Spectral function of the Tomonaga-Luttinger model revisited: Power laws and universality

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We reinvestigate the momentum-resolved single-particle spectral function of the Tomonaga-Luttinger model. In particular, we focus on the role of the momentum dependence of the two-particle interaction $V(q)$. Usually, $V(q)$ is assumed to be a constant and integrals are regularized in the ultraviolet "by hand" employing an *ad hoc* procedure. As the momentum dependence of the interaction is irrelevant in the renormalization group sense, this does not affect the universal low-energy properties of the model, e.g., exponents of power laws, if *all* energy scales are sent to zero. If, however, the momentum k is fixed away from the Fermi momentum k_F, with $|k - k_F|$ setting a nonvanishing energy scale, the details of $V(q)$ start to matter. We provide strong evidence that any curvature of the two-particle interaction at small transferred momentum q destroys power-law scaling of the momentum-resolved spectral function as a function of energy. Even for $|k - k_F|$ much smaller than the momentum-space range of the interaction the spectral line shape depends on the details of $V(q)$. The significance of our results for universality in the Luttinger liquid sense, for experiments on quasi-one-dimensional metals, and for recent results on the spectral function of one-dimensional correlated systems taking effects of the curvature of the single-particle dispersion into account ("nonlinear LL phenomenology") is discussed.

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I. INTRODUCTION

A. Luttinger liquid universality and the Tomonaga-Luttinger model

It is well established that the Tomonaga-Luttinger model (TLM) [1,2] with linear single-particle dispersion and a two-particle interaction potential $V(q)$ which is finite at vanishing momentum transfer $q = 0$ forms the infrared fixed point under renormalization group (RG) flow of a large class of gapless one-dimensional (1D) models of correlated fermions [3]. This is the essence of the much celebrated Luttinger liquid (LL) universality [4–7]. It implies that the low-temperature thermodynamic properties as well as the low-energy spectral functions of a model belonging to the LL universality class are equivalent to the ones of the TLM. Understanding the low-energy physics of the latter is thus of crucial importance. Fortunately, using bosonization [4,7,8] it is possible to derive exact and closed analytical expressions for thermodynamic observables such as the specific heat or the compressibility as well as for space-time correlation functions of the TLM. From the latter, spectral functions can be computed by Fourier transform.

The bosonization expressions for correlation functions of the TLM depending on position x and time t generically contain integrals over momenta. Within constructive bosonization, which is based on operator identities [4,7], these are naturally regularized in the ultraviolet by the momentum-space range $q_c > 0$ of the two-particle potential $V(q)$. For time-dependent correlation functions the momentum integrals cannot be performed even if a specific form of $V(q)$ is assumed [9]. However, one can show that the momentum dependence of the interaction is RG irrelevant [3]. This is employed to justify the following procedure: In the final expressions for the space-time correlation functions, $V(q)$ is routinely replaced by a constant. As a consequence, the momentum integrals become divergent in the ultraviolet. These divergences are regularized ("by hand") applying an *ad hoc* procedure [5,6,10]. We already now emphasize that this regularization is not unique. After these steps, the momentum integrals can be performed and integral-free expressions for space-time correlation functions are obtained. In field-theory inspired phenomenological bosonization procedures [6], the momentum dependence of the interaction is often neglected from the outset (even in the Hamiltonian). In correlation functions this leads to the same ultraviolet divergences as described above requiring again a regularization by hand. Similarly, the purely fermionic approach to the single-particle Green function $G^{\gtrless}(x,t)$ of the TLM by Dzyaloshinskiĭ and Larkin [11] requires an *ad hoc* ultraviolet regularization.

We thus emphasize that the integral-free expressions for a variety of time-dependent correlation functions which can be found in the literature cannot be considered as the corresponding *exact* correlation functions of the TLM. This is often acknowledged by stating that the integral-free expressions of the *ad hoc* procedure only agree to the exact ones at asymptotically large space-time distances; as discussed in Ref. [9] (see also the following), even this is incorrect when considering the decay in the directions specified by $x = \pm vt$, with v being one of the eigenmode velocities at small momentum.

B. Spectral functions of the Tomonaga-Luttinger model: The fate of power laws

We now focus on the two-point correlation function, the Green function, at temperature $T = 0$ from which the single-particle spectral function can be computed by Fourier transform. The spectral function is of particular importance as it provides forthright access to correlation effects and can directly be related to photoemission spectra. We consider the momentum-integrated function $\rho^{<}(\omega)$ $[\rho^{>}(\omega)]$, which is experimentally accessible in angular-integrated [inverse] photoemission, as well as the momentum-resolved spectral function $\rho^{<}(k,\omega)$ $[\rho^{>}(k,\omega)]$. A measurement of the latter

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requires momentum resolution. It was shown that the universal low-energy power-law suppression of $\rho^{\gtrless}(\omega) \sim |\omega|^{\alpha}$ for $\omega \to 0$, with $\alpha > 0$ is unaffected by the above *ad hoc* procedure [9]. Regardless of the details of $V(q)$, the exponent α depends on the potential at vanishing momentum transfer $V(0)$ (only), that is the constant interaction strength after the *ad hoc* procedure. Furthermore, the power-law nonanalyticity of $\rho^{\gtrless}(k_{\mathrm{F}},\omega) \sim |\omega|^{\alpha-1}$ exactly at the Fermi momentum $k = k_{\mathrm{F}}$ remains the same; depending on the size of $V(0)$ a divergence ($\alpha < 1$) or suppression ($\alpha > 1$) might occur. These findings are consistent with the RG irrelevance of the momentum dependence of $V(q)$ as in both cases *all* energy scales, that is ω *and,* in the case of momentum-resolved spectra, $v_{\mathrm{F}}(k - k_{\mathrm{F}})$ are sent to zero [3]. Here, v_{F} denotes the Fermi velocity.

The question we address here is whether or not any of the standard *ad hoc* procedures are legitimate when it comes to $\rho^{\gtrless}(k,\omega)$ as a function of ω at fixed $k - k_{\mathrm{F}} \neq 0$. Employing these to compute $\rho^{\gtrless}(k,\omega)$ of the spinless TLM, characteristic algebraic threshold nonanalyticities at $\pm v_c[k - k_{\mathrm{F}}]$ with the charge velocity v_c of the (collective, bosonic) charge eigenmodes were found. In the model with spin additional algebraic nonanalyticities appear at $\pm v_s[k - k_{\mathrm{F}}]$ with the velocity v_s of the spin modes [10–14]. The corresponding exponents can be expressed in terms of the (momentum-independent) interaction potential. For small interactions and $k - k_F < 0$, $\rho^{<}(k,\omega)$ shows *power-law singularities* at $\omega = v_c[k - k_{\mathrm{F}}]$ and $\omega = v_s[k - k_{\mathrm{F}}]$ instead of a single (Lorentzian) peak which would emerge in a Fermi liquid. This is one of the signatures of spin-charge separation regarded as a hallmark of (spinful) LLs. In Ref. [15] it was shown that these features can be found in the exact spectral function of the TLM if a boxlike potential $V(q) = V(0)\Theta(q_{\mathrm{c}}^2 - q^2)$ is assumed, as long as $|k - k_{\mathrm{F}}| < q_{\mathrm{c}}$ with the momentum-transfer cutoff q_{c}. We emphasize that using a box potential is per se not equivalent to the *ad hoc* procedure, as it can, e.g., be seen considering $\rho^{\gtrless}(k,\omega)$ for $|k - k_{\mathrm{F}}| > q_{\mathrm{c}}$ [15]. For $|k - k_{\mathrm{F}}| < q_{\mathrm{c}}$, the box potential might, however, be viewed as a unique realization of the *ad hoc* procedure (see following). Clearly, a box potential is rather special and might not even be considered as particular physical. Thus, further work for more generic $V(q)$ is required.

In Ref. [9] it was shown that the algebraic properties of the Green functions $G^{\gtrless}(x,t)$ in the space-time plane are significantly affected by the *ad hoc* procedures. It was proven that the exponent of the asymptotic decay of the Green function in the distinguished directions $x = \pm v_{c/s}t$ is not only set by $V(0)$, but in addition by a measure of the flatness of the potential at $q \to 0$, a result which cannot be obtained within any *ad hoc* procedure. Based on this and the crucial insight that the decay of $G^{\gtrless}(x,t)$ in the distinguished directions plays a central role in obtaining the power-law nonanalyticities in $\rho^{\gtrless}(k,\omega)$, the question was posed if for generic $V(q)$, $\rho^{\gtrless}(k,\omega)$ is characterized by the "thought to be universal" power laws of the *ad hoc* procedure. However, Ref. [9] lacks a definite answer.

The above-mentioned box potential is "infinitely flat" at $q \to 0$ and thus "nongeneric." The asymptotics of $G^{\gtrless}(x,t)$ for this and the *ad hoc* procedures agree and consequently also the features of $\rho^{\gtrless}(k,\omega)$ at $\omega \approx v_{c/s}[k - k_{\mathrm{F}}]$.

It is crucial to realize that a dependence of the line shape of $\rho^{\gtrless}(k,\omega)$ on the details of the interaction away from $q = 0$ including such fundamental issues as the presence or absence of algebraic nonanalyticities does not contradict the RG irrelevance of the momentum dependence of $V(q)$ in the TLM. From this, universality can only be deduced if *all* energy scales are sent to zero (see above). A fixed $k - k_{\mathrm{F}} \neq 0$ sets a scale which becomes active for all generic $V(q)$ with $V^{(n)}(q = 0) \neq 0$ for some $n \in \mathbb{N}$, where $V^{(n)}(q)$ denotes the nth derivative. Thus, $\rho^{\gtrless}(k,\omega)$ cannot be expected to be universal on general grounds. In fact, a finite scale will destroy the scale invariance of the model and thus "quantum critical" power-law scaling. In the case of the at $q = 0$ infinitely flat box potential this mechanism is not active as long as $|k - k_{\mathrm{F}}| < q_{\mathrm{c}}$.

We here supplement Ref. [9] and provide very strong evidence that the spectral function of the TLM at fixed $k - k_{\mathrm{F}} \neq 0$ and for a generic potential is not characterized by power laws. The latter are only found if $k - k_{\mathrm{F}} \to 0$. To guide the reader, we should from the outset be very precise about our understanding of "power-law scaling." We say that some real function f shows power-law scaling with exponent ξ close to $x_0 \in \mathbb{R}$ from above if $d \ln|f(x)|/d \ln(x - x_0)$ approaches ξ for $x \to x_0^+$. A similar definition can be given for power-law scaling from below. This does of course not exclude that f can to some degree be approximated by a power law or "resembles" a power law for some range of x (close to x_0) even if it does not fulfill the above criterion. We will return to this in Secs. III E and III F in which we present our results for $\rho^{\gtrless}(k,\omega)$. More generally, we show that even for $|k - k_{\mathrm{F}}| \ll q_{\mathrm{c}}$, the spectral line shape depends on the details of $V(q)$.

Although this study might be regarded as somewhat technical, or even pedantic given that issues of the momentum dependence of $V(q)$ in the TLM are virtually always nonchalantly ignored, our results have far-reaching consequences.

C. Implications of our results

The first implication of our results concerns the concept of LL universality. While $\rho^{\gtrless}(\omega)$ of *any* model from the LL universality class shows the power-law suppression of spectral weight for $\omega \to 0$ and $\rho^{\gtrless}(k_{\mathrm{F}},\omega)$, a power-law peak or suppression at the same energy, *LL universality does not predict power laws in* $\rho^{\gtrless}(k,\omega)$ for fixed $k - k_{\mathrm{F}} \neq 0$. Evidently, if this type of universality cannot be proven in the low-energy fixed point model, the TLM, it cannot be a characteristic feature of the LL universality class. This does of course not exclude that *certain models* from the LL universality class might show such power laws, however, if so for *more specific reasons than LL universality.* An obvious example for this is the TLM with box potential [15]. Other examples might be based upon the restriction of the (equilibrium) dynamics encountered in specific 1D models with an extensive number of local conserved quantities (e.g., the Hubbard model) which are often Bethe ansatz solvable [16]. A detailed discussion in which we relate our results to spectra of 1D lattice models obtained by either analytical or numerical approaches is given in Sec. IV A.

Strongly linked to this are the implications of our findings for recent results on $\rho^{\gtrless}(k,\omega)$ taking the nonlinearity of the single-particle dispersion into account [17–22] which are embedded in the framework of the so-called "nonlinear LL

phenomenology" [23]. In this, power laws are not viewed as originating from quantum critical scale invariance but rather as resulting from a Fermi-edge-singularity-like effect. In the phenomenological construction of the effective field theory including curvature effects of the dispersion, the above-described *ad hoc* regularization is employed. The spectral function is computed for this field theory. Our results obtained for *linear* LL theory raise the question as to whether the power laws found in "nonlinear LL phenomenology" are robust against a curvature of the bulk two-particle potential. We emphasize that these power laws are specific to the nonlinearity of the dispersion, which, e.g., leads to momentum-dependent exponents, and are thus different from the nonanalyticities found for the TLM treated within the *ad hoc* procedure (or, for that matter, the TLM with box potential). Again, this does of course not exclude that for *specific 1D models,* e.g., Bethe ansatz solvable lattice models, power laws with momentum-dependent exponents might be realized. More on this can be found in Sec. IV B.

Finally, our findings are of importance for the interpretation of experimental momentum-resolved spectra. Even after decades of research, none of the photoemission experiments on quasi-1D metals reporting on the observation of dispersing spin and charge peaks remain unchallenged [24]. One reason for this is that, when interpreting experimental data in the light of LL physics, the momentum-resolved spectral function obtained within the TLM employing an *ad hoc* regularization is taken paradigmatically. Crucially, we find spin and charge peaks for generic $V(q)$ and a sufficiently small amplitude of the two-particle interaction even though they are not given by power-law singularities. Our results show, however, that details are model dependent [in our case $V(q)$ dependent] and therefore nonuniversal. Thus, the detailed spectral features of the *ad hoc* regularized TLM cannot be expected to be found in experimental spectra. Further account of the relevance of our results for experimental spectra is given in Sec. IV C.

D. Structure of the paper

The rest of our paper is structured as follows. In Sec. II, we introduce the TLM and its bosonization solution. Constructive bosonization of the field operator can be used to derive a closed analytical expression for the single-particle Green function, which, however, contains a momentum integral on the right-hand side. This is discussed in Sec. III A. In Sec. III B, we introduce different versions of the *ad hoc* regularization to obtain integral-free expressions for $G^{\gtrless}(x,t)$. Sections III C and III D are devoted to the technical details of how we obtain exact spectra for a box potential and arbitrary potentials, respectively. In Sec. III E, we present results for the spectral function of the so-called g_4 model with intrabranch interaction only and different shapes of momentum dependency of the interaction. These are compared to those obtained by the *ad hoc* procedure. Section III F is devoted to the spectral function of the spinless TLM, the spinless g_2-g_4 model. In Sec. IV, we discuss the implications of our results. When alluding to the relevance of our insights for the interpretation of photoemission data, we in addition present spectral functions for the spinful TLM.

II. TOMONAGA-LUTTINGER MODEL

We here do not introduce the TLM by "deriving" it from the 1D interacting electron gas under certain assumptions (e.g., on the real-space range of the interaction; no $2k_F$ two-particle scattering processes) [7] or as the effective field theory for microscopic lattice models [6] but rather take it as a stand-alone model. It consists of independent right- ($\alpha = +$) and left-moving ($\alpha = -$) fermions with spin s, creation operators $a^\dagger_{k,\alpha,s}$, dispersion $\xi_\alpha(k) = \alpha v_F(k - \alpha k_F)$, density operators ($q \neq 0$) $\rho_{\alpha,s}(q) = \sum_k a^\dagger_{k,\alpha,s} a_{k+q,\alpha,s}$, and particle number operators $n_{k,\alpha,s} = a^\dagger_{k,\alpha,s} a_{k,\alpha,s}$. Following Luttinger [2], an infinite "Dirac sea" filled in the ground state is assumed and thus the momentum quantum number k of both particle species is unbounded. This simplifies the mathematical treatment as certain relations become operator identities [4,7] and are not only restricted to the low-energy part of the Hilbert space as in Tomonaga's approach [1]. This addition of states often requires normal ordering. The Hamiltonian for a system of length L is given by

$$\begin{aligned} H = &\sum_k \sum_{\alpha,s} \xi_\alpha(k)[n_{k,\alpha,s} - \langle n_{k,\alpha,s}\rangle_0] \\ &+ \frac{1}{2L} \sum_{\substack{q\neq 0 \\ \alpha,s,s'}} [g_{4,\parallel}(q)\delta_{s,s'} + g_{4,\perp}(q)\delta_{s,-s'}]\rho_{\alpha,s}(q)\rho^\dagger_{\alpha,s'}(q) \\ &+ \frac{1}{L} \sum_{\substack{q\neq 0 \\ s,s'}} [g_{2,\parallel}(q)\delta_{s,s'} + g_{2,\perp}(q)\delta_{s,-s'}]\rho_{+,s}(q)\rho^\dagger_{-,s'}(q). \end{aligned} \tag{1}$$

Here, $\langle \ldots \rangle_0$ denotes the (noninteracting) ground-state expectation value (normal ordering). We keep the explicit q dependence of the two-particle potential. The interbranch (g_2) and intrabranch (g_4) potentials are not necessarily equal and replace the potential $V(q)$ referred to in the Introduction. Similarly, the interaction of spin parallel ($\parallel$) and antiparallel ($\perp$; this is a confusing but standard notation [3]) particles is not necessarily the same. If the TLM is considered as the low-energy fixed point model of the LL universality class, this flexibility is required. The low-energy physics of any model from this class is characterized by four independent numbers, e.g., the two LL parameters $K_{c/s}$ and the two velocities $v_{c/s}$ [4,7]. In the TLM for a given v_F (and k_F) those are fixed by the (in general) four independent "coupling constants" $g_{i,\kappa}(q = 0)$ ($i = 2,4$; $\kappa = \parallel, \perp$) at vanishing momentum transfer (see following). Therefore, to encounter nontrivial interaction effects in the LL sense the $g_{i,\kappa}(q = 0)$ should not all be 0. We restrict ourselves to these kinds of interactions.

We assume that the Fourier transforms $g_{i,\kappa}(q)$ of the two-particle interaction are even and for $q \geqslant 0$ monotonic functions which vanish for $|q| \gg q_c$, with an interaction cutoff q_c. These requirements are physically sensible if the TLM is considered in its own right [1]. If, however, the TLM is studied as the effective low-energy model it is less clear if this assumption holds. It is thus crucial that the assumption is not essential for our main conclusions. Relaxing it would merely complicate the calculations as positive and negative momenta would have to be treated separately. We emphasize that at *no* stage of the discussion will it be necessary to introduce any further

ultraviolet cutoffs "by hand" despite the infinite (filled) Dirac sea at negative energies. In this sense, the Hamiltonian (1) represents a mathematically well-defined model.

To be more precise, the Hamiltonian in Eq. (1) defines a *whole class of models* as the four coupling functions $g_{i,\kappa}(q)$ can be arbitrarily chosen as long as the introduced requirements are fulfilled. Still, we continue to refer to this class as *the* TLM. Note that a Hamiltonian of the form Eq. (1) but with *coupling constants* $g_{i,\kappa}$ instead of *coupling functions* $g_{i,\kappa}(q)$ can often be found in the literature. The necessary ultraviolet regularization is then left implicit and frequently not uniquely defined.

We note that particle number contributions to the Hamiltonian which appear if the model is derived from the interacting 1D electron gas [4,7] are suppressed as they do not play any role for our considerations.

The spinless version of the TLM follows from Eq. (1) by dropping the spin index and keeping only $g_i(q)$ instead of $g_{i,\kappa}(q)$ for $i = 2,4$.

Bosonization of the Hamiltonian and a canonical transformation lead to [4,7]

$$H = \sum_{q\neq 0}\sum_{\nu=c,s} \omega_\nu(q)\beta_\nu^\dagger(q)\beta_\nu(q) + E_0, \tag{2}$$

with bosonic operators $\beta_\nu^{(\dagger)}(q)$ describing collective charge ($\nu = c$) and spin ($\nu = s$) excitations (spin-charge separation) as well as the ground-state energy E_0. The $\beta_\nu^{(\dagger)}(q)$ are linearly related to the densities $\rho_{\alpha,s}(q)$ of the fermions. The mode energies $\omega_\nu(q)$ are given by

$$\frac{\omega_\nu(q)}{|q|} = v_F\sqrt{\left(1+\frac{g_{4,\nu}(q)}{\pi v_F}\right)^2 - \left(\frac{g_{2,\nu}(q)}{\pi v_F}\right)^2} = v_\nu(q), \tag{3}$$

where we have introduced the renormalized *momentum-dependent* charge and spin density velocities $v_\nu(q)$ and interactions

$$g_{i,c/s}(q) = [g_{i,\parallel}(q) \pm g_{i,\perp}(q)]/2. \tag{4}$$

We already now emphasize that for momentum dependent $g_{i,\kappa}(q)$ the eigenmode dispersions will become *nonlinear*. The *linearization* of the latter, that is the replacement

$$v_\nu(q) \to v_\nu(0) = v_\nu, \tag{5}$$

is the *crucial step* in the *ad hoc* regularization procedure to derive integral-free expressions for correlation functions (see following).

In Eq. (5) we have introduced the q-independent charge and spin velocities $v_{c/s}$ relevant for the low-energy physics (all energy scales sent to 0) in the LL sense. The corresponding LL parameters $K_{c/s}$ of the TLM are obtained from

$$K_\nu(q) = \sqrt{\frac{1+g_{4,\nu}(q)/(\pi v_F) - g_{2,\nu}(q)/(\pi v_F)}{1+g_{4,\nu}(q)/(\pi v_F) + g_{2,\nu}(q)/(\pi v_F)}} \tag{6}$$

in the limit $q \to 0$. Note that $K_\nu(q) = 1$ if the interbranch interaction $g_{2,\nu}(q)$ vanishes.

The spinless version of the bosonized Hamiltonian is obtained after dropping the terms with index $\nu = s$.

When presenting our results for $\rho^{\gtrless}(k,\omega)$ we will initially focus on two special cases. The first is the spinful TLM with intrabranch interaction only, that is, $g_{2,\kappa}(q) = 0$, which is commonly referred to as the g_4 model. The second is the spinless g_2-g_4 model. They are paradigmatic for the two interaction effects characteristic for LLs: spin-charge separation and power-law scaling with interaction-dependent exponents, respectively. Proceeding this way increases the transparency of our analysis. The scenario for the general spinful model can be deduced by combining the insights of both cases; in Sec. IV C we in addition present a few explicit results for the spinful TLM.

III. MOMENTUM-RESOLVED SPECTRAL FUNCTION

A. Single-particle Green function

We are interested in the momentum-resolved spectral function. It can be computed by Fourier transforms from the greater and lesser single-particle Green function

$$iG^>_{\alpha,s}(x,t) = \langle\psi_{\alpha,s}(x,t)\psi^\dagger_{\alpha,s}(0,0)\rangle, \tag{7}$$

$$iG^<_{\alpha,s}(x,t) = \langle\psi^\dagger_{\alpha,s}(0,0)\psi_{\alpha,s}(x,t)\rangle. \tag{8}$$

The field operators $\psi^\dagger_{\alpha,s}(x)$ and $\psi_{\alpha,s}(x)$ are related in the usual way to the creation and annihilation operators in momentum space

$$\psi_{\alpha,s}(x) = \frac{1}{\sqrt{L}}\sum_k e^{ikx}a_{k,\alpha,s}. \tag{9}$$

Particle-hole symmetry of the TLM ensures [7,13]

$$G^>_{\alpha,s}(x,t) = G^<_{\alpha,s}(-x,-t) \tag{10}$$

and it is sufficient to consider the greater Green function. We note in passing that band filling is not an issue in the TLM as we consider it. However, when the model is investigated as the effective low-energy model of another 1D correlated electron model, the band filling of the latter will enter in the LL parameters and velocities characterizing the low-energy physics [7].

Furthermore, with $x \to -x$ we can go over from the Green function of right movers to the one of left movers. Therefore, we will only study $G^>_{+,s}(x,t)$. In the absence of a magnetic field, the Green function is spin independent and we thus suppress the spin index from now on.

To compute the ground-state expectation value in Eq. (7) we use constructive bosonization of the field operator [4,7]. We emphasize that this approach is based on operator identities and does not require the introduction of any cutoffs if one first considers finite systems of length L (as we will do). In more phenomenological approaches [6], a cutoff is introduced, often denoted by $1/\alpha$ and referred to as an "effective band width," which formally has to be sent to infinity. However, it is frequently kept finite artificially. This is part of one of the possible *ad hoc* ultraviolet regularizations.

The exact greater Green function of right movers for the most general spinful g_2-g_4 model is given by

$$i\,G^>_+(x,t) = i\,[G^>_+]^0(x,t)\,e^{F(x,t)}, \tag{11}$$

with

$$[G^>_+]^0(x,t) = \frac{e^{i(k_F+\pi/L)x}}{L}\exp\left\{\sum_{n=1}^{\infty}\frac{1}{n}e^{iq_n(x-v_Ft+i0^+)}\right\}. \tag{12}$$

The interaction enters in

$$F(x,t) = \frac{1}{2}\sum_{\nu=c,s}\sum_{n=1}^{\infty}\frac{1}{n}\big[e^{iq_nx}\big(e^{-i\omega_\nu(q_n)t} - e^{-iv_Fq_nt}\big) + 2\gamma_\nu(q_n)\big(\cos(q_nx)e^{-i\omega_\nu(q_n)t} - 1\big)\big], \quad (13)$$

with

$$\gamma_\nu(q) = [K_\nu(q) + 1/K_\nu(q) - 2]/4 \quad (14)$$

and $q_n = n2\pi/L$ (periodic boundary conditions). Due to the decay of the $g_{i,\kappa}(q)$ on the scale q_c, the momentum sum in $F(x,t)$ is convergent in the ultraviolet. Note in particular that for $q_n \gg q_c$, the two terms in the first line cancel each other as $\omega_\nu(q_n) \to v_Fq_n$ in this limit [compare Eq. (3)]. The term in the second line of Eq. (13) is convergent as $\gamma_\nu(q_n) \to 0$ for $q_n \gg q_c$ [compare Eqs. (14) and (6)]. For vanishing interaction, $F(x,t) = 0$. Thus, $[G_+^>]^0(x,t)$ is the noninteracting Green function. In the thermodynamic limit $L \to \infty$ it becomes

$$[G_+^>]^0(x,t) = \frac{1}{2\pi}\frac{e^{ik_Fx}}{x - v_Ft + i0^+}. \quad (15)$$

The factor $\exp(-q_n0^+)$ in Eq. (12) ensuring convergence appears naturally, and is not related to any *ad hoc* regularization. Only with this factor the exactly known $[G_+^>]^0(x,t)$ [Eq. (15)] and from this the exact noninteracting spectral function

$$[\rho_+^>(k,\omega)]^0(x,t) = \Theta(k - k_F)\delta[\omega - \xi_+(k)] \quad (16)$$

can be obtained. Due to the linear single-particle dispersion, $[G_+^>]^0(x,t)$ is of relativistic form. The greater spectral function is defined as

$$\rho_+^>(k_n,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}dt\, e^{i\omega t}\int_{-L/2}^{L/2}dx\, e^{-ik_nx}\, iG_+^>(x,t). \quad (17)$$

As $G_+^>(x,t)$ [Eqs. (11)–(13)] is an analytic function in the lower half of the complex t plane $\rho_+^>(k,\omega)$ has nonvanishing weight only for $\omega \geqslant 0$.

To compute $\rho_+^>(k,\omega)$ for arbitrary potentials in the thermodynamic limit, three nested integrals have to be performed. The integrands are slowly decaying, oscillatory, and have poles close to the real axis. Therefore, the accuracy which can be achieved when straightforwardly performing the integrals numerically, given $g_{i,\kappa}(q)$, is not sufficient to answer the question whether or not power laws can be found for $k - k_F \neq 0$. We are thus forced to proceed differently. Before presenting our approach to the exact spectral function of the TLM in Secs. III C and III D, we will describe approximate *ad hoc* procedures which were pursued in the literature to make analytical progress.

B. *Ad hoc* regularization

In the natural way of writing the *exact* Green function of the TLM (11) the noninteracting one is factorized out [9]. However, other expressions for $G_+^>(x,t)$ can be found in the literature [5,10,13,14]. These result from the following procedure. The term in the curly brackets of Eq. (12) is canceled against the last term in the first line of Eq. (13). After this step, the remaining q sum in Eq. (13) is no longer convergent in the ultraviolet. Two ways have been reported as to how to deal with this problem.

In the first (i) the remaining first term of the first line of Eq. (13) is multiplied by $\exp(-q_n0^+)$. To obtain integral-free expressions for $G_+^>(x,t)$, one then assumes that $\gamma_\nu(q_n)$ is given by $\gamma_\nu \exp(-q_n\Lambda)$ which can be reached by choosing a proper momentum dependence of the $g_{i,\kappa}(q)$. Thus, one simply selects a certain interaction potential. In *addition,* one linearizes the eigenmode dispersions $\omega_\nu(q) \to v_\nu q$ for all q. This is *not* a consequence of the special choice of the interaction potential but is rather an approximation done independently by hand. In the thermodynamic limit, one then obtains

$$[G_+^>]_{(i)}(x,t) = \frac{e^{ik_Fx}}{2\pi}\prod_{\nu=c,s}\left[\frac{1}{x - v_\nu t + i0^+}\right]^{1/2} \times\left[\frac{\Lambda^2}{(x - v_\nu t + i\Lambda)(x + v_\nu t - i\Lambda)}\right]^{\gamma_\nu/2}. \quad (18)$$

In the second *ad hoc* procedure (ii) the entire (remaining) argument of the sum in Eq. (13) is multiplied by $\exp(-q_n\Lambda)$ with a *finite*-momentum cutoff $1/\Lambda > 0$. The momentum dependence of $\gamma_\nu(q_n)$ is dropped and $\omega_\nu(q)$ is again linearized. This leads to

$$[G_+^>]_{(ii)}(x,t) = \frac{e^{ik_Fx}}{2\pi}\prod_{\nu=c,s}\left[\frac{1}{x - v_\nu t + i\Lambda}\right]^{1/2} \times\left[\frac{\Lambda^2}{(x - v_\nu t + i\Lambda)(x + v_\nu t - i\Lambda)}\right]^{\gamma_\nu/2}. \quad (19)$$

Finally, a third way (iii) to obtain an integral-free expression for $G_+^>(x,t)$ is based directly on Eqs. (11)–(13). In this, the momentum dependence of $\gamma_\nu(q_n)$ is dropped and $\omega_\nu(q)$ is linearized after the term in the square brackets in Eq. (13) was multiplied by $\exp(-q_n\Lambda)$. This way, one obtains

$$[G_+^>]_{(iii)}(x,t) = \frac{e^{ik_Fx}}{2\pi}\frac{1}{x - v_Ft + i0^+}\prod_{\nu=c,s}\left[\frac{x - v_Ft + i\Lambda}{x - v_\nu t + i\Lambda}\right]^{1/2} \times\left[\frac{\Lambda^2}{(x - v_\nu t + i\Lambda)(x + v_\nu t - i\Lambda)}\right]^{\gamma_\nu/2}. \quad (20)$$

We note that for the special case $\gamma_s = 0$ (spin-rotational-invariant interaction) and $v_s = v_F$, this is exactly the expression derived by Dzyaloshinskiĭ and Larkin [11] in a purely fermionic approach which is based on Ward identities and the closed loop theorem [25].

Obviously, all three *approximate* functions differ. However, their asymptotic behavior (power-law decay) for large space-time arguments is the same, in particular along the four special directions $x = \pm v_\nu t$. For this reason, they all lead to the same power laws in $\rho_+^>(k,\omega)$ for ω close to $\pm v_\nu(k - k_F)$ as derived in Refs. [5,9,13,14].

Similarly, for $x = 0$ all three approximate expressions have the same asymptotic behavior at large $|t|$, $\sim t^{-\alpha-1}$, with $\alpha = \gamma_c + \gamma_s$ (and are analytic functions in the lower half of the complex t plane). After a single Fourier transform from time to frequency this gives $\rho^>(\omega) \sim \Theta(\omega)\omega^\alpha$. As

discussed in Ref. [9], this power-law scaling of the momentum-integrated spectral function is also found based on the *exact* expressions (11)–(13) and for arbitrary $g_{i,\kappa}(q)$ (fulfilling the above-mentioned restrictions) with $\gamma_\nu \to \gamma_\nu(q=0)$. This universality is based on the RG irrelevance of the momentum dependence of the interaction [3]; the appearance of the power law with an exponent set by the two-particle potential at vanishing momentum transfer $q=0$ is not affected by the potential away from this point. Any of the discussed *ad hoc* procedures (i)–(iii) can thus be employed without spoiling the universal behavior. The same holds at $t=0$ but $x\neq 0$ which after Fourier transform leads to the momentum distribution function $n_+(k)$ which also shows universal power-law scaling for $k\to k_\mathrm{F}$ [5–7]. The question we address here is whether or not the *ad hoc* regularized Green functions can also be used to obtain universal results for $\rho_+^>(k,\omega)$ at $k-k_\mathrm{F}\neq 0$.

As already emphasized, the linearization of the $\omega_\nu(q)$ is the crucial step which leads to power laws in $\rho_+^>(k,\omega)$ for $k-k_\mathrm{F}\neq 0$ in the *ad hoc* procedures. In Ref. [9], it was shown using a generalization of the stationary phase method that any $q=0$ curvature of $v_\nu(q)$ affects the asymptotic behavior of $G_+^>(x,t)$ in the distinguished directions $x=\pm v_\nu t$. The decay is no longer solely given by the γ_ν and therefore not only by the $g_{i,\kappa}(0)$. However, in the *ad hoc* procedures (with constant v_ν), one can analytically show that it is the asymptotic behavior in these directions of the x-t plane which leads to the power laws at $\omega=\pm v_\nu[k-k_\mathrm{F}]$ [9]. This raises doubts that the latter are generic.

Aside from the spinful TLM we also consider its spinless version. From the above expressions for $G_+^>(x,t)$ [including those of the *ad hoc* regularization (i)–(iii)] the spinless Green function is obtained by setting $\gamma_c=\gamma_s$, $\omega_c=\omega_s$, and $v_c=v_s$. All that was said in the last two paragraphs about power-law behavior and universality remains valid in the spinless case up to the (obvious) exception that in the *ad hoc* regularized spectral function, only the two (instead of four) distinguished energies $\pm v_c(k-k_\mathrm{F})$ exist.

We note in passing that it was realized decades ago that within the *ad hoc* procedures (i) and (ii), exact spectral sum rules are not fulfilled [26–28].

C. How to compute $\rho_+^>(k,\omega)$ for a box potential

In the Introduction, we mentioned that exact results for the momentum-resolved spectral function of the TLM were derived based on Eqs. (11)–(13) assuming a box potential [15]. We will compare our results for other potentials to these. To be self-contained we here give all the formulas required to obtain $\rho_+^>(k,\omega)$ of the spinful g_4 model and the spinless g_2-g_4 model for $g_{i,\kappa}(q)=g_{i,\kappa}\Theta(q_\mathrm{c}^2-q^2)$. More details are presented for the first case which was not separately discussed in Ref. [15]. We note that for a reader primarily interested in results, it is not necessary to understand the technical details of this section in full detail.

1. Spinful g_4 model

For $g_{2,\kappa}(q)=0$ it directly follows that $\gamma_\nu(q)=0$ as $K_\nu(q)=1$; compare Eqs. (14) and (6). Equations (11)–(13) for the exact Green function thus simplify considerably. The same holds for the *ad hoc* regularized expressions (18)–(20) as $\gamma_\nu=0$. These are characterized by square-root singularities at $x=v_c t$ and $v_s t$. After Fourier transform they lead to square-root singularities in $\rho_+^>(k,\omega)$ for $\omega\to v_c[k-k_\mathrm{F}]$ and $\omega\to v_s[k-k_\mathrm{F}]$ [5,13,14]. We note that $[G_+^>]_{(\mathrm{iii})}(x,t)$ contains the additional factor $(x-v_\mathrm{F}t+i\Lambda)/(x-v_\mathrm{F}t+i0^+)$ which does not drop out as $\Lambda>0$. Further down we will discuss how this term affects the spectral properties. As the ground state of the g_4 model remains the noninteracting one [15], $\rho_+^>(k,\omega)$ has finite weight only for $k\geqslant k_\mathrm{F}$.

For the case of a box potential, it is

$$v_\nu(q)=\begin{cases} v_\nu, & q\leqslant q_\mathrm{c}\\ v_\mathrm{F}, & q>q_\mathrm{c}.\end{cases} \tag{21}$$

Setting for convenience $z=\exp\{i(2\pi/L)(x-v_\mathrm{F}t)\}$ as well as $z_\nu=\exp\{i(2\pi/L)(x-v_\nu t)\}$ and expanding the exponential function, one obtains from Eqs. (11)–(13)

$$\begin{aligned} iG_+^>(x,t) &= \frac{1}{L}e^{i(2\pi/L)(n_\mathrm{F}+1)x}\left(\sum_{l=0}^{\infty}z^l\right)\\ &\quad\times\prod_{n=1}^{n_\mathrm{c}}\left(\sum_{m=0}^{\infty}\frac{(-1/n)^m}{m!}z^{nm}\right)\\ &\quad\times\left(\sum_{j=0}^{\infty}\frac{[1/(2n)]^j}{j!}z_c^{nj}\right)\left(\sum_{l=0}^{\infty}\frac{[1/(2n)]^l}{l!}z_s^{nl}\right) \qquad (22)\\ &\overset{!}{=}\frac{1}{L}e^{i(2\pi/L)(n_\mathrm{F}+1)x}\left(\sum_{m=0}^{\infty}a_m^{(n_\mathrm{c})}z^m\right)\\ &\quad\times\left(\sum_{j=0}^{\infty}b_j^{(n_\mathrm{c})}z_c^j\right)\left(\sum_{l=0}^{\infty}b_l^{(n_\mathrm{c})}z_s^l\right), \qquad (23)\end{aligned}$$

with $q_\mathrm{c}=n_\mathrm{c}2\pi/L$. Here, $k_\mathrm{F}=(2n_\mathrm{F}+1)\pi/L$ and n_F is the index of the last occupied single-particle state; k_F thus lies in-between the last occupied and the first unoccupied one. The coefficients in Eq. (23) can be determined by a recursion relation, where $m>1$, $l\in\mathbb{N}_0$, and $i=0,\ldots,m-1$:

$$a_{lm+i}^{(m)}=\sum_{j=0}^{l}\frac{(-1/m)^j}{j!}\,a_{m(l-j)+i}^{(m-1)}, \tag{24}$$

$$b_{lm+i}^{(m)}=\sum_{j=0}^{l}\frac{[1/(2m)]^j}{j!}\,b_{m(l-j)+i}^{(m-1)}. \tag{25}$$

The initial values are given by

$$a_m^{(1)}=\sum_{j=0}^{m}\frac{(-1)^j}{j!} \tag{26}$$

and $b_m^{(1)}=(1/2)^m/m!$. The recursion can easily be performed on a computer. The double Fourier transform can be computed analytically and one obtains

$$\begin{aligned}\rho_+^>(k_n,\omega) &= \sum_{l=0}^{\tilde{n}}\sum_{j=0}^{\tilde{n}-l}a_{\tilde{n}-l-j}^{(n_\mathrm{c})}\,b_l^{(n_\mathrm{c})}\,b_j^{(n_\mathrm{c})}\\ &\quad\times\delta\left[\omega-\frac{2\pi}{L}\big((\tilde{n}-l-j)v_\mathrm{F}+lv_c+jv_s\big)\right], \qquad (27)\end{aligned}$$

with $\tilde{n} = n - (n_F + 1)$. This way, the exact spectral function $\rho_+^>(k_n,\omega)$ of the TLM with box potential can easily be computed for large but finite systems (see Sec. III E).

To obtain analytical insights, we rewrite the Green function as

$$iG_+^>(x,t) = \frac{1}{L} e^{i(2\pi/L)(n_F+1)x} \times \exp\left\{\frac{1}{2}\sum_{n=1}^{n_c}\left(\frac{z_c^n}{n}+\frac{z_s^n}{n}\right)+\sum_{n=n_c+1}^{\infty}\frac{z^n}{n}\right\}. \quad (28)$$

Thus,

$$\sum_{m=0}^{\infty} a_m^{(n_c)} z^m = 1 + \sum_{n=n_c+1}^{\infty}\frac{z^n}{n} + \frac{1}{2}\left(\sum_{n=n_c+1}^{\infty}\frac{z^n}{n}\right)^2 + \cdots \quad (29)$$

and we immediately see that $a_0^{(n_c)} = 1$, $a_m^{(n_c)} = 0$ for $1 \leqslant m \leqslant n_c$ and $a_m^{(n_c)} = 1/m$ for $n_c + 1 \leqslant m \leqslant 2n_c + 1$. For $\tilde{n} \leqslant n_c$, the simplified expression

$$\rho_+^>(k_n,\omega) = \sum_{l=0}^{\tilde{n}} b_l^{(n_c)} b_{\tilde{n}-l}^{(n_c)}\,\delta\left[\omega - \frac{2\pi}{L}\left(\tilde{n}v_s + l(v_c - v_s)\right)\right] \quad (30)$$

holds. For fixed $\tilde{n}$ and $v_s < v_c$, there is only spectral weight for $v_s[k - k_F] \leqslant \omega \leqslant v_c[k - k_F]$ (up to corrections of order $1/L$). For $v_s > v_c$, the roles of the two velocities are interchanged. We note that for $k - k_F \leqslant q_c$ the bare Fermi velocity v_F drops out.

Further analytical results can be obtained employing

$$\exp\left\{\alpha\sum_{n=1}^{n_c}\frac{1}{n}x^n\right\} = (1-x)^{-\alpha}\exp\left\{-\alpha\sum_{n=n_c+1}^{\infty}\frac{1}{n}x^n\right\} = \left[\sum_{j=0}^{\infty}\binom{-\alpha}{j}(-x)^j\right]\exp\left\{-\alpha\sum_{n=n_c+1}^{\infty}\frac{1}{n}x^n\right\}, \quad (31)$$

where $\binom{-\alpha}{j}$ is the generalized binomial coefficient. From this it follows that $b_0^{(n_c)} = 1$ and

$$b_j^{(n_c)} = (-1)^j\binom{-1/2}{j} \xrightarrow{1\ll j\leqslant n_c} \text{const} \times j^{-1/2}, \quad (32)$$

where

$$(-1)^j\binom{-\alpha}{j} \approx \frac{1}{\Gamma[\alpha]} j^{\alpha-1} \quad \text{for } j \to \infty \quad (33)$$

was used. Inserting this for energies close to $v_s[k_n - k_F]$ into Eq. (30), where l is the integer next to $(\omega - v_s[k_n - k_F])/[(2\pi/L)(v_c - v_s)]$, we obtain for $L \to \infty$ the one-sided square-root singularity ($v_c > v_s$)

$$\rho_+^>(k,\omega) \sim \Theta(\omega - v_s[k - k_F])(\omega - v_s[k - k_F])^{-1/2}. \quad (34)$$

Analogously, we get close to $v_c[k_n - k_F]$

$$\rho_+^>(k,\omega) \sim \Theta(-\omega + v_c[k - k_F])(-\omega + v_c[k - k_F])^{-1/2}. \quad (35)$$

For $v_s > v_c$, we only need to interchange the two velocities.

We have thus shown that the exact spectral function of the g_4 model with box potential shows the edge singularities also found within the *ad hoc* procedures (i)–(iii). We note in passing that for $k = k_F$, the spectral function reduces to a δ function of weight 1 located at $\omega = 0$.

2. *Spinless g_2-g_4 model*

In Sec. IV of Ref. [15], a recursive way of computing the spectral function for the full spinless TLM with box potential similar to Eqs. (24)–(27) was introduced. It is given by

$$\rho_+^>(k_n,\omega) = A^{-2\gamma_c}\sum_{r=\max\{0,-\tilde{n}\}}^{\infty}\sum_{l=0}^{\tilde{n}+r} a_{\tilde{n}+r-l}^{(n_c)} b_l^{(n_c)} c_r^{(n_c)} \times \delta\left[\omega - \frac{2\pi}{L}[(\tilde{n}+r-l)v_F + (r+l)v_c]\right], \quad (36)$$

with $A = \exp\{\sum_{n=1}^{n_c}(1/n)\}$. The coefficients $a_m^{(n_c)}$ are determined as in Eqs. (24) and (26). For $b_m^{(n_c)}$, the recursion relation reads as

$$b_{lm+i}^{(m)} = \sum_{j=0}^{l}\frac{1}{j!}\left(\frac{1+\gamma_c}{m}\right)^j b_{m(l-j)+i}^{(m-1)}, \quad (37)$$

with the initial values $b_m^{(1)} = (1+\gamma_c)^m/m!$, and for the $c_m^{(n_c)}$

$$c_{lm+i}^{(m)} = \sum_{j=0}^{l}\frac{1}{j!}\left(\frac{\gamma_c}{m}\right)^j c_{m(l-j)+i}^{(m-1)}, \quad (38)$$

with the initial values $c_m^{(1)} = \gamma_c^m/m!$.

In analogy to the considerations for the spinful g_4 model, we can infer from the behavior of the coefficients the behavior of the spectral function close to $v_c[k - k_F]$ (for $L \to \infty$). For $k - k_F > 0$, we find [15]

$$\rho_+^>(k,\omega) \sim \Theta(\omega - v_c[k - k_F])\,(\omega - v_c[k - k_F])^{\gamma_c - 1}, \quad (39)$$

that is a divergence if $\gamma_c < 1$ (not to strong interactions). In contrast to the g_4 model, $\rho_+^>$ can now also carry spectral weight for $(k - k_F) < 0$. At the threshold, we obtain [15]

$$\rho_+^>(k,\omega) \sim \Theta(-\omega - v_c[k - k_F])\,(-\omega - v_c[k - k_F])^{\gamma_c}, \quad (40)$$

that is a power-law suppression since $\gamma_c > 0$ in the full model. In the special case $k = k_F$, power-law behavior with $\rho_+^>(k_F,\omega) \propto \omega^{2\gamma_c - 1}$ is found. These threshold power laws can also be found based on the Green function of the *ad hoc* procedures (i)–(iii) discussed in Sec. III B [10,12–14].

D. How to compute $\rho_+^>(k,\omega)$ for arbitrary potentials

We now show that expressions for the Green and the spectral functions which involve recursively computed coefficients can

also be given for an arbitrary momentum dependence of the two-particle potential. For the box potential, the dispersion of the elementary charge and spin modes is piecewise linear. This changes for arbitrary potentials. As a consequence, the coefficients become time dependent and the Fourier transform with respect to time has to be performed numerically, e.g., using fast Fourier transform (FFT).

1. Spinful g_4 model

For arbitrary potentials, the spin and charge velocity are no longer piecewise momentum independent. If the potential is effectively zero for $q > \tilde{q}_c$, with a $\tilde{q}_c$ which we take sufficiently larger then the characteristic scale q_c, we can work with $g_{4,\nu}(q) \to g_{4,\nu}(q)\,\Theta(\tilde{q}_c^2 - q^2)$ for all practical purposes. Then, we can rewrite $F(x,t)$ in Eq. (13):

$$F(x,t) = \sum_{n=1}^{\tilde{n}_c} \frac{1}{n}\left(\frac{1}{2}e^{-i\omega_c(q_n)t} + \frac{1}{2}e^{-i\omega_s(q_n)t} - e^{-i(2\pi/L)nv_F t}\right) z^n, \tag{41}$$

with $z = \exp\{i(2\pi/L)x\}$, and use *time-dependent* coefficients to write

$$iG_+^>(x,t) = \frac{1}{L}\, e^{i(2\pi/L)(n_F+1)x} \left(\sum_{l=0}^{\infty} z^l\, e^{i(2\pi/L)lv_F t}\right) \prod_{n=1}^{\tilde{n}_c} \left(\sum_{m=0}^{\infty} \frac{(1/n)^m}{m!}\left[\frac{1}{2}e^{-i\omega_c(q_n)t} + \frac{1}{2}e^{-i\omega_s(q_n)t} - e^{-i(2\pi/L)nv_F t}\right]^m z^{mn}\right) \tag{42}$$

$$\stackrel{!}{=} \frac{1}{L}\, e^{i(2\pi/L)(n_F+1)x} \sum_{m=0}^{\infty} a_m^{(\tilde{n}_c)}(t)\, z^m. \tag{43}$$

The recursion relation for the $a_m^{(\tilde{n}_c)}(t)$ is given by

$$a_m^{(1)}(t) = \sum_{l=0}^{m} \frac{1}{l!}\left[e^{-i(2\pi/L)v_F t}\right]^{m-l}\left[\frac{1}{2}e^{-i\omega_c(q_1)t} + \frac{1}{2}e^{-i\omega_s(q_1)t} - e^{-i(2\pi/L)v_F t}\right]^l, \tag{44}$$

$$a_{lm+i}^{(m)}(t) = \sum_{j=0}^{l} \frac{(1/m)^j}{j!}\left[\frac{1}{2}e^{-i\omega_c(q_m)t} + \frac{1}{2}e^{-i\omega_s(q_m)t} - e^{-i(2\pi/L)mv_F t}\right]^j a_{m(l-j)+i}^{(m-1)}(t), \tag{45}$$

where in the second line $m > 1, l \in \mathbb{N}_0$, and $i = 0, \ldots, m-1$. With this representation of the Green function, the Fourier transform to momentum space can be performed analytically

$$iG_+^>(k_n,t) = \frac{1}{2\pi} a_{\tilde{n}}^{(\tilde{n}_c)}(t). \tag{46}$$

The remaining Fourier transform

$$\rho_+^>(k_n,\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, iG_+^>(k_n,t) \tag{47}$$

can be performed numerically as a FFT. Since for the finite system the spectral function consists of δ peaks, the Green function does not decay in time. Therefore, we have to multiply $iG_+^>(k_n,t)$ with a decaying function before performing the FFT. Here, we will always use the exponential function $\exp\{-\chi|t|\}$. In frequency space, this corresponds to a convolution of the spectral function with the Lorentzian $\pi^{-1}\chi/(\omega^2+\chi^2)$, i.e., each δ peak is broadened into a Lorentzian of width χ.

2. Spinless g_2-g_4 model

In the same way as for the spinful g_4 model, we can introduce recursively defined time-dependent coefficients for the spinless g_2-g_4 model:

$$a_m^{(1)}(t) = \sum_{l=0}^{m} \frac{1}{l!}[e^{-i(2\pi/L)v_F t}]^{m-l}\left[[1+\gamma_c(q_1)]e^{-i\omega_c(q_1)t} - e^{-i(2\pi/L)v_F t}\right]^l, \tag{48}$$

$$a_{lm+i}^{(m)}(t) = \sum_{j=0}^{l} \frac{(1/m)^j}{j!}\left[[1+\gamma_c(q_m)]e^{-i\omega_c(q_m)t} - e^{-i(2\pi/L)mv_F t}\right]^j a_{m(l-j)+i}^{(m-1)}(t), \tag{49}$$

and

$$b_m^{(1)}(t) = \frac{1}{m!}\left[\gamma_c(q_1)e^{-i\omega_c(q_1)t}\right]^m, \tag{50}$$

$$b_{lm+i}^{(m)}(t) = \sum_{j=0}^{l} \frac{(1/m)^j}{j!}\left[\gamma_c(q_m)e^{-i\omega_c(q_m)t}\right]^j b_{m(l-j)+i}^{(m-1)}(t). \tag{51}$$

In terms of these, we can rewrite

$$iG_+^>(k_n,t) = \frac{1}{2\pi}\exp\left\{-2\sum_{n=1}^{\tilde{n}_c}\frac{\gamma_c(q_n)}{n}\right\} \sum_{r=\max\{0,-\tilde{n}\}}^{\infty} a_{\tilde{n}+r}^{(\tilde{n}_c)}(t)\, b_r^{(\tilde{n}_c)}(t) \tag{52}$$

and the remaining Fourier transform to obtain $\rho_+^>(k_n,\omega)$ can be performed, again after multiplication with $\exp\{-\chi|t|\}$, numerically by means of a FFT.

E. Spectra of the spinful g_4 model

Based on the formulas given in Secs. III C 1 and III D 1, we are in a position to compute the exact spectral function of the spinful g_4 model for different potentials at finite system size L. For the box potential both Fourier transforms can be performed analytically, while for arbitrary potentials the time transform is performed numerically as a FFT. In this case, only broadened spectra can be obtained. We will compare the results to those derived from one of the *ad hoc* procedures of Sec. III B.

Aside from the box potential we consider ($\nu = c,s$)

$$g_{4,\nu}^{p=4}(q) = g_{4,\nu}\exp\{-(q/q_c)^4/9\} \quad (p=4), \tag{53}$$

$$g_{4,\nu}^{\text{Gauss}}(q) = g_{4,\nu}\exp\{-(q/q_c)^2\} \quad (\text{Gauss}), \tag{54}$$

$$g_{4,\nu}^{\exp}(q) = g_{4,\nu}\exp\{-3\,|q/q_c|\} \quad (\exp). \tag{55}$$

The factors in the exponential function were chosen such that, besides at $q = 0$, all potentials have the same value at $\tilde{q}_c = 3q_c$; at this momentum they have decayed to $\approx 10^{-4}$ of the $q = 0$ value and we can safely set the potentials to 0 for $q > \tilde{q}_c$. Considering larger $\tilde{q}_c$ we have verified that this does indeed not affect our results. For small momenta the potentials go as $1 - g_{4,\nu}(q)/g_{4,\nu} \sim |q/q_c|^p$, with $p = \infty$ for the box, $p = 4$ for the "$p = 4$" potential, $p = 2$ for the Gaussian potential, and $p = 1$ for the exponential potential. The exponent p is a measure for the flatness of the potential at $q = 0$; see also Ref. [9].

For definiteness in our calculations we have always chosen $g_{4,c}/(\pi v_F) = \frac{1}{2} = -g_{4,s}/(\pi v_F)$ without affecting our conclusions. In this case $v_c > v_s$. The measure for the system size is n_c. To numerically compute the recursively defined coefficients within reasonable time we choose $n_c = 5 \times 10^4$. The broadening χ is chosen such that in the broadened spectral function no effects of the individual δ peaks are visible. For the given n_c we take $\chi/(v_F q_c) = 5 \times 10^{-5}$.

In Figs. 1(a)–1(c) we show $\rho_+^>(k,\omega)$ for three different small $(k - k_F)/q_c$. The curves labeled as "box (ana)" are the results for the box potential obtained from Eq. (30) without convoluting it with a Lorentzian. The weights of the δ peaks are divided by the level spacing (which for the given parameters is $1/n_c$) and are connected to form a smooth curve. All other results [up to "broadened (iii)"; see following] were obtained from Eq. (46) multiplied with an exponentially decaying function $\exp\{-\chi|t|\}$ and transformed with a FFT; for comparison we also show the broadened spectra for the box potential ["box (FFT)"].

As can be seen from Fig. 1(a), for k very close to k_F all curves are nearly indistinguishable. The broadening of the spectra is visible by the weight "leaking out" for $\omega < v_s[k - k_F]$ and $\omega > v_c[k - k_F]$. However, already for $(k - k_F)/q_c = 0.1$ [Fig. 1(b)], the spectral function calculated with an exponentially decaying potential shows pronounced differences to the other ones. For $(k - k_F)/q_c = 0.3$, also the curve of the Gaussian potential deviates from the one of the box potential [see Fig. 1(c)]. The spectrum of the "very flat" "$p = 4$" potential still lies on top of the one obtained for the box potential; differences only appear at even larger $k - k_F$

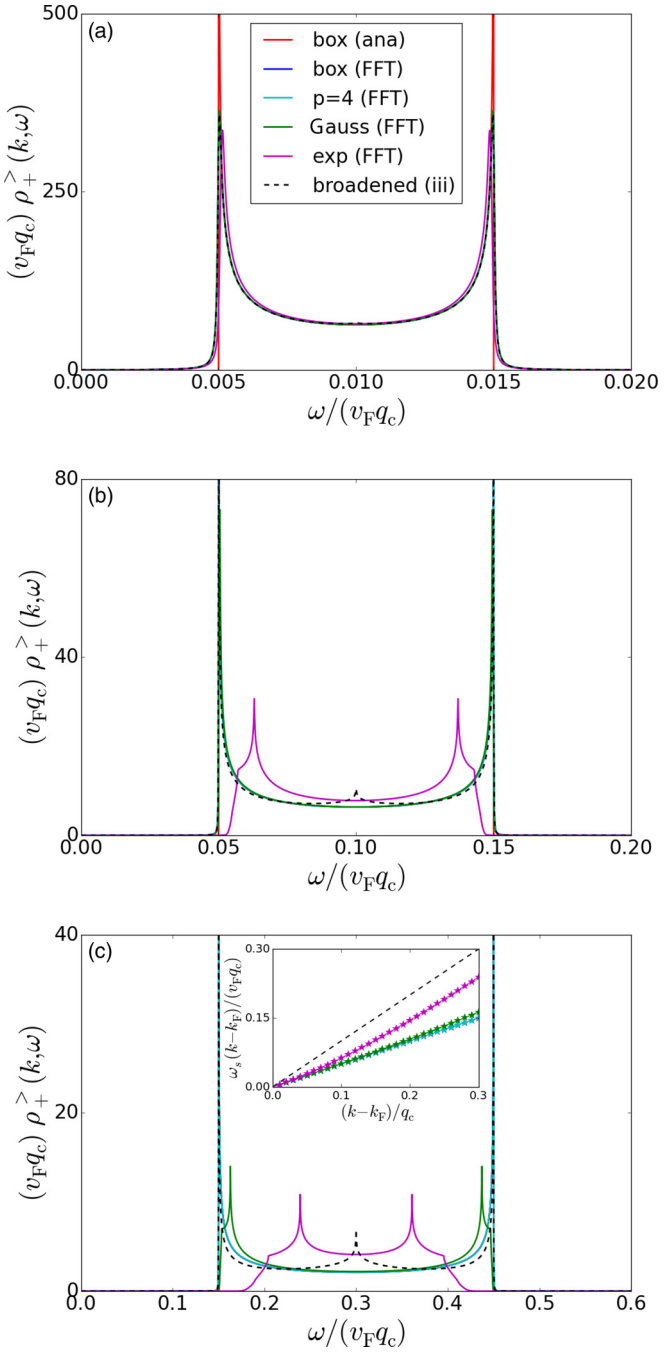

FIG. 1. Spectral function of the spinful g_4 model as a function of energy for (a) $(k - k_F)/q_c = 0.01$, (b) $(k - k_F)/q_c = 0.1$, and (c) $(k - k_F)/q_c = 0.3$. Spectra for the different potentials are shown. In addition to the FFT data with broadening $\chi/(v_F q_c) = 5 \times 10^{-5}$ we show unbroadened results for the box potential ["box (ana)"] and the broadened results of the *ad hoc* procedure (iii) ["broadened (iii)"]. The other parameters are $g_{4,c}/(\pi v_F) = \frac{1}{2} = -g_{4,s}/(\pi v_F)$ and $n_c = 5 \times 10^4$. The curves are partly hidden by others (see the text). The inset of (c) shows the position of the spin peak extracted from the data (stars) in comparison to the collective spin mode dispersion $\omega_s(k - k_F)$ (lines). The dashed line displays the unrenormalized dispersion $v_F(k - k_F)$.

(not shown). Obviously, the smaller the above introduced index p, that is the less "flat" the potential is at $q = 0$, the faster the line shape of the spectra starts to deviate from the one of the box potential when $k - k_F$ increases.

The maxima of $\rho_+^>(k,\omega)$ are located at $\omega_v(k - k_F)$, instead of at $v_v[k - k_F]$; for $|k - k_F| < q_c$ both positions are equal for the box potential. This is shown in the inset of Fig. 1(c), where the spin dispersion relation for the different potentials (full lines) is compared to the numerically determined maxima of the spectra (stars). The agreement is very good; the charge peak behaves similarly. In the limit of small $k - k_F$ and large p, the difference between $\omega_v(k - k_F)$ and $v_v[k - k_F]$ is negligible.

In the literature, the third *ad hoc* regularization procedure leading to $[G_+^>]_{(iii)}(x,t)$ [Eq. (20)] is considered "the best" one [5,28] as it leads to a $[\rho_+^>]_{(iii)}(k,\omega)$ which fulfills exact sum rules [26,27]. For $\gamma_c = \gamma_s = 0$ a closed analytical expression of the double Fourier transform can be given; see Eqs. (3.20) and (3.21) of Ref. [28]. We therefore added a graph of this analytical result convoluted with a Lorentzian of width χ as the dashed lines in Figs. 1(a)–1(c) (with $1/\Lambda \to q_c$). As mentioned above, these curves as well as the exact function for the box potential show single-sided (threshold) square-root singularities at $v_v[k - k_F]$ (for a more detailed analysis on this, see following). The approximate spectrum $[\rho_+^>]_{(iii)}(k,\omega)$ is characterized by an additional feature at $\omega = v_F[k - k_F]$ [barely visible in Fig. 1(a)] for $(k - k_F)/q_c = 0.01$); as analyzed in Ref. [28] a logarithmic divergence appears at this energy. It results from the additional factor $(x - v_F t + i\Lambda)/(x - v_F t + i0^+)$ of $[G_+^>]_{(iii)}(x,t)$ [Eq. (20)]. Obviously, the exact spectral function of the g_4 model does not display this feature for any of the considered potentials. For the box potential, which aside from the missing peak at $\omega = v_F[k - k_F]$ shows the same spectral characteristics as found in the *ad hoc* procedure, this can even be understood analytically: as seen in Eq. (30) for $k - k_F < q_c$ the information on v_F drops completely out. The logarithmic divergence of $[\rho_+^>]_{(iii)}(k,\omega)$ at $\omega = v_F[k - k_F]$ is thus an *artifact of the ad hoc regularization.* As discussed very recently, this logarithmic divergence for $k - k_F \lessapprox q_c$ turns into a power-law one if the spinful (g_2-g_4) TLM is treated within the *ad hoc* procedure (iii) [29]. This feature is an artifact of the *ad hoc* procedure as well. With Ref. [29] in mind, we emphasize that this does not exclude that for $k - k_F \gg q_c$ all the spectral weight is located around $\omega = v_F(k - k_F)$, which, in fact, is generically the case. This was discussed for the box potential in Ref. [15] and for general interactions in Ref. [27]. We here do not investigate this any further as we are exclusively interested in the spectra at small $|k - k_F|$.

In order to investigate whether or not the spectra at fixed $0 < k - k_F \ll q_c$ and for ω close to the maxima at ω_{max} show power-law scaling, we do not simply want to rely on the quality of power-law fits to the broadened data. Instead, we take the logarithmic derivative

$$\mathrm{logder}(\omega) = \frac{d \ln[\rho_+^>(k,\omega)]}{d \ln|\omega - \omega_{max}|} \tag{56}$$

of our data with ω_{max} equal to the corresponding peak positions. It is numerically approximated by centered differences and provides a very sensitive approach in the search for power laws. If for $\omega \to \omega_{max}$, logder(ω) approaches a constant ξ,

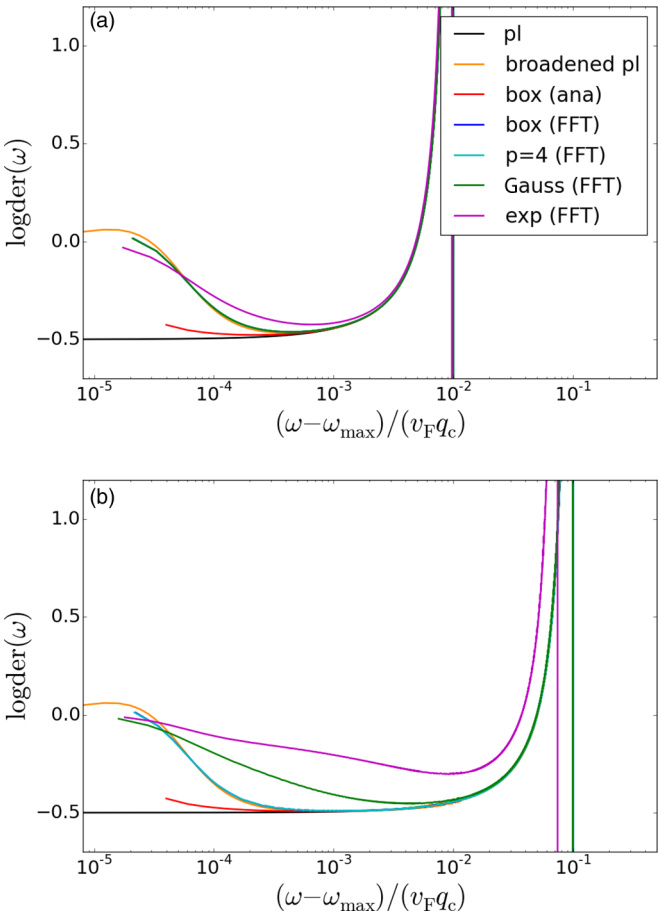

FIG. 2. Logarithmic derivative (56) of the spectral function of the spinful g_4 model close to the spin peak for (a) $(k - k_F)/q_c = 0.01$ and (b) $(k - k_F)/q_c = 0.1$. In addition to the broadened FFT data for the different potentials and the unbroadened results for the box potential ["box (ana)"], we show broadened ["broadened pl"] as well as unbroadened ["pl"] data for the product of single-sided square-root singularities [Eq. (57)]. The parameters are as in Fig. 1.

the spectral function shows power-law scaling with exponent ξ close to ω_{max} (according to the definition given in the Introduction).

In Figs. 2(a) [$(k - k_F)/q_c = 0.01$] and 2(b) [$(k - k_F)/q_c = 0.1$], we show logder(ω) close to the spin peak for the different two-particle potentials. For symmetry reasons, the behavior close to the charge peak is the same. Instead of an *ad hoc* regularized spectral function, which is spoiled by the spurious peak at $\omega = v_F(k - k_F)$, we this time present the logarithmic derivative of the simple normalized product of two single-sided square-root singularities

$$[\rho_+^>]_{pl}(k,\omega) = \frac{1}{\pi} \frac{\Theta(\omega - v_s[k - k_F])}{(\omega - v_s[k - k_F])^{1/2}} \frac{\Theta(v_c[k - k_F] - \omega)}{(v_c[k - k_F] - \omega)^{1/2}} \tag{57}$$

for reference. It is indicated by "pl". In addition, we present the logarithmic derivative of this expression convoluted with a Lorentzian of width χ, indicated by "broadened pl". The unbroadened data for the box potential, for which we analytically know that for $L \to \infty$ a square-root singularity at $v_s[k - k_F]$ exists [see Eq. (34)], very nicely follow the

pl curve down to $\omega - \omega_{\max} \approx 4 \times 10^{-4} v_F q_c$. At this energy, *finite-size corrections* destroy the power-law scaling. This can be verified by considering different n_c, that is, different system sizes. The broadened pl curve starts to deviate from the unbroadened one at $\omega - \omega_{\max} \approx 6 \times 10^{-4} v_F q_c$. From this we conclude that for the chosen n_c and χ the broadening χ cuts off the power-law scaling at slightly larger energies than the system size. This is consistent with the observation that the broadened spectrum of the box potential ["box (FFT)"] almost perfectly follows the broadened pl curve down to the much smaller scale $\omega - \omega_{\max} \approx 6 \times 10^{-5} v_F q_c$. This gives us confidence that the FFT data for the other potentials are unspoiled by both finite-size and broadening effects down to $\omega - \omega_{\max} \approx 6 \times 10^{-4} v_F q_c$. At the "high-energy" end, possible power-law scaling close to the spin peak is cut off by the charge peak. The latter is the origin of the dominant feature at $\omega - \omega_{\max} \approx 10^{-2} v_F q_c$ and $\omega - \omega_{\max} \approx 10^{-1} v_F q_c$, respectively.

While for $(k - k_F)/q_c = 0.01$ the data of the "$p = 4$" and the Gaussian potential lie perfectly on top of the "box (FFT)" data, and one is tempted to conclude that they are consistent with a square-root singularity at $\omega_{\max}$, the data for the exponential potential clearly differ and are inconsistent with such behavior. For $(k - k_F)/q_c = 0.1$ in addition the data for the Gaussian potential are incompatible with this type of singularity. For even larger $k - k_F$ (but still smaller than q_c; not shown) also the data for the $p = 4$ potential no longer follow the ones of the box potential.

The most consistent interpretation of our results is that for any potential which is not "infinitely flat" at $q = 0$, that is if $p < \infty$, the spectral function at fixed $k - k_F > 0$ does strictly speaking not show power-law scaling close to $\omega_{\max}$. The less flat the potential is, that is, the smaller p, the faster this becomes apparent as $k - k_F$ is increased. The square-root singularities found for the box potential (and for the *ad hoc* procedures) are cut off by the curvature of the potential close to $q = 0$. They can thus not be considered as universal features of the spinful g_4 model.

We note that the data for $p < \infty$ are not only inconsistent with power-law scaling when taking $\omega_{\max}$ as the point of reference. We studied the behavior relative to other distinguished energies (e.g., $v_\nu[k - k_F]$ and the, due to the broadening, apparent thresholds). For none of these do we find behavior which is consistent with power laws.

Despite this lack of power-law behavior, for all interaction potentials studied the exact spectral function of the g_4 model is still characterized by spin and charge peaks.

F. Spectra of the spinless g_2-g_4 model

Employing the formulas given in Secs. III C 2 and III D 2, we can compute the exact spectral function of the spinless TLM (the spinless g_2-g_4 model) for different forms of the potential. In the last section we saw that for the small $k - k_F$ we are interested in, the spectra for the $p = 4$ potential [Eq. (53)] barely differ from those obtained for the box potential. The same holds for the full spinless model and in this section we focus on the box potential, the Gaussian one [Eq. (54)] as well as the exponential potential [Eq. (55)]. To prevent an inflation of cases and parameters, we consider the physically reasonable situation of equal interbranch and intrabranch scattering $g_2(q) = g_4(q) = g(q)$. We choose $g(0)$ such that $v_c = 2v_F$ and $\gamma_c(0) = \frac{1}{8}$.

To compute $\rho_+^>(k,\omega)$ via $G_+^>(k,t)$ [Eq. (52)] and FFT for general potentials or via Eq. (36) for the box potential, we have to recursively compute more coefficients and perform additional sums in comparison to what was necessary in the spinless g_4 model. This increases the numerical resources required and we thus have to consider smaller system sizes compared to the latter; we choose $n_c = 2 \times 10^4$. Furthermore, at fixed n_c the energy level spacing of the spinless g_2-g_4 model is larger than that of the spinful g_4 model. To obtain smooth curves, we thus have to increase the broadening χ. We take $\chi/(v_F q_c) = 10^{-3}$.

In Figs. 3(a)–3(c), we present the *total spectral function* $\rho_+(k,\omega) = \rho_+^>(k,\omega) + \rho_+^<(k,\omega)$ for $(k - k_F)/q_c = 0, 0.01$, and 0.1. We switched to this as it simultaneously shows the photoemission as well as inverse photoemission part of the spectrum. For increasing $k - k_F$, the photoemission part loses weight quickly. Thus, in Fig. 3(c) with $(k - k_F)/q_c = 0.1$ we only present a zoom-in of the inverse photoemission part. In addition to the broadened functions for the box, Gaussian, and exponential potentials obtained by FFT we show the unbroadened one of the box potential [see Eq. (36)]. As above, to obtain the latter the weights of the δ peaks were divided by the level spacing (which for the given parameters is $4/n_c$) and are connected to form a smooth curve.

As discussed in Sec. III C 2 for $0 < k - k_F < q_c$ and the box potential $\rho_+(k,\omega)$ shows threshold power-law nonanalyticities at $\pm v_c[k - k_F]$ with exponents $\gamma_c - 1$ (for $\omega > 0$) and γ_c (for $\omega < 0$). In "box (FFT)", these are broadened. Similarly to the spinful g_4 model we observe that the smaller p, the faster the line shape starts to deviate from the one of the box potential when increasing $k - k_F$. For $(k - k_F)/q_c = 0.1$ [see Fig. 3(c)] and the exponential potential with $p = 1$, this already leads to a strongly modified distribution of the spectral weight. The deformed line shape can be understood in due detail when comparing it to the spectral function of the *spinless* g_4 model [30]. This detailed analysis is beyond the scope of this paper.

From Ref. [9] and the RG irrelevance of the momentum dependence of the two-particle potential we expect that for all potentials $\rho_+(k_F,\omega) \sim |\omega|^{2\gamma_c - 1}$. On first glance, the data of Fig. 3(a) appear to be consistent with this behavior, however, a more thorough analysis is required. In Fig. 4(a), we plot the logarithmic derivative (56) (with $\omega_{\max} = 0$) of the $\omega > 0$ broadened FFT data of Fig. 3(a) as dotted lines. While the logarithmic derivative of the unbroadened "box (ana)" data (solid line) nicely shows a plateau at the expected exponent $2\gamma_c - 1 = -\frac{3}{4}$ (for the given parameters), which is only spoiled at very small $\omega/(v_F q_c) \approx 10^{-3}$ due to finite-size effects, the dotted curves do not seem to support power-law scaling of $\rho_+(k_F,\omega)$. This also holds for the *broadened* "box (FFT)" data which establish that the *broadening destroys the power law even for* $\omega \gg \chi$.

To further analyze this surprising finding in a "controlled" setup, we took the function $f(x) = x^{-3/4}$ and convoluted it with a Lorentzian of width 10^{-3}. We indeed found that for $x \in [10^{-2}, 10^{-1}]$, the logarithmic derivative of the resulting function shows a behavior quite similar to the one of the dotted

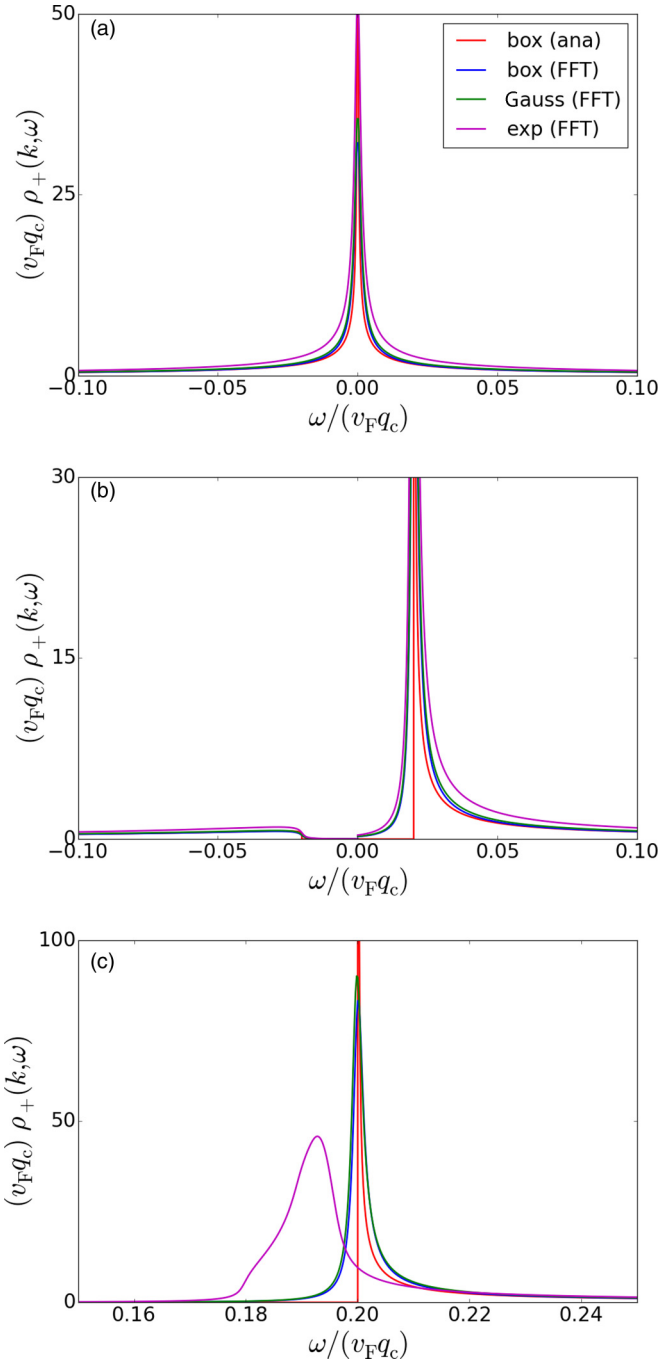

FIG. 3. Total spectral function of the g_2-g_4 model as a function of energy for (a) $(k-k_F)/q_c = 0$, (b) $(k-k_F)/q_c = 0.01$, and (c) $(k-k_F)/q_c = 0.1$. Spectra for the different potentials are shown. In addition to the FFT data with broadening $\chi/(v_F q_c) = 10^{-3}$ we show unbroadened results for the box potential ["box (ana)"]. The interaction at $q = 0$ is chosen such that $v_c = 2v_F$ and $\gamma_c(0) = \frac{1}{8}$. The system size is set by $n_c = 2 \times 10^4$.

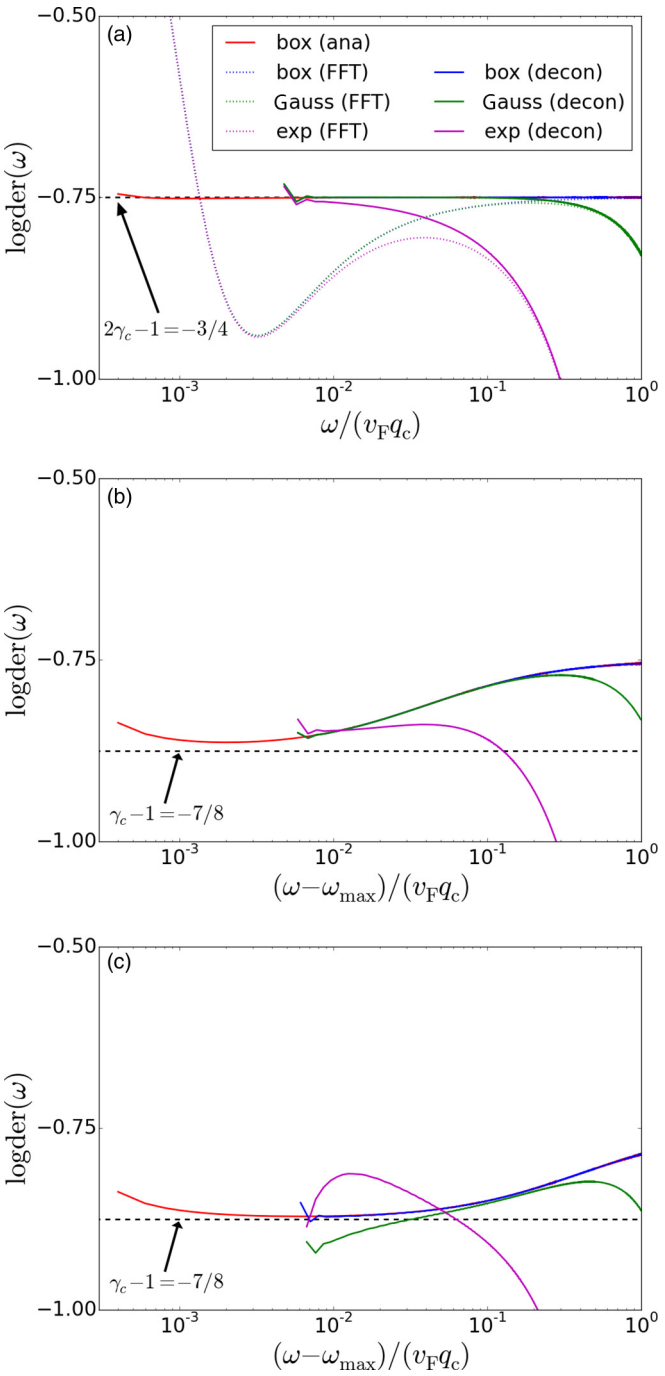

FIG. 4. Logarithmic derivative (56) of the spectral function of the spinless g_2-g_4 model close to the inverse photoemission peak for (a) $(k-k_F)/q_c = 0$, (b) $(k-k_F)/q_c = 0.01$, and (c) $(k-k_F)/q_c = 0.1$. The full lines labeled by "box (ana)" show the unbroadened results of the box potential. The dotted lines in (a) result from broadened spectra. The other full lines are obtained from the numerical spectra after a deconvolution (see the text). The dashed lines indicate the expected exponent. The parameters are as in Fig. 3.

lines in Fig. 4(a); in particular, it bends away from the expected plateau towards smaller values. For larger exponents, let us say $-\frac{1}{2}$ as in Figs. 2(a) and 2(b), this does not happen. One is thus tempted to increase the interaction strength and thus $2\gamma_c - 1$. However, larger interactions also imply larger energy level spacings of the δ peaks and thus require larger χ, which cuts off the power-law scaling at larger energies.

As a possible way out, we performed a *deconvolution* of our numerically obtained spectral function along the lines of Refs. [31,32]. The resolution was chosen to be approximately equal to the broadening χ. The deconvolution of numerical data is of course an ill-posed problem and the

corresponding spectral function can, e.g., become negative [32]. We were able to perform a stable deconvolution down to $(\omega-\omega_{\max})/(v_F q_c)\approx 10^{-2}$. In fact, the resulting spectra are *sufficiently smooth to perform logarithmic derivatives.* These are shown as solid lines in Fig. 4(a). The oscillatory behavior at the lower end signals the onset of an instability of the deconvolution. The deconvoluted data for the box potential lie exactly on top of the ones obtained from the unbroadened spectral function "box (ana)". This gives us confidence that also for the other potentials, the deconvolution can be trusted. This is further supported by the observation that the data for the Gaussian potential now clearly support power-law scaling with exponent $2\gamma_c-1$. For the exponential potential, the energy scale cutting off the power-law scaling at the *"high-energy" side* appears to be of the order 10^{-2} and thus smaller compared to the one of the box (of the order 1) and the Gaussian potential (of the order 10^{-1}). Therefore, no clear plateau is reached for the accessible energies. However, the data appear to saturate at the expected value $2\gamma_c-1$.

Along the same lines, we next investigate whether or not for $k-k_F>0$ a power law is found close to the peak on the inverse photoemission side. In Figs. 4(b) [$(k-k_F)/q_c=0.01$] and 4(c) [$(k-k_F)/q_c=0.1$], we present the logarithmic derivative of the deconvoluted spectra together with "box (ana)" data. The logarithmic derivative of the original spectra behave similar to the $k=k_F$ case and are thus not shown. As expected, the "box (ana)" data are consistent with power-law scaling with exponent $\gamma_c-1=-\frac{7}{8}$ (for the given parameters). The power law is cut off at $(\omega-\omega_{\max})/(v_F q_c)\approx 10^{-3}$ due to finite-size effects. The deconvoluted box potential data fall again exactly on top of the "box (ana)" ones indicating that the deconvolution is stable. For $(k-k_F)/q_c=0.01$, the data for the Gaussian potential are for sufficiently small $\omega-\omega_{\max}$ on top of the box potential ones [see Fig. 4(b)]. This does no longer hold for $(k-k_F)/q_c=0.1$ [see Fig. 4(c)] for which the data of the Gaussian potential are incompatible with power-law scaling. For the exponential potential with smaller p, this deviation from possible power-law behavior sets in at already smaller $k-k_F$ [see Fig. 4(b)].

As for the spinful g_4 model, the most consistent interpretation of our results is that for any potential with $p<\infty$ and fixed $k-k_F\neq 0$, the power-law scaling found for the box potential is destroyed by the curvature of the potential at $q=0$; power laws are thus nongeneric.

We note that this does not only hold when taking $\omega_{\max}$ as the point of reference. We studied the behavior relative to other distinguished energies (e.g., $v_c[k-k_F]$ and the, due to the broadening, apparent threshold). For none of these points, we find behavior which is consistent with power laws. Similarly also the behavior close to the threshold on the photoemission side is incompatible with power-law scaling if $p<\infty$.

These results imply that the spectral functions resulting from the different *ad hoc* regularizations (not shown here), which are all characterized by the threshold power laws Eqs. (39) and (40), are nongeneric.

IV. DISCUSSION

Our above results for the exact spectral function can be summarized as follows:

(1) For $k-k_F\neq 0$ and generic two-particle potentials which are not "infinitely flat" at momentum transfer $q=0$ the spectral function does not show power-law scaling close to any of the distinguished energies. Power-law behavior is generically only found if all energy scales are sent to zero, e.g., in $\rho^{>}(k_F,\omega)$ for $\omega\to 0$ [and if $g_2(0)$ is finite].

(2) The *ad hoc* regularized spectra which are commonly studied show finite $k-k_F$ power laws and can thus not be considered as generic. The origin of this nongeneric behavior is the linearization of the spin and charge dispersion relations. The *ad hoc* spectra are plagued by additional artifacts (see Sec. III E).

(3) The less flat the potential at $q=0$, that is the smaller the introduced index p, the faster the differences of the spectral line shape compared to the one of the box potential, and in many respects compared to the one of the *ad hoc* procedures, becomes apparent when $k-k_F$ is increased. For small p, e.g., the exponential potential with $p=1$, already for $(k-k_F)/q_c=0.1$ major differences are apparent [see Figs. 1(b) and 3(c)]. For $k-k_F\neq 0$, the spectral function is still characterized by the dispersing spin and charge peaks.

To individually study how the curvature of the potential at zero-momentum transfer modifies the two interaction effects of spin charge separation and power laws with interaction-dependent exponents, we have studied the spinful g_4 model and the spinless g_2-g_4 model. Combining the two limiting cases it is obvious that the same conclusions can be drawn for the full TLM, that is, the spinful g_2-g_4 model. Explicit results for the spectral function of the full TLM are presented in Sec. IV C (see Fig. 5).

We next discuss the implications of these findings.

A. Luttinger liquid universality

1. General considerations

The TLM forms the low-energy fixed point model under RG flow of a large class of 1D correlated fermion models [3]. Based on this insight, it was suggested that the power laws of the *ad hoc* regularized TLM spectral function $\rho_+^{>}(k\neq k_F,\omega)$ should be observable in other models from this class [5]. One can doubt this on general grounds as $k-k_F\neq 0$ sets a finite scale, thus breaking quantum critical scale invariance and cutting off the RG flow [9]. Our results show *explicitly* that power laws in $\rho^{>}(k\neq k_F,\omega)$ are indeed not part of LL universality; if they cannot be found in the generic TLM, they cannot expected to be a universal feature of other models.

We emphasize that LL universality does *not* imply that for a given model from the LL universality class (other than the TLM itself), one simply has to choose proper coupling functions $g_{i,\kappa}(q)$, which in any case would be unknown *a priori*, and can reproduce details of the spectral function of this model at low energies using the TLM. All that is known about the spectral properties of a model from the LL universality class is that for $\omega\to 0^+$ the scaling relations $\rho^{>}(k_F,\omega)\sim\omega^{\alpha-1}$ and $\rho^{>}(\omega)\sim\omega^{\alpha}$ hold.

When computing space-time correlation functions of the TLM other than the single-particle Green function momentum integrals of similar type as encountered here appear. To evaluate these, *ad hoc* procedures including the linearization of the collective spin and charge dispersion are commonly

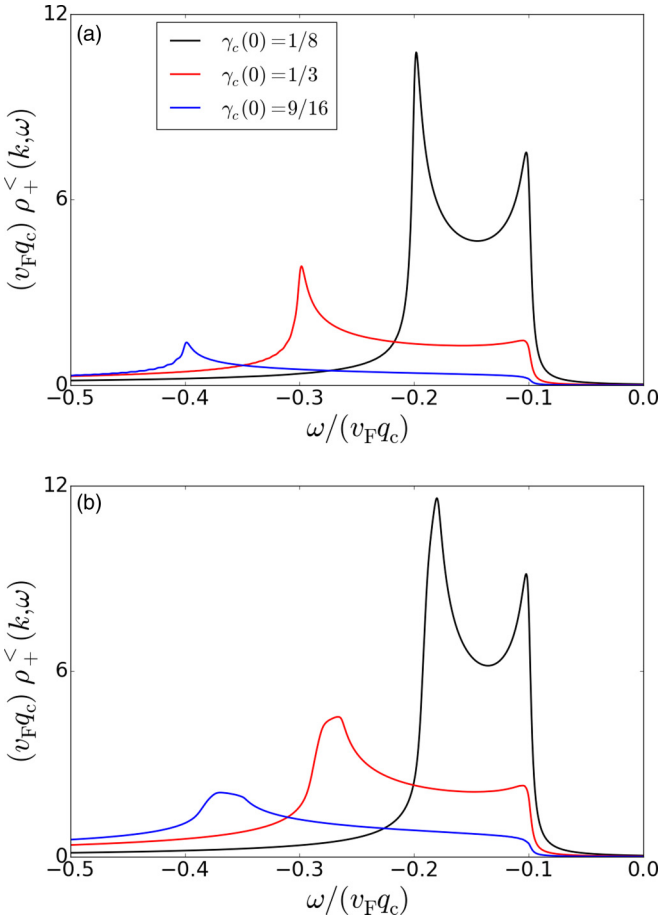

FIG. 5. Photoemission part of the spectral function of the spinful TLM at $(k-k_F)/q_c=-0.1$. A spin-independent interaction with equal interbranch and intrabranch amplitude is assumed. The interaction strength is chosen such that $\gamma_c(q=0)$ assumes the values given in the legend. The other parameters are $n_c=10^3$ (system size) and $\chi/(v_F q_c)=3\times10^{-3}$ (broadening). In (a) the spectra for the Gaussian potential are shown, in (b) the ones for the exponential potential. The small "high-energy" oscillations for larger interactions in (a) are a finite-size effect.

employed [5]. One prominent example is the density-density correlation function for momentum transfer close to $2k_F$ [10]. As for the single-particle spectral function after regularization the Fourier integrals can be performed analytically leading to power-law scaling of the susceptibility with interaction-dependent exponents close to the characteristic energies $\pm v_c[k\pm 2k_F]$. Our considerations imply that also this feature can most likely not be considered as being characteristic for LLs in general. The density response at *vanishing energy* and for $k\to\pm 2k_F$ (all energy scales are sent to 0) is, however, in general characterized by a power-law divergence (for repulsive interactions) which indicates the breakdown of linear-response theory. It is a signature of the sensitivity of a LL against single-particle perturbations with momentum transfer $2k_F$ (backscattering) [6,7]. We note that the character of the density-density correlation function with small-momentum transfer is not modified by the momentum dependence of the potentials; it is given by a δ function at energy $\omega_c(q)$ instead of $v_c|q|$.

2. *Spectra of lattice models*

Directly computing the single-particle spectral function for translational-invariant microscopic lattice models of 1D correlated fermions still poses a formidable challenge of quantum many-body theory. Two promising routes exists.

The first one is numerical in nature. Using exact diagonalization (ED), the (dynamical) density-matrix RG (DMRG), or different types of quantum Monte Carlo (QMC) approaches, valuable information on the spectral function of different models was collected. However, the search for power laws requires an exceptional energy resolution. This implies that fairly large system sizes and low temperatures must be accessible. The model most heavily investigated is the 1D Hubbard model (and variants of it). Away from half-filling it is known to fall into the LL universality class. ED is restricted to systems of a few ten lattice sites which leads to a poor energy resolution. Within so-called cluster perturbation theory, which is ED based, it was at least possible to observe spin-charge separation [33]. The latter was systematically studied using QMC [34,35]. In these studies, the finite temperature (and the required analytic continuation) turned out to be the main obstacle preventing an analysis of $\rho^>(k,\omega)$ in terms of power-law scaling. Reference [36] contains the first serious attempt in this direction. The authors use (dynamical) DMRG to obtain broadened spectra. A scaling analysis as a function of the broadening was interpreted to be consistent with power-law behavior of the charge and spin peaks. The quality of the data is, however, not good enough to either confirm or refute power laws at $k-k_F\neq 0$. Generally speaking, the numerical results for lattice models from the LL universality class are fully consistent with our conclusions.

A word of warning is in order. As our analysis of Sec. III F shows it can be very difficult to undoubtedly confirm or refute power-law behavior of broadened finite-size data. This holds even though for the TLM we can achieve a comparably high-energy resolution and obtain data which are sufficiently accurate to employ a very sensitive logarithmic derivative.

A promising analytical approach to the spectral function of 1D models is built upon the special structure of several of the standard lattice models from the LL universality class, namely, the existence of an extensive number of local integrals of motion. For this reason, several models are exactly solvable by Bethe ansatz, which, however, does not imply that the single-particle spectral function can (easily) be computed exactly [16]. In a series of papers (see Ref. [37] and references therein), $\rho^>(k,\omega)$ of the 1D Hubbard model was investigated using a "pseudofermion dynamical theory" which is based on the Bethe ansatz solution. It was reported that the spectral function contains power laws even for $k-k_F\neq 0$, with exponents which depend on $k-k_F$. This finding might be related to the "nonlinear Luttinger liquid phenomenology" briefly touched on in the following. The spectral function of another exactly solvable 1D model, namely, the Calogero-Sutherland model showing power laws with momentum-dependent exponents was interpreted in the light of this phenomenology [17].

We emphasize that our findings do not exclude that specific models from the LL universality class might show finite $k-k_F$ power laws, however, if so for *more specific reasons than LL universality*. Models with equilibrium dynamics which

are restricted by an extensive number of local conserved quantities might be examples for such behavior. This is also supported by the observation that the power laws reported on in Refs. [17,36,37] cannot only be found at low energies (for very small $|k - k_F|$), while LL theory is supposed to be applicable in this limit only.

B. Implications for the "nonlinear Luttinger liquid phenomenology"

In an extensive series of papers, reviewed in Ref. [23], an approach to study corrections to LL behavior by the curvature of the single-particle dispersion was developed. It cumulates in the so-called "nonlinear Luttinger liquid phenomenology." The approach is mainly built upon an effective field theory which is motivated by lowest-order perturbation theory. For the single-particle spectral function, this phenomenology predicts (Fermi-edge-singularity-like) power laws with momentum-dependent exponents even at $k - k_F \neq 0$. However, the field theory relies on the assumption of a momentum-independent bulk interaction (the momentum dependence of the interaction with the mobile impurity is considered) and requires *ad hoc* ultraviolet regularization. As we have shown in (linear) Luttinger liquid theory, the same assumption leads to power-law behavior which is nongeneric rather than universal. This raises the question as to whether the power laws found in nonlinear LL phenomenology are robust against a curvature of the bulk two-particle potential.

C. Implications for the interpretation of experimental spectra

When interpreting experimental angular-resolved photoemission data on quasi-1D metallic materials [24], the *ad hoc* regularized spectral function of the TLM is often taken paradigmatically. It is, e.g., expected that the distribution of the spectral weight between the spin and charge peaks as well as the spacing between the two must be exactly as in the analytical expressions given in Refs. [5,13,14,28]. A very recent example in which this leads to a putative conflict can be found in Ref. [38]. Our results for the spinful g_4 and the spinless g_2-g_4 models show that the *ad hoc* regularized spectral function cannot be considered as universal. Within the TLM, the details of the distribution of spectral weight, the peak distance, and the line shape clearly depend on the form of the two-particle potential even when considering small $|k - k_F|$. Such "details" are not part of LL universality and *expecting quantitative agreement overstresses LL theory.* We reiterate that all that is known about the $T = 0$ spectral properties from the latter is that for $\omega \to 0^+$ the scaling relations $\rho^>(k_F,\omega) \sim \omega^{\alpha-1}$ and $\rho^>(\omega) \sim \omega^\alpha$ hold. The exponent $\alpha = \gamma_c + \gamma_s$ can be expressed in terms of the LL parameters $K_{c/s}$ [see Eq. (14); $K_s = 1$ for spin-rotational-invariant systems], which also enter in other "observables" [6,7]. If they can be measured for the same system, consistency checks are possible. LL universality also makes predictions for the scaling of spectral weight as a function T. It was shown that $\rho^>(\omega = 0) \sim T^\alpha$ and $\rho^>(k_F,\omega = 0) \sim T^{\alpha-1}$ [39–41]. We note that details of the analysis of $\rho_+^>(k,\omega)$ at $T > 0$ for the TLM [40,41] beyond the above scaling relation should be taken with caution as they rely on an *ad hoc* regularization procedure.

It is tempting to employ the (spinful) TLM spectra computed with proper $g_{i,\kappa}(q)$ for comparison to experimental ones beyond the above scaling relations. This might lead to a *qualitative agreement* of certain features. One crucial generic feature we found regardless of the q dependence of the two-particle potential are dispersing spin and charge peaks, however, generically not given by power-law singularities. To illustrate this, we show the spectral function of the spinful TLM for the Gaussian and the exponential potential in Figs. 5(a) and 5(b), respectively. This time we focus on the *photoemission part* $\rho_+^<(k,\omega)$ for $(k - k_F)/q_c = -0.1$ as this is most easily accessible experimentally. The results were obtained by a straightforward generalization of the recursive procedure put forward for the spinful g_4 model (see Sec. III D 1) and the spinless g_2-g_4 model (see Sec. III D 2). The system size is set by $n_c = 10^3$ and the broadening by $\chi/(v_F q_c) = 3 \times 10^{-3}$. We consider the physically reasonable case of a spin-independent interaction with an equal amplitude of the intrabranch and interbranch parts. This implies $\omega_s(q) = v_F q$ and thus $v_s(q = 0) = v_F$. The interaction is varied such that $\gamma_c(q = 0) = \frac{1}{8}, \frac{1}{3}$, and $\frac{9}{16}$. Accordingly, $v_c(q = 0)$ increases. As for the above-discussed limiting cases of the spinful g_4 and the spinless g_2-g_4 model, the details of the spectral line shape obviously depend on the details of the two-particle potential [compare Figs. 5(a) and 5(b)]. The integral over $\rho_+(k,\omega) = \rho_+^>(k,\omega) + \rho_+^<(k,\omega)$ at fixed $k - k_F$ must be normalized to 1 which implies that for the exponential potential, less spectral weight is transferred to $\rho_+^>(k,\omega)$ (not shown) than for the Gaussian one. For the former, the "charge peak" is fairly broad and has an unusual line shape. However, the two sets of spectra also share similarities. For increasing interaction, the spin peak loses weight and deforms into a shoulderlike structure. To summarize, we expect spin-charge separation to be a robust feature of quasi-1D metals which should be observable in photoemission experiments.

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