

# Reaction diffusion dynamics and the Schryer-Walker solution for domain walls of the Landau-Lifshitz-Gilbert equation

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We study the dynamics of the equation obtained by Schryer and Walker for the motion of domain walls. The reduced equation is a reaction diffusion equation for the angle between the applied field and the magnetization vector. If the hard-axis anisotropy  $K_d$  is much larger than the easy-axis anisotropy  $K_u$ , there is a range of applied fields where the dynamics does not select the Schryer-Walker solution. We give an analytic expression for the speed of the domain wall in this regime and the conditions for its existence.

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## I. INTRODUCTION

Magnetic domain wall propagation is an active area of research both as an interesting physical phenomenon and for its possible applications in logic devices, magnetic memory elements, and others [1]. In micromagnetic theory the motion of a domain wall is described by the Landau-Lifshitz-Gilbert (LLG) equation [2,3], which cannot be solved analytically except in a few special cases. For an infinite medium with uniaxial anisotropy and an external field applied along the symmetry axis, the Schryer-Walker (SW) solution [4,5] is the best-known analytical expression for a stationary traveling domain wall. This exact solution successfully predicts many experimental results below a cutoff field. The stability of the SW solution with respect to small perturbations has been studied recently [6] using dynamical systems techniques. The analysis of the spectrum of a perturbation to the SW solution shows that it may become absolutely or convectively unstable before the breakdown field. This instability is found numerically for sufficiently large hard-axis anisotropy and for fields larger than a critical value. While this instability is in qualitative agreement with results of the numerical integrations reported in [7], it has not been confirmed experimentally. The range of physical parameters of ferromagnetic materials where domain walls are observed is wide, and as discussed in [6], it has not been fully explored. Temperature, doping, and fabrication techniques allow tailoring of the material parameters, which may vary over several orders of magnitude [8–10], and such instability may become experimentally accessible in the future [7].

In this work we study the dynamics of the equation derived by Schryer and Walker from the LLG problem. Most work on this equation has focused on the existence of an exact traveling domain wall. Here we take a different approach: the SW equation is a nonlinear partial differential equation and the existence of an exact solution does not imply that an initial condition will converge to this exact solution. Using the theory of reaction diffusion equations we find that the SW solution is not selected by the dynamics for applied fields larger than a critical value provided that the hard-axis anisotropy is sufficiently large. We give explicit analytic expressions for the transition field, the speed of the front

beyond this transition, and the conditions under which this transition occurs. Qualitatively these results agree with the findings reported in [6]; in this case, due to the simplicity of the problem, a full analytical solution is given. We conjecture that this behavior is preserved in the full LLG equation. An asymptotic analysis of the slow-time evolution of the LLG equation for large perpendicular anisotropy and small fields leads to a similar transition [11].

## II. SCHRYER-WALKER SOLUTION

For the sake of clarity we first recall the SW solution in some detail. The starting point of the calculation is the LLG equation for the magnetization. The material has magnetization  $\vec{M} = M_s \vec{m}$ , where  $M_s$  is the saturation magnetization and  $\vec{m} = (m_1, m_2, m_3)$  is a unit vector along the direction of magnetization. The dynamic evolution of the magnetization is governed by the LLG equation,

$$\frac{d\vec{M}}{dt} = -\gamma_0 \vec{M} \times \vec{H}_{\text{eff}} + \alpha \frac{\vec{M}}{M_s} \times \frac{d\vec{M}}{dt}, \quad (1)$$

where  $\vec{H}_{\text{eff}}$  is the effective magnetic field,  $\gamma_0 = |\gamma| \mu_0$ ,  $\gamma$  is the gyromagnetic ratio of the electron, and  $\mu_0$  is the magnetic permeability of vacuum. The constant  $\alpha > 0$  is the dimensionless phenomenological Gilbert damping parameter. Following SW, we consider an infinite medium with uniaxial crystalline anisotropy. The easy axis is taken to be the  $z$  axis of a Cartesian coordinate system and an external magnetic field is applied along this easy axis. The demagnetizing field is assumed to have a local representation and to depend only on  $x$ . The effective magnetic field is then given by

$$\vec{H}_{\text{eff}} = H_a \hat{z} + \frac{C_{\text{ex}}}{\mu_0 M_s^2} \frac{\partial^2 \vec{M}}{\partial x^2} + \frac{2K_u}{\mu_0 M_s^2} M_z \hat{z} - \frac{2K_d}{\mu_0 M_s^2} M_x \hat{x}, \quad (2)$$

where  $C_{\text{ex}}$  is the exchange constant,  $K_u$  the easy-axis uniaxial anisotropy, and  $K_d$  the perpendicular anisotropy.

Introducing  $M_s$  as the unit of the magnetic field and introducing the dimensionless space and time variables  $\xi = x \sqrt{K_u / C_{\text{ex}}}$  and  $\tau = \mu_0 |\gamma| M_s t$ , we rewrite Eqs. (1) and (2) in dimensionless form,

$$\frac{d\vec{m}}{d\tau} = -\vec{m} \times \vec{h}_{\text{eff}} + \alpha \vec{m} \times \frac{d\vec{m}}{d\tau}, \quad (3)$$

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with

$$\vec{h}_{\text{eff}} = h_a \hat{z} + \frac{1}{2} k_u \frac{\partial^2 \vec{m}}{\partial \xi^2} + k_u m_3 \hat{z} - k_d m_1 \hat{x}, \quad (4)$$

where  $h_a$  is the dimensionless applied field and the dimensionless numbers that have appeared are  $k_u = 2K_u/(\mu_0 M_s^2)$  and  $k_d = 2K_d/(\mu_0 M_s^2)$ . Equations (3) and (4) describe the dynamics of the problem. Next we introduce spherical coordinates for the unit magnetization vector, namely,

$$m_1 = \sin \theta \cos \varphi, \quad m_2 = \sin \theta \sin \varphi, \quad \text{and} \quad m_3 = \cos \theta. \quad (5)$$

The LLG equation then reduces to the coupled system

$$\alpha \sin \theta \dot{\varphi} + \dot{\theta} = \frac{1}{2} k_d \sin \theta \sin 2\varphi + \frac{1}{2} \frac{k_u}{\sin \theta} \frac{\partial}{\partial \xi} (\varphi_\xi \sin^2 \theta), \quad (6a)$$

$$\alpha \dot{\theta} - \sin \theta \dot{\varphi} = \frac{1}{2} k_u \theta_{\xi\xi} - h_a \sin \theta - \frac{1}{4} k_u \varphi_\xi^2 \sin 2\theta - k_u \sin \theta \cos \theta - k_d \sin \theta \cos \theta \cos^2 \varphi. \quad (6b)$$

The solution studied by Schryer and Walker is that with constant azimuthal angle  $\varphi = \varphi_0$ . With this assumption the equations above reduce to

$$\dot{\theta} = \frac{1}{2} k_d \sin \theta \sin 2\varphi_0, \quad (7a)$$

$$\alpha \dot{\theta} = \frac{1}{2} k_u \theta_{\xi\xi} - F(\theta), \quad (7b)$$

where

$$F(\theta) = \sin \theta [h_a + \cos \theta (k_u + k_d \cos^2 \varphi_0)]. \quad (8)$$

The time evolution for the polar angle  $\theta$  is governed by a reaction diffusion equation, for which a complete rigorous mathematical theory is well established. We are interested in the reversal of the magnetization induced by the applied magnetic field; therefore, as in [4,5], we assume that  $\theta_\xi$  vanishes as  $\xi \rightarrow \pm\infty$  and that  $\theta \rightarrow 0$  when  $\xi \rightarrow -\infty$  and  $\theta \rightarrow \pi$  when  $\xi \rightarrow \infty$ . Equations (7) together with the asymptotic conditions are the system studied by Schryer and Walker. For the sake of completeness we recall some of their results. The first step is to note that the angle  $\varphi_0$  is fixed through a consistency condition. In effect, multiplying (7a) by  $\alpha \theta_x$  and integrating in  $x$  between  $-\infty$  and  $+\infty$ , using (7b) and the boundary conditions, one obtains [4]

$$\sin 2\varphi_0 = -\frac{2h_a}{\alpha k_d} \equiv -\frac{h_a}{h_w}, \quad (9)$$

where the dimensionless Walker field is given by  $h_w = \alpha k_d/2$  in the present notation. It is convenient to express  $\cos^2 \varphi_0$  in terms of the applied magnetic field. Two branches exist,  $\cos^2 \varphi_0 = (1/2)(1 \pm \sqrt{1 - \sin^2 2\varphi_0})$ . When the applied field vanishes the domain wall is static and  $\varphi_0 = \pm\pi/2$ . Therefore we choose, following [5], the branch

$$\cos^2 \varphi_0 = \frac{1}{2}(1 - \sqrt{1 - \sin^2 2\varphi_0}) = \frac{1}{2}(1 - \sqrt{1 - (h_a/h_w)^2}).$$

### III. REACTION-DIFFUSION DYNAMICS

Next we consider the dynamics of Eq. (7b). In order to apply the theory of reaction diffusion equations, it is convenient to

render it in the usual form. To do so we introduce a new dependent variable  $u$  defined by  $\theta = \pi(1 - u)$ . The evolution equation for this new variable is

$$\alpha \dot{u} = D u_{\xi\xi} + f(u), \quad (10)$$

with

$$f(u) = \frac{\sin \pi u}{\pi} [h_a - (k_u + k_d \cos^2 \varphi_0) \cos \pi u], \quad D = \frac{1}{2} k_u.$$

The explicit dependence of the reaction term on the applied field is then

$$f(u) = \frac{\sin \pi u}{\pi} (h_a - R(h_a) \cos \pi u),$$

$$\text{where} \quad R(h_a) = k_u + \frac{k_d}{2} (1 - \sqrt{1 - (h_a/h_w)^2}). \quad (11)$$

Equation (10) is the well-studied reaction diffusion equation. The diffusion constant  $D = k_u/2$  and the reaction term  $f$ , which satisfies  $f(0) = f(1) = 0$ , is monostable or bistable, depending on the values of the material parameters and the applied field. In the bistable case, there is a unique domain wall. This is the exact SW solution,

$$u = \frac{2}{\pi} \arctan [e^{-\sqrt{\frac{2R(h_a)}{k_u}}(\xi - c_{\text{SW}}\tau)}],$$

$$c_{\text{SW}} = \frac{h_a}{\alpha} \sqrt{\frac{k_u}{2k_u + k_d(1 - \sqrt{1 - (h_a/h_w)^2})}}. \quad (12)$$

When the reaction term is monostable [ $f'(0) > 0$ ] there is a continuum of fronts. A small perturbation to the unstable state  $u = 0$  ( $\theta = \pi$ ) evolves into a traveling monotonic front of minimal speed  $c^*$  [12,13] that joins the stable state  $u = 1$  ( $\theta = 0$ ) to the unstable state  $u = 0$  ( $\theta = \pi$ ).

The minimal speed can be obtained from variational principles [14,15] and is bounded by [13]

$$c_{\text{KPP}} \equiv \frac{2}{\alpha} \sqrt{D f'(0)} \leq c^* \leq \frac{2}{\alpha} \sqrt{D \sup f(u)/u}. \quad (13)$$

When the asymptotic speed is exactly  $c_{\text{KPP}}$  the traveling front is called a KPP (Kolmogorov-Petrovskii-Piskunov) or pulled front. In their original paper [12] KPP proved that localized initial conditions evolve into the linearly determined speed  $c_{\text{KPP}}$  provided that  $f(u) > 0$ ,  $f'(0) > f'(u)$  for  $u \in (0, 1)$ . Less restrictive conditions and characterizations of the asymptotic speed were established in subsequent work by other authors (see [16] for a review and additional references). In the monostable case one must determine whether the front of minimal speed is the SW solution, Eq. (12), or a KPP front, of speed

$$c_{\text{KPP}} = \frac{2}{\alpha} \sqrt{\frac{k_u}{2} (h_a - R(h_a))}.$$

The analysis that follows gives the exact criterion under which the speed of the domain wall will be given by the KPP value. We show below that for an applied field  $H_- < H < H_+$ , where

$$H_\pm = \frac{2\alpha K_d [\alpha(1 + 2\kappa) \pm \sqrt{\alpha^2 - 16\kappa(\kappa + 1)}]}{\mu_0 M_s (4 + \alpha^2)}$$

$$\text{and} \quad \kappa = K_u/K_d, \quad (14)$$

the front of minimal speed, which will be selected by the dynamics, is a KPP front. This regime exists provided that  $\alpha > \sqrt{16\kappa(\kappa+1)}$ . Otherwise, the speed is given by the Schryer-Walker solution.

One can verify that the reaction term is monostable  $f'(0) > 0$  for

$$\frac{\alpha(1+2\kappa) - \sqrt{\alpha^2 - 4\kappa(\kappa+1)}}{1+\alpha^2} \leq \frac{h_a}{h_w} \leq \frac{\alpha(1+2\kappa) + \sqrt{\alpha^2 - 4\kappa(\kappa+1)}}{1+\alpha^2}.$$

This region exists only if  $\kappa \leq (\sqrt{1+\alpha^2} - 1)/2$ . In this region the speed may be given by the SW or by the KPP value. We know with certainty that the speed will be given by the KPP value when the upper and lower bounds in (13) coincide. The simplest condition to ensure this regime is to require  $f''(u) < 0$ . This condition is met provided  $h_a > 4R(h_a)$ . One can show that  $f''(u) < 0$  for an applied field in the range

$$\frac{4(\alpha(1+2\kappa) - \sqrt{\alpha^2 - 64\kappa(\kappa+1)})}{16+\alpha^2} \leq \frac{h_a}{h_w} \leq \frac{4(\alpha(1+2\kappa) + \sqrt{\alpha^2 - 64\kappa(\kappa+1)})}{16+\alpha^2}.$$

If this condition is fulfilled, the time evolution of a perturbation to the unstable state  $u = 0$  ( $\theta = \pi$ ) evolves into a monotonic traveling front of speed  $c_{\text{KPP}}$ . This criterion is sufficient but not necessary; the transition from a pushed to a pulled front may occur before the upper and lower bounds in Eq. (13) coincide. In this problem, for which there is an exact solution, we know that the transition will occur when  $c_{\text{SW}} = c_{\text{KPP}}$ . This implies that the speed of the moving front will be, in the original dimensional variables,

$$v = \begin{cases} v_{\text{SW}} & \text{if } 0 < H < H_-, \\ v_{\text{KPP}} & \text{if } H_- < H < H_+, \\ v_{\text{SW}} & \text{if } H_+ < H < H_W, \end{cases} \quad (15)$$

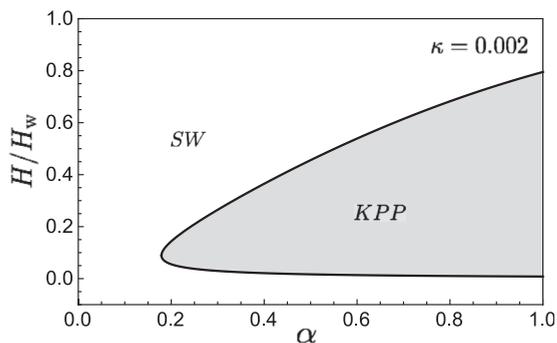


FIG. 1. Range of applied field (in units of the Walker field) versus  $\alpha$  showing the region for which the KPP regime exists for  $\kappa = 0.002$ . For low values of  $\alpha$  this regime is not present and the Schryer-Walker profile is the selected solution. As  $\kappa$  decreases the KPP regime extends to lower values of  $\alpha$ .

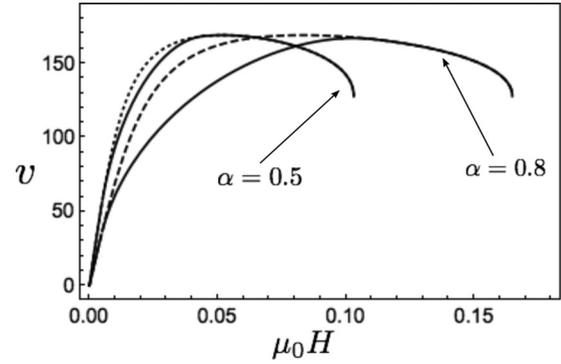


FIG. 2. The solid line is the speed of the domain wall as a function of  $\mu_0 H$  for different values of  $\alpha$ . The dashed line is the Schryer-Walker (SW) speed in the region where the speed is the KPP value. The difference between the SW speed and the KPP speed increases with  $\alpha$ .

where the Walker field is  $H_W = 2\alpha K_d / (\mu_0 M_s)$ , the limiting fields  $H_{\pm}$  are those given in (14), and

$$v_{\text{SW}} = \frac{H_a}{\alpha} \frac{\mu_0 |\gamma| \sqrt{C_{\text{ex}}}}{\sqrt{2K_u + K_d(1 - \sqrt{1 - (H/H_W)^2})}}, \quad (16)$$

$$v_{\text{KPP}} = \frac{2|\gamma| \sqrt{C_{\text{ex}}}}{\alpha M_s} \times \sqrt{\mu_0 M_s H_a - 2K_u - K_d(1 - \sqrt{1 - (H/H_W)^2})}. \quad (17)$$

This transition for the speed occurs only if  $\alpha^2 > 16\kappa(\kappa+1)$  or, equivalently, if

$$K_d > \frac{4K_u}{\sqrt{4 + \alpha^2} - 2}. \quad (18)$$

There is an explicit analytic solution for the domain wall profile in the Schryer-Walker regime, a closed-form analytic solution for the KPP front does not exist in this problem. Since the KPP regime is approached asymptotically and an analytic expression for the magnetization does not exist, one cannot calculate its speed from the identity derived in [17]. We know

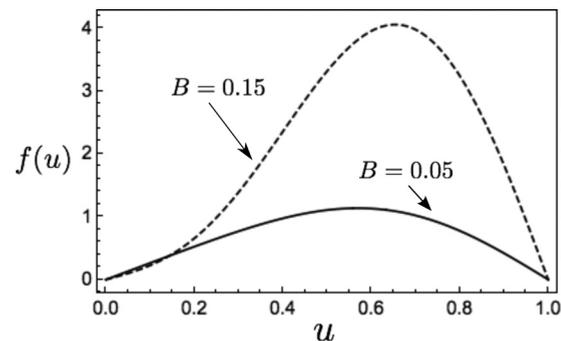


FIG. 3. Reaction term  $f(u)$  for fixed  $\alpha$  at two values of the applied field. The dashed line shows the reaction term for an applied field for which the SW speed is selected; the solid line corresponds to a field for which the KPP speed is selected as shown in Fig. 2.

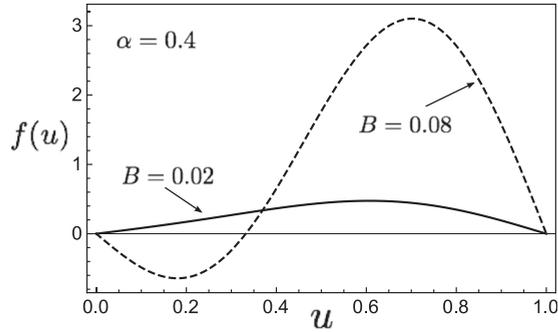


FIG. 4. As Fig. 3, for  $\alpha = 0.4$ . Reaction term  $f(u)$  for  $\alpha = 0.4$  at two values of the field. The dashed line depicts the reaction term for an applied field for which the SW solution is selected; the solid line corresponds to a field for which the KPP speed is selected. In the bistable case ( $B = 0.08$ ) the SW domain wall solution is the unique front.

that it shares the qualitative features of the Walker solution, that is, it is a monotonically decaying front joining the stable and unstable equilibrium points. In Fig. 1 the shaded region shows, for fixed  $\kappa$ , the range of applied field as a function of  $\alpha$  where the KPP regime exists. The field is expressed in units of the Walker field.

The main effect of the KPP regime is to slow the rate of increase in the speed with the applied field. In the figures below we show the speed and reaction functions  $f(u)$  for different parameter values. We fix the material parameters  $M_s = 36\,000$  A/m and  $C_{ex} = 10^{-13}$  J/m. The speed as a function of the magnetic induction  $B = \mu_0 H$  for two values of  $\alpha$ , with material parameters  $K_u = 40$  J m $^{-3}$  and  $K_d = 7500$  J m $^{-3}$ , is shown in Fig. 2. While many samples have very low values of  $\alpha$ , a large damping constant is needed for spintronics applications. Doping ferromagnetic materials with rare earth impurities has yielded values of the damping parameter up to  $\alpha = 0.8$  [18–20]. The fabrication of magnetic alloys with large perpendicular anisotropy and large damping is the subject of ongoing work [21].

The solid line is the speed given in Eq. (15) and the dashed lines are the values of the SW speed in the region where the KPP regime holds. For larger values of  $\alpha$  the rate of increase in the speed with the field is significantly slower than for the SW speed.

The different speed regimes obey the change in the reaction term as the parameters are varied. In Fig. 3 we show the reaction term for  $\alpha = 0.8$ ,  $K_u = 40$  J m $^{-3}$ , and  $K_d = 7500$  J m $^{-3}$ , for different values of the field. When the applied field  $B = \mu_0 H = 0.05$  the reaction term is of the KPP type. In both cases the equilibrium  $u = 0$  ( $\theta = \pi$ ) is unstable.

For smaller  $\alpha$  with the same values of  $K_u$  and  $K_d$  the nature of the reaction terms changes drastically with the field (Fig. 4).

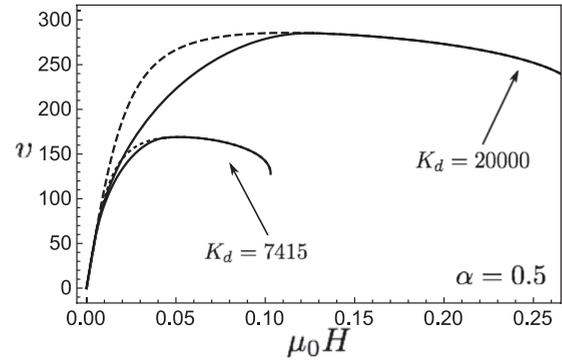


FIG. 5. As in Fig. 2, for different values of the hard-axis anisotropy.

For  $B = 0.08$  the reaction term is bistable, the states  $\theta = 0$  and  $\theta = \pi$  are stable, and the KPP regime does not exist. There is a single traveling front, the SW solution. For a small field the reaction term is of the KPP type. Not only does the speed change, but also the stability of the equilibrium  $\theta = \pi$  changes with the field.

In Fig. 5 we show the speed as a function of the field for different values of the hard-axis anisotropy. Here the presence of the KPP regime has a larger effect for larger values of  $K_d$ .

#### IV. CONCLUSION

The existence of different regimes of front propagation in the reduced system studied by Schryer and Walker that we report here follows directly from the general theory of reaction diffusion equations. It is natural to ask whether this regime arises in the full LLG equations. There is analytical evidence that it does in the case of very thin nanotubes as demonstrated in [22] and [23]. For nanowires and thin films an asymptotic expansion of the LLG equation for a large perpendicular anisotropy [11] shows the transition from the SW to the KPP solution at a small field, a transition which we identify with the transition point  $H_-$ . In these works the assumption of a fixed azimuthal angle is not imposed. While, to our knowledge, the parameter ranges for which this transition occurs have not been accessed experimentally, they may become accessible in the future.

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- [1] R. L. Stamps *et al.*, *J. Phys. D: Appl. Phys.* **47**, 333001 (2014).  
 [2] L. D. Landau and E. M. Lifshitz, *Phys. Z. Sowjetunion* **8**, 153 (1935).  
 [3] T. L. Gilbert, Ph.D. thesis, Illinois Institute of Technology 1956; partially reprinted in *IEEE Trans. Magn.* **40**, 3443 (2004).

- [4] L. R. Walker, Bell Telephone Laboratories memorandum, unpublished (1956); An account of this work is given by J. F. Dillon Jr., *Magnetism, Vol. III*, edited by G. T. Rado and H. Suhl (Academic Press, New York, 1963).  
 [5] N. L. Schryer and L. R. Walker, *J. Appl. Phys.* **45**, 5406 (1974).

- [6] B. Hu and X. R. Wang, *Phys. Rev. Lett.* **111**, 027205 (2013).
- [7] X. S. Wang, P. Yan, Y. H. Shen, G. E. W. Bauer, and X. R. Wang, *Phys. Rev. Lett.* **109**, 167209 (2012).
- [8] L. Thevenard, S. A. Hussain, H. J. von Bardeleben, M. Bernard, A. Lemaître, and C. Gourdon, *Phys. Rev. B* **85**, 064419 (2012).
- [9] W. Zhang, S. Jiang, P. K. J. Wong, L. Sun, Y. K. Wang, K. Wang, M. P. de Jong, W. G. van der Wiel, G. van der Laan, and Y. Zhai, *J. Appl. Phys.* **115**, 17A308 (2014).
- [10] Z. Xu, Y. Yin, F. Xu, Q. Xu, D. Wang, and S. Li, *J. Appl. Phys.* **117**, 17A335 (2015).
- [11] M. C. Depassier, *Europhys. Lett.* **111**, 27005 (2015).
- [12] A. N. Kolmogorov, I. G. Petrovskii and N. S. Piskunov, *Bull. Moscow Univ., Math. Mech.* **1**, 1 (1937).
- [13] D. G. Aronson and H. F. Weinberger, *Adv. Math.* **30**, 33 (1978).
- [14] K. P. Haderer and F. Rothe, *J. Math. Biol.* **2**, 251 (1975).
- [15] R. D. Benguria and M. C. Depassier, *Phys. Rev. Lett.* **77**, 1171 (1996).
- [16] W. van Saarloos, *Phys. Rep.* **386**, 29 (2003).
- [17] X. R. Wang, P. Yan, J. Lu, and C. He, *Ann. Phys.* **324**, 1815 (2009).
- [18] I. Radu, G. Woltersdorf, M. Kiessling, A. Melnikov, U. Bovensiepen, J.-U. Thiele, and C. H. Back, *Phys. Rev. Lett.* **102**, 117201 (2009).
- [19] A. Rebei and J. Hohlfeld, *Phys. Rev. Lett.* **97**, 117601 (2006).
- [20] W. Bailey, P. Kabos, F. Mancoff, and S. Russek, *IEEE Trans. Magn.* **37**, 1749 (2001).
- [21] P. He, X. Ma, J. W. Zhang, H. B. Zhao, G. Lüpke, Z. Shi, and S. M. Zhou, *Phys. Rev. Lett.* **110**, 077203 (2013).
- [22] A. Goussev, J. M. Robbins, and V. Slustikov, *Europhys. Lett.* **105**, 67006 (2014).
- [23] M. C. Depassier, *Europhys. Lett.* **108**, 37008 (2014).