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Vertex functions at finite momentum: Application to antiferromagnetic quantum criticality

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We analyze the three-point vertex function that describes the coupling of fermionic particle-hole pairs in a metal to spin or charge fluctuations at nonzero momentum. We consider Ward identities, which connect two-particle vertex functions to the self-energy, in the framework of a Hubbard model. These are derived using conservation laws following from local symmetries. The generators considered are the spin density and particle density. It is shown that at certain antiferromagnetic critical points, where the quasiparticle effective mass is diverging, the vertex function describing the coupling of particle-hole pairs to the spin density Fourier component at the antiferromagnetic wave vector is also divergent. Then we give an explicit calculation of the irreducible vertex function for the case of three-dimensional antiferromagnetic fluctuations, and show that it is proportional to the diverging quasiparticle effective mass.

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I. INTRODUCTION

Over the past several decades the theoretical description of the paramagnetic to antiferromagnetic (AFM) phase transition in metals has been an ever-growing challenge (for a review see Ref. [1]). The theoretical description of systems for which the only critical degrees of freedom are the (bosonic) spin fluctuations, while the fermionic quasiparticles (quasiparticles) are not critical, is well developed [2–4]. However, in cases where the quasiparticles acquire critical behavior, e.g., a diverging effective mass, as indicated by an apparently diverging specific heat coefficient (prominent examples of compounds showing this behavior are $CeCu_{5.9}Au_{0.1}$ [5] and $YbRh_2Si_2$ [6]), recent theoretical attempts [7–10] following the conventional field-theoretical methodology have not successfully explained experiment.

Recently a semiphenomenological theory of correlation functions $\chi(\mathbf{q},\nu)$ near an AFM quantum-critical point (QCP) has been proposed [11,12]. Since the prominent AFM fluctuations occur at a nonzero ordering wave vector $\mathbf{Q}$, the theory adopts a generalization, to nonzero wave vector, of the usual Ward identity that relates the three-point spin-density vertex function $\Lambda(\mathbf{q},\nu)$ in the limit of zero wave vector to the quasiparticle effective mass. Then, the usual weak-coupling form of the dynamical spin response function $\chi(\mathbf{q},\nu)$ acquires singular vertex corrections $\Lambda(\mathbf{Q},\nu)$ to the Landau damping term; and the coupling of bosons and fermions also gets enhanced by $\Lambda(\mathbf{Q})$. Moreover, it was shown that spin exchange energy fluctuations at small wave vector (a fluctuation combining two spin fluctuation propagators) are highly singular, and may lead to critical quasiparticles all over the Fermi surface, not only at "hot spots" [13]. A central result of the theory [11–13] is a self-consistent relation for the quasiparticle effective mass, which allows for two very different solutions, depending on initial conditions at high energy (or temperature): (1) a weak-coupling solution similar to the conventional SDW scenario [2,4] and (2) a strong-coupling solution characterized by critical quasiparticles with effective mass that diverges as a fractional power law. The results of this theory are in quite detailed agreement with experiments [5,14,15]. It is therefore natural to ask how the assumption of a singular vertex correction as adopted in the phenomenological theory [11–13] may be derived from microscopic theory. In the present paper we shall use two different approaches (Secs. III and IV) to achieve that goal.

In Sec. II we discuss generalized Ward identities, which are based on conservation laws that follow from symmetries of the Hamiltonian and which are important in the context of a class of quantum-critical phenomena.

In Sec. III we show how spin-density conservation leads to the required generalized Ward identity.

In Sec. IV we explicitly calculate the particle-hole irreducible spin-density vertex function $\Lambda(\mathbf{Q})$ at nonzero $\mathbf{Q}$ near an incommensurate antiferromagnetic QCP in the framework of the strong-coupling theory developed in Refs. [11–13] and show that the vertex diverges like the effective mass, as assumed earlier in that theory. This demonstrates that there is a closed self-consistent system of equations connecting the two singular quantities, effective mass m^* and vertex function $\Lambda(\mathbf{Q})$.

II. SYMMETRY, WARD IDENTITY, AND QUANTUM CRITICALITY

Symmetry properties are among the most important pieces of knowledge characterizing a system. The standard symmetries, related to invariance under translation in space and time, rotation in position space, spin space, or other internal spaces are well known and are used to develop methods of calculating the system's properties. In the realm of quantum many-body physics and quantum field theory, symmetry properties may be shown to give rise to useful relations among the two-particle and the one-particle Green's functions, the Ward-Takahashi identities [16,17]. These identities are usually derived by considering global symmetry transformations effected by the application of unitary operators (gauge transformations, rotations, etc.). Typically, a Ward identity relates the structure of the single-particle Green's function to a three-point vertex function $\Lambda(k;\mathbf{q},\nu)$. A standard way [18] of constructing a Ward identity is to identify a conservation law that follows from a symmetry of the Hamiltonian and then using it to simplify the equation of motion for a two-particle Green's function that contains the three-point vertex. Since we are primarily interested in an incommensurate antiferromagnetic QCP in the framework of the strong-coupling theory developed in Refs. [11–13], in what follows we shall concentrate on the consequences of spin-rotational invariance.

The Ward identities are usually applied for the limit of vanishing wave vector and frequency of an applied test field. They are therefore of limited use in characterizing fluctuations at nonzero wave vector, such as antiferromagnetic fluctuations or charge density wave fluctuations in a metal.

However, the local conservation laws are valid on all spatial and temporal scales and give rise to generalized Ward identities even at nonzero wave vector. This has already been recognized and implemented by Behn [19], who used the procedure described above and which we elaborate in Sec. III. Although these identities may be less stringent because, as we shall see, they could involve two vertex functions, one of a density type, the other of a current-density type, they nonetheless may be used to infer qualitative information. This is of particular interest if single-particle properties, such as the quasiparticle effective mass at the Fermi surface, are singular. This may happen at a quantum critical point. In metallic compounds, quantum critical points are often found to be of antiferromagnetic or charge-density wave character, which involve fluctuations of spin or charge at nonzero wave vector. Specific heat data in the neighborhood of such critical points often indicate a divergent quasiparticle effective mass. Examples are many of the heavy-fermion compounds, some of the iron-based superconductors, and possibly the cuprate superconductors. The question becomes: How does a singularity in the single-particle properties affect the two-particle vertex functions at nonzero wave vector? Here the generalized Ward identities may be useful. In the present paper the answer will be given as

$$\Lambda(k;\mathbf{Q},\nu\to 0)\sim Z^{-1}(\mathbf{k}_+)+Z^{-1}(\mathbf{k}_-), \tag{1}$$

where $Z^{-1}=1-\partial\Sigma/\partial\omega$ is proportional to the quasiparticle effective mass and $k_\pm = k\pm Q/2$ are the momenta of the incoming and outgoing legs of the three-point vertex describing momentum transfer $\mathbf{Q}$ and energy transfer ν

III. CONSERVATION LAW AND CONTINUITY EQUATION: VERTEX FUNCTION IN LINEAR RESPONSE

We consider systems of identical fermions interacting via spin-conserving density-density interactions, either on a lattice or in the continuum, e.g., the Hubbard model. To be concrete, we may take the interaction term in the form

$$\mathcal{H}_{\rm int}=U\int dr\,\Psi_\uparrow^\dagger(r)\Psi_\uparrow(r)\Psi_\downarrow^\dagger(r)\Psi_\downarrow(r), \tag{2}$$

where $\Psi_\alpha^\dagger(r),\Psi_\alpha(r)$ creates or annihilates an electron of spin $\alpha=\uparrow,\downarrow$ at location r. Later, the following derivations will be taken over for the case of electrons on a lattice within a one-band model, for which the spin is to be understood as a conserved "pseudospin," corresponding to the two-level ground-state doublet that is determined by the crystal field and the spin-orbit interaction. As a practical example we consider the response of the system to an external magnetic field $\vec{H}(r)$ that couples to the spin density $\vec{\rho}_s(r)=\sum_{\alpha,\beta}\Psi_\alpha^\dagger(r)\vec{\sigma}_{\alpha\beta}\Psi_\beta(r)$ via an interaction term

$$\mathcal{H}'=\lambda\int dr\,\vec{H}(r)\cdot\vec{\rho}_s(r). \tag{3}$$

Before showing how the spin-density vertex function, the expectation value of $\vec{\rho}_s(r,t)$ (in Heisenberg representation), controls the response to $\mathcal{H}'$, we remind that when a density operator commutes with the interaction term in the Hamiltonian, then it obeys a continuity equation, derived from $\partial\Psi/\partial t = i[\mathcal{H},\Psi]$. Since the spin density operator commutes with Eq. (2), only the kinetic energy enters the commutator and we have the familiar local conservation law (repeated greek indices are summed):

$$\frac{\partial\vec{\rho}_s(r,t)}{\partial t}-\frac{i}{2m}[\vec{\nabla}\cdot(\vec{\nabla}\Psi_\alpha^\dagger\vec{\sigma}_{\alpha\beta}\Psi_\beta-\Psi_\alpha^\dagger\vec{\sigma}_{\alpha\beta}\vec{\nabla}\Psi_\beta)]=0. \tag{4}$$

The spatial derivatives in Eq. (4) come from the commutator of ρ_s with the kinetic energy, which in momentum space is $\sum_{k,\alpha}\epsilon_k c_{k,\alpha}^\dagger c_{k,\alpha}$. For later convenience, it is useful to re-express the continuity equation in momentum space:

$$i\,\partial\vec{\rho}_q/\partial t=\sum_k(\varepsilon_{kq})c_{k_+,\alpha}^\dagger\vec{\sigma}_{\alpha\beta}c_{k_-,\beta}, \tag{5}$$

where $\vec{\rho}_q=\sum_k c_{k_+,\alpha}^\dagger\vec{\sigma}_{\alpha\beta}c_{k_-,\beta}$, with $k_\pm = k\pm q/2$ and $\varepsilon_{kq}=\epsilon_{k_+}-\epsilon_{k_-}$ In compact form, it is

$$\sum_k D(k,q;t)\,c_{k_+\alpha}^\dagger\vec{\sigma}_{\alpha\beta}c_{k_-,\beta}=0, \tag{6}$$

where the operator $D(k,q;t)$ is

$$D(k,q;t)=\frac{\partial}{\partial t}-(\varepsilon_{kq}). \tag{7}$$

The considerations above apply also to the charge response and lead to the usual charge continuity equation. For our general purposes, we want to discuss the vertex function that describes the coupling of an electron (spin or charge) density to an external perturbation such as a charge or spin density fluctuation boson field or a magnetic field. For example, in linear response, the magnetic field perturbation $\mathcal{H}'$ of Eq. (3) gives rise to a correction to the single-particle Green's function $G_{\alpha\beta}(1,2)=-\langle T\Psi_\alpha(1)\Psi_\beta^\dagger(2)\rangle$, where T is the time-ordering operator

$$\delta G_{\alpha\beta}(1,2)=\int d3\;G^{(2)}_{\alpha\beta\delta\gamma}(1,2;3,3^+)\lambda\vec{H}(3)\cdot\vec{\sigma}_{\gamma\delta}. \tag{8}$$

Here the two-particle Green's function is

$$G^{(2)}_{\alpha\beta\gamma\delta}(1,2;3,4)=-\langle T\Psi_\alpha(1)\Psi_\beta^\dagger(2)\Psi_\gamma(3)\Psi_\delta^\dagger(4)\rangle \tag{9}$$

and in Eq. (8), 3^+ means (r_3,t_3+0^+). It is seen that the right-hand side of Eq. (8) contains $\vec{\rho}_s(3)$:

$$\delta G_{\alpha\beta}(1,2)=\int d3\langle T\Psi_\alpha(1)\Psi_\beta^\dagger(2)\vec{\rho}_s(3)\rangle\cdot\lambda\vec{H}(3). \tag{10}$$

This shows how the vertex function controls the response.

Making use of the conservation law we may now derive identities relating the response function to the single-particle Green's function or its self-energy. To achieve this we re-express the two-particle Green's function in the integrand of

Eq. (10) in momentum space. It becomes

$$\sum_p G^{(2)}_{\alpha\beta\delta\gamma}(k,p;q)\vec{\sigma}_{\gamma\delta}, \tag{11}$$

where

$$G^{(2)}_{\alpha\beta\delta\gamma}(k,p;q) = -\langle T c_{k_+\alpha}(t_1)c^\dagger_{k_-\beta}(t_2)c_{p_-\delta}(t_3)c^\dagger_{p_+\gamma}(t_3^+)\rangle. \tag{12}$$

Here $k_\pm = k \pm q/2, p_\pm = p \pm q/2$ are the momenta of the particle-hole pairs entering and leaving the two-particle Green's function.

The next step is to let the operator $D(k,q;t_3)$ that appears in the conservation law, Eq. (6), act on this $G^{(2)}$. The action of the operator on the ρ_q part of $G^{(2)}$ gives zero because of the conservation law, whereas the time derivative in D acts on the step functions defined by the time ordering. The result is

$$\sum_p [\nu - \varepsilon_{pq}]G^{(2)}(k,p;q) = G(k_-) - G(k_+). \tag{13}$$

Here we have Fourier transformed in time (t_3) and on the right-hand side of Eq. (13), k and q mean $(\mathbf{k},\omega)$ and $(\mathbf{q},\nu)$, respectively. That is, $k_\pm \to (\mathbf{k} \pm \mathbf{q}/2, \omega \pm \nu/2)$ and here the sum on p includes $\int d\omega$. For the charge response, Eq. (13) holds, as the spin indices are irrelevant. We shall restore them later, when necessary.

The next steps involve the use of well-known relations among the Green's functions and associated amplitudes:

$$G^{-1}(\mathbf{k},\omega) = \omega - \epsilon_k - \Sigma(\mathbf{k},\omega) = G_0^{-1}(\mathbf{k},\omega) - \Sigma(\mathbf{k},\omega), \tag{14}$$

$$G^{(2)}(k,p;q) = G(k_+)G(k_-)\Delta(k,p;q), \tag{15}$$

$$\Delta(k,p;q) = \delta(k,p) + \Gamma(k,p;q)G(p_-)G(p_+), \tag{16}$$

$$\Lambda(k,q) = 1 + \sum_p \Gamma(k,p;q)G(p_-)G(p_+). \tag{17}$$

In the above, $\Gamma(k,p;q)$ is the four-point vertex (without external legs) and $\Lambda(k,q)$ is the three-point vertex amplitude that enters the Ward identity, We use Eq. (15) in Eq. (13), divide out $G(k_+)G(k_-)$, and find

$$\sum_p (\nu - \varepsilon_{pq})\Delta(k,p;q) = \nu - \varepsilon_{kq} - \Sigma(k_+) + \Sigma(k_-), \tag{18}$$

where $\Sigma(k)$ is the self-energy part as in Eq. (14). This result has already been anticipated in Ref. [19]; we make use of it in what follows.

We take the derivative of Eq. (18) with respect to ν and as we are interested in the behavior of $\Lambda(\mathbf{k},\omega;\mathbf{q},\nu)$ for $\nu \to 0$ and arbitrary $\mathbf{q}$, we then take $\nu = 0$ and obtain

$$\Lambda(k;\mathbf{q},0) - \sum_p \varepsilon_{pq}\left[\frac{\partial}{\partial\nu}\Gamma(k,p;q)G(p_-)G(p_+)\right]_{\nu=0} = \frac{1}{2}[Z^{-1}(\mathbf{k}_+,\omega) + Z^{-1}(\mathbf{k}_-,\omega)], \tag{19}$$

where we used the quasiparticle weight factor at $(\mathbf{k},\omega)$ defined as

$$Z^{-1}(\mathbf{k},\omega) = 1 - \frac{\partial}{\partial\omega}\Sigma(\mathbf{k},\omega). \tag{20}$$

The second term on the left-hand side of the key result Eq. (19) is the $\nu = 0$ derivative of the spin-current density vertex.

A situation of particular interest arises if the quasiparticle weight factor $Z(\mathbf{k},\omega)$ happens to be small, or even tends to vanish, implying that the effective quasiparticle mass $m^*/m = Z^{-1}(\mathbf{k},\omega)$ is large either in certain regions on the Fermi surface (so-called hot spots) or all over the Fermi surface. This will be the case in the critical regime near a quantum phase transition to, e.g., an antiferromagnetic phase. We may then conclude from the key equation (19) that the three-point vertex is enhanced approximately proportional to the effective mass enhancement. This follows from the fact that both the spin density and the spin-current density vertices are given by integrals of $\Gamma\ G(p_+)G(p_-)$ multiplied by two different weight factors, 1 and $\varepsilon_{pq}(\partial/\partial\nu)$, respectively. Although the effective mass enhancement occurs for a state near the Fermi surface, we emphasize the new result that the vertex enhancement takes place if at least one of the partners of the particle-hole pair, with momenta $\mathbf{k}_+$ or $\mathbf{k}_-$, is on the Fermi surface. In order to demonstrate that Eq. (19) does indeed imply a proportionality of Λ to Z^{-1} we consider the limit of small, but nonzero q and $\nu = 0$, when Fermi liquid theory applies. In this case the vertex function is given as $\Lambda = Z^{-1}/(1 + F_a)$, where F_a is the Landau parameter in the spin channel. In order for the Ward identity Eq. (19) to be satisfied, the current density term has to amount to $Z^{-1}F_a/(1 + F_a)$. The two contributions add up to Z^{-1}, as required by the Ward identity. In other words, for any nonzero Fermi liquid interaction F_a both terms on the left-hand side of Eq. (19) are proportional to Z^{-1}, and will therefore diverge whenever Z^{-1} diverges.

IV. VERTEX FUNCTION AT LARGE Q

We now calculate the irreducible spin-density vertex function $\Lambda(\mathbf{k},\omega;\mathbf{Q},\nu = 0)$ (called λ_Q in Ref. [13]) in the framework of the theory of critical quasiparticles near an antiferromagnetic critical point as developed in Refs. [11–13]. There it was shown that for the case of three-dimensional AFM spin fluctuations, when conventional spin density wave theory is supposed to work, a new strong-coupling regime may be accessible under certain conditions. This regime is characterized by a power-law divergence of the effective mass as a function of energy, and hyperscaling with critical exponents $z = 4$ and $\nu = 1/3$. The theory requires the particle-hole irreducible spin density vertex function at wave vector $\mathbf{Q}$ to diverge like the effective mass. As shown in Sec. III, this can be a consequence of the Ward identities. However, since the Ward identities relate the full vertex functions Λ to the effective mass (or the inverse quasiparticle weight factor $1/Z$), one may ask how the irreducible vertex, which is the quantity needed in the strong coupling theory [11–13], depends on Z. We therefore show in the following that a certain diagram contributing to the irreducible vertex correction is indeed proportional to $1/Z$, provided one assumes that this very vertex correction renormalizes the spectrum of spin fluctuations and their the coupling to quasiparticles in just the way that was assumed in the theory of critical quasiparticles.

There are two ways in which the vertex function λ_Q enters the theory: first, the spin-fluctuation spectrum is affected in

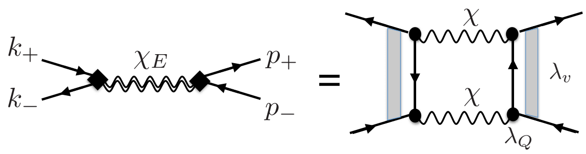

FIG. 1. Structure of the energy fluctuation. The spin fluctuations χ carry momentum Q. The black dots denote the vertex function λ_Q and the gray vertex corrections denote the small q vertex λ_v. The black diamonds on each end of χ_E denote the combination $\lambda_Q^2\lambda_v$

the Landau damping term; it acquires a factor λ_Q^2, from the renormalization of the particle-hole bubble diagram of Landau damping at each end. Thus, for the spin fluctuations

$$\text{Im}\,\chi(\mathbf{q},\nu) = \frac{N_0\lambda_Q^2\nu}{[r+(\mathbf{q}-\mathbf{Q})^2]^2 + \left(\lambda_Q^2\nu\right)^2}.$$

Here N_0 is the bare density of states, r is the dimensionless tuning parameter ($r \to 0$ at the QCP), and wave vector $\mathbf{q}$ and frequency ν are in units of k_F and ϵ_F, respectively. Second, since the coupling of the spin fluctuations to the quasiparticles also involves a factor λ_Q, each end of a spin fluctuation line receives a factor $\lambda_Q N_0^{-1}$.

The large momentum transfer involved in a scattering process of quasiparticles off AFM spin fluctuations usually takes quasiparticles into final states far from the Fermi surface, except for momenta at hot spots on the Fermi surface. The consequences of these limitations of critical scattering are often not compatible with what is observed experimentally. It was therefore suggested in Ref. [13] that simultaneous scattering off two spin fluctuations with opposite momenta, leading to small total momentum transfer, would be a more relevant process [20]. Two spin fluctuations may be thought of as an (exchange) energy fluctuation $\chi_E(\mathbf{q},\nu)$. Schematically, the energy fluctuation propagator is constructed from $G\,G\,\chi\,\chi$ (χ is the spin fluctuation propagator) as in the diagram of Fig. 1, in which the vertex corrections are shown. Not shown in Fig. 1 is a further diagram in which the two χ lines are crossed; it gives an identical contribution except that the spin structures of the two diagrams add up to give a pure density-density interaction.

In Fig. 1 the vertex function λ_v is shown at each end of χ_E. Since χ_E carries a small momentum transfer, $\lambda_v \propto 1/Z$; it is governed by by the usual Ward identity at ($\mathbf{q} \approx 0, \nu \to 0$).

The spectrum of χ_E was calculated in Ref. [13], Eqs. (2) and (3), to be

$$\text{Im}\,\chi_E(\mathbf{q},\nu) = \frac{(N_0)^3\lambda_Q^3\nu^{5/2}}{(r+q^2)^2 + \left(\lambda_Q^2\nu\right)^2}. \tag{21}$$

The corresponding self-energy due to energy fluctuation exchange is given by

$$\Sigma(\mathbf{k},\omega) = \lambda_Q^2 u^2 \int dq\; G(\mathbf{k}+\mathbf{q},\omega+\nu)\;\chi_E(\mathbf{q},\nu) \tag{22}$$

and leads to $\Sigma \propto \omega^{3/4}$; hence $Z(\omega) \propto \omega^{1/4}$. In Eq. (22), $u \propto N_0^{-1}$.

The first vertex correction diagram that corresponds to the dressing of the spin-density vertex λ_Q by energy fluctuations is

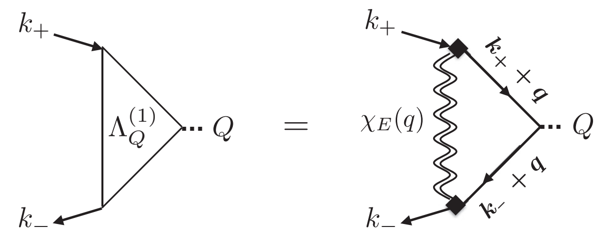

FIG. 2. Structure of the first energy fluctuation contribution to the spin-density vertex. The energy fluctuation χ_E carries (small) momentum q. The black diamonds denote the combination $\lambda_Q^2\lambda_v$, as in Fig. 1

shown in Fig. 2. It has has one energy fluctuation that bridges the vertex:

$$\lambda_Q^{(1)} = \lambda_Q^4\lambda_v^2u^4 \int dq\; G(k_+ + q)G(k_- + q)\chi_E(q). \tag{23}$$

For generic $\mathbf{p}$, one of the momenta $\mathbf{p}+\mathbf{q}+\mathbf{Q}$ will be far from the Fermi momentum, while the other $\mathbf{p}+\mathbf{q}$ is close to it (or vice versa). We may then put $G(p+q+Q) \approx 1/\epsilon_F$. What remains is the self-energy expression, Eq. (22), so that

$$\lambda_Q^{(1)}(\mathbf{k},\omega;\mathbf{Q},\nu=0) \approx \frac{\Sigma(\mathbf{k},\omega)}{\epsilon_F}. \tag{24}$$

We see that $\lambda_Q^{(1)} \to 0$ as $\omega \to 0$ and is not singular. However, singular diagrams do occur if at least three spin fluctuation lines in parallel are internal in a contribution to $\lambda_Q(\mathbf{p},\omega;\mathbf{Q},\nu=0)$ (any odd number will do). Two of these combine into an energy fluctuation. The resulting diagram, shown in Fig. 3, has a spin fluctuation and an energy fluctuation in the intermediate state, similar to the Azlamasov-Larkin diagram in the theory of superconducting fluctuations:

$$\lambda_Q^{(3)} = A\int dq\; G(p-q)T(q;Q)\chi(Q+q)\chi_E(q),$$

where $A = \lambda_Q^6\lambda_v^2u^6$ and we defined the triangle loop

$$T(q;Q) = \int dp' G(p'+q)G(p')G(p'+q+Q).$$

The quantity $T(q;Q)$ is noncritical and may be replaced by $T(q;Q) \approx N_0/\epsilon_F$. It is convenient to first calculate the

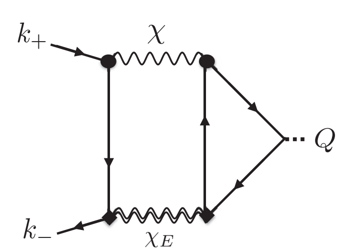

FIG. 3. Structure of the singular energy fluctuation contribution to the spin-density vertex. The energy fluctuation χ_E carries (small) momentum q. The black diamonds denote the combination $\lambda_Q^2\lambda_v$, as in Fig. 1. The black dots denote the vertex function λ_Q.

imaginary part of $\lambda_Q^{(3)}$ at temperature $T << \omega$:

$$\mathrm{Im}\lambda_Q^{(3)} \approx A_1 \int_0^\omega d\nu \int d\vec{q}\, \mathrm{Im}\chi(Q+q)\mathrm{Im}\chi_E(q) \times \mathrm{Im}G(p-q). \tag{25}$$

Here $A_1 = \lambda_Q^6 \lambda_v^2 u^6 (N_0/\epsilon_F)$. The result of the integration over the solid angle of $\vec{q}$ is $\propto 1/q$. We restrict ourselves to the critical regime $r = 0$ in χ_E. The q integration may then be performed for $\lambda_Q^2|\nu| < q^2 < \infty$:

$$\mathrm{Im}\lambda_Q^{(3)} \propto \frac{k_F^3}{N_0\epsilon_F}\lambda_Q^{11}\lambda_v^2 \int_0^\omega d\nu|\nu|^{7/2} \int q dq \frac{1}{[q^4 + (\lambda_Q^2|\nu|)^2]^2} \propto \lambda_Q^5 Z^{-2}|\omega|^{3/2},$$

where we used $\lambda_v \propto Z^{-1}$ as stated above. We now identify $\lambda_Q^{(3)} = \lambda_Q$, and solve the resulting equation for λ_Q:

$$\lambda_Q \propto Z^{1/2}|\omega|^{-3/8}. \tag{26}$$

This result may be combined with the result for Z in the strong-coupling regime which was obtained in Ref. [13], Eq. (4):

$$Z \propto \lambda_Q^5|\omega|^{3/2}. \tag{27}$$

Combining Eqs. (26) and (27) we find

$$\lambda_Q \propto Z^{-1} \propto |\omega|^{-1/4},$$

which is precisely what has been postulated in Ref. [13] on the basis of phenomenological arguments.

V. CONCLUSION

We investigated the Ward-Takahashi identity for the spin or charge density vertex amplitude that describes the response to an external probe and/or coupling to a collective mode. We showed that even when the momentum transfer entering the vertex $\vec{q}$ is nonzero, the vertex amplitude λ_q may be related to the quasiparticle weight factor Z; it therefore acquires singular behavior when Z does. Thus, our results are of use in analyzing behavior of metals near quantum critical points, where $Z \to 0$.

In particular, the result is of importance for the case of an antiferromagnetic quantum critical point, where the relevant momentum transfer $\vec{Q}$ is the ordering vector and is not small. This situation is the setting for the recent development of critical quasiparticle theory [12,13], in which the singular behavior of $\lambda_{q=Q}$ was proposed. A new feature of the present work is that it is sufficient that (at least) one of the external lines to the (three-point) vertex is on the Fermi surface, rather than requiring both to be. Therefore, the identity holds even when $\vec{q}$ does not connect two points on the Fermi surface (hot spots). This was essential in the critical quasiparticle theory to achieve good agreement with experiments on $YbRh_2Si_2$ (resistivity, specific heat, thermopower, magnetic susceptibility, magnetic Grüneisen ratio) [11,12] and $CeCu_{5.9}Au_{0.1}$ (neutron scattering, specific heat, resistivity, magnetization) [13].

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