Floquet thermalization: Symmetries and random matrix ensembles

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We investigate the role of symmetries in determining the random matrix class describing quantum thermalization in a periodically driven many-body quantum system. Using a combination of analytical arguments and numerical exact diagonalization, we establish that a periodically driven "Floquet" system can be in a different random matrix class from the instantaneous Hamiltonian. A periodically driven system can thermalize even when the instantaneous Hamiltonian is integrable. A Floquet system that thermalizes in general can display integrable behavior at commensurate driving frequencies. When the instantaneous Hamiltonian and the Floquet operator both thermalize, the Floquet problem can be in the unitary class while the instantaneous Hamiltonian is always in the orthogonal class, and vice versa. We extract general principles regarding when a Floquet problem can thermalize to a different symmetry class from the instantaneous Hamiltonian. A (finite-sized) Floquet system can even display crossovers between different random matrix classes as a function of driving frequency.

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The quantum statistical mechanics of well-isolated many-body quantum systems is drawing intense interest, driven in part by recent experimental advances in the construction, control, and measurement of such systems [1,2]. One key question involves whether—and how—such well-isolated quantum systems can thermalize. The eigenstate thermalization hypothesis (ETH) [3–5] plays a central role in these discussions. For systems with eigenstates that obey the ETH, every (ETH obeying) many-body eigenstate is individually in thermal equilibrium, in the sense that for a macroscopic system prepared in that eigenstate, the reduced density matrix for a small subregion equals the thermal density matrix, at a temperature set by the energy density in the eigenstate. These ideas have also been applied to periodically driven "Floquet" systems [6–8], which lack a conserved energy. In the absence of any conserved quantities, there arises a version of the ETH in which the reduced density matrix for a subregion is proportional to the *unit* matrix, i.e., thermalization to "infinite temperatures." Such periodically driven "Floquet" systems provide a particularly clean playground for investigations of quantum thermalization, and they have inspired much recent work [9–14]. One question that has not been asked, however, is whether all thermalizing Floquet systems are the same, or if there exist sharply distinct infinite temperature phases.

Random matrix theory [15] provides an independent and complementary approach to understanding thermalization. For systems that do thermalize, random matrix theory predicts that quantities such as eigenvalue statistics and level correlation functions should be governed by the relevant random-matrix ensemble—either one of the three traditional Wigner-Dyson ensembles, or, in the presence of particle-hole symmetry, the generalized Altland-Zirnbauer ensembles [16]. Two Floquet systems described by distinct random matrix ensembles are in sharply distinct phases, even if both are at "infinite temperature." However, the role of symmetries in determining the relevant random matrix ensemble for a Floquet system has not been explored.

In this paper, we explore the role of symmetries in Floquet thermalization. We ask the following: when a Floquet system thermalizes, can we determine the relevant random matrix ensemble by examining the symmetries of the (time-dependent) Hamiltonian? We establish by analytical arguments and numerical exact diagonalization that the answer to the above question is *no*. The Floquet problem can thermalize, displaying random matrix statistics, even when the instantaneous Hamiltonian is always trivially integrable. A thermalizing Floquet system can also display an emergent integrability at certain commensurate driving frequencies. Even when the Floquet problem and the instantaneous Hamiltonian both thermalize, they can be in different thermalizing phases. In particular, the Floquet problem can be governed by the (circular) unitary ensemble even when the instantaneous Hamiltonian is governed by the (Gaussian) orthogonal ensemble at all times, and vice versa. We discuss under what situations the Floquet problem and the instantaneous Hamiltonian can thermalize to different symmetry classes. The Floquet problem can also display crossovers between orthogonal and unitary regimes as a function of driving frequency (in addition to the well-known crossovers between thermalizing and localized regimes).

We restrict our discussion to orthogonal and unitary ensembles, leaving extensions to the symplectic and Altland-Zirnbauer classes to future work. Our results are obtained by working with "bang-bang" models, where the Hamiltonian is toggled between two discrete forms, since these provide the simplest realization of a Floquet system. However, we believe the conclusions to be generic. Our work is focused on level statistics as diagnostics of the random matrix class. We note also that unlike most work in the field, our calculations are not restricted to states in the middle of the spectrum. Indeed, we have included states near the band edge by correcting for the varying density of states, using a normalization procedure introduced in [17].

We focus on a simple model based on a chain of N spins-$1/2$ with periodic boundary conditions. The instantaneous

Hamiltonian is the generic anisotropic Heisenberg Hamiltonian with a random field,

$$H(\{J_\alpha\},\{h_\alpha\}) = \sum_{\alpha=x,y,z}\left[J_\alpha \sum_{i=1}^{N} S_i^\alpha S_{i+1}^\alpha\right] + \sum_{\alpha=x,y,z}\left[h_\alpha \sum_{i=1}^{N} c_{\alpha,i} S_i^\alpha\right], \quad (1)$$

where $S_i^\alpha = \frac{1}{2}\sigma_i^\alpha$ and where σ^α is a Pauli matrix. The coefficients $c_{\alpha,i}$ are uncorrelated and chosen according to a uniform distribution within the interval $[-1,1]$. The amplitude of the random field is set through the h_α. This model exhibits various level statistics depending on the parameters $\{J_\alpha\}$ and $\{h_\alpha\}$. Setting all parameters to zero except J_z and h_z leads to a trivially integrable model. If we now take $J_x = J_y = J_z = 1$ and $h_x = h_y = 0$, $h_z = h$, then we obtain the spin half Heisenberg model with random z fields, which is a workhorse of studies of many-body localization (MBL). This model displays a many-body localized phase (with Poisson level statistics) for large $h \gtrsim 3.5$, and a thermalizing phase [with Gaussian orthogonal ensemble (GOE) level statistics in a sector with fixed total S^z] for small h [18]. Note that the level statistics are GOE even though the time-reversal symmetry is broken by the field because of the presence of a disguised antiunitary symmetry, made up of time reversal and a rotation by π of all spins about the x axis, which leaves the Hamiltonian unchanged. Similarly, if we allow two components of the field to be nonzero (e.g., $h_x \neq 0$, $h_y \neq 0$, and $h_z = 0$), then too the level statistics are described by the GOE for small fields when the system thermalizes. The relevant antiunitary symmetry is now time reversal plus a π rotation about the z axis (i.e., $S^x \to S^x$, $S^y \to S^y$, and $S^z \to -S^z$), which leaves the Hamiltonian unchanged [19,20]. Once all three fields are nonzero, however, there is no longer any such antiunitary symmetry, and the level statistics in the thermalizing phase are described by the Gaussian unitary ensemble (GUE).

The Hamiltonian of Eq. (1) is the building block of our "bang-bang" model. We focus on the two-bang case, which is the simplest possible structure for a Floquet problem. The time-dependent τ-periodic Hamiltonian $\mathcal{H}_{2\text{bangs}}(t)$ is defined as

$$\mathcal{H}_{2\text{ bangs}}\left(0 < t < \frac{\tau}{2}\right) = H_1 = H(\{J_{\alpha,1}\},\{h_{\alpha,1}\}),$$
$$\mathcal{H}_{2\text{ bangs}}\left(\frac{\tau}{2} < t < \tau\right) = H_2 = H(\{J_{\alpha,2}\},\{h_{\alpha,2}\}), \quad (2)$$

where $\{J_{\alpha,1}\}$ and $\{h_{\alpha,1}\}$ ($\{J_{\alpha,2}\}$ and $\{h_{\alpha,2}\}$) set the $\mathcal{H}_{2\text{bangs}}(t)$ when $0 < (t \bmod \tau) < \frac{\tau}{2}$ [$\frac{\tau}{2} < (t \bmod \tau) < \tau$]. Note that the random coefficients $c_{\alpha,i}$ are *identical* for H_1 and H_2. The time evolution operator corresponding to Hamiltonian evolution over τ is

$$U(\tau) = \exp(-iH_1\tau/2)\exp(-iH_2\tau/2). \quad (3)$$

Our goal is to explore the connection between the statistics of the eigenvalues $e^{i\lambda_n}$ of $U(\tau)$ and the level statistics of the time-dependent instantaneous Hamiltonian $\mathcal{H}$. We note that the Floquet operator is governed by circular rather than Gaussian ensembles [15].

To numerically probe the level statistics, we compute the ratio of adjacent gaps. For a sorted spectrum $\{\lambda_n; \lambda_n \leqslant \lambda_{n+1}\}$, the ratio of adjacent gaps is defined as

$$r_n = \frac{\min(\lambda_n - \lambda_{n-1}, \lambda_{n+1} - \lambda_n)}{\max(\lambda_n - \lambda_{n-1}, \lambda_{n+1} - \lambda_n)}. \quad (4)$$

Depending on the level statistics, the average ratio r of adjacent gaps is $r \simeq 0.530$ for the orthogonal ensemble [9], $r \simeq 0.60$ for the unitary ensemble [9], and $r \simeq 0.386$ for a Poisson spectrum [18]. Since we are restricted to moderate matrix sizes, we also average r over different samples, denoting by $\langle r \rangle$ the corresponding ensemble-averaged value.

As a warmup, let us consider the situation in which both H_1 and H_2 are integrable by using $J_{x,1} = J_{z,2} = 1$, $h_{x,1} = h_{z,2} = h$, and all the other parameters being zero, i.e.,

$$H_1 = \sum_{i=1}^{N} S_i^x S_{i+1}^x + h c_{x,i} S_i^x, \quad (5)$$

$$H_2 = \sum_{i=1}^{N} S_i^z S_{i+1}^z + h c_{z,i} S_i^z. \quad (6)$$

As can be observed in Fig. 1, the Floquet problem thermalizes to the orthogonal ensemble for any value of τ, as long as the random fields h are weak enough. Thermalization is governed by the orthogonal ensemble because the Floquet operator (3) is invariant under the antiunitary symmetry $S^x \to S^x$, $S^y \to S^y$, and $S^z \to -S^z$ as discussed above. Note that the Floquet problem can thermalize even though the instantaneous Hamiltonian is always integrable, because the constants of motion of the instantaneous Hamiltonian change over time, and nested commutators $[H_1,H_2], [H_1,[H_2,H_1]], \ldots$ generate ever higher-order spin terms (unlike [21], where commutators only renormalize coefficients in a quadratic boson Hamiltonian) such that the Floquet Hamiltonian does *not* have an extensive number of local constants of motion. This is reminiscent of

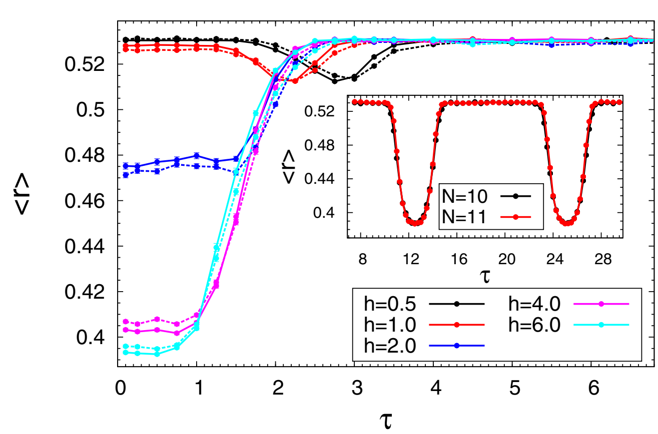

FIG. 1. Average ratio of adjacent gaps for the Floquet unitary of the two bang-models where H_1 and H_2 are given by Eqs. (5) and (6). The calculations were done for $N = 10$ spins and various amplitudes of the random field h. $\langle r \rangle$ has been averaged over 200 samples. Inset: Results for the same model at larger τ, showing an emergent integrability at commensurate frequencies $\tau = 4n\pi$. While we only show the data for $h = 0.5$, the results are identical at least up to $h = 6.0$.

single-particle quantum chaos problems involving two integrable Hamiltonians but leading to quantum chaotic behavior [22–24]. The major difference here is that our system would exhibit this feature even without an average over disorder.

For large fields, the model displays a many-body localized phase in the high-frequency limit, diagnosed by Poisson level statistics. However, the system always thermalizes below a (field-dependent) critical frequency, which is the generic behavior for Floquet systems with an MBL phase [25,26] and is related to the increasingly nonlocal response to time-dependent local perturbations [27].

Note the existence of a dip in $\langle r \rangle$ at a field-strength-dependent value of $\tau \approx 2$ [28]. In brief, the dip occurs when the Floquet zone width first becomes comparable to the many-body bandwidth (such that states start getting "folded" into the principal Floquet zone) and seems to be a universal signature of the weakened level repulsion between states that have and have not been reconstructed by the resulting many-body resonances.

Finally, note that the model discussed above actually displays *integrable* behavior at a discrete set of frequencies $\tau = 4n\pi$ (integer n). This emergent integrability illustrates the special behavior that can arise in Floquet problems at commensurate frequencies [28].

We now discuss situations in which the instantaneous Hamiltonian and the Floquet problem are both thermalizing, and we discuss the (lack of) any relation between the relevant symmetry classes for thermalization. We begin by pointing out that the Floquet problem can thermalize to the circular unitary ensemble (CUE) for all τ even when the instantaneous Hamiltonian always thermalizes to the orthogonal class. This can be achieved, e.g., in a model with

$$H_1 = \sum_{i=1}^{N} \mathbf{S}_i \cdot \mathbf{S}_{i+1} + h\left(c_{x,i} S_i^x + \frac{1}{2} c_{y,i} S_i^y\right), \qquad (7)$$

$$H_2 = \sum_{i=1}^{N} \mathbf{S}_i \cdot \mathbf{S}_{i+1} + h\left(\frac{1}{2} c_{y,i} S_i^y + c_{z,i} S_i^z\right). \qquad (8)$$

The instantaneous Hamiltonian only ever has a field along two axes, is thus invariant under an appropriate antiunitary transformation, and thus at weak disorder thermalizes to the orthogonal ensemble $\langle r \rangle \approx 0.53$. In contrast, the Floquet problem involves fields along all three axes, and it is not invariant under any such antiunitary transformation, and thus it thermalizes to the unitary ensemble (Fig. 2), at least for weak fields. For stronger fields, there exists a localized phase with Poisson statistics for high driving frequencies, which gives way to a thermalizing phase in the CUE class for low frequencies. We can understand thermalization of the Floquet problem to the unitary class as follows: in the model discussed above, for H_1 the relevant antiunitary symmetry is the improper rotation $S^z \to -S^z$, whereas for H_2 it is $S^x \to -S^x$. However, since H_1 and H_2 have different antiunitary symmetries, there is no antiunitary symmetry for $U(\tau)$. We believe that this result—that the Floquet Hamiltonian can be CUE even if the instantaneous Hamiltonian is GOE if the antiunitary symmetries change over time—is general, and not particular to two-bang models.

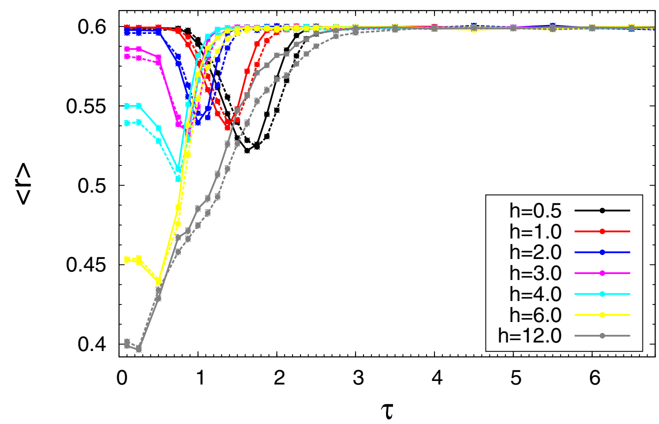

FIG. 2. $\langle r \rangle$ value for the Floquet unitary operator of the two bang-models where H_1 and H_2 are GOE and given by Eqs. (7) and (8). The calculations were done for $N = 11$ (solid lines) and $N = 10$ (dashed lines) spins and various amplitudes of the random field h. $\langle r \rangle$ has been averaged over 300 samples.

We can also have a situation in which the instantaneous Hamiltonian is always GUE but the Floquet problem is governed by the circular orthogonal ensemble (COE). In a two-bang model with equal length bangs, this happens if there is an antiunitary symmetry that exchanges H_1 and H_2. In this case, the antiunitary symmetry leaves $U(\tau)$ unchanged, up to a shift of $\tau/2$ in the origin of time, even while it changes the instantaneous Hamiltonian. A specific example is a model with all the $J_\alpha = 1$ and $h_{\alpha,1} = h_{x,2} = h_{z,2} = -h_{y,2} = h$, wherein the two Hamiltonians H_1 and H_2 are transformed into one another by the improper rotation $S^y \to -S^y$. This can be seen to have COE level statistics (Fig. 3) even through the instantaneous Hamiltonians are GUE ($\langle r \rangle = 0.6$). Again, we believe this result to be general—even if the instantaneous Hamiltonian is not invariant under any antiunitary symmetry,

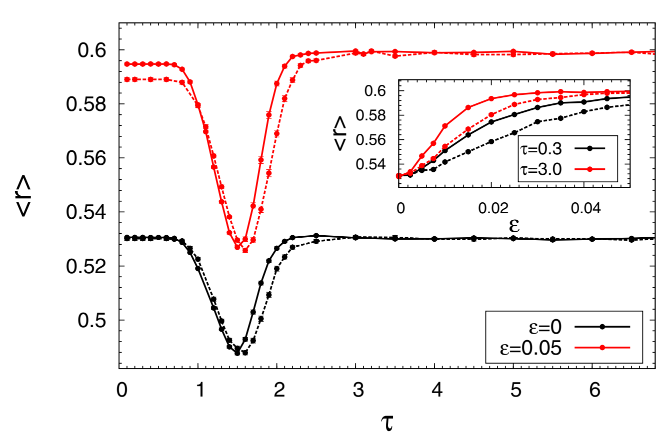

FIG. 3. The Floquet Hamiltonian of the two bang-models where H_1 and H_2 transform into one another under an antiunitary transformation, where the time spent evolving with H_1 is $\tau(1/2 - \varepsilon)$ and the evolution with H_2 is performed during $\tau(1/2 + \varepsilon)$. We consider both system sizes $N = 11$ spins (solid lines) and $N = 10$ (dashed lines) spins. The black lines corresponds to the case $\varepsilon = 0$ while the red lines are for $\varepsilon = 0.05$. The inset show the evolution of $\langle r \rangle$ when changing ε for $\tau = 0.3$ (black lines) and $\tau = 3.0$ (red lines).

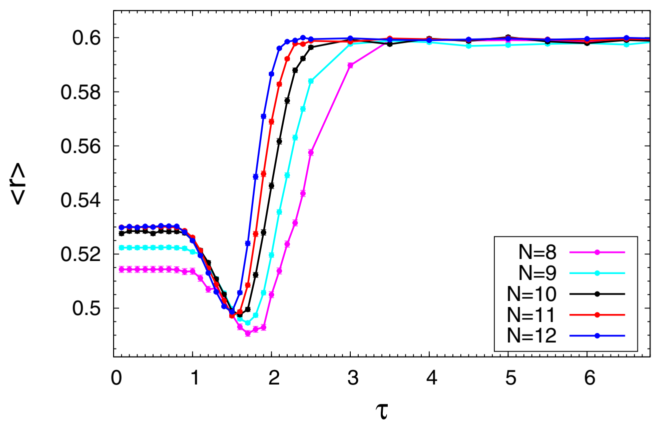

FIG. 4. Average ratio of adjacent gaps for the Floquet Hamiltonian of the two bang-models [Eq. (2)] with all $J = 1$ and $(h_{x,1},h_{y,1},h_{z,1}) = (0.5,0.5,0)$ and $(h_{x,2},h_{y,2},h_{z,2}) = (0,-0.5,0.5)$ for systems of N spins. $\langle r\rangle$ has been averaged over 200 samples.

if the Floquet operator is so invariant (up to a shift in the origin of time) then the Floquet level statistics will be COE. This may be useful numerically, since driven systems with GUE instantaneous Hamiltonians may nevertheless be represented by a completely real Floquet Hamiltonian $H_F = \frac{-i}{\tau}\ln U(\tau)$. Of course, this emergent antiunitary symmetry of the stroboscopic time evolution operator can be broken by applying H_1 and H_2 for unequal times $\tau(1/2 \pm \varepsilon)$, in which case the Floquet problem reverts to the unitary class (Fig. 3).

We now discuss situations in which the Floquet Hamiltonian displays transitions between random matrix classes as a function of driving frequency. One model that does this has all $J = 1$ and $(h_{x,1},h_{y,1},h_{z,1}) = (0.5,0.5,0)$ and $(h_{x,2},h_{y,2},h_{z,2}) = (0,-0.5,0.5)$. This has fields along all three axes so in general it should be in the unitary class, but it should have orthogonal statistics in the $\tau \to 0$ limit since $H_1 + H_2$ has a vanishing field along the y axis. What we see numerically (Fig. 4) is, however, much more striking. There is a finite regime of frequencies $\tau < \tau_c \approx 1$ over which we observe orthogonal statistics, with unitary statistics not setting in until $\tau \gtrsim 2\tau_c$. If the dip in $\langle r\rangle$ is identified with the onset of folding states into the principal Floquet zone, then the "orthogonal regime" is presumably the regime when the bandwidth is less than the Floquet zone width. The frequency window over which this is true should shrink to zero in the thermodynamic limit, since the bandwidth of an interacting system is an extensive quantity. However, the shrinking of the size of this window with system size is extremely slow (Fig. 4), and thus an appreciable "orthogonal regime" may be seen in modest-sized systems. It is interesting to note that resonances between Floquet states in different zones (in an extended zone scheme) are apparently essential to drive this $U(\tau)$ from the orthogonal to the unitary class.

Our numerical investigation of Floquet thermalization in the orthogonal and unitary symmetry classes [28] reveals the following general principles: (i) the Floquet problem can thermalize, displaying random matrix statistics, even when the instantaneous Hamiltonian is always integrable, if the instantaneous constants of motion change over time and if nested commutators of unequal time instantaneous Hamiltonians generate higher-order terms not present in the instantaneous Hamiltonian. In this case, the Floquet problem can display anomalous integrable behavior when the driving frequency is commensurate with characteristic energy scales in the instantaneous integrable Hamiltonians. (ii) A thermalizing instantaneous Hamiltonian $H(t)$ will be governed by the orthogonal ensemble iff it is invariant under an antiunitary symmetry transformation $\mathcal{T}(t)$. However, even if the instantaneous Hamiltonian is always governed by the orthogonal ensemble, the Floquet problem can be in the unitary class iff the instantaneous antiunitary $\mathcal{T}(t)$ changes as a function of time. (iii) The Floquet problem can be in the orthogonal class even if the instantaneous Hamiltonian is in the unitary class if there exists an antiunitary transformation that leaves the Floquet operator unchanged up to a shift in the origin of time. (In a two-bang model this happens if the antiunitary transformation exchanges H_1 and H_2). (iv) There can arise crossovers between orthogonal and unitary thermalization as a function of driving frequency. Our results apply to the entire spectrum, not just the states in the middle of the band. Extensions to other symmetry classes and continuously time varying Hamiltonians are left to future work.

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