Spectral correlations in finite-size Anderson insulators

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We investigate spectral correlations in quasi-one-dimensional Anderson insulators with broken time-reversal symmetry. While energy levels are uncorrelated in the thermodynamic limit of infinite wire length, some correlations remain in finite-size Anderson insulators. Asymptotic behaviors of level-level correlations in these systems are known in the large- and small-frequency limits, corresponding to the regime of classical diffusive dynamics and the deep quantum regime of strong Anderson localization. Employing nonperturbative methods and a mapping to the Coulomb-scattering problem, recently introduced by M. A. Skvortsov and P. M. Ostrovsky [JETP Lett. **85**, 72 (2007)], we derive a closed analytical expression for the spectral statistics in the classical-to-quantum region bridging the known asymptotic behaviors. We further discuss how Poisson statistics at large energies develop into Wigner-Dyson statistics as the wire-length decreases.

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I. INTRODUCTION

The spectral statistics of a quantum-mechanical system gives interesting insight into its dynamics. It is, for example, common to uncover integrable or chaotic dynamics by establishing Poisson or Wigner-Dyson statistics in the spacing of energy levels [1]. Universal spectral properties typically emerge if certain scaling limits are applied. It seems less appreciated that it is sometimes the nonuniversal corrections which store more interesting information.

Low-dimensional, disordered systems of noninteracting particles are a prominent example of such a situation. At large length scales the quantum dynamics in such systems is dominated by strong Anderson localization. Eigenstates then occupy a negligible fraction of the total system's volume, and universal Poisson statistics applies in the thermodynamic limit. It predicts the absence of correlations in disorder averages $\langle \cdots \rangle$ of the global density of states ν at different energies, $\langle \nu(\epsilon)\nu(\epsilon+\omega)\rangle \stackrel{L\to\infty}{\to} \langle \nu(\epsilon)\rangle\langle \nu(\epsilon+\omega)\rangle$. In any finite-size Anderson insulator the situation is more interesting. Nonuniversal correlations survive and give information about the system's quantum-mechanical dynamics.

Correlations of close-by levels store information on the deep quantum regime establishing in the long-time limit. The accumulation of quantum interference processes fully localizes particles at large time scales. The remaining dynamical processes are tunneling events between almost degenerate, far-distant eigenstates [2–5]. Mott's picture of resonant levels gives an intuitive explanation for the spectral correlations in this deep quantum regime: The hybridization Γ between distant localized states decays exponentially on the localization length ξ. For a given level separation ω there is thus a distance, the Mott scale l_ω, above which Γ falls below ω. Levels separated by energies larger than ω are uncorrelated. Consequently, in a d-dimensional Anderson insulator $\langle \nu(\epsilon)\nu(\epsilon+\omega)\rangle \stackrel{L\gg\xi}{\to} [1-\alpha_d(l_\omega/L)^d]\langle \nu(\epsilon)\rangle\langle \nu(\epsilon+\omega)\rangle$, and the connected correlation function of nearby levels is proportional to a power of the Mott scale, $\langle \nu(\epsilon)\nu(\epsilon+\omega)\rangle_{\rm con.} = \alpha_d(l_\omega/L)^d$, where α_d is some numerical factor.

Correlations at large level separation ω, on the other hand, give information on the dynamics on short time scales. Quantum interference processes in the short-time limit remain largely undeveloped, and spectral correlations of far-distant levels reflect classical diffusion.

The classical-to-quantum crossover of spectral correlations in finite-size Anderson insulators is unexplored. Evidently, this is because the strongly localized regime presents the strong-coupling limit of the underlying effective field theory for disordered systems [6,7] whose analysis is challenging. The situation is similar to that previously encountered in fully ergodic *chaotic* systems. Early on, it was conjectured that the classical-to-quantum crossover of spectral correlations in chaotic systems follows Wigner-Dyson statistics [8]. A proof of this "Bohigas-Giannoni-Schmit conjecture," however, turned out challenging. The reason is similar to that in Anderson insulators: arbitrary orders of quantum interference processes have to be taken into account to describe how classical correlations at large energies [9] evolve into avoided crossings at small energies. In chaotic systems this requires the summation of infinite numbers of periodic orbits and their encounters. Progress in this direction has been achieved only recently [10–12].

In the present paper we address the classical-to-quantum crossover in the spectral correlations of finite-size Anderson insulators. Concentrating on quasi-one-dimensional wires belonging to the unitary symmetry class, we derive a closed analytical expression for the connected level-level correlation function,

$$K(L,\omega) = \frac{\langle \nu(\epsilon)\nu(\epsilon+\omega)\rangle}{\langle \nu(\epsilon)\rangle\langle \nu(\epsilon+\omega)\rangle} - 1. \qquad (1)$$

Our result is valid at arbitrary level-separations ω and thus bridges the known asymptotic behaviors of correlations between close-by and far-distant levels.

The outline of the paper is as follows. Section II reviews the known asymptotic behaviors of the level-level correlation function in the classical and deep quantum regimes. In Sec. III we state our main result, i.e., spectral statistics in the classical-to-quantum crossover region, and explain in detail its derivation. In Sec. IV we present some results for the Poisson-to-Wigner-Dyson crossover of level statistics with decreasing wire length. Section V summarizes our results. Several technical details are delegated to the appendixes. Throughout the paper we set $\hbar = 1$.

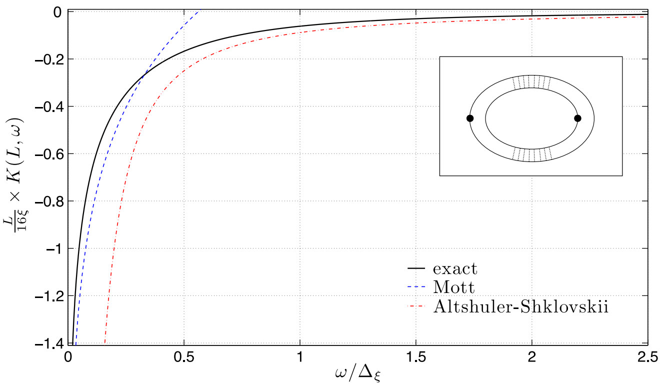

FIG. 1. Spectral correlations in a finite-size Anderson insulator, Eqs. (8) and (9). For reference we also show previously known limits (dashed lines), i.e., $K(L,\omega) = -(4\xi/L)(\Delta_\xi/2\omega)^{3/2}$ at large frequencies and $K(L,\omega) = -(8\xi/L)[\log(\Delta_\xi/\omega) - 2\gamma]$ at small frequencies [see Eq. (10)]. Inset: Feynman diagram accounting for Altshuler-Shklovskii correlations [Eq. (2)]; dashed lines represent classical diffusion modes.

II. CLASSICAL AND DEEP QUANTUM REGIMES

In this section we briefly review known asymptotic behaviors of the level-level correlation function in Anderson insulators $L \gg \xi$.

At short time scales quantum interference processes are largely undeveloped. The dynamics is not affected by weak localization corrections and remains classically diffusive. The leading level-level correlation function, Eq. (1), in this classical regime was derived by Altshuler and Shklovskii three decades ago [13]. Using diagrammatic perturbation theory they found for a quasi-one-dimensional geometry

$$K(\omega) \propto -\frac{\xi}{L}\left(\frac{\Delta_\xi}{\omega}\right)^{3/2}, \quad \omega \gg \Delta_\xi. \tag{2}$$

Here we introduce the localization length $\xi = 2\pi\nu DS$, the level spacing in a localization volume $\Delta_\xi = (2\pi\nu S\xi)^{-1}$, and ν is the average density of states. S is the wire cross section, and $D = v_F^2\tau$ denotes the diffusion constant, with v_F being the Fermi velocity and τ being the elastic scattering time. The relevant diagram is shown in the inset of Fig. 1. Noting that it contains two classical diffusion modes $D(q,\omega) \propto 1/(q^2 + i\omega)$, the ω dependence of Eq. (2) is readily understood from simple power counting $\int_{\sqrt{\omega}} dq/q^4 \sim \omega^{-3/2}$.

Equation (2) gives the leading contribution in the small parameter Δ_ξ/ω. Corrections of higher orders $O(\Delta_\xi/\omega)$ store information on quantum interference processes. These start to become relevant on time scales exceeding the classical regime $t \ll 1/\Delta_\xi$. In the nonclassical regime diffusion slows down due to weakly localizing quantum interference processes. Accumulation of these processes modifies classical diffusion, and localization eventually becomes strong as one approaches the Heisenberg time $t \sim 1/\Delta_\xi$.

In the strongly Anderson localized regime $t \gg 1/\Delta_\xi$ classical diffusion is stopped completely. The remaining dynamical processes in this deep quantum regime are probed by correlations of close-by levels $\omega \ll \Delta_\xi$. Correlation function (1) at small level separations shows logarithmic level repulsion [5,14],

$$K(\omega) \propto -\frac{\xi}{L}\log(\Delta_\xi/\omega), \quad \omega \ll \Delta_\xi. \tag{3}$$

Equation (3) neglects corrections smaller than $O(\omega/\Delta_\xi)$ and is understood within Mott's picture of resonant levels [4] already mentioned in the Introduction. Indeed, correlations of nearby levels are due to tunneling events between almost degenerate states at a distance of the Mott scale. The physics is captured by the two-level Hamiltonian [5,14]

$$H(\epsilon,\delta\epsilon,x) = \begin{pmatrix} \epsilon + \delta\epsilon & \Gamma(x) \\ \Gamma(x) & \epsilon - \delta\epsilon \end{pmatrix}, \tag{4}$$

where the hybridization $\Gamma(x) \approx \Delta_\xi e^{-x/\xi}$ accounts for a finite overlap of wave functions centered at distances $\xi \lesssim x < L$ (see also Fig. 1). Here ϵ and $\delta\epsilon$ are the mean level and level splitting, respectively, which for simplicity are both assumed uniformly distributed. Correlation function (1) for the simple model (4) reads

$$K(L,\omega) + 1 = \int_\xi^L \frac{dx}{L}\left\langle \delta\left(\omega - \sqrt{\delta\epsilon^2 + \Delta_\xi^2 e^{-\frac{2x}{\xi}}}\right)\right\rangle_{\delta\epsilon}, \tag{5}$$

where the average $\langle\cdots\rangle_{\delta\epsilon}$ is over the level splitting. For close-by levels (i.e., ω smaller than the support of the distribution of $\delta\epsilon$) integral equation (5) receives its finite contributions from distances $x \gtrsim \ell_\omega \equiv -2\xi\ln(\Delta_\xi/\omega)$, larger than the Mott scale. Subtracting the uncorrelated contribution to the level correlation function, one finds $K(L,\omega \ll \Delta_\xi) \propto \ell_\omega/L$. That is, level correlations are proportional to the configuration space volume for which hybridization between localized wave functions is strong enough to result in noticeable level repulsion.

The asymptotic behaviors of the level-level correlation function at large and small level separations, Eqs. (2) and (3), have been well established for decades [13,14]. The correlation function bridging the classical and deep quantum regimes is unknown. In the next section we derive a closed analytical expression for the latter.

III. CLASSICAL-TO-QUANTUM CROSSOVER

We next present our main result, i.e., the level-level correlation function (1) in a quasi-one-dimensional Anderson insulator. We then proceed with a detailed derivation of our result. In Sec. III B we introduce the relevant field theory. Section III C discusses a mapping from the Anderson localization problem to a Coulomb-scattering problem.

A. Results

We start out with a compact representation of the level-correlation function (1) in terms of the Green's function for the Coulomb-scattering problem,

$$K(L,\omega) = -\frac{2\pi\xi}{L}\mathrm{Re}\lim_{\mathbf{r}\to\mathbf{r}_0}\partial_\kappa^2 G_0(\mathbf{r},\mathbf{r}_0), \tag{6}$$

applicable for quasi-one-dimensional Anderson insulators, $L \gg \xi$, within the unitary symmetry class. Its derivation is given in the next sections, where we also introduce the Green's function G_0 for the nonrelativistic $3d$ Coulomb problem [15].

A general closed-form expression of the latter was derived by Hostler half a century ago [16,17],

$$G_0(\mathbf{r},\mathbf{r}') = \frac{(\partial_u - \partial_v)\sqrt{u}K_1(2\sqrt{\kappa u})\sqrt{v}I_1(2\sqrt{\kappa v})}{2\pi|\mathbf{r}-\mathbf{r}'|}. \quad (7)$$

Here we have introduced $u = r + r' + |\mathbf{r}-\mathbf{r}'|$, $v = r + r' - |\mathbf{r}-\mathbf{r}'|$, and K_m and I_m are Bessel functions of the first kind. Inserting the above Green's function (7) into Eq. (6) gives the level-level correlation function in a finite-size Anderson insulator,

$$K(L,\omega) = -\frac{16\xi}{L}\mathrm{Re}\,\mathcal{K}(4\omega/i\Delta_\xi), \quad (8)$$

where

$$\begin{aligned}\mathcal{K}(z) &= 2\partial_z K_0(\sqrt{z})I_0(\sqrt{z}) \\ &= (1/\sqrt{z})[K_0(\sqrt{z})I_1(\sqrt{z}) - K_1(\sqrt{z})I_0(\sqrt{z})]. \quad (9)\end{aligned}$$

Equations (8) and (9) are the main result of this paper. They describe how the Altshuler-Shklovskii correlations in the classical regime evolve into logarithmic level repulsion in the deep quantum regime. Poisson statistics only applies in the thermodynamic limit where the residual correlations in Anderson insulators vanish, $K(L\to\infty,\omega) = 0$. From Eq. (9) we readily extract the asymptotic correlations of far-distant and close-by levels [18],

$$K(L,\omega) = -\frac{4\xi}{L}\begin{cases}\left(\frac{\Delta_\xi}{2\omega}\right)^{\frac{3}{2}} + \frac{3}{32}\left(\frac{\Delta_\xi}{2\omega}\right)^{\frac{5}{2}} + \cdots, & \omega \gg \Delta_\xi, \\ 2\log\left(\frac{\Delta_\xi}{\omega}\right) - 4\gamma + \frac{3\pi\omega}{2\Delta_\xi} + \cdots, & \omega \ll \Delta_\xi.\end{cases} \quad (10)$$

Here $\gamma \simeq 0.577$ is the Euler-Mascheroni constant. The spectral correlation function, Eqs. (8) and (9), is shown in Fig. 1. For reference we also display the previously known asymptotic behaviors.

B. Field theory

Our derivation of representation (6) starts with a field-theory description of the level-level correlation function [6,14],

$$\begin{aligned}K(L,\omega) &= -\frac{\mathrm{Re}}{32}\int_0^L dx \int_0^L dx' \,\langle P_+[Q(x)]P_-[Q(x')]\rangle_Q, \\ P_\pm[Q] &= \mathrm{str}[(Q-\Lambda)(1\pm\Lambda)k].\end{aligned} \quad (11)$$

Here the average is with respect to the diffusive nonlinear σ-model action introduced below,

$$\langle\cdots\rangle_Q = \int \mathcal{D}Q\, e^{S_\sigma[Q]}. \quad (12)$$

Integration $\int \mathcal{D}Q$ is over 4×4 supermatrices $Q(x) = \{Q_{ss'}^{\alpha\alpha'}(x)\}$ from the supergroup $U(1,1|2)$ obeying the nonlinear constraint $Q^2(x) = \mathbb{1}$. Diagonal 2×2 top left and bottom right matrix blocks Q^{bb} and Q^{ff}, respectively, contain complex numbers. The off-diagonal blocks $Q^{\mathrm{bf,fb}}$ consist of Grassmann variables. The subscript indices $Q_{ss'}$, $s,s' = \pm$, discriminate between "retarded" and "advanced" components of the matrix field. The c-number content of the Q-field manifold reduces to the direct product of the hyperboloid $U(1,1)/[U(1)\times U(1)]$ in the bb block and the sphere $U(2)/[U(1)\times U(1)]$ in the ff block. It is parametrized by noncompact and compact variables, $1 \leqslant \lambda_{\mathrm{bb}}$ and $-1 \leqslant \lambda_{\mathrm{ff}} \leqslant 1$, respectively. The matrix $\Lambda = \{s\,\delta_{ss'}\}$ is the identity matrix in boson-fermion space but breaks symmetry in advanced-retarded space. $k = \Lambda\sigma_3^{\mathrm{bf}}$ is a diagonal matrix with σ_3^{bf} also breaking symmetry in boson-fermion space, and 'str' is the generalization of the matrix trace to graded space.

The diffusive nonlinear σ-model action reads [6,7]

$$S_\sigma[Q] = -\frac{\pi\nu S}{4}\int dx\, \mathrm{str}[2i\omega Q\Lambda + D(\partial_x Q)^2]. \quad (13)$$

It is the low-energy effective field theory for Anderson localization, here for a quasi-one-dimensional geometry and in the presence of a weak time-reversal symmetry-breaking vector potential. We refer, for a derivation of action (13), to Refs. [6,7] and here point out its structural similarity to the Hamilton function of a classical ferromagnet in an external magnetic field. The latter breaks rotational invariance of the exchange interaction, and a similar role is played by the potential $V(Q) \sim \mathrm{str}(Q\Lambda)$ in the σ-model action. $V(Q)$ breaks the invariance of the kinetic term $K(Q) \sim \mathrm{str}([\partial_x Q]^2)$ under general rotations $U(1,1|2)$. The strength of symmetry breaking is given by the level separation ω. Energies $\omega \gg \Delta_\xi$ larger than the level spacing imply strong symmetry breaking. Q fields are then pinned to the mean-field Λ, and small fluctuations can be accounted for perturbatively. A straightforward perturbative expansion at large energies gives the leading correlation function $K(\omega \gg \Delta_\xi) \propto \omega^{-3/2}$ discussed in the previous section. Once $\omega \lesssim \Delta_\xi$ falls below the level spacing, fluctuations become large, acting to restore the full symmetry of the kinetic term. A direct integration of Eq. (11) is hindered by the presence of the rotational symmetry-breaking potential $V(Q)$. Nonperturbative methods, discussed below, have to be applied to address the low-energy correlations $\omega \lesssim \Delta_\xi$.

C. Nonperturbative solution

We next derive the alternative representation of the correlation function (11) in terms of the Green's function for the Coulomb-scattering problem [15]. To this end we recall that one can map the integral (11) to a set of equivalent differential equations [6,19]. Indeed, identifying Q with the coordinate of a multidimensional quantum particle and the wire coordinate with time, Eq. (13) reads as the Feynman path integral of a particle with kinetic energy $K(Q)$ moving in the potential $V(Q)$. Alternatively to calculating the path integral, one can solve the corresponding "Schrödinger equation," known as transfer-matrix equations.

Similar to a radial potential in quantum mechanics, the high degree of symmetry of the potential (V is invariant under similarity transformations of Q, leaving Λ invariant) reduces the effective dimensionality of the problem. As detailed in Appendix A, correlation function (11) reduces to an integral over the c-number variables λ_{bb} and λ_{ff}. The integrand is expressed in terms of a ground- and an excited-state wave function of the underlying Schrödinger equation. As the Laplace operator on the Q-matrix manifold has a rather complex structure, closed solutions of the latter are not available.

Progress in this direction has been made in a recent work by Skvortsov and Ostrovsky [15]. There the authors elaborate on a connection between localization in quasi-one-dimensional systems within the unitary class to scattering in a Coulomb potential. Changing from angle to "Coulomb" coordinates,

$$\lambda_{\rm ff} = (r - r_1)/2, \quad \lambda_{\rm bb} = (r + r_1)/2, \tag{14}$$

a major simplification occurs when the latter are understood as elliptic coordinates of a three-dimensional problem with cylindrical symmetry,

$$r = \sqrt{z^2 + \rho^2}, \quad r_1 = \sqrt{(z-2)^2 + \rho^2}. \tag{15}$$

Here (ρ,φ,z) are the usual cylindrical coordinates. Following the outlined procedure (and leaving details to Appendix B) the correlation function (11) becomes

$$K(L,\omega) = \frac{\xi^2 \text{Re}}{2\pi L^2} \int_0^{\frac{L}{\xi}} dt' \int \frac{d\mathbf{r}}{r} \Phi_0(\mathbf{r}, L/\xi - t') \Phi_1(\mathbf{r}, t'), \tag{16}$$

where Φ_0 and Φ_1 are the ground- and excited-state wave functions, respectively. The latter are solutions of the following transfer-matrix equations in Coulomb coordinates,

$$\left(\frac{1}{r_1 r} \partial_t - \partial_{\mathbf{r}}^2 + \frac{2\kappa}{r} \right) \Phi_0(\mathbf{r},t) = 0, \tag{17}$$

$$\left(\frac{1}{r_1 r} \partial_t - \partial_{\mathbf{r}}^2 + \frac{2\kappa}{r} \right) \Phi_1(\mathbf{r},t) = \frac{1}{r} \Phi_0(\mathbf{r},t). \tag{18}$$

Here $\partial_{\mathbf{r}}$ is the three-dimensional Laplace operator, $\kappa = \omega/(4i\Delta_\xi)$, and $t = x/\xi$. Equations (17) and (18) are supplemented by the boundary conditions $\Phi_0(\mathbf{r},0) = 1/r_1$ and $\Phi_1(\mathbf{r},0) = 0$. In the strongly localized regime of interest, homogeneous solutions of the transfer-matrix equations give the leading contribution to Eq. (16). Indeed, inhomogeneous solutions decay exponentially from the boundaries, and their contributions to the integral (16) can be neglected for $L \gg \xi$. Dropping t dependencies of the wave functions, Eqs. (17) and (18) become spherically symmetric. The corresponding boundary conditions read $\Phi_0(\mathbf{r}_0) = 1/r_1$ and $\Phi_1(\mathbf{r}_0) = 0$, with $\mathbf{r}_0 = (0,0,2)^t$. The reduction to a problem of a higher degree of symmetry is a key simplification which allows for an analytical calculation of (16) to leading order in the small parameter ξ/L.

Following Ref. [15], we introduce the zero-energy Green's function for the nonrelativistic $3d$ Coulomb problem,

$$\left(\partial_{\mathbf{r}}^2 - \frac{2\kappa}{r} \right) G_0(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'). \tag{19}$$

Imposing the usual boundary condition $G_0(\mathbf{r},\mathbf{r}') \overset{\mathbf{r}\to\mathbf{r}'}{\mapsto} \delta(\mathbf{r} - \mathbf{r}')$, the homogeneous solution to Eq. (17) with the required boundary condition affords the representation [15]

$$\Phi_0(\mathbf{r}) = -4\pi G_0(\mathbf{r},\mathbf{r}_0). \tag{20}$$

Similarly, it is verified that the convolution

$$\Phi_1(\mathbf{r}) = -\int \frac{d\mathbf{r}'}{r'} G_0(\mathbf{r},\mathbf{r}') \Phi_0(\mathbf{r}') \tag{21}$$

is a t-independent solution of Eq. (18) with the required boundary condition. Inserting solutions (20) and (21) into (16), one confirms that [20]

$$\begin{aligned} K(L,\omega) &= -\frac{8\pi\xi}{L} \text{Re} \int d\mathbf{r} \int d\mathbf{r}' \, G_0(\mathbf{r}_0,\mathbf{r}) \\ &\quad \times \frac{1}{r} G_0(\mathbf{r},\mathbf{r}') \frac{1}{r'} G_0(\mathbf{r}',\mathbf{r}_0) \\ &= -\frac{2\pi\xi}{L} \text{Re} \lim_{\mathbf{r}\to\mathbf{r}_0} \partial_\kappa^2 G_0(\mathbf{r},\mathbf{r}_0). \end{aligned} \tag{22}$$

This completes our derivation of the spectral correlation function (1) in terms of the Green's function for the Coulomb-scattering problem.

IV. FROM POISSON TO WIGNER-DYSON STATISTICS

We next address how spectral correlations in a finite-size Anderson insulator turn into Wigner-Dyson correlations as the wire length is reduced. We recall that in the fully ergodic quantum dot limit level correlations follow Wigner-Dyson statistics $K(L \to 0,\omega) = -(\Delta/\omega\pi)^2 \sin^2(\omega\pi/\Delta)$. Here $\Delta = (\nu SL)^{-1}$ is the level spacing of the wire of length L. Noting that $\Delta/\Delta_\xi \sim \xi/L$, one can thus study how Altshuler-Shklovskii correlations at $\omega \gg \Delta_\xi$ evolve into Wigner-Dyson correlations as the wire length decreases.

Correlations at arbitrary ratios L/ξ can be derived from the inhomogeneous transfer-matrix equations (17) and (18). For levels separated by $\omega \gg \Delta_\xi$ the potential V pins the wave functions to the region $r_1 \ll 1$ enforced by the boundary conditions. We can thus approximate for the ground-state wave function

$$\left(\frac{1}{2r_1} \partial_t - \partial_{\mathbf{r}}^2 + \kappa \right) \Phi_0(\mathbf{r},t) = 0. \tag{23}$$

The radial symmetry of Eq. (23) substantially simplifies the problem. Starting out from the ansatz $\Phi(\mathbf{r},t) = e^{-F(\kappa,t)r_1}/r_1$ the function F satisfies $\partial_t F + 2F^2 - 2\kappa = 0$. Employing the boundary condition $F(\kappa,0) = 0$, one then finds

$$\Phi_0(\mathbf{r},t) = \frac{1}{r_1} e^{-\sqrt{\kappa}\tanh(2\sqrt{\kappa}t) r_1}. \tag{24}$$

Equation (24) interpolates between the known limits at small and large ratios L/ξ. Indeed, $\tanh(2\sqrt{\kappa}t)$ approximates to 1 and $2\sqrt{\kappa}t$ for ω/Δ much larger and smaller than ξ/L, respectively. This reflects the typical decaying and oscillating behaviors of the ground-state wave function in the two limits. A similar calculation, detailed in Appendix C, gives for the level-level correlation function

$$\begin{aligned} K(L,\omega) &= \frac{\Delta\xi \, \text{Im}}{\pi L} \int_0^{\frac{L}{\xi}} dt' \left(1 - e^{-H^{-1}(\omega/i\Delta_\xi, L/\xi, t')}\right) \\ &\quad \times \partial_\omega H(\omega/i\Delta_\xi, L/\xi, t'), \end{aligned} \tag{25}$$

with

$$H(z,t,t') = \frac{\cosh(\sqrt{z}t')\cosh(\sqrt{z}(t'-t))}{\sqrt{z}\sinh(\sqrt{z}t)}. \tag{26}$$

We emphasize that Eqs. (25) and (26) hold for $\omega \gg \Delta_\xi$ and arbitrary ratios L/ξ.

The analytical result (25) and (26) is displayed in Fig. 2. It shows how the Altshuler-Shklovskii correlations in long wires evolve into the Wigner-Dyson correlations as one approaches

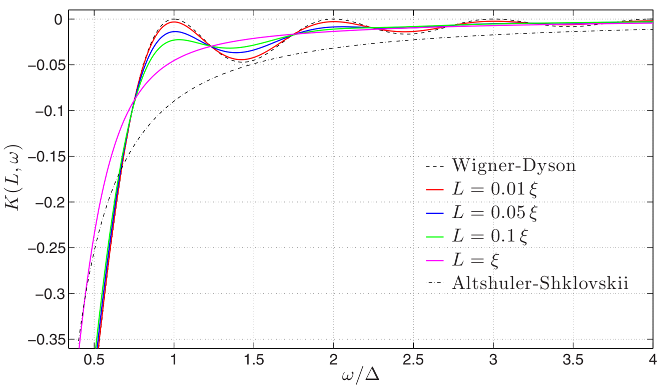

FIG. 2. Level-level correlations [Eqs. (25) and (26)] for different ratios L/ξ in the Poisson-to-Wigner-Dyson crossover region. For comparison we also show Wigner-Dyson and Althsuler-Shklovskii correlations (dashed and dash-dotted lines).

the quantum-dot limit. Unfortunately, we do not know the corresponding result for correlations between levels separated by $\omega \lesssim \Delta_\xi$. The transfer-matrix equations in the Poisson-to-Wigner-Dyson crossover region then lack rotational symmetry, and we were not able to find analytical solutions.

V. SUMMARY AND DISCUSSION

In this paper we have derived the leading level-level correlations in quasi-one-dimensional Anderson insulators with broken time-reversal symmetry. While energy levels are uncorrelated in the thermodynamic limit correlations remain in finite-size Anderson insulators.

The correlations of far-distant and nearby levels reflect the dynamics in the classical diffusive regime and the deep quantum regime of strong Anderson localization. They have been well established for decades [13,14]. This paper discusses the previously unknown correlations at arbitrary level separations. Specifically, our result describes how Altshuler-Shklovskii correlations at large separations turn into logarithmic level repulsion at small separations. Only in the limit of infinite wire length do the correlations vanish, in accordance with universal Poisson statistics expected in the nonergodic system.

We further discuss how Altshuler-Shklovskii correlations in Anderson insulators turn into Wigner-Dyson correlations with decreasing wire length. A corresponding analysis for correlations between close-by levels remains an open problem.

Finally, we would like to put our results into the context of previous works. Spectral correlations of quasi-one-dimensional disordered systems in the Wigner-Dyson-to-Poisson crossover have been addressed in Ref. [14]. From numerical solutions of the relevant transfer-matrix equations a qualitative understanding of the level-level correlation function in the Wigner-Dyson-to-Poisson crossover was obtained. Local correlations in the density of states within a localization volume were derived in Ref. [15]. The authors find the exact ground-state wave function of the homogeneous transfer-matrix equation by mapping the equation to the Coulomb-scattering problem. It is shown that correlations of different eigenfunctions are different in quasi- and strictly one-dimensional geometries [3]. Correlation functions for the *global* density of states (discussed in this work) and the *local* density of states (discussed in Ref. [15]) both have representations in terms of the Coulomb Green's function. A similar relation has been observed in Ref. [21] in the context of parametric correlations, where it was connected to a symmetry in the σ model. Reference [22] derives a perturbative expansion for the local density of states correlation function at small frequencies. Its general expression, e.g., reproduces statistics of single localized wave functions and predicts reentrant behavior at the Mott scale (similar to that in strictly $1d$ chains [3]). It would be interesting to investigate how the findings reported in the present work are obtained within this approach. Furthermore, extensions of the discussed results to other symmetry classes remain an open problem.

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APPENDIX A: TRANSFER-MATRIX EQUATIONS

For self-consistency we state the level-level correlation function in a quasi-one-dimensional wire. In the unitary symmetry class the latter depends on the c-number variables $\lambda_{\rm bb}, \lambda_{\rm ff}$. Following Refs. [6,14,19], one finds

$$K(L,\omega) = \frac{\xi^2}{L^2}\mathrm{Re}\int_0^{\frac{L}{\xi}} dt' \int_1^\infty d\lambda_1 \int_{-1}^1 \frac{d\lambda}{\lambda_1 - \lambda} \times \Psi_0(\lambda_1,\lambda,L/\xi - t')\Psi_1(\lambda_1,\lambda,t'). \quad \text{(A1)}$$

For notational convenience we here introduce $\lambda_{\rm bb} \equiv \lambda_1$ and $\lambda_{\rm ff} \equiv \lambda$, and Ψ_0 and Ψ_1 are the ground-state and excited-state wave functions, respectively. The latter follows the transfer-matrix equations

$$(\partial_t + 2\hat{H}_0)\Psi_0(\lambda_1,\lambda,t) = 0, \quad \text{(A2)}$$

$$(\partial_t + 2\hat{H}_0)\Psi_1(\lambda_1,\lambda,t) = (\lambda_1 - \lambda)\Psi_0(\lambda_1,\lambda,t), \quad \text{(A3)}$$

where we introduced $t = x/\xi$. The Hamilton operator reads $\hat{H}_0 = \Delta_Q + \hat{V}$, where

$$\Delta_Q = -\frac{(\lambda_1 - \lambda)^2}{2}\left(\partial_\lambda \frac{1-\lambda^2}{(\lambda_1-\lambda)^2}\partial_\lambda + \partial_{\lambda_1}\frac{\lambda_1^2 - 1}{(\lambda_1 - \lambda)^2}\partial_{\lambda_1}\right) \quad \text{(A4)}$$

is the Laplace operator on the Q-field manifold and

$$\hat{V} = -\frac{i\omega}{4\Delta_\xi}(\lambda_1 - \lambda) \quad \text{(A5)}$$

is the symmetry-breaking potential. The above equations should be solved with boundary conditions for an open wire,

$$\Psi_0(\lambda_1,\lambda,0) = 1, \quad \Psi_1(\lambda_1,\lambda,0) = 0. \quad \text{(A6)}$$

APPENDIX B: CYLINDRICAL COORDINATES

Employing elliptic coordinates ($\lambda_{\rm bb} \equiv \lambda_1$, $\lambda_{\rm ff} \equiv \lambda$)

$$\lambda = (r - r_1)/2, \quad \lambda_1 = (r + r_1)/2, \tag{B1}$$

$$r = \sqrt{z^2 + \rho^2}, \quad r_1 = \sqrt{(z-2)^2 + \rho^2}, \tag{B2}$$

the Hamilton operator takes the form

$$\hat{H}_0 = -\frac{r_1^2 r}{2}\left[\partial_z^2 + \partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{i\omega}{2\Delta_\xi r}\right]\frac{1}{r_1}. \tag{B3}$$

Notice that the differential operator is the usual Laplacian in cylindrical coordinates (ρ,φ,z), acting on cylindrical symmetric functions.

It is then convenient to make the ansatz $\Psi_i = r_1\Phi_i$ and to express the corresponding Schrödinger equations in three-dimensional coordinates

$$\left(\partial_t - r_1 r\partial_{\mathbf{r}}^2 + 2r_1\kappa\right)\Phi_0(\mathbf{r},t) = 0, \tag{B4}$$

$$\left(\partial_t - r_1 r\partial_{\mathbf{r}}^2 + 2r_1\kappa\right)\Phi_1(\mathbf{r},t) = r_1\Phi_0(\mathbf{r},t). \tag{B5}$$

Here $\kappa = \omega/(4i\Delta_\xi)$, and we recall that the boundary conditions read $\Phi_0(\mathbf{r},0) = 1/r_1$ and $\Phi_1(\mathbf{r},0) = 0$. The integration measure transforms into the three-dimensional volume element of cylindrical symmetric functions,

$$d\lambda \wedge d\lambda_1 = \frac{\rho}{rr_1}dz\wedge d\rho \mapsto \frac{d\mathbf{r}}{2\pi r r_1}, \tag{B6}$$

where $d\mathbf{r} = dz \wedge \rho d\rho \wedge d\varphi$. The level-level correlation function thus takes the form

$$K(L,\omega) = \frac{\xi^2 \mathrm{Re}}{2\pi L^2}\int_0^{\frac{L}{\xi}} dt' \int \frac{d\mathbf{r}}{r}\Phi_0(\mathbf{r},L/\xi - t')\Phi_1(\mathbf{r},t'). \tag{B7}$$

APPENDIX C: POISSON-TO-WIGNER-DYSON CROSSOVER

Level-level correlations in systems of arbitrary ratios L/ξ can be derived from the inhomogeneous transfer-matrix equations. For levels separated by $\omega \gg \Delta_\xi$ the potential V pins the ground-state wave function to the region $r_1 \ll 1$ enforced by the boundary condition. We may thus approximate Eq. (17) by

$$\left(\partial_t - 2r_1\partial_{\mathbf{r}}^2 + 2r_1\kappa\right)\Phi_0(\mathbf{r},t) = 0. \tag{C1}$$

Equation (C1) with boundary condition $\Phi_0(\mathbf{r},0) = 1/r_1$ is solved by $\Phi_0(\mathbf{r},t) = e^{-\sqrt{\kappa}\tanh(2\sqrt{\kappa}t)r_1}/r_1$. Similarly, one may verify that for $\omega \gg \Delta_\xi$ the excited-state wave function

$$\Phi_1(\mathbf{r},t) = -\tfrac{1}{2}\partial_\kappa\Phi_0(\mathbf{r},t) \tag{C2}$$

satisfies the transfer-matrix equation

$$\left(\partial_t - 2r_1\partial_{\mathbf{r}}^2 + 2r_1\kappa\right)\Phi_1^0(\mathbf{r},t) = r_1\Phi_0(\mathbf{r},t), \tag{C3}$$

with boundary condition $\Phi_1(\mathbf{r},0) = 0$. Inserting Φ_0 and Φ_1 into the level-level correlation function (B7), one arrives at Eqs. (26) and (25) in the main text. Notice that in the quantum-dot limit $\Phi_0(\mathbf{r},t) = e^{-2\kappa r_1 t}/r_1$ and $\Phi_1(\mathbf{r},t) = te^{-2\kappa r_1 t}$. This results in the Wigner-Dyson correlations $K(L,\omega) = -(\Delta/\omega\pi)^2\sin^2(\omega\pi/\Delta)$ applicable at arbitrary ratios ω/Δ. That is, the restriction $\omega \gg \Delta_\xi \sim \Delta L/\xi$ becomes irrelevant in the limit $L \ll \xi$.

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