Consistent bosonization-debosonization. I. A resolution of the nonequilibrium transport puzzle

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We critically reexamine the bosonization-debosonization procedure for systems including certain types of localized features (although more general scenarios are possible). By focusing on the case of a tunneling junction out of equilibrium, we show that the conventional approach gives results that are not consistent with the exact solution of the problem even at the qualitative level. We identify inconsistencies that can adversely affect the results of all types of calculations. We subsequently show a way to avoid these and proceed consistently. The extended framework that we develop here should be widely applicable.

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I. INTRODUCTION

It is well known that quantum physics gets richer and more peculiar as one considers reduced-dimensionality scenarios. Such scenarios are nowadays far from being esoteric. From highly anisotropic and artificially layered materials, to nanostructures, to confined ultracold atomic gases, examples abound of what was once a playground for theorists but modern experimental techniques turn into a practical reality. The physics of one-dimensional systems provides an example in which strong quantum effects together with interactions and restricted kinematics modify the expectations we bring with us from our more familiar three-dimensional world [1,2]. A case in point, the successful paradigm of the Landau Fermi liquid generically breaks down and gives rise to a new type of quantum fluid known as the Luttinger liquid [3]. A technical stepping stone on which the generality of this new paradigm rests is the technique known as bosonization.

The term bosonization refers to the practical possibility of describing the excitations of fermionic systems via a description based on bosonic degrees of freedom. The key observation is that for a fermionic one-dimensional system with strictly linear dispersion and no cutoff, the excitations at constant fermion number are particle-hole pairs that can be used to construct bosonic operators which completely capture the full excitation spectrum; such a view is known as the constructive approach [4]. The conceptual advantage of the constructive point of view is that it highlights the fact that bosonization is an exact correspondence between the two systems. There are also various complementary presentations based on the matching of correlators and known as the field-theoretic or the hydrodynamic approaches (these are, for example, more amenable to the conceptual description of the phenomenology of Luttinger liquids) [1,2,5].

Since the conditions of linear dispersion with large bandwidth and conserved particle number are all natural approximations for systems at sufficiently low temperatures (much lower than the bandwidth and the energy range of deviation from linearity), the applicability of bosonization is ubiquitous for all types of one-dimensional systems. Moreover, since within its applicability conditions the bosonization mapping is exact, it can equally well be used in both equilibrium and out-of-equilibrium situations. In particular, it provides a fertile ground for the study of transport phenomena in a variety of settings.

A. Transport problems and bosonization

Due to the versatility of the technique, a sizable fraction of the current theoretical studies of transport in one-dimensional settings rests on the use of bosonization. If the applied voltages are sufficiently low (in the same sense as discussed above for the temperature), one can use bosonization even if some biases are large compared with other characteristic energy scales in the system and the problem falls outside the linear-response regime. Bosonization thus provides us in many a case with a powerful way of addressing strongly nonequilibrium transport problems.

Given the vast array of possible experimental situations, there are many types of setups to be considered and one has to proceed in a class by class basis [6]. We shall restrict ourselves to the situation in which the leads are Fermi liquids in different equilibria (as is typical of Landauer-style setups; cf. Refs. [7,8]) and the nonequilibrium situation is confined to a zero-dimensional system (i.e., at most a Hilbert space with just a few degrees of freedom) adjoining them via point contacts. In particular, we shall focus on the important example of tunneling junctions of different types and steady-state conditions.

II. CASE STUDY: A SIMPLE JUNCTION OUT OF EQUILIBRIUM

How to set up the bosonization formalism so that it remains valid under nonequilibrium conditions is a delicate procedure, often used but not so very often discussed in the literature. We shall focus on the important case of nonequilibrium steady states and we will expand on the details via a particular example that will also serve to highlight certain subtleties whose resolution will be the main focus of this paper.

We choose to study the problem of a point-contact junction for noninteracting spin-1/2 fermions formed between two leads separated by a tunneling barrier. This situation is captured via a standard tunneling Hamiltonian [9]. Provided the interactions in the leads are screened so one can describe them with a noninteracting model, the problem can be reduced to one dimension via the use of symmetries [10] or, alternatively, introducing a lattice regularization and applying a Lanczos-Haydock recursion [11] which is valid even in the presence of disorder. One ends up with a semi-infinite chain with the junction or impurity attached to its boundary. In a continuum description (after introducing an appropriate bandwidth cutoff and linearizing the spectrum) we have

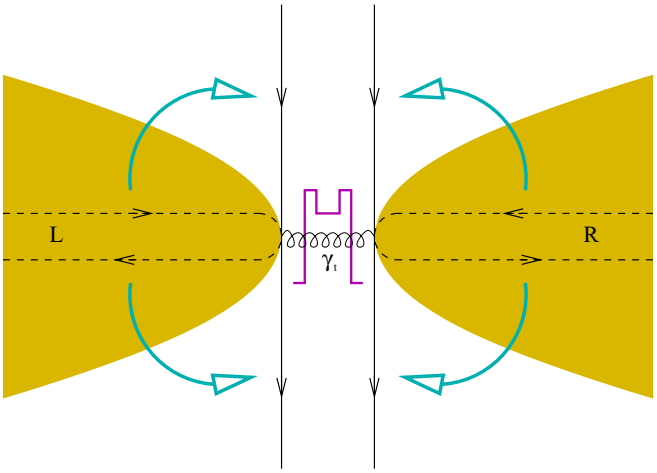


FIG. 1. Schematic representation of the mapping procedure. When each lead can be described as fermions on a half line and there are no interactions (or at least no backscattering), one can unfold the space into a full line with only one type of mover. These are called chiral fermions and the final setting is in many ways similar to that which presents itself naturally in setups involving quantum-Hall edge states. By convention, one sets the junction at the point $x = x_0 = 0$ (with $\hat{x}$ being the vertical axis in the figure). In the present case to be studied, the junction is simply a potential barrier modeled with a tunneling-overlap matrix element γ_t. More in general, one could have a more complicated tunneling system (such as a double barrier with resonant levels in between) in order to describe tunneling through nanostructures such as quantum dots. The unfolding procedure can also clearly be generalized to three-terminal settings and more [15].

two degrees of freedom which can be called the incoming and outgoing electrons, that move towards or away from the boundary, respectively. These are defined for $r > 0$ and obey the boundary condition $\psi^\dagger_{\rm in}(r = 0,t) = \psi^\dagger_{\rm out}(r = 0,t)$; we omitted internal indexes for brevity. One can introduce the new operator $\psi^\dagger(r,t) = \theta(-r)\psi^\dagger_{\rm in}(-r,t) + \theta(r)\psi^\dagger_{\rm out}(r,t)$ defined in the whole axis, where $\theta(r)$ is the Heaviside step function. These are now chiral fermions and the junction problem has been mapped to two chiral-fermion leads adjoined at a point (see Fig. 1). These standard transformations are commonly used in the context of (quantum-)impurity problems [12,13] (see also Refs. [14] for a related description in the context of numerical renormalization). Notice that we could equally well exchange $\psi^\dagger_{\rm in}$ and $\psi^\dagger_{\rm out}$ while defining $\psi^\dagger$, which means that we have the freedom to work with either right or left movers and which type to choose is a matter of convention. In what follows, after the mapping to a one-dimensional model, we will refer to the spatial axis as the $\hat{x}$ axis. The open-circuit (open-junction) boundary conditions now read $\psi^\dagger(x = 0^-,t) = \psi^\dagger(x = 0^+,t)$.

A. Setting of the problem and direct solution

In Hamiltonian language, the model we shall focus on is given by

$$H = \sum_{\sigma,\ell}\left(\int \mathcal{H}^0_\ell\, dx + H_{\rm tun}\right), \tag{1}$$

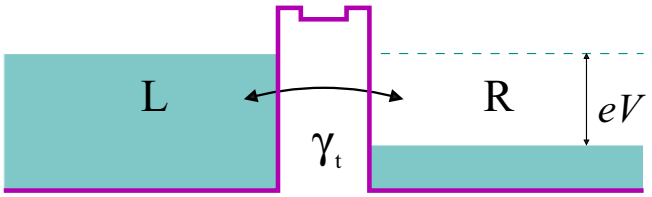


FIG. 2. Schematic depiction of the setting in which two Fermi seas kept at different chemical potentials with a difference given by $eV = \mu_L - \mu_R$ are connected via quantum tunneling across the potential barrier that separates them. The tunneling across the classically forbidden region is modeled by a tight-binding matrix overlap γ_t that can be taken to be energy independent in certain cases (in particular, the characteristic energy scale of dependence of the transmission coefficient, $|\gamma_t|^2$, has to be much larger than both eV and k_BT [8]). Here we assume that the barrier region does not allow for internal states; a situation when that happens will be discussed elsewhere [16].

where the Hamiltonian (densities) describing the leads and the Hamiltonian for the tunneling across a barrier modeled as a quantum point contact between the two leads [17], respectively, are

$$\mathcal{H}^0_\ell = v_F\, \psi^\dagger_{\sigma\ell}(x,t)(-i\partial_x)\psi_{\sigma\ell}(x,t), \tag{2a}$$

$$H_{\rm tun} = -\gamma_t\, \psi^\dagger_{\sigma\ell}(0,t)\psi_{\sigma\bar{\ell}}(0,t). \tag{2b}$$

Here $\psi_{\sigma\ell}(x,t)$ are spin-1/2 ($\sigma = \uparrow,\downarrow$) chiral fermions in the Heisenberg representation that are obtained after "unfolding" the two leads as described above. Notice that the leads are modeled with an exactly linear dispersion and v_F is the Fermi velocity. The tunneling matrix element, γ_t, that characterizes the barrier is taken to be energy independent and the local fields at $x = 0$ are understood in the sense of the *local-action* formalism used for the calculations as explained below. We took here γ_t to be real for notational simplicity (though that is not necessary and we will write more general expressions later). Going beyond the standard (physically motivated) regime, we will consider $|\gamma_t|$ to be arbitrarily large. To fully define the physical situation, we still need to describe the (nonequilibrium) state of the system. We assume now that at a much earlier time the connection between the two leads was established and that there is a battery keeping a constant chemical-potential difference between the two leads. Let us call μ_ℓ the chemical potential of lead $\ell = {\rm L,R} = \mp 1$, such that $\mu_L - \mu_R = eV$ and the full potential drop takes place in the junction region (see Fig. 2 for a sketch of the physical configuration). The information about these chemical potentials will enter into the distribution functions for each lead. Under these conditions we know, by design, that the system would have reached some nonequilibrium steady state [18] and can be described with a time-translationally invariant action (this is only important for the particular solutions we discuss below, but not for the more general conclusions that we reach).

1. Transport characteristics

One of the reasons for defining the problem as we did is that it is amenable to an exact solution. How to find the

transport characteristics is well known and there exist a number of standard ways of going about it. So we shall be brief but give a complete summarized account in order to highlight notations and conventions. Our approach is to integrate out the degrees of freedom in the leads that are not directly active in the tunneling process (i.e., away from $x = 0$) and thus derive a local action for the problem [2]. When doing that, one obtains the diagonal matrix elements as momentum-space integrals of the two-point Green's functions regularized as principal-value integrals. Notice this is consistent with the normal order and the regularization of the diagonal terms in the action, which are needed for the bosonization treatments that are the focus of this work. One can capture the nonequilibrium situation by using a standard Schwinger-Keldysh formalism (see Refs. [19] for examples). At the moment we neglect the spin, which will just give a factor of 2 at the end, and choose the following spinor basis in Keldysh space (here we follow the same notation as in Ref. [20], but we reorder the basis):

$$\Psi = \begin{pmatrix} \psi_{\rm L}^{\kappa=-} & \psi_{\rm L}^{\kappa=+} & \psi_{\rm R}^{\kappa=-} & \psi_{\rm R}^{\kappa=+} \end{pmatrix}^T, \tag{3}$$

where the index κ labels the Keldysh-contour branch following the "minus-means-forward" convention [21]. Let us define $\gamma_{\rm t} = 2v_{\rm F}t$ and use the result for the local inverse Green's function of the junction [20],

$$G^{-1}(\omega) = -2i\,v_{\rm F}\begin{pmatrix} -s_{\rm L} & s_{\rm L}-1 & it & 0 \\ s_{\rm L}+1 & -s_{\rm L} & 0 & -it \\ it^* & 0 & -s_{\rm R} & s_{\rm R}-1 \\ 0 & -it^* & s_{\rm R}+1 & -s_{\rm R} \end{pmatrix}, \tag{4}$$

where $s_\ell = s_\ell(\omega) \equiv 1 - 2f(\frac{\omega-\mu_\ell}{T_\ell}) = \tanh\frac{\omega-\mu_\ell}{2T_\ell}$ and T_ℓ is the temperature of each lead; $f(x) = 1/(e^x+1)$ is the Fermi function. The current can be computed according to

$$\begin{aligned} \hat{I} &= \partial_t\frac{\Delta N}{2} = \frac{i}{2}[H,\Delta N] = \frac{i}{2}[H_{\rm tun}, N_{\rm R} - N_{\rm L}] \\ &= i\ell\gamma_{\rm t}\psi_\ell^\dagger(0,t)\psi_{\bar\ell}(0,t) \Rightarrow \ell\gamma_{\rm t}G_{\bar\ell\ell}^{-+}(\delta t = 0) \end{aligned} \tag{5}$$

(we shall follow the convention in which sums over varying indexes are implicit). Thus, restoring the complex conjugate tunneling amplitude $\gamma_{\rm t}^*$, we have $I = \langle\hat{I}\rangle = -\int\frac{d\omega}{2\pi}[\gamma_{\rm t}G_{\rm RL}^{-+} - \gamma_{\rm t}^*G_{\rm LR}^{-+}]$, where the expectation value is evaluated via a choice of appropriate Green's functions in the Schwinger-Keldysh formalism. Next one can proceed to invert the inverse of the Green's function and find the necessary expressions for the integrand,

$$\gamma_{\rm t}G_{\rm RL}^{-+} - \gamma_{\rm t}^*G_{\rm LR}^{-+} = \frac{2|t|^2(s_{\rm L}-s_{\rm R})}{(1+|t|^2)^2}. \tag{6}$$

More explicitly, and including now the spin-degeneracy factor, we have

$$\begin{aligned} I &= \frac{4|t|^2}{(1+|t|^2)^2}\int_{-\infty}^{+\infty}[s_{\rm L}(\omega)-s_{\rm R}(\omega)]\frac{d\omega}{2\pi} \\ &= \frac{4|t|^2}{\pi(1+|t|^2)^2}\int_{-\infty}^{+\infty}[f_{\rm R}(\omega)-f_{\rm L}(\omega)]d\omega. \end{aligned} \tag{7}$$

This expression has a standard form and is intuitively appealing, as the integrand selects a window (smeared by the temperature) that is 1 in the frequency interval between the two chemical potentials and zero outside of it. The integral can be carried out in general, but we will be taking the zero-temperature limit ($T_{\rm L} = T_{\rm R} = T \to 0$) for simplicity. In that limit, both s_ℓ become sign functions (f_ℓ become step functions) and the integration is trivial:

$$I \underset{T\to0}{\longrightarrow} \frac{4|t|^2eV}{\pi(1+|t|^2)^2}. \tag{8}$$

This gives the particle current, and as always one needs to multiply by $(-e)$ to get the electric current instead. The result is standard [17] and the fact that the response is exactly linear in V to all orders is a property of the linear spectrum of the model.

B. Bosonizing in the steady state

On the one hand, to study a problem using bosonization, one of the first things to do is to factor out the fast modes [1]. On the other hand, to study a problem in which a finite voltage bias is present, one of the first things to do is to introduce it into the calculations (for instance, via a careful treatment of the interaction picture [22]). Here, we need to take care of both things, so it is better to discuss them in the more formal unified language of gauge transformations. (See, though, Ref. [23] for an approach in terms of scattering states that provides an alternative to ours but is ultimately equivalent [24].)

As discussed already in the introduction, bosonization is a rewriting of the excitation spectrum in terms of bosonic degrees of freedom. As such, it does not capture the information about the reference state or ground state (which is a Fermi-Dirac sea of noninteracting fermions). Technically, one would say that what one knows how to bosonize is the normal-ordered Hamiltonian in which the vev (vacuum expectation value) has been subtracted. The type of nonequilibrium situation we are considering here presents thus a problem, because one knows in principle how to normal-order for each lead, but only in an open-junction configuration. The subsequent inclusion of the tunneling term constitutes a delicate task. A systematic way of carrying this out starts by using time-dependent gauge transformations to map the finite-bias problem into a zero-bias one but with explicitly time-dependent couplings.

Let us first switch to Lagrangian language (to fully capture the effects of a time-dependent gauge transformation), in which the system is described by the Lagrangians (densities):

$$\begin{aligned} \mathcal{L}_\ell^0 &= \psi_{\sigma\ell}^\dagger(x,t)(i\partial_t)\psi_{\sigma\ell}(x,t) - \mathcal{H}_\ell^0 \\ &= \psi_{\sigma\ell}^\dagger(x,t)(i\partial_t + iv_{\rm F}\partial_x)\psi_{\sigma\ell}(x,t), \end{aligned} \tag{9a}$$

$$L_{\rm tun} = -H_{\rm tun} = \gamma_{\rm t}\psi_{\sigma\ell}^\dagger(0,t)\psi_{\sigma\bar\ell}(0,t). \tag{9b}$$

We can now make the following field transformation $\psi_{\sigma\ell}(x,t) = e^{-i\mu_\ell t}\tilde\psi_{\sigma\ell}(x,t)$ (notice that some authors follow as an alternative a prescription of including a time dependence related to the lead chemical potentials into the respective Klein factors when bosonizing the model [25]). This

gives

$$\mathcal{L}^0_\ell = \tilde{\psi}^\dagger_{\sigma\ell}(x,t)(i\partial_t + \mu_\ell + iv_F\partial_x)\tilde{\psi}_{\sigma\ell}(x,t), \quad (10a)$$

$$\begin{aligned} L_{\rm tun} &= e^{i(\mu_\ell-\mu_{\bar\ell})t}\gamma_t\tilde{\psi}^\dagger_{\sigma\ell}(0,t)\tilde{\psi}_{\sigma\bar\ell}(0,t) \\ &= e^{ieVt}\gamma_t\tilde{\psi}^\dagger_{\sigma L}(0,t)\tilde{\psi}_{\sigma R}(0,t) \\ &\quad + e^{-ieVt}\gamma_t^*\tilde{\psi}^\dagger_{\sigma R}(0,t)\tilde{\psi}_{\sigma L}(0,t) \end{aligned} \quad (10b)$$

(where, in the last line, we restored explicitly the complex conjugate γ_t^*). An important point is that now the distribution functions in the Keldysh action do not contain information about the chemical potentials any longer [26]. Next we subtract the vev's of each lead which, by assumption, are the same as those in the absence of the tunneling term (this is where the Landauer prescription [8] enters the calculation) and drop the now ineffectual chemical potential terms. For a noninteracting problem this is equivalent to factoring out the fast oscillations in each lead according to an additional field transformation: $\tilde{\psi}_{\sigma\ell}(x,t) = e^{ik_F^\ell x}\check{\psi}_{\sigma\ell}(x,t)$, with $k_F^\ell = \mu_\ell/v_F$ for this linear-dispersion case, and then subtracting the same (infinite) constant for all leads. So we are naturally left with the normal-ordered formulation of the problem,

$$\mathcal{L}^0_\ell = :\check{\psi}^\dagger_{\sigma\ell}(x,t)(i\partial_t + iv_F\partial_x)\check{\psi}_{\sigma\ell}(x,t):, \quad (11a)$$

$$L_{\rm tun} = e^{ieVt}\gamma_t\check{\psi}^\dagger_{\sigma L}(0,t)\check{\psi}_{\sigma R}(0,t) + e^{-ieVt}\gamma_t^*\check{\psi}^\dagger_{\sigma R}(0,t)\check{\psi}_{\sigma L}(0,t). \quad (11b)$$

At this point we lost all the information about any absolute-energy reference, but we still have the information about the potential drop encoded in the time-dependent phase of the tunneling term (cf. Fig. 2). Given the infinite-bandwidth setting, we are also in a situation in which space is naturally to be regarded as half filled. Now one is ready to bosonize the problem following the standard procedure.

1. Abelian bosonization and standard transformations

The Abelian-bosonization recipe is by now textbook material [1,2,5,27] and there is no need to present the details here. There exist though a number of different conventions, which can bring in some confusion at times. Our notation and conventions follow closely the review article in Ref. [4] (which in turn is based on the constructive presentation given earlier by Haldane [3]), with the only difference of factors of $1/\sqrt{2\pi}$ that are needed in order to have a more standard normalization for the real-space Fermi-field anticommutators [28].

In order to bosonize we go back to the Hamiltonian formulation

$$\mathcal{H}^0_\ell = :\check{\psi}^\dagger_{\sigma\ell}(x,t)(-iv_F\partial_x)\check{\psi}_{\sigma\ell}(x,t):, \quad (12a)$$

$$\begin{aligned} H_{\rm tun} &= -e^{ieVt}\gamma_t\check{\psi}^\dagger_{\sigma L}(0,t)\check{\psi}_{\sigma R}(0,t) \\ &\quad - e^{-ieVt}\gamma_t^*\check{\psi}^\dagger_{\sigma R}(0,t)\check{\psi}_{\sigma L}(0,t), \end{aligned} \quad (12b)$$

and we proceed to bosonize according to $\mathcal{H}^0_\ell$, which is akin to working in the interaction picture (with $H_{\rm tun}$ taken as the interaction term) [29]. We shall follow the bosonization prescription [30]

$$\check{\psi}_{\sigma\ell}(x,t) = \frac{1}{\sqrt{2\pi a}}F_{\sigma\ell}(t)e^{-i\phi_{\sigma\ell}(x,t)}, \quad (13)$$

where the $F_{\sigma\ell}(t)$ are the so-called Klein factors and a is a short-distance regulator [3]. We shall not include subleading $1/L$ corrections in the bosonization formulas, because infinite size is the appropriate limit for a description of the leads in a Landauer-style transport setup to describe a steady state; as a bonus, this keeps formulas shorter. In terms of these bosons the Hamiltonian density for the leads can be shown to take the usual form,

$$\mathcal{H}^0 = \sum_\ell \mathcal{H}^0_\ell = \frac{v_F}{4\pi}\sum_{\sigma=\uparrow,\downarrow;\,\ell={\rm L,R}} :[\partial_x\phi_{\sigma\ell}(x,t)]^2: . \quad (14)$$

One of the main advantages of the bosonic description is that with it one can more easily recombine degrees of freedom in order to, for instance, separate the effects of charge and spin dynamics (phenomena such as spin-charge separation are thus very naturally described with the use of bosonization). Using the standard, physically motivated, rotated boson basis $\phi_{\sigma\ell} = \frac{1}{2}(\phi_c + \sigma\phi_s + \ell\phi_l + \sigma\ell\phi_{sl})$, where $\sigma,\ell = \pm 1$ when entering as multiplying factors, the noninteracting Hamiltonian density retains its quadratic form,

$$\mathcal{H}^0 = \frac{v_F}{4\pi}\sum_{\nu=c,s,l,sl} :[\partial_x\phi_\nu(x,t)]^2: , \quad (15)$$

and, as usual, the Klein factors drop out from these terms. We shall refer to these "physical" sectors as *charge*, *spin*, *lead* (or flavor), and *spin-lead* (or spin-flavor), respectively. We will see how they naturally reorganize the information about the physics of tunneling transport.

Let us now bosonize the tunneling term, rotate the bosons into the physical sectors, and make some standard simplifications (the sum over σ is implicit and the fields are evaluated at $x = 0$):

$$\begin{aligned} H_{\rm tun} &= -e^{ieVt}\frac{\gamma_t}{2\pi a}F^\dagger_{\sigma L}F_{\sigma R}e^{i\phi_{\sigma L}}e^{-i\phi_{\sigma R}} \\ &\quad - e^{-ieVt}\frac{\gamma_t^*}{2\pi a}F^\dagger_{\sigma R}F_{\sigma L}e^{i\phi_{\sigma R}}e^{-i\phi_{\sigma L}} \end{aligned} \quad (16a)$$

$$\begin{aligned} &= -e^{ieVt}\frac{\gamma_t}{2\pi a}F^\dagger_{\sigma L}F_{\sigma R}e^{-i(\phi_l+\sigma\phi_{sl})} \\ &\quad - e^{-ieVt}\frac{\gamma_t^*}{2\pi a}F^\dagger_{\sigma R}F_{\sigma L}e^{i(\phi_l+\sigma\phi_{sl})}. \end{aligned} \quad (16b)$$

To proceed further we need to take care of the mapping of Klein factors. We anticipate no subtleties coming from these, but we carry out a careful treatment nevertheless so as to show that explicitly. The most rigorous way to proceed is by identifying relations between different bilinears of old and new Klein factors, and fixing the four arbitrary phases that appear [31–33]:

$$F^\dagger_{\uparrow R}F_{\downarrow R} = F^\dagger_{sl}F^\dagger_s , \quad (17a)$$

$$F^\dagger_{\uparrow L}F_{\downarrow L} = F_{sl}F^\dagger_s , \quad (17b)$$

$$F^\dagger_{\uparrow R}F_{\uparrow L} = F^\dagger_{sl}F^\dagger_l , \quad (17c)$$

$$F^\dagger_{\uparrow R}F^\dagger_{\uparrow L} = F^\dagger_c F^\dagger_s . \quad (17d)$$

All the rest of the Klein-factor bilinear relations can be derived from these. In particular, in order to simplify $H_{\rm tun}$ we will need the following ones:

$$F^\dagger_{\uparrow \rm R} F_{\uparrow \rm L} = F^\dagger_{sl} F^\dagger_{l}\,, \tag{17e}$$

$$F^\dagger_{\downarrow \rm R} F_{\downarrow \rm L} = F^\dagger_{l} F_{sl}\,, \tag{17f}$$

$$F^\dagger_{\uparrow \rm L} F_{\uparrow \rm R} = F_{l} F_{sl}\,, \tag{17g}$$

$$F^\dagger_{\downarrow \rm L} F_{\downarrow \rm R} = F^\dagger_{sl} F_{l} \tag{17h}$$

(where the last two are simply the Hermitian conjugate of the first two). Notice that, as one should have expected by looking at the boson fields and comparing Eqs. (16a) and (16b), the right-hand sides involve only the *lead* and *spin-lead* Klein factors. The tunneling Hamiltonian density can thus be further rewritten referring only to the "physical" sectors. One can then undo the steps of the bosonization procedure and *debosonize* (also called reverse bosonization or refermionization) in order to arrive again at a problem written in terms of Fermi fields. Using the standard debosonization prescription, $\breve{\psi}_\nu(x,t) = \frac{1}{\sqrt{2\pi a}} F_\nu(t) e^{-i\phi_\nu(x,t)}$, which parallels the one we used for bosonizing in the first place, we arrive at

$$\mathcal{H}^0_\nu = :\breve{\psi}^\dagger_\nu(x,t)(-i v_{\rm F}\partial_x)\breve{\psi}_\nu(x,t):\,, \tag{18a}$$

$$H_{\rm tun} = -[e^{ieVt}\gamma_{\rm t}\breve{\psi}_l(0,t) + e^{-ieVt}\gamma^*_{\rm t}\breve{\psi}^\dagger_l(0,t)] \times[\breve{\psi}_{sl}(0,t) - \breve{\psi}^\dagger_{sl}(0,t)]. \tag{18b}$$

We find that the tunneling term involves only the *lead* and *spin-lead* sectors, while the *charge* and *spin* sectors have decoupled from the tunneling process.

The new problem, defined by $H = \int \mathcal{H}^0 + H_{\rm tun}$, can now be regarded as arising from an original problem with the voltage acting as a chemical-potential shift of the *lead* fermions only ($\nu = l$). In other words, if we consider the problem given by

$$\mathcal{H}^0 = \sum_\nu \mathcal{H}^0_\nu = \psi^\dagger_\nu(x,t)(-i v_{\rm F}\partial_x)\psi_\nu(x,t), \tag{19a}$$

$$H_{\rm tun} = -[\gamma_{\rm t}\psi_l(0,t) + \gamma^*_{\rm t}\psi^\dagger_l(0,t)][\psi_{sl}(0,t) - \psi^\dagger_{sl}(0,t)], \tag{19b}$$

where the chemical potential is set as $\mu_{\nu=l} = -(eV)$ and is zero for all sectors $\nu \neq l$, this can be connected with the debosonized problem of interest following equivalent steps to those we presented above via the combined transformation $\psi_l(x,t) = e^{ieV(t-x/v_{\rm F})}\breve{\psi}_l(x,t)$. Moreover, this "parent" problem can be seen to be unique (i.e., there is only one way to eliminate the time dependence from the tunneling term by reintroducing chemical potentials into the problem).

C. Indirect solution using conventional bosonization-debosonization

One of the goals of a bosonization-debosonization program (BdB for short), as exemplified above, is to achieve a simplification of the problem at hand that would not be so easy otherwise. (There could be other alternative or additional motivations for bosonizing, such as carrying out a renormalization-group analysis that is more easily done in the bosonic language; see Ref. [1] for examples.) Indeed, transformations like the one introduced by the simple rotation of the bosonic basis would be hardly evident if one were to express them directly in terms of the old and new fermions instead. The example that we picked is special, because we are able to solve it exactly already in the original formulation and even in an out-of-equilibrium setting. However, the BdB program is, in most other cases, crucial for simplifying the problems and being able to find solutions either exact or approximate.

In the case of our simple junction problem, the BdB program does indeed show some apparent simplifications. A simple glance at the final form of $H_{\rm tun}$ shows that only the *lead* and *spin-lead* sectors are involved in the transport while the other two sectors (*spin* and *charge*) do not participate. This provides a certain economy of description that we will discuss further below. For now, our immediate goal in this section is to recompute the I-V characteristics of the junction.

1. Recalculation of transport after conventional BdB

We need again the operator expression of the current, but now in terms of the new fermionic degrees of freedom. One can translate it from the expression we gave above [see Eq. (5)] using BdB or, equivalently, it can be recomputed directly in terms of the new fields:

$$\hat{I} = \partial_t \frac{\Delta N}{2} = i\left[H, \frac{\Delta N}{2}\right] = i[H_{\rm tun}, N_{\nu=l}] = -i[\psi^\dagger_{sl}(0,t) - \psi_{sl}(0,t)][\gamma_{\rm t}\psi_l(0,t) - \gamma^*_{\rm t}\psi^\dagger_l(0,t)]. \tag{20}$$

Notice that this time the spin degeneracy is already included implicitly in the formalism. Thus, $I = \langle\hat{I}\rangle$ is given as

$$I = -i\gamma_{\rm t}(\langle\psi^\dagger_{sl}(0,t)\psi_l(0,t)\rangle - \langle\psi_{sl}(0,t)\psi_l(0,t)\rangle) + i\gamma^*_{\rm t}(\langle\psi^\dagger_{sl}(0,t)\psi^\dagger_l(0,t)\rangle - \langle\psi_{sl}(0,t)\psi^\dagger_l(0,t)\rangle).$$

Next we calculate the necessary Green's function elements using the same procedure as in Sec. II A 1. However, this time we need to introduce a Nambu structure due to the presence of *anomalous* processes in $H_{\rm tun}$. As a result, we adopt the following spinor basis (including also the Keldysh indexes and with the frequencies restricted to the positive semiaxis only in order to avoid double counting):

$$\Psi(\omega) = (\psi^-_l(\omega) \quad \psi^+_l(\omega) \quad \psi^{\dagger-}_l(\bar{\omega}) \quad \psi^{\dagger+}_l(\bar{\omega}) \quad \psi^-_{sl}(\omega) \quad \psi^+_{sl}(\omega) \quad \psi^{\dagger-}_{sl}(\bar{\omega}) \quad \psi^{\dagger+}_{sl}(\bar{\omega}))^T. \tag{21}$$

We write the local inverse Green's function of H^0, using the fact that all nonequilibrium Green's functions (i.e., advanced, retarded, and Keldysh components) are diagonal in the Nambu basis. The only change required for the time-reversed Nambu component, as compared with the time-forward one, is to define $\bar{s}_v \equiv \tanh \frac{\omega+\mu_v}{2T_v}$ for ω as given in the argument of the spinor (and we will be taking the temperature to be uniform, $T_v = T_{\rm emp}$). Including also the contribution of $H_{\rm tun}$, the local inverse Green's function for the junction is thus given by

$$G^{-1}(\omega) = -2i\upsilon_{\rm F}\begin{pmatrix} -s_l & s_l-1 & 0 & 0 & it^* & 0 & -it^* & 0 \\ s_l+1 & -s_l & 0 & 0 & 0 & -it^* & 0 & it^* \\ 0 & 0 & -\bar{s}_l & \bar{s}_l-1 & it & 0 & -it & 0 \\ 0 & 0 & \bar{s}_l+1 & -\bar{s}_l & 0 & -it & 0 & it \\ it & 0 & it^* & 0 & -s_{sl} & s_{sl}-1 & 0 & 0 \\ 0 & -it & 0 & -it^* & s_{sl}+1 & -s_{sl} & 0 & 0 \\ -it & 0 & -it^* & 0 & 0 & 0 & -\bar{s}_{sl} & \bar{s}_{sl}-1 \\ 0 & it & 0 & it^* & 0 & 0 & \bar{s}_{sl}+1 & -\bar{s}_{sl} \end{pmatrix}. \tag{22}$$

We invert the matrix, identify the relevant matrix elements, and replace them into the expression for the current. After some algebra one gets

$$I = \frac{|t'|^2}{(1+|t'|^2)}\int_0^{+\infty}[s_l(\omega)-\bar{s}_l(\omega)]\frac{d\omega}{2\pi}, \tag{23}$$

where $t' = 2t$. The integral can be done in general, but in the zero-temperature limit reduces to

$$I \underset{T_v\to 0}{\longrightarrow} \frac{|t'|^2 eV}{\pi(1+|t'|^2)} = \frac{(1+|t'|^2)}{4}\frac{4|t'|^2 eV}{\pi(1+|t'|^2)^2}. \tag{24}$$

We see that the result we obtained for the current shows several discrepancies from the one in Sec. II A 1. Such differences need to be understood.

III. THE NONEQUILIBRIUM TRANSPORT PUZZLE

We have carefully chosen the nonequilibrium junction problem so that it meets all the requirements for bosonization to be an exact operator correspondence between fermions and bosons (cf. Ref. [4]). All the transformations we carried out are thus rigorous and the discrepancy between the results of Secs. II A 1 and II C 1 is not only unexpected but also unwelcome. There has to be an inconsistency somewhere and, given that the result of the direct solution is standard and can be reobtained in a number of alternative ways, everything seems to indicate that the problem has to be with the indirect solution. Moreover, the actual transport calculation of the indirect solution proceeded in a very similar way to the case of the direct one. As a result, the reason for the discrepancies is likely not in there, but in the preceding BdB-based mapping used to rewrite the junction problem in terms of the new fermionic degrees of freedom.

Before furthering the analysis, let us first catalog the discrepancies between the two solutions:

(1) To match the solutions one needs to arbitrarily correct the tunneling matrix element of the indirect solution by a factor of 2 (namely, $t' \mapsto t$) in order to make it look closer to the exact direct solution.

(2) There is a overall factor of 4 difference between the two solutions (the indirect solution would need to be multiplied by 4 to match with the direct solution).

(3) There is also an additional factor of $(1+|t|^2)$ in the numerator of the indirect solution that cancels one power from the denominator and introduces a further discrepancy with the exact direct solution.

These three discrepancies are present no matter which method we use for the final transport calculation (they all yield the same result). We highlighted them by looking at the zero-temperature limit, but it is easy to see that they are also exactly the same at finite temperature. Additionally, very similar discrepancies can be seen to be present in equilibrium thermodynamic calculations using a Matsubara formalism (see the Appendix). Thus, the puzzle is not restricted only to transport, but it is more evident in transport calculations.

Motivated specially by the third entry from the list of discrepancies, one could imagine expanding the results of the direct and indirect solutions in powers of t. It is clear that big differences will show up as soon as one goes beyond leading order in the tunneling matrix element for both calculations. We therefore expect to be able to gain some insight by studying the problem using perturbation theory in t.

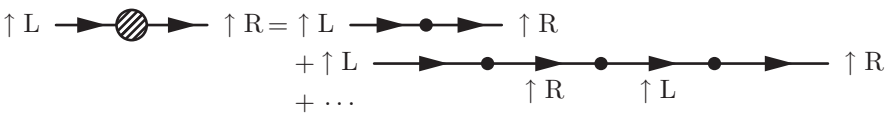
A. A diagrammatic diagnosis

Let us start by setting up a dictionary for processes allowed by the different vertexes in $H_{\rm tun}$. There are four of those, given by the two possible spin orientations and the two possible directions of tunneling. Since our BdB program rests neither on the SU(2) invariance nor on the Hermiticity of $H_{\rm tun}$, we can, in principle, set the four corresponding matrix elements to different constants and thus individually trace each process thorough the BdB procedure to construct the dictionary given in the table below. Alternatively, one can construct the dictionary by looking at the changes operated by the different *graph vertexes* on the fermion numbers of the different sectors (which is essentially the construction that is used to identify the different Klein-factor bilinears [31]). The translation between the fermionic structure of the vertexes in terms of "old" (original) and "new" fermions is thus given by

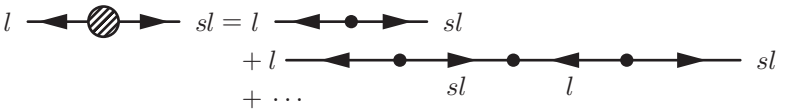
Simple-junction Graph-vertex Dictionary			
Original Fermions		New Fermions	
$\psi^\dagger_{\uparrow \mathrm{R}}\psi_{\uparrow \mathrm{L}}$	$\uparrow \mathrm{L} \longrightarrow\bullet\longrightarrow \uparrow \mathrm{R}$	$\psi^\dagger_{sl}\psi^\dagger_{l}$	$l \longleftarrow\bullet\longrightarrow sl$
$\psi^\dagger_{\downarrow \mathrm{R}}\psi_{\downarrow \mathrm{L}}$	$\downarrow \mathrm{L} \longrightarrow\bullet\longrightarrow \downarrow \mathrm{R}$	$\psi^\dagger_{l}\psi_{sl}$	$sl \longrightarrow\bullet\longrightarrow l$
$\psi^\dagger_{\uparrow \mathrm{L}}\psi_{\uparrow \mathrm{R}}$	$\uparrow \mathrm{R} \longrightarrow\bullet\longrightarrow \uparrow \mathrm{L}$	$\psi_{l}\psi_{sl}$	$sl \longrightarrow\bullet\longleftarrow l$
$\psi^\dagger_{\downarrow \mathrm{L}}\psi_{\downarrow \mathrm{R}}$	$\downarrow \mathrm{R} \longrightarrow\bullet\longrightarrow \downarrow \mathrm{L}$	$\psi^\dagger_{sl}\psi_{l}$	$l \longrightarrow\bullet\longrightarrow sl$

Notice that the second two lines are the Hermitian conjugate of the first two. We can refer to them as (i)–(iv) from top to bottom. Now in order to calculate the current we need to find the fully dressed vertexes. We can proceed to dress them by carrying out a perturbative expansion in H_{tun} (the Keldysh structure is not important for the present argument and will be suppressed for the sake of clarity).

Let us consider, for instance, the dressing of vertex (i) in terms of the original fermions. It proceeds by alternating vertexes (i) and (iii) at different orders of expansion. Up to third (the first nontrivial) order we have

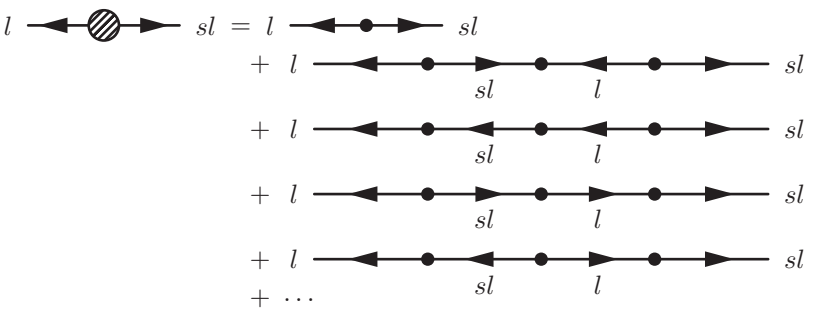


which in terms of the new fermions translates according to our dictionary into

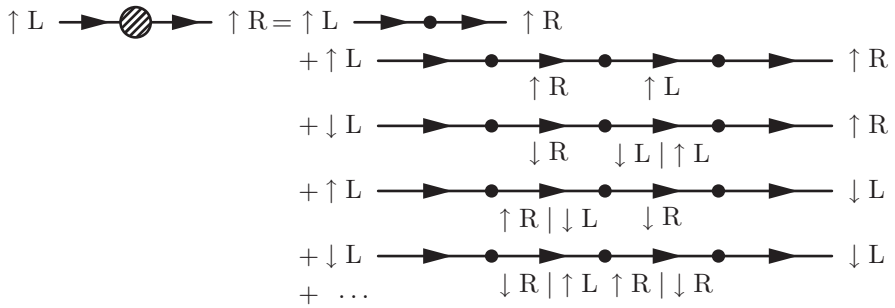


For spin-down, the diagrams in terms of the original fermions are exactly the same with the obvious label replacement $(\uparrow) \to (\downarrow)$. This corresponds to alternating vertexes (ii) and (iv) at different orders of expansion. After translation to the new-fermions language one just changes the labels according to $(\downarrow \mathrm{L}) \to (sl)$ and $(\downarrow \mathrm{R}) \to (l)$, but this time the arrows of the fermion propagators stay unchanged (no anomalous processes are involved in this case, exactly the opposite from the example above with spin-up).

Difficulties arise when we start directly from the new-fermions language and proceed to dress the vertex in question. This is so because we have additional (and, we shall claim, unphysical) ways of introducing contractions. Consider, for instance, again the case of vertex (i). One would proceed to dress it as follows:



The four third-order processes correspond to vertex insertions (i-iii-i), (ii-iv-i), (i-ii-iv), and (ii-i-iv), respectively, which, according to our dictionary, translated back in terms of the original fermions, read as follows:



The last three contractions are not allowed in the original-fermions framework, as they require spin flip and some even L ↔ R exchange (as indicated by the inner labelings). Moreover, they do not even dress the correct vertex (as indicated by the outer labelings). From a practical point of view, one may notice that while we deal with four distinct types of original fermions (↑ L, ↓ L, ↑ R, and ↓ R), we deal with only two types of new fermions (l and sl). We conclude that the more compact description achieved after the BdB-based mapping introduces the possibility of spurious processes that should not have been there. These are processes that mix vertexes (i) and (iii) with vertexes (ii) and (iv), which in terms of the original fermions cannot happen due to spin conservation. This clearly hints at the possibility that, in the new-fermions framework, the *spin* sector should not really be decoupled after all.

IV. CONSISTENT APPROACH TO BOSONIZATION-DEBOSONIZATION

We need to revisit the transformations in the BdB-based mapping used above, with the goal of finding the source of the discrepancies with respect to the direct calculations. In particular, one needs to be careful about the fact that the tunneling term is not normal ordered (since the procedure of subtracting the vev is not well defined for the processes in $H_{\rm tun}$ for they are not diagonal in fermion "internal indexes").

A. Keys to consistency

We proceed to study again the bosonization of the tunneling term but taking care of *not combining exponentials*. If we start from Eq. (16a) and perform the change of basis for the bosons, we arrive at

$$\begin{aligned} H_{\rm tun} = &-e^{ieVt}\frac{\gamma_{\rm t}}{2\pi a}F^{\dagger}_{\sigma{\rm L}}F_{\sigma{\rm R}}e^{i(\phi_c+\sigma\phi_s-\phi_l-\sigma\phi_{sl})/2}\\ &\times e^{-i(\phi_c+\sigma\phi_s+\phi_l+\sigma\phi_{sl})/2}\\ &-e^{-ieVt}\frac{\gamma^*_{\rm t}}{2\pi a}F^{\dagger}_{\sigma{\rm R}}F_{\sigma{\rm L}}e^{i(\phi_c+\sigma\phi_s+\phi_l+\sigma\phi_{sl})/2}\\ &\times e^{-i(\phi_c+\sigma\phi_s-\phi_l-\sigma\phi_{sl})/2}. \end{aligned}$$

We will now, on the one hand, combine the exponentials in which the bosons appear with the same sign (we are prompted to do this by a study of the corresponding operator product expansions, OPEs, and by the consistency with the mapping of the Klein factors [31,32]). On the other hand, we will be careful not to combine the exponentials in which the bosonic exponents appear with opposite signs (prompted by the suspicion, from our perturbative analysis, that the $\nu = c,s$ sectors should not completely decouple from the tunneling process). We will discuss the *charge* and *spin* sectors carefully momentarily; for now we debosonize in the *lead* and *spin-lead* sectors only (using the same prescription that was introduced above). The tunneling term takes the following form (all the fields are evaluated at $x = 0$ and at time t):

$$\begin{aligned} H_{\rm tun} = &-e^{ieVt}\gamma_{\rm t}\check{\psi}_l\check{\psi}_{sl}e^{i\phi_c/2}e^{-i\phi_c/2}e^{i\phi_s/2}e^{-i\phi_s/2}\\ &-e^{ieVt}\gamma_{\rm t}\check{\psi}^{\dagger}_{sl}\check{\psi}_l e^{i\phi_c/2}e^{-i\phi_c/2}e^{-i\phi_s/2}e^{i\phi_s/2}\\ &-e^{-ieVt}\gamma^*_{\rm t}\check{\psi}^{\dagger}_{sl}\check{\psi}^{\dagger}_l e^{i\phi_c/2}e^{-i\phi_c/2}e^{i\phi_s/2}e^{-i\phi_s/2}\\ &-e^{-ieVt}\gamma^*_{\rm t}\check{\psi}^{\dagger}_l\check{\psi}_{sl}e^{i\phi_c/2}e^{-i\phi_c/2}e^{-i\phi_s/2}e^{i\phi_s/2}, \end{aligned}$$

which is the same as before but with the addition of the extra exponential factors.

A pragmatic way to proceed in order to debosonize in the *charge* and *spin* sectors as well is by replacing the vertex products by lattice-like fermionic densities according to the prescription

$$e^{\pm i\phi_{c,s}/2}e^{\mp i\phi_{c,s}/2} \mapsto \tilde{n}^{\pm}_{c,s}\,. \tag{25}$$

These new objects (to be defined and discussed more in detail below) can be interpreted as particle and hole densities for new fermionic degrees of freedom in the *charge* and *spin* sectors. They shall be considered in their "eigenbasis" and they have eigenvalues 0 or 1 and 1 or 0, respectively and correspondingly. This is the central result of the *consistent* way to debosonize and the (almost) final form of the tunneling Hamiltonian is

$$\begin{aligned} H_{\rm tun} = &-e^{ieVt}\gamma_{\rm t}\tilde{n}^{+}_c\tilde{n}^{+}_s\check{\psi}_l\check{\psi}_{sl} - e^{ieVt}\gamma_{\rm t}\tilde{n}^{+}_c\tilde{n}^{-}_s\check{\psi}^{\dagger}_{sl}\check{\psi}_l\\ &-e^{-ieVt}\gamma^*_{\rm t}\tilde{n}^{+}_c\tilde{n}^{+}_s\check{\psi}^{\dagger}_{sl}\check{\psi}^{\dagger}_l - e^{-ieVt}\gamma^*_{\rm t}\tilde{n}^{+}_c\tilde{n}^{-}_s\check{\psi}^{\dagger}_l\check{\psi}_{sl}\,. \end{aligned}$$

It can be easily seen that the inclusion of the $\tilde{n}$ factors naturally avoids the mixing of graph vertexes (i) and (iii) with graph vertexes (ii) and (iv), exactly as was concluded to be necessary in the diagrammatic discussion of the previous section (Sec. III A). Notice also that, in the same vein, these factors also stop us from being able to rewrite $H_{\rm tun}$ in terms of Majorana-fermion combinations.

In the next two subsections we provide some additional rationale, but those readers that want to skip some of the technical discussion can jump ahead to the last subsection of this section (Sec. IV D) and see how we are now able to recover exactly the results of the direct calculation (which can be taken as a pragmatic justification for the procedure).

B. Matters of regularization

The exponentials of bosonic fields of the type $e^{i\lambda\phi_\nu}$ are central objects in the bosonization formalism known as *vertex operators*. The bosonization prescription tells us that $\psi_\nu^\dagger \propto e^{i\phi_\nu}$ (with $\lambda = 1$) while normal-ordered densities are bosonized according to : $\psi_\nu^\dagger\psi_\nu :=\frac{1}{2\pi}\partial\phi_\nu$. The consistency between these two prescriptions can be checked by bosonizing the non-normal-ordered case, $\psi_\nu^\dagger\psi_\nu = e^{i\phi_\nu}e^{-i\phi_\nu}/2\pi a$, and expanding the right-hand side by using known results for the OPEs of vertex operators [4].

However, when bosonizing, oftentimes our aim is to change basis from the spin-and-lead (or spin-and-flavor) states to a basis that separates physical sectors (*charge*, *spin*, *lead*, and *spin-lead*) because some of the physics will simplify by doing that (this is the transformation that we performed in $H_{\rm tun}$). Proceeding formally for each vertex operator of a density operator [using $\phi_{\sigma\ell}(x) = \sum_n \phi_{\nu_n}/2$, where the ν_n label the physical sectors and we absorbed minus signs that are not important for this part of the discussion], we have

$$e^{i\phi_{\sigma\ell}}e^{-i\phi_{\sigma\ell}} = \prod_n e^{i\phi_{\nu_n}/2}e^{-i\phi_{\nu_n}/2}, \tag{26a}$$

$$[1 + a\,\partial\phi_{\sigma\ell} + \cdots] \approx 1 + \sum_n a\,\partial\phi_{\nu_n}/2 + \cdots, \tag{26b}$$

$$1 + \delta n_{\sigma\ell} + \cdots \approx 1 + \sum_n \delta n_{\nu_n}/2 + \cdots, \tag{26c}$$

where in the third line we introduced lattice-like density fluctuations, $\delta n_{\sigma\ell} \equiv a\,\partial\phi_{\sigma\ell}$ and $\delta n_{\nu_n} \equiv a\,\partial\phi_{\nu_n}$, to stress that they need to be small in order to connect to the first line. (A standard view is to treat a as a control parameter for the expansions in the second line.) Therefore, these transformations are consistent if bosonization is treated as an expansion around a half-filled ground state (in a real-space picture). While the bosonization identities are precise, some manipulations might not hold when the deviations from the local half-filled state are large. If a particular problem, as is the case of some transport problems like the one that we are studying, forces us to consider large δn fluctuations, then we need to proceed with caution while expanding.

One solution is to expand around a different state, which can be achieved via a linear transformation. Consider the following vertex OPE at some $x = x_0$ (the position of the junction or impurity) and treat a as an expansion parameter (not necessarily small) [34]:

$$e^{i\phi_{\sigma\ell}}e^{-i\phi_{\sigma\ell}} \approx 1 + a\,\partial\phi_{\sigma\ell} + \cdots \equiv 1 + (1 + a\,\partial\tilde\phi_{\sigma\ell}) + \cdots.$$

This serves as a definition of a *shifted* set of bosons, $\tilde\phi_{\sigma\ell}$, which are used to expand around a differently filled state (unit-filling in this case) and need to obey $\partial\tilde\phi_{\sigma\ell} = \partial\phi_{\sigma\ell} - 1/a$. Reintroducing the x dependence from the OPE before taking the $a \to 0$ limit [i.e., replacing $1/a \mapsto \pi\delta(x - x_0)$], and integrating this relation one gets $\tilde\phi_{\sigma\ell}(x) = \phi_{\sigma\ell}(x) - \frac{\pi}{2}\,\mathrm{sgn}(x - x_0)$, up to an additive constant. The new bosons have identical commutation relations and OPEs except at $x = x_0$ due to the presence of these solitonic shifts.

Expanding around $a\partial\phi_{\sigma\ell} = 1$ is equivalent to expanding around $a\partial\tilde\phi_{\sigma\ell} = 0$ and we can use small-variable expansions in terms of the latter. For the kind of vertex products we are considering (at $x = x_0$) we have

$$e^{i\phi_{\sigma\ell}}e^{-i\phi_{\sigma\ell}} \approx 2\left(1 + \frac{a}{2}\partial\tilde\phi_{\sigma\ell} + \cdots\right) + \cdots \tag{27a}$$

$$\approx 2\sqrt{1 + a\partial\tilde\phi_{\sigma\ell}} + \cdots \tag{27b}$$

$$\approx 2\sqrt{e^{i\tilde\phi_{\sigma\ell}}e^{-i\tilde\phi_{\sigma\ell}}}, \tag{27c}$$

where to get to the second line we used a Taylor expansion for the square root of a binomial (the first two lines are strictly equivalent to the order that is given explicitly; their connection can be regarded as a sort of partial resummation that is also consistent with a further study of other vertex OPEs that we carried out as well). Alternatively, applying the same vertex-vertex OPE, but in reverse, to the parentheses in the first line of the equation above we have

$$e^{i\phi_{\sigma\ell}}e^{-i\phi_{\sigma\ell}} \approx 2\,e^{i\tilde\phi_{\sigma\ell}/2}e^{-i\tilde\phi_{\sigma\ell}/2}. \tag{28}$$

This implies that, generically,

$$e^{i\phi/2}e^{-i\phi/2} \approx \sqrt{e^{i\phi}e^{-i\phi}}, \tag{29}$$

as will be proven below by working to all orders without resorting to OPEs [see Eq. (33)].

There is a delicate point regarding the proper normalization (or scaling of the coupling constants) of non-normal-ordered terms as those in $H_{\rm tun}$. This is more easily understood considering vertex products diagonal in internal indexes. Using the consistent identities derived above, we can proceed as follows:

$$e^{i\phi_{\sigma\ell}}e^{-i\phi_{\sigma\ell}} \approx 2\sqrt{e^{i\tilde\phi_{\sigma\ell}}e^{-i\tilde\phi_{\sigma\ell}}} \tag{30a}$$

$$\approx \frac{1}{2}\sqrt{16\prod_n e^{i\tilde\phi_{\nu_n}/2}e^{-i\tilde\phi_{\nu_n}/2}} \tag{30b}$$

$$\approx \frac{1}{2}\sqrt{\prod_n 2\sqrt{e^{i\tilde\phi_{\nu_n}}e^{-i\tilde\phi_{\nu_n}}}} \tag{30c}$$

$$\approx \frac{1}{2}\prod_n \sqrt{e^{i\phi'_{\nu_n}}e^{-i\phi'_{\nu_n}}} \tag{30d}$$

$$\approx \frac{1}{2}\prod_n e^{i\phi'_{\nu_n}/2}e^{-i\phi'_{\nu_n}/2}, \tag{30e}$$

where (i) in the first line we shifted the bosons away from half filling; (ii) from the first to the second line we did a change of basis; (iii) going to the third line we used Eq. (29); (iv) from the third to the fourth line we shifted the bosons back to half filling and we also distributed the overall square root; and (v) finally we redistributed the square root between the two vertex operators again using Eq. (29). Notice the introduction of the primes (in ϕ'_ν) to distinguish this case

when the change of basis is done with the $\tilde{\phi}$'s from the case when it was done directly with the original ϕ's. The primes will be dropped when a comparison is not being done and the case in point is clear from the context (this notational variation is used in this subsection only).

What we found is that if the change of bosonic basis is done in terms of *shifted bosons*, then a prefactor of $1/2$ appears for proper normalization (and we shall make this conclusion extensive to nondiagonal products as well). This kind of normalization changes, or rescaling of couplings, is common in bosonization treatments and can often be traced to subtle differences in regularization schemes. In particular, shifting the bosons is equivalent to acting with so-called *boundary-condition changing operators* [2,35], which are a known source for "coupling-constant redefinitions" (for another example, also involving a relative factor of 2, the reader can look at Sec. 2 of Appendix A in Ref. [31]). To summarize our result, we should contrast the differences between Eq. (30e) and the one we presented at the start of this subsection in Eq. (26a). To develop some intuition, let us introduce the lattice-like notation $2n_\nu \approx 1 + \delta n_\nu$, so that, near half filling, the left-hand side is close to 1 and near maximum filling it is close to 2. We will also use the notation $\sqrt{2}\tilde{n}_\nu \approx \sqrt{2n_\nu} \approx \sqrt{1+\delta n_\nu}$ (these will be made more precise in the next subsection). The two BdB-mapping relations can then be rewritten as

$$2n_{\sigma\ell} \approx \prod_n \sqrt{2}\tilde{n}_{\nu_n} \quad \text{when} \quad 2n_{\sigma\ell} \approx 1, \tag{31a}$$

$$2n_{\sigma\ell} \approx \frac{1}{2}\prod_n \sqrt{2}\tilde{n}_{\nu_n} \quad \text{when} \quad 2n_{\sigma\ell} \approx 2 \text{ (or 0)}, \tag{31b}$$

where we highlighted that they are useful in different regimes. Which one, or when each of the two, should be used needs to be judged depending on the problem that is being solved (and that is part of what we mean by a *consistent* use of BdB-based transformations). We argue that the junction problem requires the use of the second one, because the physics of tunneling calls for the consideration of unit-size particle-number fluctuations at the junction ($n_{\sigma\ell} = 0 \leftrightarrow 1$).

It is instructive to see how these two different regimes (i.e., half filling versus maximum/minimum filling) are connected in our formalism to a change of boundary conditions for the new fermions after the BdB-based mapping. We start from the continuum boundary conditions, $\psi^\dagger_{\sigma\ell}(0^-) = \psi^\dagger_{\sigma\ell}(0^+)$ for all $\sigma\ell$. After changing basis in the intermediate bosonic language of the ϕ's, we get to the new fields with $\psi^\dagger_{\nu_n}(0^-) = \psi^\dagger_{\nu_n}(0^+)$ for all $\nu_n = c,s,l,sl$, as naturally expected. If we do the change of basis with the $\tilde{\phi}$'s instead, the resulting boundary conditions are different. In the *charge* sector, from the definition of the $\tilde{\phi}$'s it follows that $\tilde{\phi}_c = \phi_c - \pi\,\mathrm{sgn}(x)$ (where we went back to $x_0 = 0$), and thus $\phi'_c = \tilde{\phi}_c + \frac{\pi}{2}\,\mathrm{sgn}(x) = \phi_c - \frac{\pi}{2}\,\mathrm{sgn}(x)$. From there it follows that $\psi^\dagger_c(0^-) = -\psi^\dagger_c(0^+)$. For the other sectors ($\nu_n \neq c$), we simply have $\tilde{\phi}_{\nu_n} = \phi_{\nu_n}$, and thus $\phi'_{\nu_n} = \tilde{\phi}_{\nu_n} + \frac{\pi}{2}\,\mathrm{sgn}(x) = \phi_{\nu_n} + \frac{\pi}{2}\,\mathrm{sgn}(x)$, so that, in a different way, we still get that $\psi^\dagger_{\nu_n}(0^-) = -\psi^\dagger_{\nu_n}(0^+)$.

Remarkably, these antiperiodic boundary conditions parallel what Affleck calls "strong-coupling boundary conditions" in the context of the boundary-conformal-field-theory approach to quantum-impurity problems (see Eq. (1.29) of Ref. [13]). The name is because these are the type of boundary conditions needed in the strong-coupling limit of those problems. What these boundary conditions actually do is to decouple the band-fermion degrees of freedom at $x = x_0$ from the rest of the bulk; that way they are not tied to half filling (or other) conditions and they are available to couple them (strongly) to the impurity. In our case, we shall in general need those degrees of freedom to be available (even if there is no impurity) to participate unrestrainedly in transport situations.

We shall thus refer in our context more generically to *consistent boundary conditions* (CBCs). These depend on the problem at hand and in the particular example studied here they turn out to be antiperiodic boundary conditions. Notice that the need for a factor of $1/2$ as discussed above can be seen as the practical manifestation of the boundary conditions that were (implicitly) adopted. Let us also mention that the use of CBCs does not modify the form of the kinetic part of the action [i.e., when rewriting Eq. (15) in terms of $\phi \to \phi'$]. The solitons that we introduce with ϕ' will induce in $\mathcal{H}^0$ additional slips of 2π localized to a length scale of a around x_0, but since that is the limit of length resolution and the bosonic fields are compact with radius 2π, those contributions consistently drop out.

C. Tunneling of new fermions

It is now a matter of a delicate but ultimately simple replacement to finish the debosonization of $H_{\rm tun}$ in the *charge* and *spin* sectors. We (re)introduce the following definitions (all fields are at $x_0 = 0$ and time t):

$$\begin{aligned}\sqrt{2}\tilde{n}^+_{c,s} &\equiv e^{i\phi_{c,s}/2}e^{-i\phi_{c,s}/2} \\ &\equiv \sqrt{1 + a\,\partial\phi_{c,s}} = \sqrt{e^{i\phi_{c,s}}e^{-i\phi_{c,s}}} \\ &\equiv \sqrt{2\pi a \psi^\dagger_{c,s}\psi_{c,s}} = \sqrt{2n^+_{c,s}}\,, \end{aligned} \tag{32a}$$

$$\begin{aligned}\sqrt{2}\tilde{n}^-_{s} &\equiv e^{-i\phi_{s}/2}e^{i\phi_{s}/2} \\ &\equiv \sqrt{1 - a\,\partial\phi_{c,s}} = \sqrt{e^{-i\phi_{s}}e^{i\phi_{s}}} \\ &\equiv \sqrt{2\pi a \psi_{s}\psi^\dagger_{s}} = \sqrt{2n^-_{s}}\,. \end{aligned} \tag{32b}$$

Thus $\tilde{n}^+_{c,s}$ ($\tilde{n}^-_s$) are simply the square roots of the particle (hole) density of *charge* or *spin* fermions at the site of the junction. For a physical picture, one could think of them as corresponding to a single lattice site after a lattice discretization with πa as the lattice constant, even though that is not the type of regularization adopted when bosonizing [3]. One can explicitly check consistency by calculating their squares via the equal-time, *full* operator product. For that we need to write the bosons in terms of their creation and annihilation components, $\phi(x) = \varphi^\dagger(x) + \varphi(x)$, which obey $[\varphi(x),\varphi^\dagger(x')] = -\ln(1 - e^{-\frac{2\pi}{L}[i(x-x')+a]})$ (see Ref. [4] for the notational convention to point-split the product and

normal-order the vertex operators):

$$
\begin{aligned}
[\tilde{n}]^2 &= \frac{1}{2} e^{i\phi(x)/2} e^{-i\phi(x')/2} e^{i\phi(x)/2} e^{-i\phi(x')/2} \\
&= \sqrt{\frac{\pi a}{2L}} e^{\frac{i}{2}\varphi^\dagger(x)} e^{\frac{i}{2}\varphi(x)} e^{-\frac{i}{2}\varphi^\dagger(x')} e^{-\frac{i}{2}\varphi(x')} \\
&\quad \times e^{\frac{i}{2}\varphi^\dagger(x)} e^{\frac{i}{2}\varphi(x)} e^{-\frac{i}{2}\varphi^\dagger(x')} e^{-\frac{i}{2}\varphi(x')} \\
&= \sqrt{\frac{\pi a}{2L}} \left(\frac{1 - e^{-\frac{2\pi}{L}[i(x-x')+a]}}{1 - e^{-\frac{2\pi}{L}[i(x'-x)+a]}} \right)^{1/4} \\
&\quad \times \sqrt{1 - e^{-\frac{2\pi}{L}a}}\, e^{i\varphi^\dagger(x)} e^{i\varphi(x)} e^{-i\varphi^\dagger(x')} e^{-i\varphi(x')} \\
&\approx \frac{\pi a}{L} e^{i\varphi^\dagger(x)} e^{i\varphi(x)} e^{-i\varphi^\dagger(x')} e^{-i\varphi(x')} \\
&= \frac{1}{2} e^{i\phi(x)} e^{-i\phi(x')} \\
&= n, \qquad (33)
\end{aligned}
$$

where ϕ stands for either $\pm\phi_c$ or $\pm\phi_s$.

The squares of the $\tilde{n}$'s have the properties that $(n^{\pm}_{c,s})^2 = n^{\pm}_{c,s}$ and $n^{\pm}_{c,s} n^{\mp}_{c,s} = 0$; we shall refer to these as *idempotence* and *co-nilpotence*, respectively (notice that if an operator on a finite Hilbert space is idempotent, one of its square roots is the operator itself). In addition, the sum of their squares resolves the identity, $n^{+}_{c,s} + n^{-}_{c,s} = 1$. Thus, as we will see below, they can be consistently assigned the eigen-expectation-values 0 or 1, as if $\langle \tilde{n}^{\pm}_{c,s} \rangle \mapsto \sqrt{\langle n^{\pm}_{c,s} \rangle}$.

Let us introduce the notation $\gamma_{t\sigma} \equiv \gamma_t \tilde{n}^{+}_{c} \tilde{n}^{\sigma}_{s} = \gamma_t (\sqrt{2}\tilde{n}^{+}_{c})(\sqrt{2}\tilde{n}^{\sigma}_{s})/2$ (where the factor of $1/2$ at the end is included for a coupling-constant rescaling in accordance with our discussion in the previous subsection). The consistent form of the tunneling term is then more compactly rewritten as

$$
\begin{aligned}
H_{\rm tun} = &- e^{ieVt} \gamma_{t\uparrow} \check{\psi}_l \check{\psi}_{sl} - e^{ieVt} \gamma_{t\downarrow} \check{\psi}^\dagger_{sl} \check{\psi}_l \\
&- e^{-ieVt} \gamma^*_{t\uparrow} \check{\psi}^\dagger_{sl} \check{\psi}^\dagger_l - e^{-ieVt} \gamma^*_{t\downarrow} \check{\psi}^\dagger_l \check{\psi}_{sl} \,. \qquad (34)
\end{aligned}
$$

And we can finally gauge out the applied voltage from the explicit time dependence, as we discussed already for the conventional procedure, by using $\psi_l(x,t) = e^{ieV(t-x/v_F)} \check{\psi}_l(x,t)$. This gives

$$
H_{\rm tun} = -[\gamma_{t\uparrow} \psi_l - \gamma^*_{t\downarrow} \psi^\dagger_l] \psi_{sl} - \psi^\dagger_{sl} [\gamma_{t\downarrow} \psi_l - \gamma^*_{t\uparrow} \psi^\dagger_l].
$$

Notice that we are not able to combine the fields into Majorana components, as we did in the conventional framework, due to the spin dependence acquired by $\gamma_{t\sigma}$. We see how this time the spin plays a role and starts to show up clearly, as expected from our diagrammatic analysis of the problem.

D. A resolution of the puzzle

We are now ready to recompute the indirect solution to the transport problem after debosonizing consistently. Revising the expression for the current we find

$$
\begin{aligned}
I = &- i\gamma_{t\uparrow} \langle \psi_l(0,t) \psi_{sl}(0,t) \rangle - i\gamma_{t\downarrow} \langle \psi^\dagger_{sl}(0,t) \psi_l(0,t) \rangle \\
&+ i\gamma^*_{t\uparrow} \langle \psi^\dagger_{sl}(0,t) \psi^\dagger_l(0,t) \rangle + i\gamma^*_{t\downarrow} \langle \psi^\dagger_l(0,t) \psi_{sl}(0,t) \rangle .
\end{aligned}
$$

We adopt the same conventions as before for the definition of the Keldysh-Nambu spinor basis and make also the same redefinitions of the couplings to factor out the Fermi velocity. The expression for the inverse Green's function is like in Eq. (22) with the addition of the spin index into the tunneling terms (which is straightforward, since all the Nambu-off-diagonal components acquire $\sigma = \uparrow$ while the Nambu-diagonal components go with $\sigma = \downarrow$).

It should be remarked that unpaired $\psi^{[\dagger]}_{c,s}$ fields do not enter in the tunneling term and appear only in the kinetic one (as bilinears). As a result, any connected perturbative expansion in $H_{\rm tun}$ does not involve the *charge* and *spin* sectors and the $\tilde{n}^{\pm}_{c,s}$ can be treated as c-numbers, (restoring the Gaussianity of the problem). Due to global gauge invariance for each lead, the final expressions involve always the squares of the $\tilde{n}$'s and can thus be simplified thanks to their idempotence and co-nilpotence. An alternative equivalent calculational procedure is to set the $\tilde{n}$'s to their different eigen-expectation-values, to do the calculation, and to trace over all such values (not average over, because they are not exactly conserved quantities). This second path is shorter and makes more explicit the connection with the direct solution.

The result one gets for the I-V characteristics, by following the procedure outlined above, is what one was hoping for:

$$
\begin{aligned}
I &= \frac{4|t|^2}{(1+|t|^2)^2} \int_0^{+\infty} [s_l(\omega) - \bar{s}_l(\omega)] \frac{d\omega}{2\pi} \\
&\xrightarrow[T_v \to 0]{} \frac{4|t|^2 eV}{\pi (1+|t|^2)^2} . \qquad (35)
\end{aligned}
$$

When comparing with the result of the direct calculation, given in Eqs. (7) and (8), the matching is now exact and all the discrepancies are gone. Namely, (i) the $t \mapsto t/2$ correction is not required as it happened naturally courtesy of the CBCs; (ii) the spin degeneracy arises automatically and the correct overall prefactor arises also naturally; (iii) the extra factor of $(1+|t|^2)$ in the numerator is not present.

V. CONCLUSION AND PROSPECTS

By focusing on a case study in which bosonization is rigorously applicable and, not less importantly, exact calculations are possible and enable detailed comparisons, we were able to uncover some subtleties of the bosonization-debosonization procedure that had quite strong implications. Besides directly comparing the mathematical expressions as we have been doing, it is instructive to compare the two results graphically. To that end, we plot in Fig. 3 the two indirect solutions for the differential conductance ($G = dI/dV$) computed conventionally and consistently. As expected, the two results only agree in the limit of $t \to 0$. Expanding for small t, the two results coincide to order $O(t^2)$ and start to disagree in the coefficient of the t^4 term (with the conventional result being larger by a factor of 2). This is as expected from the diagrammatic analysis. Let us remark that, for $t > 1$, the differential conductance computed conventionally not only lacks the $t \leftrightarrow 1/t$ duality of the exact result, but it does not even go to zero for $t \to \infty$. In that limit, one should have expected a resonating-tunneling bond at the site of the junction to trap an electron (for each spin) and thus block the passage

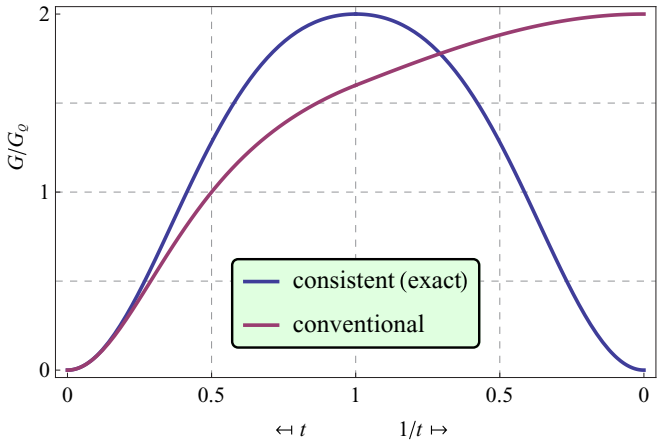


FIG. 3. Comparison of the differential conductance for the simple junction calculated both consistently (or directly) and conventionally. Notice the unusual convention for the horizontal axis in order to highlight the $t \leftrightarrow 1/t$ duality of the problem. The vertical axis is in units of the single-channel quantum of conductance, $G_Q = e^2/h$, and $G = 2G_Q$ is the quantum limit for this problem.

of the current. A different way to describe it is by appealing to a tight-binding picture. The Hamiltonian for the two sites linked by t needs to be diagonalized first when t is the largest scale in the problem. One finds bonding and antibonding states that, in the $t \to \infty$ limit, will be always occupied and always empty, respectively. The rest of the leads are relatively weakly coupled to these two states and not able to change their fillings and thus not able to produce a current. Instead of agreeing with this picture, the whole curve for the conventional I-V characteristics resembles the result for a diode-like asymmetric junction with a non-Hermitian Hamiltonian [20], in which the formation of a resonating-tunneling bond is precluded by the model. This behavior alone could have been a clear indication that there are problems with the conventional way of calculating (even if one did not have a direct solution to compare with).

The possible ramifications of our findings are many. A large number of the calculations done in the past, for example any problems sharing similarities with the one considered here (i.e., involving a junction, an impurity, or simply a boundary), will need to be reexamined critically. More generally, problems involving backward scattering or other types of nondiagonal interactions or processes need to be reconsidered for possible changes. Not all past results will be significantly affected though. For instance, on the one hand, the (weak-coupling) renormalization-group analysis of the effects of a "classical impurity" in a Tomonaga-Luttinger liquid [36] requires the knowledge of the impurity-potential β function to leading order only, at which consistent and conventional calculations could be expected to (at least roughly) coincide; as our present results have shown it is the case if there are no interactions. There will be differences, but those would be expected in the finer details, probably appear at the next-leading order, and a calculation would be needed to determine them. On the other hand, the implications for the case of "quantum impurities" will be more dramatic. To put things in perspective, the good news is that we were able to provide a clear procedure, in the form of the $\tilde{n}$ factors, to bosonize and debosonize a large class of models *consistently*.

This paper was focused on the motivation and presentation of the formalistic details. In the future we will look at more involved examples of greater physical significance. We already started to reexamine some salient cases, and in the next paper we shall focus on the important case of transport through quantum impurities in Fermi liquids [16].

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APPENDIX: JUNCTION THERMODYNAMICS

The subtleties with the BdB-based mapping are quite generic and not restricted to nonequilibrium situations. Let us briefly compute the junction contribution to the free energy (or grand potential) and thus, indirectly, all thermodynamic quantities. We define the junction contribution in the same way as is done for impurity models: as the difference between the full thermodynamic potentials with the junction closed and open, respectively. There is no voltage applied to the junction.

1. Direct calculation

We start with the original (old) fermions and neglect the spin which will just give a factor of 2 at the end. Here we follow the procedure and notations as in Ref. [32]. We will use, for convenience, a Nambu structure [otherwise we need to introduce $\mathrm{sgn}(\omega_n)$ in the diagonal entries], but we do not need to use Keldysh and, instead, we will use Matsubara formalism and the following spinor basis:

$$\Psi(\omega_n) = (\psi_{\mathrm{L}}(\omega_n) \quad \psi_{\mathrm{L}}^\dagger(-\omega_n) \quad \psi_{\mathrm{R}}(\omega_n) \quad \psi_{\mathrm{R}}^\dagger(-\omega_n))^T.$$

Let us use again the definition $\gamma_t = 2v_F t$ and use the standard result for the local inverse Green's function for the leads to write the local inverse Green's function for the whole junction:

$$G^{-1}(\omega_n) = -2iv_F \begin{pmatrix} 1 & 0 & it & 0 \\ 0 & 1 & 0 & -it \\ it^* & 0 & 1 & 0 \\ 0 & -it^* & 0 & 1 \end{pmatrix}. \tag{A1}$$

We compute the junction contribution to the thermodynamic potential via the standard method of "integrating over the coupling constant":

$$\Delta\Omega = \Omega - \Omega_0 = \int_0^1 \frac{d\xi}{\xi} \langle \xi\, H_{\mathrm{tun}} \rangle_\xi. \tag{A2}$$

Introducing the action determinant

$$D(\omega_n, \xi) \equiv \det G_\xi^{-1}(\omega_n) = |t|^4\xi^4 + 2|t|^2\xi^2 + 1 = (|t|^2\xi^2 + 1)^2, \tag{A3}$$

we can use the formula

$$\Delta\Omega = -\int_0^1 d\xi \, \frac{1}{\beta} \sum_{n\geqslant 0} \frac{\partial_\xi D(\omega_n,\xi)}{D(\omega_n,\xi)}.$$

Since $D(\omega_n,\xi) = D(\xi)$ does not depend on frequency for the problem at hand, we factor out the divergent sum and indicate it as $\delta_{\tau=0} \equiv \frac{2}{\beta}\sum_{n\geqslant 0} 1$. We have

$$\begin{aligned}\Delta\Omega &= -\frac{\delta_{\tau=0}}{2}\int_0^1 d\xi \, \partial_\xi \ln D(\xi) \\ &= -\delta_{\tau=0}\ln(1+|t|^2) \\ &\underset{\times\text{spin}}{\longrightarrow} -2\delta_{\tau=0}\ln(1+|t|^2), \end{aligned} \tag{A4}$$

and we want to compare it with the result after the BdB-based transformations.

2. Conventional indirect calculation

Let us work in terms of the new fermions and adopt the following spinor basis:

$$\Psi(\omega_n) = (\psi_l(\omega_n) \quad \psi_l^\dagger(-\omega_n) \quad \psi_{sl}(\omega_n) \quad \psi_{sl}^\dagger(-\omega_n))^T.$$

With the same definitions, the local inverse Green's function for the junction is

$$G^{-1}(\omega_n) = -2iv_{\mathrm{F}}\begin{pmatrix} 1 & 0 & -it^* & it^* \\ 0 & 1 & -it & it \\ -it & -it^* & 1 & 0 \\ it & it^* & 0 & 1 \end{pmatrix}. \tag{A5}$$

This time the action determinant reads

$$D(\omega_n,\xi) = 4|t|^2\xi^2 + 1, \tag{A6}$$

and applying the same formulas we find

$$\Delta\Omega = -\frac{\delta_{\tau=0}}{2}\ln(1+4|t|^2). \tag{A7}$$

But notice that if we "correct" the coupling constant we get

$$\Delta\Omega \underset{t\mapsto t/2}{\longrightarrow} -\frac{\delta_{\tau=0}}{2}\ln(1+|t|^2). \tag{A8}$$

We see that (i) the same "correction" as in the transport calculation is needed; (ii) we again lack an overall factor of 4, but (iii) the extra factor of $(1+|t|^2)$ is not an issue this time (but notice that the logarithm would turn powers into factors). Because of the last point, a perturbative analysis is not effective to pinpoint the source of the discrepancies the way it is for transport calculations.

Nota bene: Working in the consistent approach and using the (non-number-eigenstates) half-filled basis $(|0\rangle \pm |1\rangle)/\sqrt{2}$ in both the *charge* and *spin* sectors, if we have $\langle\tilde{n}_c\rangle = \langle\tilde{n}_s^\sigma\rangle \equiv 1/\sqrt{2}$ then the "correction" of the coupling constant reappears explicitly (but due to the use of CBCs). Moreover, tracing over the c and s sectors gives the missing factor of 4. One is thus able to recover the direct result with a calculation which does not differ much from the conventional one at the level of the local inverse Green's function, but in an *ad hoc* way.

3. Consistent indirect calculation

Let us repeat the calculation but introducing $\gamma_{\mathrm{t}} \to \gamma_{\mathrm{t}\sigma} = \gamma_{\mathrm{t}}\tilde{n}_c\tilde{n}_s^\sigma$ (recall we divided by 2 since we need to use CBCs). Using the appropriately modified result for the local inverse Green's function for the junction, one finds the following action determinant:

$$\begin{aligned} D(\omega_n,\xi) &= 1 + (2t_\uparrow^* t_\uparrow + 2t_\downarrow^* t_\downarrow)\xi^2 + [t_\downarrow^2(t_\downarrow^*)^2 + t_\uparrow^2(t_\uparrow^*)^2]\xi^4 \\ &= (|t|^2\xi^2 + 1)^2, \end{aligned} \tag{A9}$$

where the last expression is valid for either eigen-expectation-value of n_s. One thus recovers the same expression as in the direct calculation in the original-fermions language [cf. Eq. (A3)]. All the ensuing results are thus identical. Notice the factor of 2 for spin will be contributed by tracing over eigenstates of $\tilde{n}_s$, while on the *charge* sector only the $\langle\tilde{n}_c\rangle = 1$ subspace contributes.

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