ising symmetry are added to gap out the edge states. These two mass terms break Z_2 symmetry in opposite ways. A non-Abelian bound state with quantum dimension $d_{q_g} = \sqrt{2}$ is localized at each Ising domain wall. For a conventional Z_2 SET phase, such an Ising mass domain wall will trap an Abelian bound state with quantum dimension 1.

4. Gauging symmetry in unconventional SET phases

In this section, we provide a simple pictorial argument, which shows that gauging symmetry in an Abelian unconventional SET phase will lead to a non-Abelian topological order. In particular, we would like to show that in the new topological order obtained by gauging the symmetry, the quantum dimensions of certain quasiparticle excitations are larger than 1. Therefore, the topological order obtained by gauging symmetry must be non-Abelian. Although we will use $G_s = Z_2$ symmetry as an illustration, this argument naturally generalizes to other finite unitary symmetry groups.

Let us assume two distinct quasiparticle types a and b in an Abelian SET phase are exchanged under unitary Z_2 symmetry operation $\boldsymbol{g}$. Once Z_2 symmetry $\boldsymbol{g}$ is gauged, Z_2 symmetry flux q_g becomes a deconfined excitation in the system. Hence, the new ground state after gauging symmetry $\boldsymbol{g}$ is a condensate of closed loops (or sting-net condensate [38]), which are trajectories of a symmetry flux q_g and its antiparticle $\bar{q}_g$ before they annihilate each other (see Fig. 5). Whenever such a closed loop is created, symmetry $\boldsymbol{g}$ is implemented in the region enclosed by the loop which will transform a quasiparticle a inside the loop into b. After gauging symmetry $\boldsymbol{g}$, since these closed loops (string nets) are condensed in the new ground state, quasiparticles a and b are not separately deconfined excitations anymore. Instead, their quantum superposition

$$\alpha = a + b$$

becomes a well-defined finite-energy excitation after gauging Z_2 symmetry $\boldsymbol{g}$.

More precisely, e.g., the excited-state wave function with spatially separated quasiparticles $\{\alpha(\mathbf{r}_1),\alpha(\mathbf{r}_2),\dots\}$ (we use ... to denote other quasiparticles) is a quantum

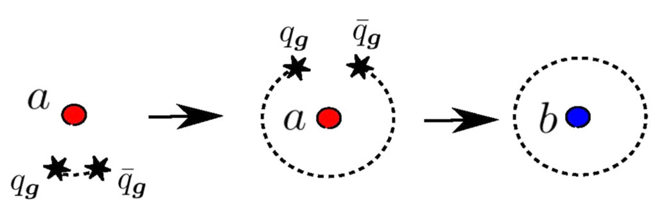

FIG. 5. The process in which a symmetry flux q_g and its antiparticle $\bar{q}_g$ are created out of the vacuum, dragged around a quasiparticle a, and then annihilated. The dashed line denotes the trajectory of q_g and $\bar{q}_g$ from their creation to annihilation. After this process, symmetry $\boldsymbol{g}$ is implemented in the region inside the closed loop (dashed line), and hence quasiparticle a is transformed into b under $\boldsymbol{g}$ symmetry operation.

superposition of four amplitudes

$$|\alpha(\mathbf{r}_1),\alpha(\mathbf{r}_2),\dots\rangle = |a(\mathbf{r}_1),a(\mathbf{r}_2),\dots\rangle + |a(\mathbf{r}_1),b(\mathbf{r}_2),\dots\rangle + |b(\mathbf{r}_1),a(\mathbf{r}_2),\dots\rangle + |b(\mathbf{r}_1),b(\mathbf{r}_2),\dots\rangle.$$

Therefore, by definition the quantum dimension of new quasiparticle α is $\mathcal{D}_\alpha = \sqrt{4} = 2$, after gauging unitary Z_2 symmetry $\boldsymbol{g}$. In the example we studied in this paper, this new quasiparticle is listed on the fifth row of Table VII.

Similarly, it is straightforward to show that after gauging unconventional Z_N symmetry which permutes N distinct anyons, a new quasiparticle α with quantum dimension $\mathcal{D}_\alpha = N > 1$ will emerge. This demonstrates that gauging an unconventional SET phase will inevitably lead to non-Abelian topological orders.

In previous sections, we use the Chern-Simons approach to study SET phases which are nonchiral, i.e., there are no gapless edge excitations in the absence symmetry. Chern-Simons approach also applies to chiral topological phases, which have gapless edge modes even in the absence of symmetry. These chiral phases have a nonzero chiral central charge [44] c_- and quantized thermal Hall effect [57], which necessarily breaks time-reversal symmetry. In the following, we will use Laughlin states as illustrative examples of chiral SET phases.

C. Classifying bosonic Laughlin state at $\nu = \frac{1}{2k}$ ($k \in \mathbb{Z}$) with onsite Z_2 symmetry

A Laughlin state [5] at filling fraction $\nu = 1/m$ is described by $\mathbf{K} \simeq m$ in effective theory (1). When $m =$ even it describes a bosonic topological order, while $m =$ odd corresponds to a fermionic state. Such an effective theory also describes chiral spin liquids [48,80]. Here, we start from the simplest case, i.e., $m = 2$. It has twofold GSD on torus, corresponding to two different types of quasiparticles (or two superselection sectors): boson 1 and semion s. Under a unitary Z_2 symmetry, a semion always transforms into a semion, hence, we do not expect any unconventional SET phases where two inequivalent quasiparticles exchange under Z_2 operation. Again, due to the existence of nontrivial Z_2 SPT phase of bosons in (2+1)D, we use the following 3×3 matrix:

$$\mathbf{K} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{32}$$

in (1) to represent a generic $\nu = \frac{1}{2}$ Laughlin state with Z_2 symmetry. The group compatibility conditions (18) for symmetry transformations (15) are ($\eta^g = 1$ for unitary Z_2 symmetry)

$$(\mathbf{W}^g)^2 = 1_{3\times3}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = (\mathbf{W}^g)^T \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \mathbf{W}^g,$$
$$(1_{3\times3} + \mathbf{W}^g)\delta\vec{\phi}^g = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \mathbf{n}\pi,\ \ \mathbf{n} \in \mathbb{Z}^3. \tag{33}$$

The inequivalent solutions to the above conditions are $\mathbf{W}^g = 1_{3\times3}$ and

$$\delta\vec{\phi}^g = \left(\frac{i_1\pi}{2}, \pi, i_3\pi\right)^T. \tag{34}$$

In this case, they correspond to four different SET phases with $i_1 = 0,1$ and $i_3 = 0,1$, as summarized in Table IV with $k = 1$. This can be easily understood according to Criterion II since they lead to four distinct topological orders when we gauge the unitary Z_2 symmetry.

Following the Chern-Simons approach to gauge the unitary Z_2 symmetry described in Appendix B, we obtain the following "gauged" topological order:

$$\mathbf{K}_g^{-1} = \mathbf{M}^T\mathbf{K}^{-1}\mathbf{M},\ \mathbf{M} = \begin{pmatrix} 1 & 0 & \frac{i_1}{2} \\ 0 & 1 & \frac{i_3}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

and, hence,

$$\mathbf{K}_g = \begin{pmatrix} 2 & -i_1 & 0 \\ -i_1 & -2i_3 & 2 \\ 0 & 2 & 0 \end{pmatrix}. \tag{35}$$

Take SET phase No. 3 ($i_1 = 1$, $i_3 = 0$) in Table IV, for example, its gauged topological order contains the following quasiparticles in (35):

$$\gamma \equiv \begin{pmatrix} 0 \\ 0 \\ \gamma \end{pmatrix},\ \theta_\gamma = \frac{\gamma^2}{8}\pi,\ \gamma = 0,1,\dots,7 \tag{36}$$

with $(1,0,0)^T \simeq (0,0,2)^T$ and $(0,1,0)^T \simeq (0,0,4)^T$. The new quasiparticles, i.e., Z_2 symmetry fluxes with $\gamma =$ odd, in (36) have topological spins $\pm e^{i\pi/8}$.

Meanwhile, for SET phase No. 4 in Table IV, its gauged theory has the following quasiparticle contents in (35):

$$\gamma \equiv \begin{pmatrix} 0 \\ 0 \\ \gamma \end{pmatrix},\ \theta_\gamma = \frac{5\gamma^2}{8}\pi,\ \gamma = 0,1,\dots,7, \tag{37}$$

where $(1,0,0)^T \simeq (0,0,2)^T$ and $(0,1,0)^T \simeq (0,0,4)^T$. One can easily show that $i_3 = 0,1$ in (35) correspond to distinct topological orders, e.g., the new quasiparticles or Z_2 symmetry fluxes with $\gamma =$ odd in (37) in SET phase No. 4 have topological spins $\pm e^{i5\pi/8}$, in contrast to $\pm e^{i\pi/8}$ in phase No. 3.

In a complete parallel fashion, we can study generic conventional Z_2 symmetry-enriched even-denominator Laughlin state at $\nu = 1/(2k), k \in \mathbb{Z}$. Without loss of generality, a $\nu = 1/(2k)$ Laughlin state with unitary Z_2 symmetry is represented by

$$\mathbf{K} = 2k \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2k & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{38}$$

It has $2k$ different quasiparticles (or superselection sectors) labeled as

$$Q_a = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix},\ h_{Q_a} = \frac{\theta_{Q_a}}{2\pi} = \frac{a^2}{2k},\ a = 0,1,\dots,2k-1.$$

The group compatibility conditions (18) for symmetry transformations (15) have the following inequivalent solutions:

$$\mathbf{W}^g = 1_{3\times 3},\ \ \delta\vec{\phi}^g = \left(\frac{i_1\pi}{2k},\pi,i_3\pi\right)^T, \tag{39}$$

where $i_{1,3} = 0,1$. The solution $i_1 = 2$ represents the same SET phase as $i_1 = 0$, according to Corollary II in the Criteria. Comparing with the $\nu = \frac{1}{2}$ bosonic Laughlin state case, we can see there is a universal structure for all bosonic Lauglin state with $\mathbf{K} \simeq 2k$, $k =$ odd. To be specific, for a $\nu = 1/2k$ ($k =$ odd) bosonic Laughlin state, there are four different Z_2 SET phases as summarized in Table IV. The quasiparticles (or edge chiral bosons) transform as

$$\phi \xrightarrow{g} \phi + \left(\frac{i_1\pi}{2k},\pi,i_3\pi\right)^T,\ \ i_{1,3} = 0,1 \tag{40}$$

under conventional Z_2 operation.

Again following the Chern-Simons approach to gauge the unitary Z_2 symmetry described in Appendix B, we obtain the following topological order

$$\mathbf{K}_g^{-1} = \mathbf{M}^T\mathbf{K}^{-1}\mathbf{M},\ \mathbf{M} = \begin{pmatrix} 1 & 0 & \frac{i_1}{2} \\ 0 & 1 & \frac{i_3}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

and, hence,

$$\mathbf{K}_g = \begin{pmatrix} 2k & -i_1 & 0 \\ -i_1 & -2i_2 & 2 \\ 0 & 2 & 0 \end{pmatrix},\ \ i_{1,3} = 0,1. \tag{41}$$

All these four Abelian topological orders obtained by gauging symmetry have $|\det \mathbf{K}_g| = 8k$-fold GSD on a torus.

After gauging the Z_2 symmetry, we obtain new quasiparticles which are the Z_2 symmetry fluxes $\{q_g\}$. A generic Z_2 symmetry flux is represented by $q_g = (i_1,i_3,1)^T/2 + \mathbf{l}$, $\mathbf{l} \in \mathbb{Z}$. Its topological spin is given by $\exp(2\pi i h_{q_g})$, where

$$\begin{aligned} h_{q_g} &= \frac{\theta_{q_g}}{2\pi} = \frac{1}{2}q_g^T\mathbf{K}^{-1}q_g \\ &= \frac{1}{2}\left(\frac{\delta\vec{\phi}^g}{2\pi}\right)^T\mathbf{K}\frac{\delta\vec{\phi}^g}{2\pi} + \frac{1}{2}\mathbf{l}^T\mathbf{K}^{-1}\mathbf{l} + \frac{1}{2\pi}\mathbf{l}^T\delta\vec{\phi}^g \\ &= \frac{i_1^2}{16k} + \frac{l_1^2 + i_1 l_1}{4k} + \frac{i_3}{4} + \frac{l_2 + i_3 l_3}{2} \quad \text{mod } 1. \end{aligned} \tag{42}$$

Its mutual statistics with the fundamental Laughlin quasihole $p \equiv (1,p_2,p_3)^T$ is

$$\tilde{\theta}_{q_g,p} = \frac{\pi(i_1 + 2l_1)}{2k} + \pi(p_2 + i_3 p_3) \quad \text{mod } 2\pi \tag{43}$$

as summarized in Table IV.

It is straightforward to check that No. 3 and No. 4 are different SET phases since they lead to distinct topological orders after gauging Z_2 symmetry (Corollary II). For example, certain Z_2 symmetry flux in phase No. 3 has topological spin $e^{i\pi/8k}$, while such Z_2 symmetry flux does not exist in phase No. 4 when $k =$ odd.

When $k =$ even, on the other hand, No. 3 and No. 4 states actually lead to the same Z_2 SET phase. A careful analysis reveals that $i_3 = 0$ and 1 actually belong to the same SET phase when $i_1 = 1$ in Sec. III C. They are related by gauge transformation (24) as follows:

$$\mathbf{X} = \begin{pmatrix} 1 & 0 & -1 \\ 2k & 1 & -k \\ 0 & 0 & 1 \end{pmatrix},\ \ \mathbf{X}^T\mathbf{K}\mathbf{X} = \mathbf{K},$$

$$\delta\vec{\phi}^g = \begin{pmatrix} \frac{\pi}{2k} \\ \pi \\ \pi \end{pmatrix} \simeq \begin{pmatrix} \frac{\pi}{2k} \\ 0 \\ \pi \end{pmatrix} \simeq \begin{pmatrix} \frac{\pi}{2k} + \pi \\ -k\pi \\ \pi \end{pmatrix} = \mathbf{X}^{-1}\delta\vec{\phi}^g$$

when $k =$ even. Therefore, with $k =$ even there are only three different conventional $\nu = \frac{1}{2k}$ Laughlin states enriched by a unitary Z_2 symmetry, as summarized in Table V.

In general, there are also many unconventional $\nu = 1/2k$ Laughlin states, where distinct superselection sectors exchange under Z_2 symmetry operation. We leave the classification of these unconventional Laughlin states to future study. A few interesting examples are discussed in [81].

D. Fermionic Laughlin state at $\nu = \frac{1}{2k+1}$ ($k \in \mathbb{Z}$) with Z_2^f symmetry is unique

Now, let us turn to the simplest fermionic Laughlin state with $\mathbf{K} \simeq 3$. It has threefold GSD on a torus and anyon excitations with statistics $\theta = \pi/3, -2\pi/3$. We consider the following matrix in effective theory (1):

$$\mathbf{K} = 3 \oplus \mathbf{K}_t, \tag{44}$$

where $\mathbf{K}_t$ generically take the form of (13) and (14). Notice that for a fermion system with only $Z_2^f = \{\boldsymbol{e},\boldsymbol{g} = (-1)^{N_f}\}$ (fermion number parity) symmetry, there is no nontrivial SPT phases [30,83] in (2+1)D, which hosts gapless edge excitations protected by Z_2^f symmetry. This fact suggests that $\mathbf{K} = 3$ is enough to describe a $\nu = \frac{1}{3}$ fermionic Laughlin state with only Z_2^f symmetry. Now for such a 1×1 matrix $\mathbf{K} = 3$, the group compatibility conditions (18) becomes (note that $\mathbf{P} = 2$ for fermions)

$$\mathbf{W}^g = 1,\ \ 2\delta\phi^g = \frac{2\pi}{6}n,\ \ n \in \mathbb{Z} \tag{45}$$

for a unitary Z_2 symmetry $\boldsymbol{g}$. The gauge-inequivalent solutions are $\delta\phi^g = \frac{n}{6}\pi$ with $n \in \mathbb{Z}$. However, notice fermions in this system have gauge charge 3 in (1), or alternatively it is represented by $e^{3i\phi}$ on the edges (5) and (6). Under the Z_2^f operation $\boldsymbol{g}$ each fermion obtains a -1 sign, which means $3\delta\phi^g = \pi$ mod 2π and n must be even in (45). Therefore, the quasiparticle (chiral boson) transforms as

$$\phi \xrightarrow{g=(-1)^{N_f}} \phi + \frac{2n+1}{3}\pi,\ \ n \in \mathbb{Z}. \tag{46}$$

According to Corollary II on smooth sewing condition between edges, we know different integer $n \in \mathbb{Z}$ above correspond to the same Z_2^f SET phase. As a result, when only fermion number parity (Z_2^f symmetry) is conserved, the Laughlin $\nu = \frac{1}{3}$ state of fermions is unique.

It is straightforward to see that after gauging the Z_2^f symmetry, we obtain an Abelian topological order

$$\mathbf{K}_g = 3 \times 4 = 12. \tag{47}$$

In fact, the above conclusion is true for any fermionic Laughlin state at $\nu = 1/(2k+1)$, $k \in \mathbb{Z}$, with conserved fermion number parity. In the presence of Z_2^f symmetry, it is unique with $\mathbf{K} = 2k+1$. After gauging the Z_2^f symmetry, we obtain an Abelian topological order $\mathbf{K}_g = 4(2k+1)$.

E. Z_2 spin liquids with onsite $Z_2 \times Z_2$ symmetry

In the end we turn to a Z_2 spin liquid $\mathbf{K} \simeq \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ in the presence of $Z_2 \times Z_2$ symmetry. The symmetry group $G_s = Z_2 \times Z_2 = \{\boldsymbol{e},\boldsymbol{g}_1,\boldsymbol{g}_2,\boldsymbol{g}_1\boldsymbol{g}_2\}$ consists of two generators $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$, satisfying the following algebra:

$$\boldsymbol{g}_1^2 = \boldsymbol{g}_2^2 = (\boldsymbol{g}_1\boldsymbol{g}_2)^2 = \boldsymbol{e}. \tag{48}$$

In an integer-spin [82] system, these two generators $\boldsymbol{g}_{1,2}$ can correspond to, e.g., spin rotations along $\hat{x}(\boldsymbol{g}_1)$ and $\hat{z}(\boldsymbol{g}_2)$ direction by an angle of π. Naturally, the π-spin rotation along the $\hat{y}$ direction corresponds to group element $\boldsymbol{g}_1\boldsymbol{g}_2$.

Here, we will not attempt to fully classify all $Z_2 \times Z_2$ symmetry-enriched Z_2 spin liquids. Instead, we focus on one nontrivial example, where spinons transform projectively under $Z_2 \times Z_2$ symmetry, in the sense that under 2π-spin rotation along any $(\hat{x},\hat{y},\hat{z})$ direction the spinon (or electric charge e) obtains a phase -1, just like a half-integer spin. On the other hand, the vison (or magnetic vortex m) transforms trivially under the spin rotations. Such a SET phase can be easily realized by, e.g., Schwinger boson [84] representation of Z_2 spin liquids, for integer spin-S ($S = 0,1,2,\dots$):

$$\mathbf{S} = \frac{1}{2}\begin{pmatrix} b_\uparrow^\dagger \\ b_\downarrow^\dagger \end{pmatrix}^T \vec{\sigma} \begin{pmatrix} b_\uparrow \\ b_\downarrow \end{pmatrix}, \tag{49}$$

where $\vec{\sigma}$ are Pauli matrices. The following constraint needs to be enforced for each spin

$$b_\uparrow^\dagger b_\uparrow + b_\downarrow^\dagger b_\downarrow = 2S \tag{50}$$

to guarantee $\mathbf{S}^2 = S(S+1)$ for a spin-S system. Once the bosons $b_{\uparrow/\downarrow}$ form a pair superfluid (but not a superfluid) with $\langle bb\rangle \neq 0$ (but $\langle b\rangle = 0$), the resulting spin-S state after projection into the physical Hilbert space (50) is a Z_2 spin liquid [16,85]. Its spinon excitations $b_{\uparrow/\downarrow}$ carry half-spin each, hence transforming projectively under SO(3) (and hence $Z_2 \times Z_2$) spin rotations. By "transforming projectively" we simply mean that after all three symmetry operations in (48) which equals identity operation $\boldsymbol{e}$, all spinons obtain -1 phase instead of remaining invariant (or transforming linearly). In the following, we will show such a $Z_2 \times Z_2$ SET phase can be captured in the Chern-Simons approach.

Starting from a 4×4 matrix $\mathbf{K}_0 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to describe Z_2 spin liquid, for clarity we first perform a $GL(4,\mathbb{Z})$ gauge transformation (24) on $\mathbf{K}_0$:

$$\mathbf{K} = \mathbf{X}^T\mathbf{K}_0\mathbf{X} = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$

$$\mathbf{X} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad \det \mathbf{X} = 1. \tag{51}$$

We study Z_2 spin liquid with $Z_2 \times Z_2$ spin rotational symmetry in the above representation $\mathbf{K}$. Notice that

$$\mathbf{K}^{-1} = \frac{1}{2}\mathbf{K} = \begin{pmatrix} 0 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ -1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix}.$$

Apparently, the first two components ($\phi_{1,2}$ in the edge chiral boson context) can be regarded as spinons, which obey semionic mutual statistics with the last two components ($\phi_{3,4}$), i.e., the visons. Two spinons (visons) are mutually local w.r.t. each other as indicated by (3).

In a $Z_2 \times Z_2$ symmetry group (48), the group compatibility conditions for symmetry transformations $\{\mathbf{W}^{g_{1,2}},\delta\vec{\phi}^{g_{1,2}}\}$ in (15) are

$$(\mathbf{W}^{g_{1,2}})^2 = (\mathbf{W}^{g_1}\mathbf{W}^{g_2})^2 = 1_{4\times 4},$$
$$(1_{4\times 4} + \mathbf{W}^{g_{1,2}})\delta\vec{\phi}^{g_{1,2}} = 2\pi\mathbf{K}^{-1}\mathbf{n}_{1,2},$$
$$(1_{4\times 4} + \mathbf{W}^{g_1}\mathbf{W}^{g_2})(\delta\vec{\phi}^{g_2} + \mathbf{W}^{g_2}\delta\vec{\phi}^{g_1}) = 2\pi\mathbf{K}^{-1}\mathbf{n},$$

where $\mathbf{n}_1,\mathbf{n}_2,\mathbf{n} \in \mathbb{Z}^4$. Among all inequivalent solutions to these group compatibility conditions (48), the following one

$$\mathbf{W}^{g_1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \delta\vec{\phi}^{g_1} = \begin{pmatrix} \pi/2 \\ \pi/2 \\ 0 \\ 0 \end{pmatrix},$$
$$\mathbf{W}^{g_2} = 1_{4\times 4}, \quad \delta\vec{\phi}^{g_2} = (\pi/2, -\pi/2,0,0)^T,$$
$$\mathbf{n}_1 = \mathbf{n} = (0,0,0,1)^T, \ \mathbf{n}_2 = (0,0,-1,0)^T$$

corresponds to such an integer-spin Z_2 spin liquid where spinons transform projectively under the $Z_2 \times Z_2$ spin rotations. To be specific, the quasiparticles transform as

$$\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} \xrightarrow{g_1} \begin{pmatrix} \phi_2 + \pi/2 \\ \phi_1 + \pi/2 \\ -\phi_3 \\ \phi_4 \end{pmatrix},$$
$$\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} \xrightarrow{g_2} \begin{pmatrix} \phi_1 + \pi/2 \\ \phi_2 - \pi/2 \\ \phi_3 \\ \phi_4 \end{pmatrix}. \tag{52}$$

Indeed, each spinon ($\phi_{1,2}$) acquires -1 phase after every 2π-spin rotation, while visons ($\phi_{3,4}$) transform trivially.

Notice that in such a SET phase there are no symmetry-protected gapless edge states, i.e., generically all edge states are gapped in the presence of $Z_2 \times Z_2$ symmetry. Specifically, the following backscattering terms can be added to the edge actions (5) and (6) without breaking symmetry

$$\mathcal{L}_{\text{Higgs}} = C_3\cos(2\phi_3) + C_4\cos(2\phi_4).$$

All the chiral boson modes $\{\phi_i|i = 1,2,3,4\}$ on the edge will be gapped out by this term.

F. Condition for symmetry-protected edge states in SET phases

In this section, we briefly comment on the symmetry-protected edge states in all SET phases discussed above. First of all for a chiral topological order, such as Laughlin state [5] at filling fraction $\nu = 1/m$, its edge excitations have net chirality $|n_+ - n_-| \neq 0$ and hence cannot be destroyed even in the absence of any symmetry. These chiral topological orders are featured by quantized thermal Hall transport [57].

On the other hand, the edge excitations of a nonchiral topological order have both right and left movers and can be fully gapped out in the absence of any symmetry [86], such as in Z_2 spin liquids $\mathbf{K} \simeq \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ and double-semion theory $\mathbf{K} \simeq \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$. In the presence of global symmetry G_s, they might have symmetry-protected gapless edge modes, if the edge backscattering terms are forbidden by symmetry. To be specific, the backscattering terms are typically Higgs terms (8). The edge states will be fully gapped, if and only if each chiral boson field ϕ_i is either pinned at a classical minimal by the Higgs terms or does not commute with at least one Higgs term [30].

Take Z_2 spin liquids with $\mathbf{K} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, for example, either $A_1 \cos(2\phi_1 + \alpha_1)$ or $A_2 \cos(2\phi_2 + \alpha_2)$ could gap out chiral boson fields $\phi_{1,2}$ since $[\phi_1(x),\phi_2(y)] \neq 0$. Similarly, either $A_3 \cos(\phi_3 + \alpha_3)$ or $A_4 \cos(\phi_4 + \alpha_4)$ could gap out chiral bosons $\phi_{3,4}$. When these terms are not allowed by symmetry, there could be gapless excitations on the edge protected by symmetry. The symmetry-protected edge modes in conventional Ising symmetry-enriched Z_2 spin liquids are summarized in Table II. Among the four different conventional SET phases, No. 2 and No. 4 support symmetry-protected gapless edge modes.

SET phase No. 4 provides an interesting example. Here, both electric and magnetic particles transform projectively under the global Z_2 symmetry. Hence, the edge perturbations $A_1 \cos(2\phi_1 + \alpha_1)$ and $A_2 \cos(2\phi_2 + \alpha_2)$ which attempt to condense them are both disallowed, leading to a symmetry-protected edge state. Although symmetry does allow one backscattering term, such as $\cos(2\phi_1 + \phi_3 + \alpha_{13})$ in SET phase No. 4, the rest part (generated by $\phi_2 - \phi_4$ and ϕ_1) still remains gapless and is responsible for central charge $c = 1$.

For unconventional Ising symmetry-enriched Z_2 spin liquids summarized in Table III, there are always Ising symmetry-protected Majorana edge modes with central charge $c = \frac{1}{2}$ mod 1.

For double-semion theory with $\mathbf{K} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, on the other hand, either $A_+ \cos(2\phi_1 + 2\phi_2 + \alpha_+)$ or $A_- \cos(2\phi_1 - 2\phi_2 + \alpha_-)$ can fully gap out chiral bosons $\phi_{1,2}$. Meanwhile again either $A_3 \cos(\phi_3 + \alpha_3)$ or $A_4 \cos(\phi_4 + \alpha_4)$ could gap out chiral bosons $\phi_{3,4}$. When symmetry forbids these terms on the edge, there will be gapless edge excitations. The results are summarized in Table VI for Ising symmetry-enriched double-semion theories. Among the six different SET phases, only No. 3, No. 4, and No. 6 host symmetry-protected gapless edge states. They all have central charge $c = 1$.

An immediate observation from results in Tables II–VI is summarized as follows:

Conjecture II. If every $\boldsymbol{g}$ symmetry flux $\{q_{\boldsymbol{g}}\}$ in a nonchiral SET phase carries nontrivial topological spin ($h_{q_{\boldsymbol{g}}} \neq 0$ mod 1) after gauging the symmetry, this SET phase must support $\boldsymbol{g}$ symmetry-protected gapless edge states.

There is a natural physical picture behind this conclusion. Symmetric edge states can always be obtained when we start from a symmetry-breaking edge. By proliferating or condensing the defects of the broken symmetry, such as domain walls for unitary Z_2 symmetry $\boldsymbol{g}$ here, one can restore symmetry on the edge [63] and obtain a gapped symmetric edge. The symmetry defects or domain walls on the edge are nothing but $\boldsymbol{g}$ symmetry flux discussed earlier, as illustrated by Fig. 4. If all $\boldsymbol{g}$ symmetry fluxes carry nontrivial topological spin, i.e., none of the symmetry defects on the edge obey bosonic statistics [63], we cannot condense them to restore the symmetry. Hence, a gapped symmetric edge is not possible in this situation, and there must be $\boldsymbol{g}$ symmetry-protected gapless edge states.

IV. CONCLUDING REMARKS

In summary, we have presented a general framework to study (2+1)D symmetry-enriched topological phases with Abelian topological order, using the Chern-Simons approach. It allows us to implement generic onsite unitary (or antiunitary) symmetry in an Abelian topologically ordered phase in (2+1)D to differentiate whether two states belong to the same SET phase or not, and to gauge a unitary (discrete) Abelian symmetry and extract the resultant topological order. Symmetry-protected edge states are also easily captured in this framework. Based on this general formulation, we classify all different SET phases in a series of examples, including Z_2 spin liquids with time-reversal symmetry (Table I), Z_2 spin liquids with unitary Ising symmetry (Tables II and III), double-semion theory with unitary Ising symmetry (Table VI), bosonic Laughlin states with unitary Ising symmetry (Tables IV and V), and others. We also show that (odd-denominator) fermionic Laughlin states with only conserved fermion number parity (Z_2^f symmetry) are *unique*. Consequences of gauging symmetries and measurable effects, such as gapless edge states, are also discussed for these SET phases.

A number of directions remain. Can the approach applied be extended to spatial symmetries? Can we extend this framework to symmetry-enriched non-Abelian topological orders in (2+1)D and SET phases in (3+1)D? While SPT phases form an Abelian group, it is presently unclear if the set of SET phases have additional structure. We leave these questions to future work.

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APPENDIX A: INTRODUCTION TO $GL(N,\mathbb{Z})$

$GL(N,\mathbb{Z})$ is the group of all $N\times N$ unimodular matrices. All $GL(N,\mathbb{Z})$ matrices can be generated by the following basic transformations ($i\neq j$):

$$T^{(i,j)}_{a,b}=\delta_{a,b}+\delta_{a,i}\delta_{b,j},$$
$$S^{(i,j)}_{a,b}=\delta_{a,b}(1-\delta_{a,i})(1-\delta_{a,j})+\delta_{a,j}\delta_{b,i}-\delta_{a,i}\delta_{b,j},$$
$$D_{a,b}=\delta_{a,b}-2\delta_{a,N}\delta_{b,N}. \tag{A1}$$

$T^{(i,j)}\mathbf{K}$ will add the jth row of matrix $\mathbf{K}$ to the ith row of $\mathbf{K}$, while $S^{(i,j)}\mathbf{K}$ will exchange the ith and jth rows of $\mathbf{K}$ with a factor of -1 multiplied on the ith row. $D\mathbf{K}$ will just multiply the Nth row of $\mathbf{K}$ by a factor of -1. $\mathbf{K}T^{(i,j)}$, $\mathbf{K}S^{(i,j)}$, and $\mathbf{K}D$ correspond to similar operations to columns (instead of rows). A subgroup of $GL(N,\mathbb{Z})$ with determinant $+1$ is called $SL(N,\mathbb{Z})$ and it is generated by $\{T^{(i,j)},S^{(i,j)}\}$.

As a simple example when $N=2$, group $GL(2,\mathbb{Z})$ is generated by the following basic transformations:

$$S=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\ T=\begin{pmatrix}1&1\\0&1\end{pmatrix},\ D=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{A2}$$

The following results will be useful:

$$T^n=\begin{pmatrix}1&n\\0&1\end{pmatrix},\ (-STS)^n=\begin{pmatrix}1&0\\-n&1\end{pmatrix},\ n\in\mathbb{Z}.$$

APPENDIX B: CHERN-SIMONS APPROACH TO GAUGE A UNITARY SYMMETRY

In this appendix, we discuss how to obtain the (intrinsic) topological order by gauging the unitary symmetry [29] in an Abelian SET phase. We will restrict ourselves to "conventional" SET phases, characterized by data $[\mathbf{K},\{\eta^{\boldsymbol{g}}=+1,\mathbf{W}^{\boldsymbol{g}}=1_{N\times N},\delta\vec{\phi}^{\boldsymbol{g}}|\boldsymbol{g}\in G_s\}]$. In these cases, the chiral bosons ϕ_I only acquire U(1) phase factors $\phi_I\to\phi_I+\delta\phi_I^{\boldsymbol{g}}$ after unitary symmetry operation $\boldsymbol{g}\in G_s$. When we couple the quasiparticles to a gauge field (with gauge group G_s), the following gauge flux

$$\epsilon^{0\mu\nu}\partial_\mu a^I_\nu(\mathbf{r},t)=\delta\phi^{\boldsymbol{g}}_I\delta(\mathbf{r}-\mathbf{r}^{(0)})$$

becomes deconfined excitations in the system. Since in a Chern-Simons theory (1) gauge charges are always accompanied by gauge fluxes by the following equation of motion:

$$j^\mu_I=\frac{\epsilon^{\mu\nu\lambda}}{2\pi}\sum_J\mathbf{K}_{I,J}\partial_\nu a^J_\lambda.$$

Clearly, this quasiparticle carries gauge charge vector $\mathbf{K}\delta\vec{\phi}^{\boldsymbol{g}}/(2\pi)$. We coin such new excitations emerged after gauging unitary symmetry $\boldsymbol{g}$ as "$\boldsymbol{g}$ *symmetry fluxes*" and denote them as $\{q_{\boldsymbol{g}}\}$. A generic $\boldsymbol{g}$ symmetry flux corresponds to gauge charge vector $\frac{\mathbf{K}\delta\vec{\phi}^{\boldsymbol{g}}}{2\pi}+\mathbf{l}$ where $\mathbf{l}\in\mathbb{Z}^N$ since it can always combine with any gapped excitation ($\forall\,\mathbf{l}\in\mathbb{Z}^N$) in the original (ungauged) SET phase.

The new topological order $\mathbf{K}_g$ obtained by gauging symmetry must include these new excitations in its quasiparticle content. More precisely, the quasiparticle content of topological order $\mathbf{K}_g$ is expanded by all the integer vectors as well as multiples of vectors $\{\mathbf{K}\delta\vec{\phi}^{\boldsymbol{g}}/(2\pi)\}$ for $q_{\boldsymbol{g}}$ particles

$$\mathbf{l}'=\mathbf{l}+\sum_{\boldsymbol{g}}n_{\boldsymbol{g}}\frac{\mathbf{K}\delta\vec{\phi}^{\boldsymbol{g}}}{2\pi},\ \ \mathbf{l}\in\mathbb{Z}^N,\ n_{\boldsymbol{g}}\in\mathbb{Z} \tag{B1}$$

where $\{\boldsymbol{g}\}$ denote the generators of symmetry group G_s that are gauged. Also, we can identify the new matrix $\mathbf{K}_g$ which contains all these quasiparticles in its spectrum.

The above procedures work for all discrete Abelian symmetries. Gauging continuous Abelian symmetries [i.e., direct product of U(1) and discrete Z_n groups] symmetry can also be done conveniently in the Chern-Simons approach. To be specific, to gauge each U(1) subgroup we coupled the physical degrees of freedom to a dynamical U(1) gauge field, which increases the dimension of $\mathbf{K}$ matrix by 1. However, we will restrict ourselves to discrete Abelian symmetries for all examples in this paper.

In the following, we work on one example to demonstrate this gauging procedure. We consider all four SET phases in Table II, with $\mathbf{K}=\left(\begin{smallmatrix}0&2\\2&0\end{smallmatrix}\right)\oplus\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and symmetry transformation $\mathbf{W}^{\boldsymbol{g}}=1_{4\times4}$, $\delta\vec{\phi}^{\boldsymbol{g}}=\pi(i_1/2,i_2/2,1,i_4)^T$ with $i_{1,2,4}=0,1$. According to (B1) we know here a generic quasiparticle is labeled by gauge charge vector

$$\mathbf{l}'=\mathbf{M}\mathbf{l};\quad \mathbf{M}=\begin{pmatrix}i_2/2&1&0&0\\i_1/2&0&1&0\\i_4/2&0&0&1\\1/2&0&0&0\end{pmatrix},\ \mathbf{l}\in\mathbb{Z}^4.$$

Since this new Abelian topological order is determined by the statistics of its quasiparticles, we immediately obtain

$$(\mathbf{l}'_1)^T\mathbf{K}^{-1}\mathbf{l}'_2=\mathbf{l}_1^T\mathbf{K}_g^{-1}\mathbf{l}_2,\ \ \mathbf{l}'_\alpha=\mathbf{M}\mathbf{l}_\alpha$$

and therefore

$$\mathbf{K}_g^{-1}=\mathbf{M}^T\mathbf{K}^{-1}\mathbf{M}=\begin{pmatrix}\frac{i_1i_2+2i_4}{4}&i_1/4&i_2/4&1/2\\i_1/4&0&1/2&0\\i_2/4&1/2&0&0\\1/2&0&0&0\end{pmatrix}$$
$$\Rightarrow\mathbf{K}_g=\begin{pmatrix}0&0&0&2\\0&0&2&-i_2\\0&2&0&-i_1\\2&-i_2&-i_1&-2i_4\end{pmatrix}. \tag{B2}$$

Clearly from (2) and (3) we know the statistical angle of new quasiparticle $q_{\boldsymbol{g}}\equiv\frac{\mathbf{K}\delta\vec{\phi}^{\boldsymbol{g}}}{2\pi}+\mathbf{l}$ is

$$\theta_{q_g}=\pi q_{\boldsymbol{g}}^T\mathbf{K}^{-1}q_{\boldsymbol{g}}=\pi\left(\frac{\delta\vec{\phi}^{\boldsymbol{g}}}{2\pi}\right)^T\mathbf{K}\frac{\delta\vec{\phi}^{\boldsymbol{g}}}{2\pi}+\pi\mathbf{l}^T\mathbf{K}^{-1}\mathbf{l}+\mathbf{l}^T\delta\vec{\phi}^{\boldsymbol{g}}$$
$$=\pi\left(\frac{i_1i_2+2i_4}{4}+l_1l_2+\frac{i_1l_1+i_2l_2}{2}+l_3+i_4l_4\right) \tag{B3}$$

as summarized in Table II. In addition to their (self-)statistics, another important character of these "$\boldsymbol{g}$ symmetry fluxes" $\{q_{\boldsymbol{g}}\}$ is their mutual statistics with the original quasiparticles in the (ungauged) SET phase. Here, e.g., a generic electric charge is represented by gauge charge vector $e\equiv(1,2e_2,e_3,e_4)^T$ with $e_i\in\mathbb{Z}$, and its mutual statistics with $\boldsymbol{g}$ symmetry flux

$q_g \equiv \frac{\mathbf{K}\delta\vec{\phi}^g}{2\pi} + \mathbf{l}$ is

$$\tilde{\theta}_{q_g,e} = \left(\frac{i_1}{2} + l_2 + e_2 i_2 + e_3 + e_4 i_4\right)\pi. \tag{B4}$$

Meanwhile, a generic magnetic vortex $m = (2m_1,1,m_3,m_4)^T$ has mutual statistics

$$\tilde{\theta}_{q_g,m} = \left(\frac{i_2}{2} + l_1 + m_1 i_1 + m_3 + m_4 i_4\right)\pi \tag{B5}$$

with $\boldsymbol{g}$ symmetry flux q_g. Topological spin [44] $\exp(2\pi i h_q)$, the Berry phase obtained by adiabatically rotating a quasiparticle q by 2π, is an important character of a (2+1)D topological order. In Abelian topological orders, the topological spin $\exp(2\pi i h_{q_g})$ has a one-to-one correspondence to the self-statistics (B3) of a quasiparticle in unit of 2π:

$$h_{q_g} = \frac{\theta_{q_g}}{2\pi} = \frac{i_1 i_2 + 2i_4}{8} + \frac{i_1 l_1 + i_2 l_2}{4} + \frac{l_1 l_2 + l_3 + i_4 l_4}{2}. \tag{B6}$$

All these statistical properties are summarized in Table II.

For the unconventional SET phases, e.g., in our case with $G_s = Z_2$, the Z_2 symmetry would exchange quasiparticles that belong to different superselection sectors in Abelian topological order $\mathbf{K}$. Gauging this kind of Z_2 symmetry will in general lead to $U(1)^N \rtimes Z_2$ Chern-Simons theory [66], which describes non-Abelian topological orders in relation to Z_2 orbifold conformal field theory [87,88].

APPENDIX C: CLASSIFYING DOUBLE-SEMION THEORY WITH ONSITE Z_2 SYMMETRY

Double-semion theory [37,38] is a "twisted" Z_2 gauge theory in (2+1)D, with Abelian topological order described by $\mathbf{K} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$. Due to the presence of nontrivial bosonic Z_2 SPT phase in (2+1)D, again here we use a 4×4 matrix

$$\mathbf{K} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{C1}$$

to capture all the different Z_2 symmetry-enriched double-semion theory. Such a theory has the following quasiparticle contents in its spectra:

$$1 \simeq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \simeq \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \simeq \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} \simeq \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \simeq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$s \simeq \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \bar{s} \simeq \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, b \simeq \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \tag{C2}$$

where s and $\bar{s}$ represent semion and antisemion, respectively, and b is the bound state of a semion and an antisemion. b has bosonic (self-)statistics (2) but mutual semion (antisemion) statistics with s ($\bar{s}$). Here, $\{1,s,\bar{s},b\}$ represent the four superselection sectors of double-semion theory. Any two quasiparticles differing by a local excitation $\simeq 0$ belong to the same superselection sector.

Now, let us consider the implementation of unitary $G_s = Z_2$ symmetry on double-semion theory. We have group compatibility condition (18) for symmetry transformation (15) on quasiparticles:

$$(\mathbf{W}^g)^2 = 1_{4\times 4},$$
$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = (\mathbf{W}^g)^T \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \mathbf{W}^g,$$
$$(1_{4\times 4} + \mathbf{W}^g)\delta\vec{\phi}^g = \pi \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix} \mathbf{n}, \ \ \mathbf{n} \in \mathbb{Z}^4. \tag{C3}$$

We consider [89] the solutions to (C3) with $\mathbf{W}^g = 1_{4\times 4}$. Due to Criterion I, naively there are eight distinct solutions to (C3): $\delta\vec{\phi}^g = \pi(i_1/2, i_2/2, 1, i_4)^T$ with $i_{1,2,4} = 0,1$. However a careful analysis reveals the following gauge equivalence between certain solutions:

$$\delta\vec{\phi}^g_{(1)} = \begin{pmatrix} \pi/2 \\ 0 \\ \pi \\ 0 \end{pmatrix} \simeq \begin{pmatrix} \pi/2 \\ 0 \\ \pi \\ \pi \end{pmatrix} \simeq \begin{pmatrix} -\pi/2 \\ \pi \\ \pi \\ \pi \end{pmatrix} = \mathbf{X}^{-1}\delta\vec{\phi}^g_{(1)},$$

$$\delta\vec{\phi}^g_{(2)} = \begin{pmatrix} 0 \\ \pi/2 \\ \pi \\ 0 \end{pmatrix} \simeq \begin{pmatrix} 0 \\ \pi/2 \\ \pi \\ \pi \end{pmatrix} \simeq \begin{pmatrix} \pi \\ -\pi/2 \\ \pi \\ \pi \end{pmatrix} = \mathbf{X}^{-1}\delta\vec{\phi}^g_{(2)},$$

$$\mathbf{X} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & -2 & 0 & 1 \end{pmatrix}, \ \ \mathbf{X}^T\mathbf{K}\mathbf{X} = \mathbf{K}.$$

As a result, there are only six gauge-inequivalent solutions of $\delta\vec{\phi}^g$ to (C3), as summarized in Table VI.

Following Appendix B, we briefly discuss the consequence of gauging the unitary Z_2 symmetry in the double-semion theory. Since the symmetry transformation is a U(1) phase shift $\delta\vec{\phi}^g = \pi(i_1/2, i_2/2, 1, i_4)^T$ as shown in Table VI, the quasiparticle content in the new topological order obtained by gauging Z_2 symmetry is expanded by gauge charge vector:

$$\mathbf{l}' = \mathbf{M}\mathbf{l}; \quad \mathbf{M} = \begin{pmatrix} i_1/2 & 1 & 0 & 0 \\ -i_2/2 & 0 & 1 & 0 \\ i_4/2 & 0 & 0 & 1 \\ 1/2 & 0 & 0 & 0 \end{pmatrix}, \mathbf{l} \in \mathbb{Z}^4$$

and therefore

$$\mathbf{K}_g^{-1} = \mathbf{M}^T\mathbf{K}^{-1}\mathbf{M} = \begin{pmatrix} \frac{i_1^2 - i_2^2 + 4i_4}{8} & i_1/4 & i_2/4 & 1/2 \\ i_1/4 & 1/2 & 0 & 0 \\ i_2/4 & 0 & -1/2 & 0 \\ 1/2 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \mathbf{K}_g = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & -i_1 \\ 0 & 0 & -2 & i_2 \\ 2 & -i_1 & i_2 & -2i_4 \end{pmatrix}. \quad \text{(C4)}$$

Take No. 5 as an example with $\delta\vec{\phi}^{g} = \pi(1/2,1/2,1,0)^T$ (or $i_1 = i_2 = 1$, $i_4 = 0$), the Abelian topological order obtained by gauging Z_2 symmetry is

$$\mathbf{K}_g \simeq \mathbf{X}^T\mathbf{K}_g\mathbf{X} = \begin{pmatrix} 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix},$$

$$\mathbf{X} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -2 & 2 & 1 & 1 \end{pmatrix} \in GL(4,\mathbb{Z}).$$

Notice that

$$\begin{pmatrix} 8 & 0 \\ 0 & -2 \end{pmatrix} \simeq \begin{pmatrix} 0 & 4 \\ 4 & 6 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 0 & -8 \end{pmatrix} \simeq \begin{pmatrix} 0 & 4 \\ 4 & 2 \end{pmatrix}.$$

Again, from (2) and (3) we can obtain the (self-)statistics of $\boldsymbol{g}$ symmetry flux $q_{\boldsymbol{g}} = \frac{\mathbf{K}\delta\vec{\phi}^{g}}{2\pi} + \mathbf{l}$, $\mathbf{l} \in \mathbb{Z}^N$, as

$$\theta_{q_g} = \pi q_{\boldsymbol{g}}^T \mathbf{K}^{-1} q_{\boldsymbol{g}} = \pi \left(\frac{\delta\vec{\phi}^{g}}{2\pi}\right)^T \mathbf{K}\frac{\delta\vec{\phi}^{g}}{2\pi} + \pi \mathbf{l}^T\mathbf{K}^{-1}\mathbf{l} + \mathbf{l}^T\delta\vec{\phi}^{g}$$
$$= \pi\left(\frac{i_1^2 - i_2^2}{8} + \frac{l_1^2 - l_2^2 + i_4 + i_1 l_1 + i_2 l_2}{2} + i_4 l_4 + l_3\right). \quad \text{(C5)}$$

Its mutual statistics with original quasiparticles $s \equiv (1,2s_2,s_3,s_4)$ and $\bar{s} \equiv (2\bar{s}_1,1,\bar{s}_3,\bar{s}_4)$ are

$$\tilde{\theta}_{q_g,s} = \pi\left(\tfrac{i_1}{2} + l_1 + i_2 s_2 + s_3 + i_4 s_4\right),$$
$$\tilde{\theta}_{q_g,\bar{s}} = \pi\left(\tfrac{i_2}{2} + i_1\bar{s}_1 - l_2 + \bar{s}_3 + i_4\bar{s}_4\right). \quad \text{(C6)}$$

The topological spin of new quasiparticle $q_{\boldsymbol{g}}$ is given by $\Theta_{q_g} \equiv \exp(2\pi i h_{q_g})$ where

$$h_{q_g} = \frac{\theta_{q_g}}{2\pi} = \frac{i_1^2 - i_2^2}{16} + \frac{l_1^2 - l_2^2 + i_4 + i_1 l_1 + i_2 l_2}{4} + \frac{i_4 l_4 + l_3}{2}. \quad \text{(C7)}$$

Unlike others, for SET phases No. 3, No. 4, and No. 6, it is not easy to find a $GL(4,\mathbb{Z})$ transformation (11) on $\mathbf{K}_g$ matrix (C4) to reduce it to a simpler form. For example, one can only show for SET phase No. 3

$$\mathbf{K}_g \simeq \mathbf{X}^T\mathbf{K}_g\mathbf{X} = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix},$$

$$\mathbf{X} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in GL(4,\mathbb{Z})$$

and for SET phase No. 4

$$\mathbf{K}_g \simeq \mathbf{X}^T\mathbf{K}_g\mathbf{X} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix},$$

$$\mathbf{X} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in GL(4,\mathbb{Z}).$$

However, a one-to-one correspondence between the quasiparticle contents of two seemingly different $\mathbf{K}_g$ matrices can be established. For example, there are 16 different superselection sectors (or 16 quasiparticle types) for Abelian topological order $\mathbf{K}_g$ in (C4), obtained by gauging Z_2 symmetry in No. 3 SET phase ($i_1 = 1 = i_4$, $i_2 = 0$):

$$\begin{pmatrix} \gamma_1 + 4\gamma_2 \\ 0 \\ \gamma_1 + \gamma_2 \\ 0 \end{pmatrix} \text{in (C4)} \Leftrightarrow \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \text{in} \begin{pmatrix} 8 & 0 \\ 0 & -2 \end{pmatrix},$$
$$\gamma_1 = 0,1,\ldots,7, \ \ \gamma_2 = 0,1.$$

For No. 4 SET phase ($i_2 = 1 = i_4$, $i_1 = 0$):

$$\begin{pmatrix} 4\gamma_1 + 3\gamma_2 \\ \gamma_1 + \gamma_2 \\ 0 \\ 0 \end{pmatrix} \text{in (C4)} \Leftrightarrow \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \text{in} \begin{pmatrix} 2 & 0 \\ 0 & -8 \end{pmatrix},$$
$$\gamma_2 = 0,1,\ldots,7, \ \ \gamma_1 = 0,1.$$

The Abelian topological order obtained by gauging Z_2 symmetry in No. 6 SET phase ($i_2 = i_4 = i_1 = 1$) is characterized by the following matrix:

$$\mathbf{K}_g \simeq \mathbf{X}^T\mathbf{K}_g\mathbf{X} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \quad \text{(C8)}$$

$$\mathbf{X} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \det \mathbf{X} = 1.$$

It has 16 different types of quasiparticles:

$$\vec{\gamma} \equiv \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ 0 \\ 0 \end{pmatrix} \text{in (C4)}, \quad \gamma_{1,2} = 0,1,2,3.$$

Among them, four have bosonic self-statistics ($\theta = 0 \mod 2\pi$), six with semionic statistics ($\theta = \frac{\pi}{2} \mod 2\pi$), and the other six with antisemionic statistics ($\theta = -\frac{\pi}{2} \mod 2\pi$). In the above basis, the 16×16 modular $\mathcal{S}$ matrix [43,44] of this Abelian topological order is given by

$$\mathcal{S}_{\vec{\gamma},\vec{\gamma}'} = \frac{1}{4}\exp\left[\frac{\pi i}{2}\left(2\sum_{a=1}^{2}\gamma_a\gamma_a' + \gamma_1\gamma_2' + \gamma_2\gamma_1'\right)\right]. \quad \text{(C9)}$$

APPENDIX D: DISCUSSIONS ON Z_2 SYMMETRY-ENRICHED Z_2 GAUGE THEORIES

First, we discuss the results for conventional SET phases with onsite Z_2 symmetry, and their relation to Dijkgraaf-Witten gauge theories [42] in (2+1)D. For Z_2 spin liquids $\mathbf{K} \simeq \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$, with onsite Z_2 symmetry we obtain four different conventional Z_2 SET phases as summarized in Table II. For double-semion theory $\mathbf{K} \simeq \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$, with onsite Z_2 symmetry we obtain six different conventional Z_2 SET phases as summarized in Table VI. Here, we make some connection between these SET phases and Z_2 symmetry-enriched Z_2 gauge theories obtained in previous studies [19,21].

In Ref. [19] the exact soluble lattice models for eight different $G_s = Z_2$ symmetry-enriched $G_g = Z_2$ gauge theories are obtained. They correspond to group cohomology $\mathcal{H}^3[G_s \times G_g,\text{U}(1)] = \mathcal{H}^3[Z_2 \times Z_2,\text{U}(1)] = \mathbb{Z}_2^3$. Among these eight different SET phases, four come from Z_2 spin liquids with onsite $G_s = Z_2$ symmetry, and the other four from double-semion theory with onsite $G_s = Z_2$ symmetry. They are nothing but No. 1, No. 2, and No. 3 in Table II, together with No. 1, No. 2, No. 5, and No. 6 in Table VI. In fact, two different models constructed in Ref. [19], labeled by (010) and (110) in Table II, belong to the same SET phase (No. 3 in Table II).

On the other hand, Ref. [21] claimed 12 different Z_2 symmetry-enriched Z_2 topological orders, among which 6 are Z_2 spin liquids and the others are double-semion theories. It was conjectured that different G_s symmetry-enriched G_g gauge theory (with gauge group G_g) are classified by group cohomology $\mathcal{H}^{d+1}[G,\text{U}(1)]$ or Dijkgraaf-Witten G gauge theory [42] in d-spatial dimensions, where G is an extension of symmetry group G_s by gauge group G_g (in other words $G/G_s = G_g$). When $G_s = G_g = Z_2$ we have $G = Z_2 \times Z_2$ or $G = Z_4$. The number $12 = 2^3 + 4$ is associated with $\mathcal{H}^3[Z_2 \times Z_2,\text{U}(1)] \oplus \mathcal{H}^3[Z_4,\text{U}(1)] = \mathbb{Z}_2^3 \oplus \mathbb{Z}_4$. The proposed eight different SET phases from $\mathcal{H}^3[Z_2 \times Z_2,\text{U}(1)]$ are the same as those in Ref. [19], which are discussed earlier. After gauging the $G_s = Z_2$ symmetry, these eight different SET phases lead to Abelian topological orders described by a 4×4 matrix [21]

$$\mathbf{K}(n_1n_2n_3) = \begin{pmatrix} 2n_1 & 2 & n_2 & 0 \\ 2 & 0 & 0 & 0 \\ n_2 & 0 & 2n_3 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \tag{D1}$$

where $n_1,n_2,n_3 = 0,1$. It is not difficult to check that these eight different SET phases labeled by $(n_1n_2n_3)$ have the following correspondence with our results:

$$\mathbf{K}(000) = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix} \Leftrightarrow \text{No. 1 in Table II},$$

$$\mathbf{K}(100) \simeq \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \Leftrightarrow \text{No. 2 in Table II},$$

$$\mathbf{K}(010) \simeq \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix} \Leftrightarrow \text{No. 3 in Table II},$$

$$\mathbf{K}(110) \simeq \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix} \Leftrightarrow \text{No. 3 in Table II}$$

and

$$\mathbf{K}(001) \simeq \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix} \Leftrightarrow \text{No. 1 in Table VI},$$

$$\mathbf{K}(101) \simeq \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \Leftrightarrow \text{No. 2 in Table VI},$$

$$\mathbf{K}(011) \simeq \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix} \Leftrightarrow \text{No. 5 in Table VI},$$

$$\mathbf{K}(111) \simeq \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix} \Leftrightarrow \text{No. 6 in Table VI}.$$

The other four SET phases proposed in Ref. [21] are associated to group cohomology $\mathcal{H}^3[Z_4,\text{U}(1)] = \mathbb{Z}_4$. Reference [21] asserted that after gauging the Z_2 symmetry they lead to Abelian Z_4 topological orders described by

$$\mathbf{K}(m_1) = \begin{pmatrix} 2m_1 & 4 \\ 4 & 0 \end{pmatrix}, \quad m = 0,1,2,3.$$

We found that these four different SET phases have overlap with the previous seven SET phases associated to $\mathcal{H}^3[Z_2 \times Z_2,\text{U}(1)] = \mathbb{Z}_2^3$: they turn out to be

$$\mathbf{K}(m_1 = 0) = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix} \Leftrightarrow \text{No. 3 in Table II},$$

$$\mathbf{K}(m_1 = 2) \simeq \begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix} \Leftrightarrow \text{No. 4 in Table II}.$$

and

$$\mathbf{K}(m_1 = 3) \simeq \begin{pmatrix} 8 & 0 \\ 0 & -2 \end{pmatrix} \Leftrightarrow \text{No. 3 in Table VI},$$

$$\mathbf{K}(m_1 = 1) \simeq \begin{pmatrix} 2 & 0 \\ 0 & -8 \end{pmatrix} \Leftrightarrow \text{No. 4 in Table VI}.$$

We want to emphasize that when the onsite unitary Z_2 symmetry is gauged, a Z_2 spin liquid and another double-semion theory could result in the same "gauged" topological order. For Z_2 spin liquid No. 2 in Table II and double-semion theory No. 1 in Table VI, it is straightforward to see $\mathbf{K}(100) \simeq \mathbf{K}(001)$. Similarly, for Z_2 spin liquid No. 3 in Table II and double-semion theory No. 5 in Table VI, with $\mathbf{X}_{1,2} \in GL(4,\mathbb{Z})$ we have

$$\mathbf{X}_1^T \cdot \mathbf{K}(010) \cdot \mathbf{X}_1 = \mathbf{X}_2^T \cdot \mathbf{K}(110) \cdot \mathbf{X}_2 = \begin{pmatrix} 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\mathbf{X}_1 = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{X}_2 = \begin{pmatrix} 2 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ -2 & -2 & -1 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

and one can further show $\mathbf{K}(110) \simeq \mathbf{K}(011)$. In fact, the eight different $\mathbf{K}$ matrices $\mathbf{K}(n_1n_2n_3)$ describe only five different Abelian topological orders. Since these eight theories correspond to different Dijkgraaf-Witten [42] theories, i.e., gauge theories with different topological terms specified by $\mathcal{H}^3[Z_2\times Z_2,\mathrm{U}(1)]=Z_2^3$, this also implies that *different Dijkgraaf-Witten theories based on a particular gauge group can share the same topological order, and correspond to the same SET phase*. Further information regarding which particles comprise electric charges and magnetic vortices is required to uniquely define those phases.

APPENDIX E: VERTEX ALGEBRA APPROACH TO GAUGE A UNITARY SYMMETRY

The Chern-Simons approach to gauge a unitary symmetry, introduced in Appendix B, applies to all cases where we obtain an Abelian topological order after gauging the symmetry. Thus, for many "conventional" SET phases we can gauge its unitary symmetry and obtain an Abelian topological order in the Chern-Simons approach. For "unconventional" SET phases (and certain conventional ones, e.g., in Sec. III E), such as those summarized in Table III, gauging a unitary (e.g., Z_2) symmetry will result in non-Abelian topological orders. In the case $G_s=Z_2$ as discussed in this work, these non-Abelian topological orders are described by $\mathrm{U}(1)^N\rtimes Z_2$ Chern-Simons theory [66]. In these unconventional cases, the Chern-Simons approach introduced previously is not enough. In order to obtain the full structure [such as topological spin $\exp(2\pi i h)$ of quasiparticles and modular $\mathcal{S}$ matrix associated with quasiparticle statistics] of these non-Abelian topological orders, here we introduce a vertex algebra approach to gauge the unitary symmetry. It applies to both the conventional and unconventional SET phases and in the following we will demonstrate its power by two examples: conventional and unconventional Z_2 symmetry-enriched Z_2 spin liquids.

1. The vertex algebra formalism, and application to conventional SET phases

The vertex algebra approach [45,90,91] is based on the close connection [52,88] between the bulk topological order [described by (2+1)D topological field theory] and its boundary excitations [described by (1+1)D conformal field theory] in two spatial dimensions. Let us take Z_2 spin liquid (25) for an example. The edge effective theory (5) contains two branches of chiral bosons $\{\phi_{1,2}\}$, which could be reformulated by a $c=1$ $\mathrm{U}(1)\times\mathrm{U}(1)$ Gaussian model with a holomorphic and antiholomorphic part:

$$\varphi(x+i\tau)\equiv\varphi(z)=\phi_1(x,t)+\phi_2(x,t),\quad \bar\varphi(x-i\tau)\equiv\bar\varphi(\bar z)=\phi_1(x,t)-\phi_2(x,t).$$

The Gaussian model has Lagrangian density $\mathcal{L}_{\text{Gaussian}}=\frac{1}{2\pi}\partial\varphi(z)\bar\partial\varphi(z)=\frac{1}{8\pi}|\vec\nabla\varphi|^2$, yielding the following correlation function

$$\langle\varphi(z)\varphi(w)\rangle=-\ln(z-w) \tag{E1}$$

and $\langle\bar\varphi(\bar z)\bar\varphi(\bar w)\rangle=-\ln(\bar z-\bar w)$. The free boson field φ has compactification radius $R=2$ for Z_2 spin liquid (25) so that periodicity $\varphi\sim\varphi+2\pi R$ holds. In general, for $\mathbf{K}=\begin{pmatrix}0&N\\N&0\end{pmatrix}$ in (5) the associated compactification radius of scalar boson

$$\sqrt{\frac{2}{N}}\varphi(z)=\phi_1(x,t)+\phi_2(x,t),\quad \sqrt{\frac{2}{N}}\bar\varphi(\bar z)=\phi_1(x,t)-\phi_2(x,t) \tag{E2}$$

is $R=\sqrt{2N}$. The allowed physical excitations must be compatible with $2\pi R$ periodicity of bosons and they are [92]

$$V_k(z)=e^{ik\varphi(z)/\sqrt{2N}},\ \ k=0,1,\dots,2N-1 \tag{E3}$$

for holomorphic part (and similarly $\bar V_k$ for antiholomorphic part). These $2N$ vertex operators are primary fields of the holomorphic U(1) conformal field theory (CFT) and they form different representations of the conformal algebra. From (E1) one can see they have the following (radial-ordered) operator product expansion [93,94] (OPE):

$$e^{i\alpha\varphi(z)}e^{i\beta\varphi(w)}=(z-w)^{\alpha\beta}e^{i(\alpha+\beta)\varphi(w)}+\dots \tag{E4}$$

for $\alpha+\beta\neq0$. There is an energy-momentum tensor $T=-\frac{1}{2}(\partial\varphi)^2$ which generates the conformal transformation of the vertex algebra, so that any primary field $P(z)$ has the following OPE with energy-momentum tensor:

$$T(z)P(w)=\frac{h_P}{(z-w)^2}P(w)+\frac{1}{z-w}\partial P(w)+\dots, \tag{E5}$$

where h_P is the scaling dimension of primary field P. Apparently, the vertex operator $\exp[i\alpha\varphi(z)]$ has scaling dimension $h_\alpha=\frac{1}{2}\alpha^2$. Another primary field is the current operator $j(z)\equiv i\partial\varphi(z)$ which has scaling dimension $h_j=1$. In addition, we have

$$e^{i\alpha\varphi(z)}e^{-i\alpha\varphi(w)}=\frac{1}{(z-w)^{\alpha^2}}+\frac{\alpha j(w)}{(z-w)^{\alpha^2-1}}+\dots.$$

These OPEs imply the following *fusion rules* of primary fields:

$$e^{i\alpha\varphi}\times e^{i\alpha\varphi}=1+j\ \ (\alpha\neq0),\quad e^{i\alpha\varphi}\times e^{i\beta\varphi}=e^{i(\alpha+\beta)\varphi}\ \ (\alpha\neq-\beta).$$

Similar results hold for antiholomorphic $\bar\varphi(\bar z)$ part, only that all scaling dimensions change sign for their antiholomorphic counterparts.

A natural question is among all these primary fields: Which ones appear in the physical edge spectra of the topologically ordered phase? There are a few physical principles to follow. First of all, every physical edge excitation ($e^{i\sum_i l_i\phi_i}$, $l_i\in\mathbb{Z}$) must be a primary field. Second, there are electron operators (or the microscopic local degrees of freedom $e^{i\sum_{i,j}l_i\mathbf{K}_{i,j}\phi_j}$, $l_i\in\mathbb{Z}$) which is local with respect to all other edge excitations. In the context of vertex algebra, two operators A and B are local w.r.t. each other if and only if in OPE

$$A(z)B(w)=\frac{f_{A,B}^C C(w)}{(z-w)^{\alpha_{A,B}}}+O((z-w)^{1-\alpha_{A,B}}),\quad \alpha_{A,B}\in\mathbb{Z} \tag{E6}$$

where C is also a primary field and $f_{A,B}^C$ is a structure constant. For example, before we gauge the Z_2 symmetry, for Abelian topological order $\mathbf{K}=\begin{pmatrix}0&N\\N&0\end{pmatrix}$ the electron operator is

$$e^{iN(l_1\phi_1+l_2\phi_2)}=\exp\left[i\sqrt{\frac{N}{2}}[(l_1+l_2)\varphi(z)+(l_1-l_2)\bar{\varphi}(\bar{z})]\right],\quad l_{1,2}\in\mathbb{Z}. \tag{E7}$$

It is straightforward to check that all allowed quasiparticles (local w.r.t. the above electron operator) have the following form:

$$e^{i(l_1\phi_1+l_2\phi_2)}=\exp\left[i\frac{(l_1+l_2)\varphi(z)+(l_1-l_2)\bar{\varphi}(\bar{z})}{\sqrt{2N}}\right],\ l_{1,2}\in\mathbb{Z}.$$

Lastly, any two primary fields differing by an electron operator are regarded as the same (or belong to the same superselection sector).

Now, let us go back to Z_2 spin liquids with $\mathbf{K}=\begin{pmatrix}0&2\\2&0\end{pmatrix}\oplus\begin{pmatrix}0&1\\1&0\end{pmatrix}$, which have four branches of chiral bosons $\{\phi_i,\ 1\leqslant i\leqslant 4\}$. We can introduce free bosons $\varphi_1(z),\bar{\varphi}_1(\bar{z})$ for chiral bosons $\phi_{1,2}$ as in (E2) with $N=2$, and free bosons $\varphi_2(z),\bar{\varphi}_2(\bar{z})$ for chiral bosons $\phi_{3,4}$ as in (E2) with $N=1$. In other words, we have

$$\begin{pmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{pmatrix}=\begin{pmatrix}1/2&1/2&&\\1/2&-1/2&&\\&&1/\sqrt{2}&1/\sqrt{2}\\&&1/\sqrt{2}&-1/\sqrt{2}\end{pmatrix}\begin{pmatrix}\varphi_1\\\bar{\varphi}_1\\\varphi_2\\\bar{\varphi}_2\end{pmatrix}.$$

Before gauging the unitary Z_2 symmetry, the four superselection sectors (or four types of different quasiparticles) correspond to

$$\begin{aligned}&1\sim\bar{j}_1(\bar{z})\sim j_1(z)\sim e^{2i\varphi_1(z)}\sim e^{2i\bar{\varphi}_1(\bar{z})}\sim e^{i[\varphi_1(z)\pm\bar{\varphi}_1(\bar{z})]}\sim\bar{j}_2(\bar{z})\sim j_2(z)\sim e^{i\frac{\varphi_2(z)\pm\bar{\varphi}_2(\bar{z})}{\sqrt{2}}}\sim e^{\sqrt{2}i\varphi_2(z)}\sim e^{\sqrt{2}i\bar{\varphi}_2(\bar{z})},\\&e\sim e^{i\frac{\varphi_1(z)+\bar{\varphi}_1(\bar{z})}{2}},\ m\sim e^{i\frac{\varphi_1(z)-\bar{\varphi}_1(\bar{z})}{2}},\ f\sim e^{i\varphi_1(z)}\sim e^{i\bar{\varphi}_1(\bar{z})}.\end{aligned} \tag{E8}$$

Now, after gauging the conventional Z_2 symmetries in Table II, as discussed in Appendix B, a new type of quasiparticle q_g becomes deconfined excitations:

$$q_g\sim e^{i\sum_{I,J}\phi_I\mathbf{K}_{I,J}\delta\bar{\phi}_J^g/2\pi}\sim\exp\left[i\frac{(i_1+i_2)\varphi_1+(i_2-i_1)\bar{\varphi}_1+\sqrt{2}(1+i_4)\varphi_2+\sqrt{2}(i_4-1)\bar{\varphi}_2}{4}\right],$$

where $i_{1,2,4}=0,1$ in $\delta\vec{\phi}^g$. Notice that when such a Z_2 symmetry flux q_g is deconfined, we have to modify the previous definition of electron operators 1 in (E8). The new electron operator is defined as anything that is local w.r.t. quasiparticles $\{e,m,f,q_g\}$. With this new definition for electron operators, we can track down all the inequivalent quasiparticles (superselection sectors) and obtain the full structure of the topological order obtained by gauging Z_2 symmetry. One can easily check this approach indeed reproduces Table II, consistent with the result of Chern-Simons approach.

In the vertex algebra context, the scaling dimension h of a quasiparticle determines its topological spin $\Theta\equiv\exp(2\pi ih)$, the Berry phase obtained by self-rotating a quasiparticle adiabatically by 2π. On the other hand, the mutual statistics of quasiparticles A and B is given by $\tilde{\theta}_{A,B}=-2\pi\alpha_{A,B}$ in OPE (E6).

2. Application to unconventional SET phases

For an unconventional SET phase, e.g., where two inequivalent quasiparticles (e and m in Z_2 spin liquid) are exchanged under Z_2 symmetry operation as summarized in Table III, a non-Abelian topological order is obtained by gauging the Z_2 symmetry. Here, we apply the vertex algebra approach to extract the full structure of these non-Abelian topological orders.

First, let us review some known results, discussed in detail in Refs. [66,92]. When the unconventional Z_2 symmetry $\{\mathbf{W}^g=\begin{pmatrix}0&1\\1&0\end{pmatrix},\delta\vec{\phi}^g=0\}$ is gauged for Abelian topological order $\mathbf{K}=\begin{pmatrix}0&N\\N&0\end{pmatrix}$, the resultant topological order is described by $U(1)\times U(1)\rtimes Z_2$ Chern-Simons theory (coined "twisted" Z_N gauge theory in [92]), which has GSD $=(N^g/2)[N^g+1+(2^{2g}-1)(N^{g-1}+1)]$ on a genus-g Riemann surface. It contains $2N$ different quasiparticles with quantum dimension $d=1$, another $2N$ quasiparticles with $d=\sqrt{N}$, and $N(N-1)/2$ quasiparticles with $d=2$. Under the unconventional Z_2 symmetry operation, two superselection sectors $e\leftrightarrow m$ exchange and so do chiral bosons $\phi_1\leftrightarrow\phi_2$. Therefore, in the context of vertex algebra (E2) the antiholomorphic free boson $\bar{\varphi}\to-\bar{\varphi}$ under Z_2 symmetry operation! After this unconventional Z_2 symmetry is gauged for Z_2 spin liquids ($N=2$), we obtain a non-Abelian topological order whose quasiparticle content has an antiholomorphic part (from $\bar{\varphi}$) given by Z_2 orbifold

CFT [87,88] with compactification radius $R=2$. It has been shown that Z_2 orbifold CFT is equivalent to Ising $\times$ Ising (or Ising^2) CFT [87]. In each Ising CFT, there are three different quasiparticles: vacuum (or boson) 1, fermion ψ, and the "disorder" field [95] σ with the following fusion rules:

$$\psi\times\psi=1,\ \psi\times\sigma=\sigma,\ \sigma\times\sigma=1+\psi. \tag{E9}$$

Both 1 and ψ have quantum dimension 1 while disorder operator σ has quantum dimension $\sqrt{2}$. Their scaling dimensions are 0, $\frac{1}{2}$, and $\frac{1}{16}$. Therefore, the Z_2 orbifold CFT, equivalent to the direct product of two copies of Ising CFTs, contains $9=3\times 3$ inequivalent quasiparticles (superselection sectors). The quasiparticle contents of the Z_2 orbifold CFT are summarized in the first three columns of TABLE VII.

Now, let us get back to our cases of Z_2 spin liquids with unconventional onsite Z_2 symmetry. There are two such SET phases as summarized in Table III. After gauging the unitary Z_2 symmetry, they both lead to non-Abelian topological orders with nine inequivalent quasiparticles (superselection sectors). In the vertex algebra context, they share the same antiholomorphic ($\bar\varphi_1$) part which gives rise to the non-Abelian quasiparticles. However, their different holomorphic parts discriminate these two SET phases. A key issue in determining the quasiparticle contents is as follows: Which quasiparticles are identical (or belong to the same superselection sector), after the symmetry is gauged?

In the vertex algebra context, once we fix the electron operator $1\sim ?$ (or the vacuum/trivial sector) which is local w.r.t. all quasiparticles, the full structure of inequivalent quasiparticles is determined. So, the above issue becomes the following question: How to determine the electron operators in the vertex algebra, once we gauge the unitary symmetry? The answer lies in the following physical principle:

If in the original SET phase, two quasiparticles belong to the same superselection sector (i.e., they are equivalent) and transform in the same way under a unitary symmetry, then they belong to the same superselection sector after the unitary symmetry is gauged.

To be specific, if two quasiparticles q_A and q_B belong to the same superselection sector and transform in the same way under unitary symmetry, then after gauging the symmetry, quasiparticle $q_Aq_B^\dagger\sim 1$ ($q_B^\dagger$ is the antiparticle of q_B) belongs to the trivial sector. For instance, in SET phase No. 1 in Tables III and VII, the following two quasiparticles belong to the the trivial sector and are both odd under Z_2 symmetry $\boldsymbol{g}$:

$$\bar{j}_1\sim e^{i\phi_3}=e^{i\frac{\varphi_2+\bar\varphi_2}{\sqrt{2}}}$$

and they are both their own antiparticles. Besides, the following two fermions also belong to the same superselection sector and are both even under Z_2 symmetry:

$$\bar{f}^1=\cos(\phi_1-\phi_2)=\cos(\bar\varphi_1)\sim e^{i(\phi_1+\phi_2)}=e^{i\varphi_1}.$$

Both of them are also their own antiparticles. Therefore, we have the following definitions of electron operators (or trivial sector) as shown in Table VII:

$$1\sim \bar{j}_1e^{i\frac{\varphi_2+\bar\varphi_2}{\sqrt{2}}}\sim \bar{f}^1e^{i\varphi_1}.$$

This enables us to obtain all the nine inequivalent quasiparticles (superselection sectors) as summarized in Table VII, for the non-Abelian topological order acquired by gauging Z_2 symmetry in these SET phases.

As discussed earlier, in the vertex algebra approach, the mutual statistics of two quasiparticles A and B is given in their OPE (E6) by $\mathcal{S}_{A,B}=\exp(i\tilde\theta_{A,B})=\exp(-2\pi i\alpha_{A,B})$. If quasiparticles A and B lead to more than one fusion channel, the corresponding entry $\mathcal{S}_{A,B}=0$ vanishes in the modular $\mathcal{S}$ matrix. Besides, scaling dimensions $\{h_q\}$ of quasiparticles $\{q\}$ determine their topological spins $\mathcal{T}_{A,B}=\delta_{A,B}\exp(2\pi i h_A)$, which correspond to the modular $\mathcal{T}$ matrix. So, we can extract all the topological properties of the non-Abelian topological orders, obtained by gauging Z_2 symmetry in SET phases.

The modular $\mathcal{S}$ matrix in the basis q_a ($0\leqslant a\leqslant 8$, see Table VII) of the nine different quasiparticles (superselection sectors) is $\mathcal{S}_{\text{No.5}}=$

$$\frac{1}{4}\begin{pmatrix}
1 & 1 & 1 & 1 & 2 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2}\\
1 & 1 & 1 & 1 & 2 & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2}\\
1 & 1 & 1 & 1 & -2 & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & \sqrt{2}\\
1 & 1 & 1 & 1 & -2 & \sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2}\\
2 & 2 & -2 & -2 & 0 & 0 & 0 & 0 & 0\\
\sqrt{2} & -\sqrt{2} & -\sqrt{2} & \sqrt{2} & 0 & 0 & 2 & 0 & -2\\
\sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & 0 & 2 & 0 & -2 & 0\\
\sqrt{2} & -\sqrt{2} & -\sqrt{2} & \sqrt{2} & 0 & 0 & -2 & 0 & 2\\
\sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & 0 & -2 & 0 & 2 & 0
\end{pmatrix}$$

for gauged unconventional SET phase No. 5. Meanwhile, the $\mathcal{T}$ matrix is a diagonal unitary matrix $\mathcal{T}_{a,b} = \delta_{a,b}\exp(2\pi i h_a)$, where h_a gives the topological spin $\Theta_a = \exp(2\pi i h_a)$ of quasiparticle q_a shown in Table VII. To be specific, we have

$$\mathcal{T}_{\text{No.5}} = \begin{pmatrix} 1 &&&&&&&& \\ & 1 &&&&&&& \\ && -1 &&&&&& \\ &&& -1 &&&&& \\ &&&& 1 &&&& \\ &&&&& e^{i\pi/8} &&& \\ &&&&&& e^{-i\pi/8} && \\ &&&&&&& -e^{i\pi/8} & \\ &&&&&&&& -e^{-i\pi/8} \end{pmatrix}.$$

For SET phase No. 6, after gauging the unitary Z_2 symmetry we have its modular $\mathcal{S}$ matrix as $\mathcal{S}_{\text{No.6}} =$

$$\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 & 2 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & 1 & 1 & 1 & 2 & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 1 & 1 & 1 & 1 & -2 & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & \sqrt{2} \\ 1 & 1 & 1 & 1 & -2 & \sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} \\ 2 & 2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & -\sqrt{2} & -\sqrt{2} & \sqrt{2} & 0 & 0 & -2 & 0 & 2 \\ \sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & 0 & -2 & 0 & 2 & 0 \\ \sqrt{2} & -\sqrt{2} & -\sqrt{2} & \sqrt{2} & 0 & 0 & 2 & 0 & -2 \\ \sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & 0 & 2 & 0 & -2 & 0 \end{pmatrix}$$

and its $\mathcal{T}$ matrix as

$$\mathcal{T}_{\text{No.6}} = \begin{pmatrix} 1 &&&&&&&& \\ & 1 &&&&&&& \\ && -1 &&&&&& \\ &&& -1 &&&&& \\ &&&& 1 &&&& \\ &&&&& e^{i5\pi/8} &&& \\ &&&&&& e^{i3\pi/8} && \\ &&&&&&& -e^{i5\pi/8} & \\ &&&&&&&& -e^{3i\pi/8} \end{pmatrix}.$$

Clearly, after gauging the unconventional Z_2 symmetry, $\delta\vec{\phi}^g = (0,0,\pi,0)^T$ and $\delta\vec{\phi}^g = (\pi/2,\pi/2,\pi,0)^T$ lead to the same non-Abelian topological order since they belong to the same SET phase. They share the same $\mathcal{S}$ and $\mathcal{T}$ matrices, differing by a relabel of quasiparticles in Table VII. For example, quasiparticle q_5 in $\delta\vec{\phi}^g = (0,0,\pi,0)^T$ case corresponds to quasiparticle q_6 in $\delta\vec{\phi}^g = (\pi/2,\pi/2,\pi,0)^T$ case. Similarly, two cases $\delta\vec{\phi}^g = (0,0,\pi,\pi)^T$ and $\delta\vec{\phi}^g = (\pi/2,\pi/2,\pi,\pi)^T$ lead to the same non-Abelian topological order, by gauging the unconventional Z_2 symmetry.

It is easy to verify that they satisfy the following consistency conditions [44] for modular transformations:

$$(\mathcal{S}\mathcal{T})^3 = \Theta \cdot \mathcal{S}^2, \ \ \mathcal{S}^4 = 1, \tag{E10}$$

where the U(1) phase factor Θ is defined as

$$\Theta \equiv d_a^2 e^{2\pi i h_a} \Big/ \sqrt{\sum_a d_a^2} = e^{2\pi i c_-/8}. \tag{E11}$$

d_a and h_a correspond to the quantum dimension and topological spin $\exp(2\pi i h_a)$ of quasiparticle q_a, respectively. c_- is the chiral central charge of the edge excitations of the topological ordered phase. Both non-Abelian topological orders in Table VII have $c_- = 0$ and hence $\Theta = 1$. In fact, for both non-Abelian topological orders (No. 5 and No. 6) summarized in Table VII, their modular $\mathcal{S}$ and $\mathcal{T}$ matrices satisfy $\mathcal{S}^2 = (\mathcal{S}\mathcal{T})^3 = 1_{9\times 9}$.

Starting from a Z_2 gauge theory (Z_2 spin liquid or double-semion theory) with unitary Z_2 symmetry, once the symmetry is gauged, a resultant $Z_2 \times Z_2$ gauge theory is expected [21,42]. The above non-Abelian topological orders can be regarded as unconventional $Z_2 \times Z_2$ gauge theories, related to Kitaev's 16-fold way classification [44] of Z_2 gauge theories in (2+1)D. In particular, they are associated with Z_2 gauge theories where fermions having an odd Chern number ($\nu =$ odd) couple to Z_2 gauge fields. Notice that before gauging the symmetry, all SET phases have nonchiral edge excitations with chiral central charge $c_- = 0$. As a result, we expect that after gauging the Z_2 symmetry their edge states remain nonchiral and should be gapped due to backscattering in a generic situation. Indeed, in all the "gauged" non-Abelian topological orders in Table VII, a Z_2 gauge theory with fermion Chern number ν is always accompanied by its time-reversal counterpart $\bar{\nu} \equiv 16 - \nu \mod 16$ through a direct product.

Specifically, in [44] Kitaev introduced a 16-fold way classification of (2+1)D Z_2 gauge theories, describing fermions coupled to a Z_2 gauge field. When the Chern number ν of fermions changes by 16, one ends up with the same Z_2 gauge theory. Specifically when $\nu =$ odd, associated Z_2 gauge theory contains three inequivalent quasiparticles: vaccum (or boson) 1, fermion ψ (ε in Kitaev's notation [44]), and vortex σ. Their fusion rules are the same as (E9), i.e., those in Ising anyon theory [43]. Their quantum dimensions are

$$d_1 = d_\psi = 1,\ \ d_\sigma = \sqrt{2}.$$

The topological spin $\exp(2\pi i h)$ of these quasiparticles is given by

$$h_1 = 0,\ \ h_\psi = \frac{1}{2},\ \ h_\sigma = \frac{\nu}{16}.$$

Therefore, when $\nu = 1$ this corresponds to the Ising anyon theory. When a direct product of a Z_2 gauge theory with Chern number ν (we denote this Z_2 gauge theory by ν) and its time-reversal counterpart $\bar{\nu} = 16 - \nu$ is made, one can combine the fermion ψ in ν and the vortex $\bar{\sigma}$ in $\bar{\nu}$ to form a new vortex operator, which has scaling dimension $\frac{1}{2} - \frac{\nu}{16} = \frac{8-\nu}{16}$. Therefore, one can clearly see the following two seemingly different direct products

$$\nu \otimes (16 - \nu) \simeq (8 - \nu) \otimes (8 + \nu) \tag{E12}$$

lead to the same topological order. As a result, SET phases No. 5 and No. 6 in Table III lead to two distinct non-Abelian topological orders ($\nu = 1,7$ and $\nu = 3,5$), by gauging the unitary Z_2 symmetry.

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