Phase diagram of Model C in the parametric space of order parameter and space dimensions

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The scaling behavior of Model C describing the dynamical behavior of the n-component nonconserved order parameter coupled statically to a scalar conserved density is considered in d-dimensional space. Conditions for the realization of different types of scaling regimes in the (n,d) plane are studied within the field-theoretical renormalization group approach. Borders separating these regions are calculated on the base of high-order RG functions using ϵ expansions as well as by fixed dimension d approach with resummation.

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I. INTRODUCTION

According to the dynamical universality hypothesis the dynamical properties of physical systems in the vicinity of their critical points can be grouped into universality classes (similar to static ones) irrespective of details of their microscopic dynamical behavior. There, in addition to the global parameters of a system, a crucial role is played by the behavior of the relevant slow variables, namely order parameter and secondary densities associated with conservation laws. Their dynamical behavior is described by a set of equations of motion of Langevin type with Gaussian noise terms caused by remaining microscopic degrees of freedom [1–3].

The coupling between the order parameter and secondary densities is also important. In the simplest case such coupling is realized within the so-called Model C [1,4], where a nonconserved order parameter is coupled via a static term only to a scalar conserved density. Being quite simple, the model can be used to describe different physical systems. In particular, a lattice model of intermetallic alloys [5], the supercooled liquids [6], and layers on solid substrates [7] are described by models with nonconserved order parameter and additional coupled conserved density. Systems containing annealed impurities with long relaxational times [8] manifest certain similarity with the Model C as well. It was argued that relativistic scalar field theory in 2+1 dimensions is consistent with Model C dynamics [9]. Recently, Model C was applied for description of the solid-liquid like phase transition [10]. Numerical simulations of Model C critical dynamics were performed for an Ising antiferromagnet with conserved full magnetization and nonconserved staggered magnetization [11,12] and also for an Ising magnet with conserved energy [13].

In order to describe critical properties of the system it is standard now to apply renormalization group (RG) methods. The dynamical universality class of Model C was studied first within the dynamical version of perturbative field-theoretical RG [2,3]. The behavior of Model C with an n-component order parameter was analyzed by $\epsilon = 4 - d$ expansion in different regions of (n,d) plane in the first order [4] and subsequently in two-loop order [14,15]. Results of Ref. [14] were corrected by later calculations [16].

According to the two-loop results [16] three different regimes are observed within $(n,\epsilon = 4 - d)$ plane (see Fig. 1): (1) *decoupled regime* (region $\mathcal{D}$), where the secondary density decouples from the order parameter and therefore the order parameter dynamics is appropriately described by Model A with the dynamical critical exponent $z = 2 + c\eta$, where η is the critical exponent of the pair correlation function and c is some coefficient, while the dynamical exponent for the secondary density is $z_m = 2$; (2) *weak scaling regime* (region $\mathcal{W}$), where the order parameter and the secondary density scale differently, the order parameter with $z = 2 + c\eta$, and the secondary density with $z_m = 2 + \alpha/\nu$, where α and ν are critical exponents of the specific heat and of the correlation length correspondingly; (3) *strong scaling regime* (region $\mathcal{S}$), where the order parameter and the secondary density scale both with the critical exponent $z = z_m = 2 + \alpha/\nu$ of the conserved density. The borderlines obtained within two-loop approximations [16] are shown in Fig. 1(a) by orange curves: the dot dashed curve means border between $\mathcal{D}$ and $\mathcal{W}$, while the dotted one means border between $\mathcal{W}$ and $\mathcal{S}$.

There is a question about the existence of one more region (4) of the so-called *anomalous scaling regime* (region $\mathcal{A}$) discovered within one-loop order [4] for d near 4 for $2 < n < 4$. In this region the order parameter behaves much faster than the secondary density and scaling is questionable. The region $\mathcal{A}$ was shown to be an artifact of the ϵ expansion within correct two-loop calculations [16].

An alternative approach for investigating critical dynamics is the nonperturbative RG (NPRG). This approach is based on the use of the exact RG equation for an effective action [17,18]. It is developed intensively now and is applied to describe scaling properties of different classical and quantum models. In particular, it is successful in studies of the critical properties of $O(n)$ models [17,19–21], models with different types of disorder [22], systems with complex symmetries [23–25], critical dynamics near equilibrium [26], reaction-diffusion processes [27], and fully developed turbulence [28] to mention some examples.

Recently the NPRG approach was also applied to study Model C dynamics [29]. These results show changes concerning the borders separating different regimes in the (n,d) plane (shown by black curves in Fig. 1) as well as they report the existence of a new region $\mathcal{A}$ (shown by dot dashed black curve in Fig. 1).

In this paper, to elucidate the discrepancy between both RG approaches we reconsider Model C within the perturbative RG approach. Special attention is paid to the borderlines separating different scaling regimes in the (n,d) plane. We use results of

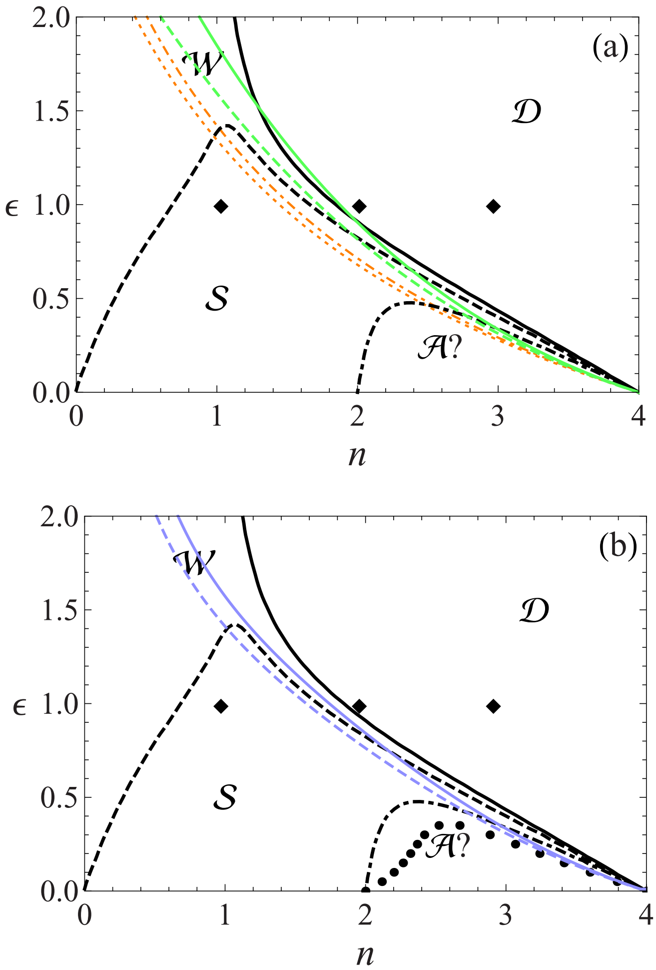

FIG. 1. Phase diagram for Model C in the plane $(n,\epsilon = 4 - d)$ obtained (a) within the perturbative field-theoretical approach by resummation of the ϵ expansion and (b) due to resummation of the RG functions at fixed d, in comparison with the nonperturbative results [29]. Solid lines separate the decoupled region $\mathcal{D}$ from the weak scaling region $\mathcal{W}$; dashed lines separate the weak scaling region $\mathcal{W}$ from the strong scaling region $\mathcal{S}$. Green curves (a): Padé-Borel resummation of ϵ expansions, data of this paper; blue curves (b): fixed dimension approach based on resummed five-loop RG functions, data of this paper; black curves: nonperturbative RG analysis, data from Ref. [29]. Two-loop order results of Ref. [16] are presented as well in (a) by dot dashed and dotted orange curves. The anomalous region $\mathcal{A}$ predicted by the nonperturbative RG is shown separated by the black dot-dashed line (data from Ref. [29]); see the text for the meaning of the black dots. The diamonds mark locations of three-dimensional Ising, XY, and Heisenberg models in (n,d) plane.

high orders of perturbation theory completed by resummation procedures. Our results show the qualitative shift up in values of ϵ for fixed n as well as the larger region $\mathcal{W}$ [see green curves in Fig. 1(a) and blue curves in Fig. 1(b)] in comparison with two-loop order calculations of Ref. [16] [see orange curves in Fig. 1(a)]. Therewith this outcome qualitatively supports these results of the NPRG study [29] (see black curves in Fig. 1) obtained for $n > 1$ and $\epsilon < 1$, showing them trustable. However, our results do not confirm the existence of region $\mathcal{A}$ found in NPRG calculations [29] (see black dot-dashed curve in Fig. 1). There are also severe differences with NPRG results in the rest of the $(n,\epsilon = 4 - d)$ plane. In particular, the nonmonotonic behavior of the borderline between regions $\mathcal{W}$ and $\mathcal{S}$ (see black dashed curve in Fig. 1) for $n < 1$ is not observed within our approach. That indicates that NPRG treatments require improvements also concerning Model A.

The rest of the paper is organized as follows. In Sec. II we present dynamical Model C and its RG description. Then we analyze the conditions for realization of different regimes of critical dynamics. We present results obtained on the base of high-order ϵ expansion in Sec. III. We devote Sec. IV to results obtained by resummation of static five-loop RG functions of minimal subtraction scheme. We end the paper with a discussion and a conclusion in Sec. V.

II. MODEL AND ITS RG DESCRIPTION

Model C introduced to study the influence of the energy conservation on dynamical critical phenomena [4] contains a nonconserved n-component order parameter $\vec{\varphi}_0 = (\varphi_{0,1},\varphi_{0,2},\ldots,\varphi_{0,n})$ and a conserved scalar secondary density m_0. The equations of motion for $\vec{\varphi}_0$ and m_0 are the following:

$$\frac{\partial \varphi_{i,0}}{\partial t} = -\mathring{\Gamma}\frac{\partial \mathcal{H}}{\partial \varphi_{i,0}} + \theta_{\varphi_i}, \quad i = 1\ldots n, \tag{1}$$

$$\frac{\partial m_0}{\partial t} = \mathring{\lambda}\nabla^2\frac{\partial \mathcal{H}}{\partial m_0} + \theta_m . \tag{2}$$

The order parameter relaxes and the conserved density diffuses with kinetic coefficients $\mathring{\Gamma}$, $\mathring{\lambda}$ correspondingly. The stochastic forces θ_{φ_i}, θ_m obey the Einstein relations:

$$\langle\theta_{\varphi_i}(x,t)\theta_{\varphi_j}(x',t')\rangle = 2\mathring{\Gamma}\delta(x - x')\delta(t - t')\delta_{ij}, \tag{3}$$

$$\langle\theta_m(x,t)\theta_m(x',t')\rangle = -2\mathring{\lambda}\nabla^2\delta(x-x')\delta(t-t'), \tag{4}$$

ensuring that static critical properties of the system in d dimensions are described by the equilibrium effective static functional $\mathcal{H}$:

$$\mathcal{H} = \int d^d x\left\{\frac{1}{2}[|\nabla\vec{\varphi}_0|^2 + \mathring{\tilde{r}}|\vec{\varphi}_0|^2] + \frac{\mathring{\tilde{u}}}{4!}|\vec{\varphi}_0|^4 + \frac{1}{2}m_0^2 + \frac{1}{2}\mathring{\gamma}m_0|\vec{\varphi}_0|^2 - \mathring{h}m_0\right\}, \tag{5}$$

where $\mathring{\tilde{r}}$ is connected with relative temperature distance to the critical point, $\mathring{\tilde{u}}$ and $\mathring{\gamma}$ are static coupling constants, and $\mathring{h}$ is an external field.

Integrating out the secondary density one reduces (5) to the usual Ginzburg-Landau-Wilson model with new parameters $\mathring{u}$ and $\mathring{r}$ expressed by the model parameters $\mathring{\tilde{r}}$, $\mathring{\tilde{u}}$, $\mathring{\gamma}$, and $\mathring{h}$ via the relations

$$\mathring{r} = \mathring{\tilde{r}} + \mathring{\gamma}\mathring{h}, \quad \mathring{u} = \mathring{\tilde{u}} - 3\mathring{\gamma}^2. \tag{6}$$

A field-theoretical RG description of the model defined by (1)–(5) is presented in detail in Ref. [16]. In the following subsections we only recall its main points.

A. RG functions

The field-theoretical RG dynamical approach is based on appropriate Lagrangians incorporating equations of motion for the corresponding dynamical models [30]. Several schemes are available for renormalization of those Lagrangians. Here we use results of the minimal subtraction scheme [31], in which the renormalization of Model C is well known [16].

Within the minimal subtraction scheme, one introduces renormalization factors Z_a, $a = \{\{\alpha\},\{\delta\}\}$, leading to the renormalized parameters $\{\alpha\} = \{u,\gamma,\Gamma,\lambda\}$ and renormalized densities $\{\delta\} = \{\varphi,\tilde{\varphi},m,\tilde{m}\}$, where $\tilde{\varphi}$, $\tilde{m}$ are auxiliary field densities introduced to obtain the corresponding Lagrangian for the dynamics defined by Eqs. (1)–(4).

The behavior of the parameters under renormalization is described by the flow equations

$$\ell\frac{d\{\alpha\}}{d\ell} = \beta_{\{\alpha\}}, \tag{7}$$

where ℓ is the flow parameter describing the effective critical behavior. The β functions for the static parameters have the following explicit form:

$$\beta_u(u) = uf_u(u)=u(\epsilon + \zeta_\varphi(u) + \zeta_u(u)), \tag{8}$$

$$\beta_\gamma(u,\gamma) = \gamma f_\gamma(u,\gamma) = \gamma\left(\frac{\epsilon}{2} + \zeta_{\varphi^2}(u) + \frac{\gamma^2}{2}B_{\varphi^2}(u)\right), \tag{9}$$

where the function $B_{\varphi^2}(u)$ is obtained from the additive renormalization A_{φ^2} for the specific heat:

$$B_{\varphi^2}(u) = \mu^\epsilon Z^2_{\varphi^2}\mu\frac{d}{d\mu}\left(Z^{-2}_{\varphi^2}\mu^{-\epsilon}A_{\varphi^2}\right). \tag{10}$$

Here, μ is the scale parameter and factor Z_{φ^2} renormalizes the vertex with φ^2 insertion.

RG functions ζ_a in (8), (9) describing the critical properties are obtained from Z factors:

$$\zeta_a(\{\alpha\}) = -\frac{d\ \ln Z_a}{d\ \ln\mu}. \tag{11}$$

Relations between the renormalization factors lead to relations between the corresponding ζ functions (for details see [16]). In consequence for the description of the critical dynamics one needs only ζ functions of the coupling ζ_u, the order parameter ζ_φ, the auxiliary field $\zeta_{\tilde{\varphi}}$, φ^2 insertion ζ_{φ^2}, and also function B_{φ^2}. In particular, the ζ function of the ratio

$$w = \frac{\Gamma}{\lambda} \tag{12}$$

characterizing the time scales of two dynamical densities is related to the above ζ functions:

$$\zeta_w(u,\gamma,w) = \tfrac{1}{2}\zeta_\varphi(u) - \tfrac{1}{2}\zeta_{\tilde{\varphi}}(u,\gamma,w) - \gamma^2 B_{\varphi^2}(u). \tag{13}$$

The dynamical β function for the introduced time scale ratio w reads then

$$\begin{aligned}\beta_w(u,\gamma,w) &= w\,\zeta_w(u,\gamma,w)\\ &= w\left(\tfrac{1}{2}\zeta_\varphi(u) - \tfrac{1}{2}\zeta_{\tilde{\varphi}}(u,\gamma,w) - \gamma^2 B_{\varphi^2}(u)\right).\end{aligned} \tag{14}$$

This ratio w can take values from zero to infinity. Therefore in order to work in the space of finite parameters it turns out to be useful to introduce the parameter $\rho = w/(1+w)$. Then instead of the flow equation (7) for w the flow equation for ρ arises:

$$\ell\frac{d\rho}{d\ell} = \beta_\rho(u,\gamma,\rho), \tag{15}$$

where according to (14)

$$\begin{aligned}\beta_\rho(u,\gamma,\rho) &= \rho(\rho-1)\zeta_w(u,\gamma,\rho)\\ &= \rho(\rho-1)(\zeta_\Gamma(u,\gamma,\rho) - \zeta_\lambda(u,\gamma)).\end{aligned} \tag{16}$$

In (16) ζ_Γ, ζ_λ are the RG functions describing renormalization of the corresponding kinetic coefficients.

The static RG functions of Model C are known now in high orders. Within the minimal subtraction scheme five-loop results for functions β_u, ζ_φ, ζ_{φ^2} are accessible [32]. The function $B_{\varphi^2}(u)$ is derived within the five-loop order approximation too [33], while the only dynamical function $\zeta_{\tilde{\varphi}}$ is known only within two-loop order [16]. These expressions will serve us below to analyze the phase diagram of Model C in the parametric space of order parameter and space dimensions.

B. Fixed points and their stability

The asymptotic critical behavior is analyzed from the knowledge of the fixed points (FPs) of the flow equations (7). A FP $\{\alpha^*\} = \{u^*,\gamma^*,\rho^*\}$ is defined as a simultaneous zero of the β functions (8), (9), and (16). Equations (8) and (9) give static FPs, whereas Eq. (16) defines the existence regions of the different possible dynamical FPs. Checking the structure of (16) one can see that it has three different FPs (at least within two-loop order): $\rho^* = 0$, $\rho^* = 1$, and $\rho^* = \rho_C$ $(0 < \rho_C < 1)$. The first two solutions exist for any n and d, whereas the solution $\rho^* = \rho_C$ is found from $\zeta_w(u^*,\gamma^*,\rho^*) = 0$ and exists only in a certain region of the (n,d) plane. The stability of these FPs is defined by (22) below. The FP which is stable and accessible from the initial conditions corresponds to the specific behavior at the critical point. Being calculated in this FP, $\{\alpha^*\}$, the RG functions ζ_Γ, ζ_λ define dynamical critical exponents of the order parameter and of the conserved density via relations

$$z = 2 + \zeta_\Gamma(u^*,\gamma^*,\rho^*), \tag{17}$$

$$z_m = 2 + \zeta_\lambda(u^*,\gamma^*). \tag{18}$$

A FP is stable if all eigenvalues ω_i of the stability matrix $\partial\beta_{\alpha_i}/\partial\alpha_j$ calculated at this FP have positive real parts. The values of ω_i indicate also how fast the renormalized model parameters reach their fixed point values. From the structure of the β functions (8), (9), and (16) we conclude that the stability of any FP with respect to the parameters u, γ, and ρ is determined solely by the derivatives of the corresponding β functions:

$$\begin{aligned}\omega_u(u^*) &= \left.\frac{\partial\beta_u(u)}{\partial u}\right|_{\{\alpha^*\}},\\ \omega_\gamma(u^*,\gamma^*) &= \left.\frac{\partial\beta_\gamma(u,\gamma)}{\partial\gamma}\right|_{\{\alpha^*\}},\\ \omega_\rho(u^*,\gamma^*,\rho^*) &= \left.\frac{\partial\beta_\rho(u,\gamma,\rho)}{\partial\rho}\right|_{\{\alpha^*\}}.\end{aligned} \tag{19}$$

The FP coordinates of Model C as well as their stability were established in Ref. [16]. Here we present FPs which, being stable for some values of n and d, describe scaling regimes of Model C in the corresponding regions of (n,d) plane; see Fig. 1 for explicit description. FP $\{u_H,0,0\}$ describes the situation when the conserved density is decoupled from the order parameter (region $\mathcal{D}$), while FP $\{u_H,\gamma_C,0\}$ corresponds to the weak scaling regime, where the order parameter and the conserved density scale with different exponents (region $\mathcal{W}$). The strong scaling regime (region $\mathcal{S}$) is described by the FP $\{u_H,\gamma_C,\rho_C<1\}$, which exists only at certain values n and d, as was noted already. In this regime both quantities have the same critical exponent $z=z_m$, that is the consequence of $\zeta_w(u_H,\gamma_C,\rho_C<1)=0$; therefore at this FP the ζ functions for the kinetic coefficients are equal to each other [see Eqs. (16) and (17),(18)]. Region $\mathcal{A}$, where the behavior of the secondary density is much slower than the behavior of the order parameter, corresponds to the stability region of the FP $\{u_H,\gamma_C,1\}$. However it appears to be unstable within perturbative field-theoretical RG [16]; therefore region $\mathcal{A}$ does not exist within the perturbative field-theoretical RG.

It is known (see e.g. [16]) that ω_γ governs the stability of FPs $\{u_H,0,0\}$ and $\{u_H,\gamma_C,0\}$. Depending on n and d the stability exponent ω_γ will be positive for $\{u_H,0,0\}$ and negative for $\{u_H,\gamma_C,0\}$ or *vice versa*. Therefore, the condition of vanishing ω_γ defines a border between stability regions for FP $\{u_H,0,0\}$ and FP $\{u_H,\gamma_C,0\}$:

$$\omega_\gamma(u_H,0)=0. \tag{20}$$

It can be shown [16] that this condition is equivalent to the vanishing of the specific heat exponent α:

$$\alpha(n,\epsilon)=0. \tag{21}$$

Therefore, this condition gives a borderline $n_\alpha(\epsilon)$ [or $\epsilon_\alpha(n)$] in the parametric space (n,d) between the regions $\mathcal{D}$ and $\mathcal{W}$.

The border between the stability of the FPs $\{u_H,\gamma_C,0\}$ and $\{u_H,\gamma_C,\rho_C\}$ is governed by the stability exponent with respect to ρ:

$$\omega_\rho(u^*,\gamma^*,\rho^*)=(1-2\rho^*)\zeta_w(u^*,\gamma^*,\rho^*) + \rho^*(1-\rho^*)\frac{\partial\zeta_w(u,\gamma,\rho)}{\partial\rho}\bigg|_{\{\alpha^*\}}. \tag{22}$$

Note that for $\rho^*=1$ the transient exponent diverges as $\ln(1-\rho^*)$. Thus $\rho^*=1$ is nowhere stable at least in two-loop approximation [see Eq. (85) in the first reference of [16]]. If $\zeta_w(u^*,\gamma^*,\rho^*)$ is zero the FP is marginal. Therefore, values of n and d at which the condition

$$\omega_\rho(u_H,\gamma_C,0)=0 \tag{23}$$

is satisfied give us the border between stability regions of the FPs $\{u_H,\gamma_C,0\}$ and $\{u_H,\gamma_C,\rho_C\}$. It was shown [16] that this condition is equivalent to the condition

$$c\eta=\frac{\alpha}{\nu}, \tag{24}$$

in all orders of perturbation theory, where $c\eta=z-2$ is a part of the dynamical critical exponent of Model A. Thus (24) defines a border $n_1(\epsilon)$ [or $\epsilon_1(n)$] between the weak scaling region $\mathcal{W}$ described by the FP $\{u_H,\gamma_C,0\}$ and the strong scaling region $\mathcal{S}$ described by the FP $\{u_H,\gamma_C,\rho_C\}$.

Conditions (21) and (24) are valid in all orders of perturbation theory and they include critical exponents of $O(n)$ symmetrical model. Critical exponents are universal quantities depending only on the global characteristics as space dimension d and order parameter dimension n. Within RG critical exponents can be calculated as functions of d and n. Therefore studying the dependence of critical exponents on d and n we can extract from (21) and (24) the borderlines between regions $\mathcal{D}$, $\mathcal{W}$, $\mathcal{S}$ in (n,d) plane. Whereas condition (21) is purely static and separates the region, where the static coupling γ is relevant or it is not, condition (24) is a dynamical condition. It separates in the region where the static coupling γ is relevant and the order parameter follows the Model A dynamics from the region where the genuine Model C dynamics is present. It is therefore possible to improve the two-loop result for these two borderlines by using higher order field-theoretical results for the ϕ^4 theory and Model A alone.

There are two alternative ways to analyze perturbative RG functions in order to get universal quantities. Within the first approach one applies ϵ expansion to obtain the corresponding quantities in a form of series in ϵ and then to evaluate them at the value of interest. Within the second way of analysis one fixes the space dimension d to a certain value and then directly solves the system of equations for FPs numerically, the so-called *fixed dimension approach*. In the next two sections we use these approaches to analyze conditions (21) and (24) numerically.

III. HIGH-ORDER ϵ EXPANSIONS FOR BORDER LINES

Conditions (21) and (24) include static critical exponents as well as the dynamical critical exponent of Model A for the $O(n)$ symmetrical model. These quantities are known now within high orders of perturbative field-theoretical RG. Therefore, we can use these expressions to study conditions (21) and (24). In particular, the condition determining $\epsilon_\alpha(n)$ (21) coincides with the condition for the marginal dimension n_c of a weakly diluted $O(n)$ model according to the Harris criterion [34,35]. We can use the known ϵ expansion for $n_c\equiv n_\alpha$ obtained on the base of five-loop minimal subtraction RG functions [35]:

$$n_\alpha=4-4\epsilon+4.707199\epsilon^2-8.727517\epsilon^3+20.878373\epsilon^4. \tag{25}$$

Now we consider condition (24). Note that Model A quantity, $c\eta$, for a system with $O(n)$ symmetrical order parameter is now known in four-loop order within the minimal subtraction RG scheme [36], while expressions for static critical exponents for this model are obtained in the next fifth order [32]. To be consistent, we restrict ourselves only to the four-loop expressions for the static exponents. Substituting them into (24) and then keeping the coefficients as functions of n and reexpanding (24) in ϵ we obtain n_1 in the form

$$n_1=4-4.181523\epsilon+4.751724\epsilon^2-8.701434\epsilon^3. \tag{26}$$

Formally, numerical values for n_α (25) and n_1 (26) at given space dimension may be obtained by fixing the value of ϵ. However, the series for RG functions are known to be of asymptotic nature [37,38]; therefore, some resummation procedure should be applied to extract reliable information on their basis. Different resummation procedures were successfully applied for numerical analysis of ϵ expansions obtained for some marginal dimensions (stability borders) of static field-theoretical models [35,39,40]. To get borderlines from (25) and (26) we can apply to them a Padé-Borel resummation technique. Such resummation procedure has been successfully used in various tasks of theory of critical phenomena [41]. The procedure is based on the integral Borel transformation [42], and uses an extrapolation by means of a Padé approximant as an intermediate step [43]. Starting from the initial sum S of $L+1$ terms $S(x)=\sum_{i=0}^{L} a_i x^i$ we construct its Borel image

$$S^{\mathrm{B}}(xt)=\sum_{i=0}^{L}\frac{a_i(xt)^i}{i!}. \tag{27}$$

Subsequently we extrapolate the Borel image (27) by a rational Padé approximant:

$$S^{\mathrm{B}}(xt)\Rightarrow[L-1/1](xt)=\frac{\sum_{i=0}^{L-1} b_i(xt)^i}{(1+c_1xt)}, \tag{28}$$

with the coefficients b_i,c_1 expressed in terms of the initial coefficients a_i and the denominator linear in xt to avoid problems with multiple poles.

The resummed function S^{Res} is finally obtained by the inverse Borel transform:

$$S^{\mathrm{Res}}(x)=\int_0^{\infty} dt\ \exp(-t)[L-1/1](xt). \tag{29}$$

Applying (27)–(29) with $x=\epsilon$ to (25) and then fixing the value of ϵ we can obtain the corresponding numerical value n_α, while doing the same for (26) we can get the numerical value of n_1. However, note that expansion (26) is one order shorter than (25). To obtain borderlines within the same order we neglect the ϵ^4 term for n_α (25). Therefore, applying the Padé-Borel procedure (27)–(29) to (25) without the last term and to (26) and changing ϵ from 0 to 2 we get borderlines $n_\alpha(\epsilon)$ and $n_1(\epsilon)$ given by the green solid and dashed curves respectively in Fig. 1(a). As one can see, the region for the weak scaling regime is larger, compared to the two-loop order results [16], and in qualitative accordance with the NPRG results [29]. Such borderlines are very close to the nonperturbative results in the vicinity of $\epsilon=1$.

IV. RESUMMATION OF THE RG FUNCTIONS AT FIXED SPACE DIMENSION

A. Borderlines

As it was already noted, static RG functions are known up to the five-loop order within the minimal subtraction RG scheme [32], as well as the factorial divergence of their coefficients in coupling u being established [37]. Therefore, we apply the procedure (27)–(29) with x meaning u now to the five-loop static RG function $f_u(u)$, as well as to the polynomials $f_\gamma(\gamma,u)-\gamma^2B_{\varphi^2}(u)/2=\frac{\epsilon}{2}+\zeta_{\varphi^2}(u)$ and $\zeta_\varphi(u)/u^2$. Note that in our calculation we use the two-loop expression for $B_{\varphi^2}(u)$: $B_{\varphi^2}(u)=n/2$.

Although we work in *fixed dimension approach*, in order to get the corresponding borderlines we fix the value of n and then solve a system of equations. In particular, solving the system of equations $f_u(u_H)=0$ and (20), where the corresponding functions are substituted by their resummed counterparts for fixed values of n, we find the stability borderline $\epsilon_\alpha(n)$ separating in the (n,ϵ) plane the decoupled scaling region $\mathcal{D}$ from weak scaling region $\mathcal{W}$. The result is shown in Fig. 1(b) by a blue solid curve.

In a similar way, solving the system of equations $f_u(u_H)=0$, $f_\gamma(u_H,\gamma)=0$, and (23), containing resummed functions for fixed values of n we find the stability border $\epsilon_1(n)$ separating in the (n,ϵ) plane the weak scaling region $\mathcal{W}$ from the strong scaling region $\mathcal{S}$. The result is given in Fig. 1(b) by a blue dashed curve.

The agreement of these results with those of the NPRG approach [29] (black curves in Fig. 1) is worse than for the ϵ expansion for the borderlines [green curves in Fig. 1(a)]. Nevertheless the qualitative shift up in values of ϵ for fixed n is present in the region $n>1$ and $\epsilon<1$ compared to the two-loop order result. Also the region $\mathcal{W}$ for the weak scaling regime is wider than within the two-loop approximation [16]. A possible existence of an anomalous regime $\mathcal{A}$ is considered in the next subsection.

B. Stability of the FP describing strong scaling regime and possible existence of region $\mathcal{A}$

The existence of the region $\mathcal{A}$ in (n,d) plane is connected with the behavior of the FP $\{u_H,\gamma_C,\rho_C\}$. For some values of n at small ϵ the value of ρ_C has the tendency to go to unity, approaching the FP $\{u_H,\gamma_C,1\}$, which was shown to be unstable within the two-loop approximation [16]. Also it was proven that in this loop order the value of ρ_C being very close to 1 never reaches this value. Nevertheless, note that $\rho\to1$ means the time scale ratio $w\to\infty$; therefore, the behavior of the order parameter is much faster than that of the conserved density. However, the dynamical critical exponents for the order parameter and for the conserved density remain the same for all values of FP $\{u_H,\gamma_C,\rho_C\}$.

We check these results of Ref. [16] looking for FP solutions of the beta functions (8), (9), and (16) with all nonzero coordinates $\{u^*,\gamma^*,\rho^*\}$ using the same resummed five-loop functions as in the subsection above. We calculate also the critical exponent ω_ρ for this FP. Our results for ρ_C and ω_ρ obtained at fixed $\epsilon=0.25$ as functions of n are shown in Fig. 2.

Figure 2 qualitatively supports the situation observed already within the two-loop order [16]. The value of ρ_C numerically is always less than 1; however one does not see this within a resolution of Fig. 2, while ω_ρ is always positive. That means stability of the FP $\{u_H,\gamma_C,\rho_C\}$ in the full region of existence. These results were expected, because only two-loop expressions for dynamical RG functions are accessible: taking into account high-loop orders for static functions only does not lead to considerable changes of the results for the dynamical values.

However looking at our solution for the time scale ratio in the region $\mathcal{A}$ found in the NPRG approach we observe the following. Increasing n from the small values to larger ones at fixed dimension up to the boundary where the decoupling region is reached, ρ_C increases first and then decreases

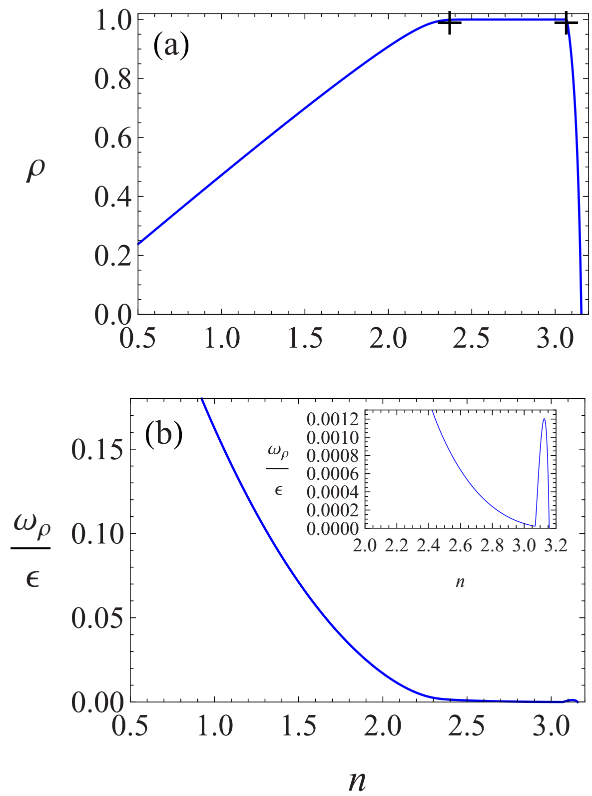

FIG. 2. Values of ρ_C calculated from $\zeta(u_H,\gamma_C,\rho_C)=0$ (a) as well as $\omega_\rho(u_H,\gamma_C,\rho_C)$ calculated from (22) (b) at $\epsilon=0.25$ as functions of the order parameter dimension n obtained on the base of the resummed five-loop static RG functions. The inset depicts the right bottom part of the corresponding picture. The marks "+" denote artificial region for $\epsilon=0.25$ shown by black dots in Fig. 1.

showing a maximal value. This maximum is always smaller than one; however below $\epsilon=0.4$ this maximum is of order $\rho_C\approx 0.99$. Decreasing ϵ further ρ seems to develop a plateau value at almost 1 over a certain region of n (see Fig. 2), which is qualitatively in agreement with the existence of a region $\mathcal{A}$.

To be specific we define a region where ρ_C rises to the value of almost one. Let us take the values n when ρ_C is crossing 0.999 as a left part of a border for this artificial region. Then we choose then position of the "beak" shown in the inset of Fig. 2 as the right part of the border. Data obtained in this way are shown in Fig. 1 by black dots for values ϵ from 0.05 till 0.35 with a step size of 0.05. For $\epsilon=0$ we plot one-loop values $n=2$ and $n=4$. This region is also marked by "+" in the upper picture of Fig. 2. As one can see in Fig. 1 this artificial region turns out to be roughly similar to region $\mathcal{A}$ found in the NPRG analysis [29]; however, increasing the limiting value for border from $\rho=0.999$ to larger values below 1 one shifts the left borderline to larger values of n. In the limit $\rho\to 1$ this region of course disappears in our approach.

V. DISCUSSION AND CONCLUSIONS

In this study we have reexamined perturbative field-theoretical results obtained for Model C within two-loop order without resummation. In order to do this we have used already known five-loop RG expressions for the field-theoretical $O(n)$ model, as well as ϵ expansions known within high-loop orders for marginal dimension of the diluted $O(n)$ model and for the dynamical critical exponent of Model A. In particular, using four-loop ϵ expansion for $c\eta$ we have derived the ϵ expansion for borderline n_1, that separates weak and strong scaling regimes of Model C up to $O(\epsilon^4)$. Our result is compared with the result of NPRG approach [29] obtained recently for the Model C dynamics.

Our analysis qualitatively supports NPRG results [29] for $\epsilon\leqslant 1$ and $n\geqslant 1$, leading to the wider region for weak scaling regime and larger values $\epsilon_\alpha(n)$ and $\epsilon_1(n)$ in comparison with results obtained on the base of two-loop RG functions without resummation. However there are striking deviations in other regions of the (n,ϵ) plane. Note that φ^4 models are not appropriate for investigations for $d<3$, because of a different physics in low dimensions. Moreover, for $\epsilon>1$ (that is $d<3$) it is obvious that ϵ expansion does not work because the expansion parameter ϵ now is not small. While in the NPRG study [29] only the local potential approximation with scale dependent constant at gradient term (LPA$'$) was used with the truncation of a local potential of φ^4 type. This approximation relies on the assumption that correlation functions with large number of legs have a small impact on the RG flow as those with fewer legs, as well as on the assumption that the anomalous dimension is small. This is not the case for the low-dimensional systems ($d<3$); therefore, high-order field expansions should be used to go below $d=3$.

Finite truncations within the NPRG approach have another unpleasant effect: they induce a residual dependence of the physical quantities on the choice of a cutoff function used to suppress low-momentum fluctuations. The NPRG approach of Ref. [29] uses a sharp θ cutoff, which, although nonanalytical in its nature, leads to analytical expressions of nonperturbative β functions. However other cutoff functions like a power law or an exponential can be used. For instance, data of a NPRG study of the two-dimensional $O(2)$ model with the help of an optimized exponential cutoff function are in very good agreement with universal features of the Kosterlitz-Thouless transition [19]. Probably the nonmonotonic behavior of the borderline between regions $\mathcal{W}$ and $\mathcal{S}$ obtained in Ref. [29] (black dashed curve in Fig. 1) is a consequence of the use of the θ-cutoff function. In any case it is interesting to have data from other choices of a cutoff function. The check of Model C behavior at $n\to 0$ proposed in Ref. [29] on the base of Monte Carlo simulations for SAW models in fractal dimensions is not reliable, since the similarity between SAW and $O(n\to 0)$ model was proven only for the static case [44]. Moreover, the relation between the noninteger dimension arising due to the analytic continuation in field theories (e.g., via ϵ expansion) and fractal dimension is not straightforward [45].

Conditions that govern the borders between regions $\mathcal{D}$ and $\mathcal{W}$ (21) as well as between $\mathcal{W}$ and $\mathcal{S}$ (24) are valid to all orders of perturbation theory. Therefore, it can be also checked within the NPRG approach on the base of a simpler $O(n)$ model and dynamical Model A. However Model A critical dynamics was studied within NPRG for a scalar order parameter only [26]. Moreover, despite recent NPRG studies of $O(n)$ models in fractional dimensions [20,21] the condition for a vanishing exponent α was not a subject of interest.

The most intriguing point and a genuine Model C feature is that NPRG results are in favor of the existence of an anomalous region $\mathcal{A}$. Within this approach region $\mathcal{A}$ is described by a stable solution with $\gamma \neq 0$ and $\rho = 1$, as well as with dynamical critical exponents $z < z_m$. This result requires its confirmation by studies with more elaborate truncations as well as with other choices of a cutoff function. Concerning the perturbative field-theoretical RG approach, region $\mathcal{A}$ can be checked only if high-loop order dynamical RG functions will be accessible. They can possibly lead to new solutions for dynamical FPs or a change of stability of the present found FPs. However one should be careful trusting new solutions obtained only in high orders of perturbations, because they might be controversial (see, e.g., Ref. [46], where one can find a discussion about conflicting results of perturbed field-theoretical RG and NPRG approaches for frustrated magnets).

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