Spin transfer torque and dc bias magnetic field effects on the magnetization reversal time of nanoscale ferromagnets at very low damping: Mean first-passage time versus numerical methods

D. J. Byrne,[1] W. T. Coffey,[2] W. J. Dowling,[2] Y. P. Kalmykov,[3] and S. V. Titov[4]
[1]*School of Physics, University College Dublin, Belfield, Dublin 4, Ireland*
[2]*Department of Electronic and Electrical Engineering, Trinity College Dublin, Dublin 2, Ireland*
[3]*Laboratoire de Mathématiques et Physique (EA 4217), Université de Perpignan Via Domitia, F-66860, Perpignan, France*
[4]*Kotel'nikov Institute of Radio Engineering and Electronics of the Russian Academy of Sciences, Vvedenskii Square 1, Fryazino, Moscow Region 141120, Russia*

(Received 16 September 2015; revised manuscript received 26 November 2015; published 9 February 2016)

Spin transfer torque and bias field effects on the magnetization reversal time of a nanoscale ferromagnet are investigated in the very-low-damping regime via the energy-controlled diffusion equation. That equation is rooted in a generalization of the Kramers escape rate theory for point Brownian particles in a potential to the magnetic relaxation of a macrospin. Using the mean first-passage method, the reversal time is then evaluated in closed integral form for a nanomagnet with the free-energy density given in the standard form of superimposed easy-plane and in-plane easy-axis anisotropies with the dc bias field along the easy axis. The results completely agree with those yielded by independent numerical methods.

DOI: 10.1103/PhysRevB.93.064413

I. INTRODUCTION

On account of the spin transfer torque (STT) effect [1–6], the magnetization of a nanoscale ferromagnet *may be altered by spin-polarized currents* because an electric current with spin polarization in a ferromagnet has an associated flow of angular momentum [3,7], thereby exerting a macroscopic spin torque. The phenomenon underpins the novel subject of spintronics [7,8], i.e., current-induced control over magnetic nanostructures. Common place applications include very-high-speed current-induced magnetization switching by reversing the orientation of magnetic bits [3,9] and using spin-polarized currents to manipulate steady-state microwave oscillations [9]. These are maintained via the steady-state magnetization precession due to STT representing the conversion of dc input current into an ac output voltage [3]. Now, in STT devices, the thermal fluctuations cannot be ignored at finite temperatures because they lead to mainly noise-induced switching at currents far less than the critical switching current without noise as well as introducing randomness into the precessional orbits [10]. This phenomenon has been corroborated by many experiments (e.g., [11]) demonstrating that STT near room temperature alters thermally activated switching processes, which then exhibit a pronounced dependence on both material and geometrical parameters. Therefore, accurate solutions of generic STT models at finite temperatures are necessary both to assess properly such theories and to achieve further improvements in the design and interpretation of experiments, particularly due to the manifold practical applications in spintronics, random access memory technology, and so on.

The archetypal model (Fig. 1) of a STT device is a nanostructure comprising two magnetic strata labeled the *free* and *fixed* layers and a nonmagnetic conducting spacer. The fixed layer is much more strongly pinned along its orientation than the free one. On passing an electric current through the fixed layer it becomes spin polarized which, as it encounters the free layer, induces a STT. Thus the magnetization $\mathbf{M}$ of the free layer is altered. Both ferromagnetic layers are assumed to be uniformly magnetized [3,6]. Although this approximation cannot explain all observations of the magnetization dynamics in spin-torque systems, nevertheless many qualitative features needed to interpret experimental data are satisfactorily reproduced. Now, the current-induced magnetization dynamics in the free layer may be described by the Landau-Lifshitz-Gilbert-Slonczewski equation including thermal fluctuations. This is simply the Landau-Lifshitz-Gilbert equation [12] including the STT augmented by a random magnetic field $\mathbf{h}(t)$ so that it becomes a Langevin equation [3,6,7,13], viz.,

$$\dot{\mathbf{M}} = -\gamma[\mathbf{M}\times(\mathbf{H}+\mathbf{h})] + \frac{\gamma}{M_S}[\mathbf{M}\times[\mathbf{M}\times\mathbf{I}_S]] + \frac{\alpha}{M_S}[\mathbf{M}\times\dot{\mathbf{M}}]. \tag{1}$$

Here the random field $\mathbf{h}(t)$ is treated as spatially isotropic Gaussian white noise with the properties

$$\overline{h_i(t_1)} = 0, \quad \overline{h_i(t_1)h_j(t_2)} = 2D\delta_{ij}\delta(t_1-t_2), \tag{2}$$

where the indices $i,j = 1,2,3$ in Kronecker's delta δ_{ij} and h_i correspond to the Cartesian axes X, Y, Z of the laboratory coordinate system $OXYZ$, $\delta(t)$ is the Dirac-delta function, $D = \alpha kT/(v\gamma\mu_0 M_S)$ is the diffusion coefficient, γ is the gyromagnetic-type constant, α is a dimensionless damping parameter, M_S is the saturation magnetization, v is the particle volume, $\mu_0 = 4\pi\times10^{-7}\,\mathrm{J\,A^{-2}\,m^{-1}}$ in SI units, and kT is the thermal energy. The overbar means the statistical average over an ensemble of magnetic moments $\boldsymbol{\mu}(t) = v\mathbf{M}(t)$, which all have at time t the *same* sharp value $v\mathbf{M}$, these sharp values subsequently being regarded as random variables. The effective magnetic field $\mathbf{H}$ comprising the anisotropy and external applied fields is defined as

$$\mathbf{H} = -\frac{1}{\mu_0}\frac{\partial V}{\partial \mathbf{M}}, \tag{3}$$

where V is the free-energy density of the free layer constituting a conservative potential. The STT-induced field $\mathbf{I}_S$ is given by

$$\mathbf{I}_S = \frac{1}{\mu_0}\frac{\partial \Phi}{\partial \mathbf{M}}. \tag{4}$$

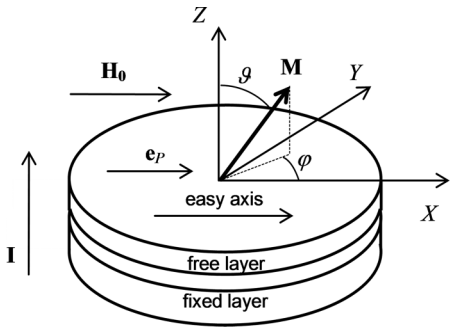

FIG. 1. Geometry of the problem: A STT device consists of two ferromagnetic strata labeled the *free* and *fixed* layers, respectively, and a normal conducting spacer all sandwiched on a pillar between two Ohmic contacts [3,6]. Here **I** is the spin-polarized current; **M** is the magnetization of the free layer; $\mathbf{H}_0$ is the dc bias magnetic field. The magnetization of the fixed layer is directed along the unit vector $\mathbf{e}_P$.

Here, Φ is the nonconservative potential due to the spin-polarized current, which is taken in the simplest form as

$$\Phi(\mathbf{M}) = J\frac{kT}{vM_S}(\mathbf{e}_P \cdot \mathbf{M}). \tag{5}$$

Here $J = b_P\hbar I/(|e|kT)$ is the dimensionless STT parameter, I is the current which is taken as positive if electrons flow from the free into the fixed layer, e is the electronic charge, $\hbar$ is Planck's reduced constant, and b_P is a parameter determined by the spin-polarization factor P [1]. In tandem with the magnetic Langevin equation (1), one has the Fokker-Planck equation (FPE) for the probability density function $W(\vartheta,\varphi,t)$ of orientations of **M** on the unit sphere, viz., [6,14]

$$\frac{\partial W}{\partial t} = \mathrm{L}_{\mathrm{FP}}W, \tag{6}$$

where L_{FP} is the Fokker-Planck operator, which is given explicitly, e.g., in Ref. [6]. Invariably STT effects on the magnetization relaxation exist only at low damping, $\alpha \ll 1$, because the magnitude of these effects is governed by the ratio J/α. For $\alpha \geqslant 1$ the STT term in Eq. (1) does not influence the reversal process at all because it is negligible compared to the damping and random field terms. In nanomagnets the most important region of damping is the very-low-damping (VLD) range, $\alpha \ll 1$, because both experimental and theoretical estimates suggest small values of α of the order of 0.001–0.1 (see, e.g., Refs. [6,15–17]).

During the last decade, various analytical and numerical approaches to the calculation of the measurable parameters of STT devices such as the magnetization reversal (or switching) time, etc., via the magnetic Langevin and/or Fokker-Planck equations including STT have been developed [6,7,10,14,18–30]. These mainly generalize methods originally developed for zero STT [13,31–36] such as stochastic dynamics simulations (e.g., Refs. [25–29]) and extensions of the mean first-passage time (MFPT) method (e.g., Refs. [20,21]) in the Kramers escape rate theory [37,38]. The generalization of the MFPT to the magnetic relaxation of macrospins with *nonseparable* and *nonadditive* Hamiltonians in the VLD limit can also be accomplished [7,18,21,39] just as for point particles and rigid rotators, where the Hamiltonians are *separable* and *additive* [13,37,38]. Here, the pronounced time separation between *fast* precessional and *slow* energy changes in *lightly* damped closed phase space trajectories (called Stoner-Wohlfarth orbits [7]) at energies near the barrier energy has been exploited in Refs. [6,7,18,21] to formulate a one-dimensional Fokker-Planck equation for the energy distribution function essentially similar to that derived by Kramers [37] in the problem of the very-low-damping (VLD) noise-activated escape rate from a potential well. The Stoner-Wohlfarth orbits and steady precession along such an orbit of constant energy occurs if the spin torque cancels out the dissipative torque [cf. Eq. (1)]. The origin of the orbits arises from the structure of the anisotropy potential allowing one to define a nonconservative "effective" potential with damping- and current-dependent potential barriers between stationary self-oscillatory states (limit cycles) of the magnetization and, hence, to estimate the reversal (switching) time τ between these states via the MFPT method (for reviews of this method see Refs. [35,38]). Using this method, it has been demonstrated [18,36] how the reversal time for spins can be evaluated in the VLD limit both for *zero and nonzero STT*. In particular, for *zero STT*, the analytic equation for the VLD reversal time has been derived from the energy-controlled diffusion equation in Ref. [39] via the MFPT for *arbitrary* free-energy density V. In particular, these results have allowed us to treat dc bias field effects in the magnetization reversal time in nanomagnets [39]. Likewise, for *nonzero STT*, the VLD reversal time in nanomagnets has been evaluated [21a,b] via the MFPT for the nonconservative potential Φ, Eq. (5), in the absence of a dc bias field. Here we shall extend the results of Refs. [21] and [39] treating simultaneously STT and dc bias field effects in nanomagnets, when the free-energy potential V of the free layer is given in the standard form of superimposed easy-plane and in-plane easy-axis anisotropies [40] [see Eq. (37) below]. As far as the verification of the MFPT formulas for the reversal time with nonzero STT is concerned, no comparison with calculations based on the solutions of the Fokker-Planck equation or on numerical simulations of the Langevin dynamics has ever been given. Thus, another purpose of this paper is to demonstrate that the MFPT approach as applied completely agrees with these independent methods yielding an accurate description of STT and dc bias field effects in the magnetization reversal time of nanoscale ferromagnets.

The paper is arranged as follows. In Sec. II, we give a short historical overview of application of the MFPT approach in the escape rate theory in evaluating the reversal time of the magnetization in nanomagnets for zero STT. In Sec. III, we present the basic equations describing STT effects in the stochastic spin dynamics in the VLD regime. In Sec. IV, we derive in quadratures a general equation for the VLD reversal time for nanomagnets using the energy-controlled diffusion equation for spins in substantially the same manner as for zero STT [39]. Here, we also demonstrate that in the high-barrier approximation, $\Delta E \gg 1$, our result reduces to the asymptotic solution of Klik and Gunther [34] [Eq. (13) below], thus reconciling their solution with that from the Kramers theory. By way of illustration of our general results, which are valid for an arbitrary free energy, we determine in Sec. V the VLD reversal time of the free-energy anisotropy potential

taken in the standard form of superimposed easy-plane and in-plane easy-axis anisotropies [40]. In Sec. VI, we outline an independent numerical method of the calculation of the reversal time via matrix continued fractions. In Sec. VII, we compare our analytical results obtained via the MFPT with numerical calculations. Appendixes A–C contain the details of the calculations.

II. REVERSAL TIME FOR ZERO STT

The reversal or switching time τ, which is the longest relaxation time of the magnetization, may be defined as the inverse of the smallest nonvanishing eigenvalue λ_1 of the Fokker-Planck operator in Eq. (6) [32,36]. The reversal time τ may then be estimated using three different approaches: (i) Brownian dynamics simulations via the magnetic Langevin equation (1), (ii) numerical solutions of the Fokker-Planck equation (6), and (iii) analytical solutions of Eqs. (1) and (6) such as those yielded by escape rate theory, the MFPT method, etc. (for a review see Ref. [36]). These complementary approaches allow one to evaluate τ over wide ranges of temperature, damping, etc. In particular, numerical methods and escape rate theory are very useful for the determination of τ at low and high potential barriers, respectively. However, they have considerable limitations: for example, escape rate theory cannot be used at low barriers, $\Delta E \leqslant 1$, while numerical methods encounter substantial computational difficulties in the VLD range, $\alpha \ll 1$.

The magnetic Langevin equation (1) with zero STT, i.e., at $\Phi \equiv 0$, was originally used by Brown [32,33] for the theoretical treatment of the magnetization reversal in magnetic nanoparticles. His primary objective was to securely anchor Néel's conjectures [31] concerning the nature of the superparamagnetic relaxation of single-domain ferromagnetic particles within the framework of the theory of stochastic processes. In particular, Brown [32,33] found damping-dependent correction factors for the magnetization reversal time τ in magnetic nanoparticles originally calculated by Néel [31] via transition-state theory (TST) as

$$\tau \sim \tau^{\mathrm{TST}} = f_A^{-1} e^{\Delta E}. \tag{7}$$

Here f_A is the so-called attempt frequency associated with the *gyromagnetic precession* frequency of $\mathbf{M}$ in the potential well A and ΔE is the dimensionless (normalized by the thermal energy kT) potential barrier. We recall that the TST reversal time τ^{TST} is independent of damping and represents the lower bound of τ because all dissipation to the bath is ignored in that time [36]. The Néel-Brown theory is in effect an adaptation of the Kramers theory [37,38] originally given for point Brownian particles to magnetization relaxation governed by a gyromagneticlike equation. Hence, the verification of that theory in the nanomagnet context nicely illustrates the Kramers conception of a thermal relaxation process over a potential barrier arising from the shuttling action of the Brownian motion. In his earliest calculations of the reversal time of the magnetization in magnetic nanoparticles, Brown [32] confined himself to uniaxial nanomagnet subjected to a dc external magnetic field $\mathbf{H}_0$ applied *along* the easy (here Z) axis of the magnetization, where the free-energy density is given by

$$V(\vartheta) = -K(\cos^2\vartheta + 2h\cos\vartheta). \tag{8}$$

Here K is an anisotropy constant and $h = \mu_0 M_{\mathrm{S}} H_0/(2K)$ is the external field parameter. In this *axially symmetric situation*, since no dynamical coupling between the longitudinal and the transverse modes of motion exists, the longitudinal relaxation is governed by a *single* state variable, namely, the polar angle ϑ of $\mathbf{M}$. The second state variable, namely, the azimuthal angle φ gives rise only to a steady precession of $\mathbf{M}$. By recognizing this fact, Brown obtained from Eq. (6) a Fokker-Planck equation in ϑ only, viz. [32],

$$\frac{\partial W}{\partial t} = \frac{1}{2\tau_{\mathrm{N}}\sin\vartheta}\frac{\partial}{\partial\vartheta}\left[\sin\vartheta\left(\frac{\partial W}{\partial\vartheta} + \frac{vW}{kT}\frac{\partial V}{\partial\vartheta}\right)\right], \tag{9}$$

where $\tau_{\mathrm{N}} = (1+\alpha^2)/(2\gamma^2 D)$ is the free diffusion time of the magnetization. Equation (9) has the form of the Smoluchowski equation for the overdamped rotation of the Brownian particle in a liquid [13]. However, in contrast to that equation, Eq. (9) for spins is valid for *all values* of the damping parameter α including the VLD range because it follows from the *axial symmetry* of $V(\vartheta)$ and not from the overdamped rotational Brownian motion. For *axially symmetric* potentials, Brown [32] and others (see Ref. [36] for a review) have developed various techniques such as variational methods, MFPT, etc., for the calculation of the reversal time of uniaxial nanomagnets. As an example, we mention Brown's well-known high-barrier asymptotic formula for the reversal time, which at $h = 0$ becomes in the VLD limit [32]

$$\tau_{\mathrm{as}}^{\mathrm{VLD}} = \frac{\tau^{\mathrm{TST}}}{4\alpha\sqrt{\pi\sigma}}\left(1 + \frac{1}{\sigma} + \cdots\right), \tag{10}$$

where $\tau^{\mathrm{TST}} = \pi\mu_0 M_{\mathrm{S}} e^{\sigma}/(K\gamma)$ and $\sigma = vK/(kT)$ is the dimensionless barrier height parameter.

In the context of the MFPT approach, the magnetization reversal time τ^{VLD} of a uniaxial nanomagnet can be evaluated via an equation for the MFPT $\tau(\vartheta)$, viz. [13,35,38],

$$\mathrm{L}_{\mathrm{FP}}^{\dagger}\tau(\vartheta) = -1, \tag{11}$$

with the appropriate boundary condition. Here $\mathrm{L}_{\mathrm{FP}}^{\dagger}$ is the adjoint Fokker-Planck operator associated with Eq. (9). The MFPT is the average time needed to reach the barrier point C for the *first* time from a starting point ϑ inside the initial potential well (domain of attraction). In particular, for $V(\vartheta)$ from Eq. (8) with $h = 0$, i.e., when $V(\vartheta)$ is a symmetric bistable potential with minima at $\vartheta = 0$ and $\vartheta = \pi$ and a maximum at $\vartheta = \pi/2$, the MFPT τ^{VLD} from the minimum A at $\vartheta_A = \pi$ to reach the barrier point C at $\vartheta_C = \pi/2$ is given by the exact analytic equation [13,35]

$$\begin{aligned}\tau^{\mathrm{VLD}} &= \frac{v\mu_0 M_{\mathrm{S}}}{\alpha\gamma kT}\int_{\pi/2}^{\pi}\frac{e^{-\sigma\cos^2\vartheta}}{\sin\vartheta}\int_{\vartheta}^{\pi} e^{\sigma\cos^2\vartheta'}\sin\vartheta' d\vartheta' d\vartheta \\ &= \frac{v\mu_0 M_{\mathrm{S}}}{2\alpha\gamma kT}\sqrt{\frac{\pi}{\sigma}}\int_{-1}^{0}\frac{[\mathrm{erfi}(\sqrt{\sigma}) - \mathrm{erfi}(z\sqrt{\sigma})]e^{-\sigma z^2}}{1-z^2}dz,\end{aligned} \tag{12}$$

where

$$\mathrm{erfi}(x) = \frac{1}{\sqrt{\pi}} \int_0^x e^{t^2} dt$$

is the imaginary error function. For high barriers, $\sigma \gg 1$, the MFPT τ^{VLD} from Eq. (12) is closely approximated by Brown's asymptotic equation (10).

The calculation of the MFPT of nanomagnets in the VLD range via the Fokker-Planck equation (6) for *nonaxially symmetric free-energy potentials* $V(\vartheta,\varphi)$, even at zero STT, is very complicated due to the mathematical difficulties encountered, which arise because more than one space variable is involved. Nevertheless, the VLD magnetization reversal time τ^{VLD} was evaluated via the MFPT by Klik and Gunther [34] in the high-barrier limit, $\Delta E \gg 1$. They derived, via the MFPT originally developed for point particles by Matkowsky *et al.* [41], the magnetization reversal time from an individual well A over a saddle point C with a *single* escape path for *nonaxially symmetric free-energy densities* $V(\vartheta,\varphi)$, viz. [34],

$$\tau_{\mathrm{as}}^{\mathrm{VLD}} = \frac{\tau^{\mathrm{TST}}}{\alpha S_{E_C}}. \tag{13}$$

Here τ^{TST} is the TST reversal time given by Eq. (7) and S_{E_C} is the dimensionless action at the saddle-point energy E_C given by

$$S_{E_C} = \frac{v}{kT} \oint_{E=E_C} \left(\frac{1}{\sin\vartheta} \frac{\partial V}{\partial \varphi} d\vartheta - \sin\vartheta \frac{\partial V}{\partial \vartheta} d\varphi \right). \tag{14}$$

The contour integral in Eq. (14) is taken along the critical energy trajectory, or separatrix, on which the magnetization may reverse by passing through the saddle point C. The critical energy is the energy required by a spin to just escape the well and the separatrix delineates the bounded precessional motion in the well from that outside it. In the VLD regime, the system is only *very lightly* coupled to the bath so that the energy loss per cycle of the almost-periodic noisy motion of the magnetization on the saddle-point energy (escape) trajectory is much less than the thermal energy, $\alpha S_{E_C} \ll 1$, so that $\tau_{\mathrm{as}}^{\mathrm{VLD}} \gg \tau^{\mathrm{TST}}$ for VLD [39]. Everywhere the tacit assumption is made that the separatrix lies infinitesimally near to the *closed* noiseless and undamped trajectory with the energy E_C. We remark that like point Brownian particles, in the escape rate problem as it pertains to spins *three* regimes of damping appear [36–38], where each one arises as a direct consequence of the particular asymptotic method involved in the solution of the Fokker-Planck equation, namely, VLD $\alpha S_{E_C} \ll 1$, intermediate-to-high damping (IHD) $\alpha S_{E_C} \gg 1$, and a more or less critically damped turnover range $\alpha S_{E_C} \sim 1$. In each range, the damping dependence of the escape rates, reversal time, etc., differ substantially. Furthermore, the VLD reversal time for spins [Eq. (13), etc.] can be obtained [39] from the energy-controlled diffusion equation just for point particles. The interested reader can find a detailed discussion and appropriate formulas in Refs. [13,36–39].

The methods of evaluating the reversal time originally developed for zero STT can be generalized to take into account STT effect [21]. In the VLD limit, the calculations can be accomplished using the energy-controlled diffusion equation and the MFPT [7,39] as follows.

III. ENERGY-CONTROLLED DIFFUSION EQUATION IN THE VLD REGIME

The dynamics of the magnetization $\mathbf{M}$ in the presence of the STT may be very different from those with zero STT. In particular, due to the STT the Gilbert damping torque may be overcome so that the reversal of the magnetization becomes possible *in the absence* of thermal fluctuations. As far as the joint action of the STT and thermal fluctuations are concerned, the overall situation, albeit more complicated, is in some way reminiscent of that occurring in the resistively shunted junction (RSJ) model of a Josephson junction which is an electric analog of the motion of a Brownian particle in a tilted periodic potential [13,42,43]. Now just as the bias current in the junction, which constitutes a *nonconservative* electrical source giving rise to the motion in a tilted cosine periodic potential, ensures that the stationary distribution is no longer the Boltzmann distribution, in like manner the stationary distribution of magnetization orientations in a ferromagnet subjected to spin-polarized current is no longer a Boltzmann one. Rather it depends both on the spin-polarized current and damping analogous to the dependence of the stationary distribution in the RSJ model on the bias current (or tilt) parameter. Moreover, the damping and external current parameters now govern the reversal time so that the effect of the STT may be as much as several orders of magnitude [6].

As we have already mentioned in the Introduction, the stochastic spin dynamics in the VLD limit (which is the case of our interest) may be described via the energy-controlled diffusion equation for spins derived by Apalkov and Visscher [18] and others [6,7,19]. By analogy with Kramers' derivation [37] of the energy-controlled diffusion equation for point particles, one may parametrize the instantaneous magnetization direction $\mathbf{M} = \mathbf{M}(E,\phi)$ by the *slow* dimensionless energy variable $E = vV/(kT)$ and the *fast* phase variable ϕ [7]: In the VLD limit, the energy varies very slowly compared to ϕ. For the *slightly damped precession* in the presence of the random field $\mathbf{h}(t)$ with white noise properties given by Eq. (2), the Langevin equations for the *random variables* E and ϕ can be written as [7] (see Appendix A)

$$\frac{dE}{dt} = F_1 + (\mathbf{g}_1 \cdot \mathbf{h}), \tag{15}$$

$$\frac{d\phi}{dt} = F_2 + (\mathbf{g}_2 \cdot \mathbf{h}), \tag{16}$$

where

$$F_1(E,\phi) = -\frac{v\alpha\mu_0}{\gamma kT M_{\mathrm{S}}} |\dot{\mathbf{M}}_{\mathrm{pr}}|^2 + \frac{v\mu_0}{kT M_{\mathrm{S}}} (\mathbf{I}_{\mathrm{S}} \cdot [\mathbf{M} \times \dot{\mathbf{M}}_{\mathrm{pr}}]), \tag{17}$$

$$F_2(E,\phi) = 2\pi f_E \left[1 - \frac{\gamma M_{\mathrm{S}}}{|\dot{\mathbf{M}}_{\mathrm{pr}}|^2} (\mathbf{I}_{\mathrm{S}} \cdot \dot{\mathbf{M}}_{\mathrm{pr}}) \right], \tag{18}$$

$$\mathbf{g}_1(E,\phi) = \frac{2\pi v\mu_0 f_E}{kT} \frac{\partial \mathbf{M}}{\partial \phi}, \tag{19}$$

$$\mathbf{g}_2(E,\phi) = -\frac{2\pi v\mu_0 f_E}{kT} \frac{\partial \mathbf{M}}{\partial E}. \tag{20}$$

Here f_E is the frequency of the precession in the potential well at a given energy E, which can be calculated explicitly via the corresponding precession period $P_E = 1/f_E$ by taking a closed line integral along a Stoner-Wohlfarth orbit of constant energy E, viz. [7],

$$P_E = \gamma^{-1} \oint_E \frac{([\mathbf{H}\times\mathbf{M}]\cdot d\mathbf{M})}{|[\mathbf{H}\times\mathbf{M}]|^2}.$$

Furthermore, $\dot{\mathbf{M}}_{\rm pr}$ in Eqs. (15) and (16) must be understood in the purely conservative sense as

$$\dot{\mathbf{M}}_{\rm pr} = \gamma[\mathbf{H}\times\mathbf{M}]. \tag{21}$$

Equations (15) and (16), which are the Langevin equations with multiplicative noise interpreted in the Stratonovich sense [13,43], describe the precession of the magnetization subject to *weak* damping and STT torques and weak thermal fluctuations.

Apalkov and Visscher [18] and others [6,7,19,39] were then able to derive, via the Langevin equations (15) and (16), the Fokker-Planck equation for the probability density function $W(E,\phi,t)$ in energy-phase space [see Eq. (A17) in Appendix A]. Since in the VLD regime, the energy E diffuses very slowly over time, i.e., is almost constant, while in contrast, the phase ϕ varies rapidly, the ϕ dependence may be eliminated. This is accomplished by averaging $W(E,\phi,t)$ along a *closed* trajectory of the energy ultimately yielding the energy-controlled diffusion equation for the probability density function $W(E,t)$ in energy space, viz. [7,18] (see Appendix A for detail),

$$\frac{\partial W}{\partial t} = \alpha\frac{\partial}{\partial E}\left[S_E\left(1+\frac{\partial}{\partial E}\right)f_E W\right] - \frac{\partial}{\partial E}(V_E f_E W), \tag{22}$$

where the dimensionless action S_E and the dimensionless work V_E done by the STT are given by

$$S_E = \frac{v\mu_0}{\gamma kTM_{\rm S}}\int_0^{1/f_E} |\dot{\mathbf{M}}_{\rm pr}|^2 dt, \tag{23}$$

$$V_E = \frac{J}{M_{\rm S}^2}\int_0^{1/f_E} (\mathbf{e}_p\cdot[\mathbf{M}\times\dot{\mathbf{M}}_{\rm pr}])dt. \tag{24}$$

In the derivation of Eq. (22), it has been also assumed that the damping parameter α is independent of $\mathbf{M}$. However, the results may be also generalized to magnetization-dependent damping $\alpha(\mathbf{M})$ [7]. We remark that S_E and V_E given by Eqs. (23) and (24) are closely related to the corresponding terms in Langevin equation (15) for the energy variable, namely,

$$-\frac{v\alpha\mu_0}{\gamma kTM_{\rm S}}|\dot{\mathbf{M}}_{\rm pr}|^2 \quad\text{and}\quad \frac{v\mu_0}{kTM_{\rm S}}(\mathbf{I}_{\rm S}\cdot[\mathbf{M}\times\dot{\mathbf{M}}_{\rm pr}]);$$

these terms take into account, respectively, the energy dissipation due to the damping and the pumping of the energy into the system due to the spin-polarized current and determine the magnetization reversal process and Stoner-Wohlfarth orbits for the steady-state precession of the magnetization [7]. We also remark that the energy-controlled diffusion equation (22) for spins is very similar but not identical to that for point Brownian particles in a potential $V(x)$, viz. [13,37,38],

$$\frac{\partial W}{\partial t} = \alpha\frac{\partial}{\partial E}\left[S_E\left(1+\frac{\partial}{\partial E}\right)f_E W\right]. \tag{25}$$

The differences lie in the presence of the STT term and in the definitions of the damping coefficient α and of the action S_E. For point particles, they are $\alpha = \zeta/m$ and

$$S_E = \frac{\sqrt{2m}}{kT}\oint_E \sqrt{kTE - V(x)}dx = \frac{m}{kT}\int_0^{1/f_E}|\dot{x}|^2 dt.$$

Here $E = [m\dot{x}^2/2 + V(x)]/(kT)$ is the normalized energy; x and m define the position and mass of a particle, respectively; ζ is the drag coefficient; and $f_E = \partial E/\partial S_E$ is the librational frequency in the potential well [13]. For zero STT when $V_E = 0$, the energy-controlled diffusion equation (22) for spins has the same form as Eq. (25) for particles.

IV. REVERSAL TIME IN THE VLD LIMIT

In order to evaluate the magnetization reversal time from Eq. (22) via the MFPT, we consider an assembly of classical spins in a potential well with a minimum at point A. In the true VLD case, $\alpha S_E \ll 1$, where the energy loss per cycle of a precessing spin is very much less than the thermal energy, the energy trajectories diffuse very slowly so that they do not differ significantly from those of the undamped precessional motion in a well. Then because of thermal fluctuations, on a noisy trajectory near the saddle energy the spin may have enough energy to escape over the potential barrier at the saddle point C. The energy-controlled diffusion equation for spins, Eq. (22), represents the continuity equation

$$\frac{\partial W}{\partial t} + \frac{\partial J_P}{\partial E} = 0, \tag{26}$$

where J_P is the probability current. Now, like the Kramers calculation [37] for particles (see also [38], Sec. II D, and [39]), we consider the quasistationary solution $W_0(E)$ of Eq. (26). Here with $\dot{W}_0 = 0$ and $J_P(E) = J_P = \text{const}$ representing a steady injected current of spins to replenish those continually being lost at a saddle point C, we then find that the quasistationary distribution $W_0(E)$ satisfies the first-order linear differential equation, viz.,

$$J_P = -\alpha S_E\left(1 - \frac{V_E}{\alpha S_E} + \frac{\partial}{\partial E}\right)f_E W_0. \tag{27}$$

Next, considering the behavior of $W_0(E)$ at E_C and assuming that $W_0(E_C) = 0$, i.e., all spins that reach the barrier go over, we have the particular solution of Eq. (27) as

$$W_0(E) = \frac{J_P}{\alpha f_E}\int_E^{E_C}\frac{1}{S_{E'}}e^{E'-E-\int_E^{E'}\frac{V_{E''}}{\alpha S_{E''}}dE''}dE'. \tag{28}$$

In order to find the population N in the well A, we integrate the quasistationary distribution $W_0(E)$ over the domain of the well energy so that

$$\begin{aligned} N &= \int_{E_A}^{E_C} W_0(E)dE \\ &= \frac{J_P}{\alpha}\int_{E_A}^{E_C}\frac{e^{-E}}{f_E}\int_E^{E_C}\frac{1}{S_{E'}}e^{E'-\int_E^{E'}\frac{V_{E''}}{\alpha S_{E''}}dE''}dE'dE. \end{aligned} \tag{29}$$

We then have via the flux-over-population method [38,39] the characteristic MFPT $\tau^{\rm VLD} = N/J_P$ from a potential well

with minimum energy E_A over the saddle point C, namely,

$$\tau^{\mathrm{VLD}} = \frac{1}{\alpha}\int_{E_A}^{E_C}\frac{e^{-E}}{f_E}\int_{E}^{E_C}\frac{1}{S_{E'}}e^{E'-\int_E^{E'}\frac{V_{E''}}{\alpha S_{E''}}dE''}dE'dE \tag{30}$$

or after integrating by parts,

$$\tau^{\mathrm{VLD}} = \frac{1}{\alpha}\int_{E_A}^{E_C}\frac{1}{S_E}e^{E-\frac{1}{\alpha}\int_{E_A}^{E}\frac{V_{E''}}{S_{E''}}dE''} \times \int_{E_A}^{E}\frac{1}{f_{E'}}e^{-E'+\frac{1}{\alpha}\int_{E_A}^{E'}\frac{V_{E''}}{S_{E''}}dE''}dE'dE. \tag{31}$$

τ^{VLD} is the time to reach a separatrix from the point A *provided that all spins there are absorbed*, which is the boundary condition that W vanishes at $E = E_C$. The inverse of τ^{VLD} also determines the escape rate from the well. We emphasize that for the calculation of τ^{VLD} from Eq. (30) only a knowledge of the *deterministic dynamics* is required; i.e., S_E, V_E, and f_E in Eq. (30) are always calculated via the deterministic Larmor equation (21). For zero STT, $J = 0$, Eq. (30) can be reduced to Eq. (4.48a) derived by Hänggi *et al.* [38] (noting that $dE = f_E dS_E$) while Eq. (31) reduces to that given in Ref. [39].

The quadrature solution, Eq. (31), is valid for *all barrier heights* including low barriers. However, in the high-barrier limit, Eq. (31) can be considerably simplified. Indeed, the main contribution to the inner integral of Eq. (31) comes from near the bottom of the well because the negative exponential dominates the integral in that region. Furthermore, the precession frequency now satisfies $f_{E_A} \approx f_A$, where f_A is the well precession frequency, which is independent of E in the paraboloid approximation for the potential near the bottom of the well. Thus

$$\int_{E_A}^{E}\frac{1}{f_{E'}}e^{-E'+\frac{1}{\alpha}\int_{E_A}^{E'}\frac{V_{E''}}{S_{E''}}dE''}dE' \simeq \frac{1}{f_A}\int_{E_A}^{\infty}e^{-E}dE = \frac{e^{-E_A}}{f_A}. \tag{32}$$

In contrast, the main contribution to the outer integral of Eq. (31) comes from the positive exponential factor dominating the integrand near the saddle point C of the potential. Therefore, noting Eq. (23) and using the approximation $S_{E'} \approx S_{E_C}$, where

$$S_{E_C} = \frac{v\mu_0}{\gamma kTM_{\mathrm{S}}}\int_0^{1/f_{E_C}}|\dot{\mathbf{M}}_{\mathrm{pr}}|^2_{E=E_C}dt = \frac{v}{kTM_{\mathrm{S}}^2}\oint_{E=E_C}([\mathbf{M}\times\boldsymbol{\nabla}V]\cdot d\mathbf{M}), \tag{33}$$

we have

$$\int_{E_A}^{E_C}\frac{1}{S_E}e^{E-\frac{1}{\alpha}\int_{E_A}^{E}\frac{V_{E''}}{S_{E''}}dE''}dE \approx \frac{e^{-\frac{1}{\alpha}\int_{E_A}^{E_C}\frac{V_{E''}}{S_{E''}}dE''}}{S_{E_C}}\int_{-\infty}^{E_C}e^{E'}dE' = \frac{1}{S_{E_C}}e^{E_C-\frac{1}{\alpha}\int_{E_A}^{E_C}\frac{V_E}{S_E}dE}. \tag{34}$$

Here the contour integral in Eq. (33) is taken along the critical energy trajectory on which the magnetization may reverse by passing through the saddle point C. Using Eqs. (32) and (34) in Eq. (31) yields Eq. (13), namely,

$$\tau_{\mathrm{as}}^{\mathrm{VLD}} \sim \frac{e^{\Delta E}}{\alpha f_A S_{E_C}}, \tag{35}$$

where the effective barrier ΔE, which now depends on both the spin-polarized current and damping coefficient, is given by

$$\Delta E = E_C - E_A - \frac{1}{\alpha}\int_{E_A}^{E_C}\frac{V_E}{S_E}dE. \tag{36}$$

In spherical polar coordinates $(\mathbf{e}_r,\mathbf{e}_\vartheta,\mathbf{e}_\varphi)$ [44], where $\mathbf{u} = \mathbf{M}/M_{\mathrm{S}} = \mathbf{e}_r$, $d\mathbf{u} = \mathbf{e}_\vartheta d\vartheta + \mathbf{e}_\varphi \sin\vartheta d\varphi$, and $\boldsymbol{\nabla}V = \mathbf{e}_\vartheta\partial_\vartheta V + \mathbf{e}_\varphi \csc\vartheta\partial_\varphi V$, S_{E_C} from Eq. (33) becomes Eq. (14). Hence, for zero STT, Eq. (35) predicts in the low-temperature limit exactly the same reversal time as Eq. (13) of Klik and Gunther [34]. In order to evaluate τ^{VLD} from Eq. (35), we require only explicit equations for E_A, E_C, f_A, S_E, and V_E. The method of calculation of the precession frequency f_A and the well and saddle energies E_A and E_C is described in Refs. [13,33,36], while S_E and V_E can be calculated analytically or numerically by solving Eq. (21) as in the zero STT case [39] (see Appendix B).

V. VLD REVERSAL TIME FOR SUPERIMPOSED EASY-PLANE AND IN-PLANE EASY-AXIS ANISOTROPIES

By way of practical illustration of Eq. (31), here we calculate the reversal time of the magnetization for the free-energy density taken in the standard form of superimposed easy-plane and in-plane easy-axis anisotropies with the dc bias field $\mathbf{H}_0$ along the easy X axis, viz. [6,40],

$$E = \sigma\left(\delta u_Z^2 - u_X^2 - 2hu_X\right) = \sigma(\delta\cos^2\vartheta - \sin^2\vartheta\cos^2\varphi - 2h\sin\vartheta\cos\varphi). \tag{37}$$

Here $\sigma = v\mu_0 M_{\mathrm{S}}^2 D_\parallel/(kT)$ is the dimensionless anisotropy parameter, which can also be regarded as an inverse temperature parameter; $h = H_0/(2M_{\mathrm{S}}D_\parallel)$ is the external field parameter; $\delta = D_\perp/D_\parallel$ is the dimensionless biaxiality parameter; while $D_\parallel$ and $D_\perp$ account for both demagnetizing and magnetocrystalline effects. We shall also suppose for simplicity that the unit vector $\mathbf{e}_P$, which defines the orientation of the magnetization of the *fixed* layer (see Fig. 1), is applied along the easy (X) axis, so that Eq. (5) for the nonconservative potential Φ becomes

$$\Phi = (JkT/v)u_X = (JkT/v)\sin\vartheta\cos\varphi.$$

In general, the normalized free-energy density $E(\vartheta,\varphi)$ from Eq. (37) has two nonequivalent *wells* and two *equivalent saddle points* (see Fig. 2).

For the biaxial anisotropy potential, Eq. (37), the gyromagnetic equation (21) can be written in terms of the Cartesian components (u_X, u_Y, u_Z) of the unit vector $\mathbf{u} = \mathbf{M}/M_{\mathrm{S}}$ as

$$\tau_0\dot{u}_X(t) = \delta u_Z(t)u_Y(t), \tag{38}$$

$$\tau_0\dot{u}_Y(t) = -[(1+\delta)u_X(t) + h]u_Z(t), \tag{39}$$

$$\tau_0\dot{u}_Z(t) = [u_X(t) + h]u_Y(t), \tag{40}$$

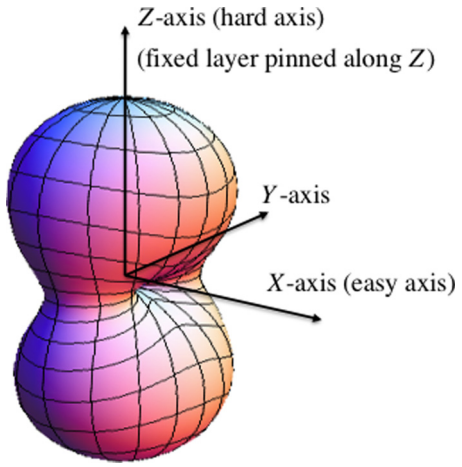

FIG. 2. Free-energy potential V of the free layer presented in the standard form of superimposed easy-plane and in-plane easy-axis anisotropies, Eq. (37).

where a characteristic time τ_0 is given by

$$\tau_0 = \frac{1}{2\gamma M_S D_\parallel} = \frac{\tau_N}{\sigma(\alpha + \alpha^{-1})}.$$

The solutions of Eqs. (38)–(40) are subject to the obvious constraints of

$$u_X^2 + u_Y^2 + u_Z^2 = 1, \tag{41}$$

and energy conservation

$$\varepsilon = \delta u_Z^2 - u_X^2 - 2hu_X, \tag{42}$$

where $\varepsilon = E/\sigma$ is the normalized free energy. Equations (38)–(42) can then be used to calculate S_ε, V_ε, and f_ε as explained in Appendix B.

If $|h| < 1$, the potential from Eq. (37) has two *nonequivalent* wells with minima $\varepsilon_A = -1 \pm 2h$ at $u_X = \mp 1$ and two equivalent saddle points $\varepsilon_C = h^2$ at $u_X = -h$ (see Fig. 2). Thus, we must define two individual MFPTs, namely, τ_+ from the deeper well around the energy minimum at $u_X = +1$ ($-1 - 2h \leqslant \varepsilon \leqslant h^2$) and τ_- from the shallow well around the energy minimum at $u_X = -1$ ($-1 + 2h \leqslant \varepsilon \leqslant h^2$). These times are unequal in general, i.e., $\tau_+ \neq \tau_-$, so that Eq. (31), as specialized to the potential Eq. (37), becomes

$$\tau_\pm = \frac{\sigma^2}{\alpha} \int_{-1\mp 2h}^{h^2} \frac{1}{S_\varepsilon^\pm} e^{\sigma\varepsilon - \frac{\sigma}{\alpha}\int_{-1\mp 2h}^{\varepsilon} \frac{V_{\varepsilon'}^\pm}{S_{\varepsilon'}^\pm} d\varepsilon'} \times \int_{-1\mp 2h}^{\varepsilon} \frac{1}{f_{\varepsilon'}^\pm} e^{-\sigma\varepsilon' + \frac{\sigma}{\alpha}\int_{-1\mp 2h}^{\varepsilon'} \frac{V_{\varepsilon''}^\pm}{S_{\varepsilon''}^\pm} d\varepsilon''} d\varepsilon' d\varepsilon. \tag{43}$$

Here the energy-dependent functions f_ε^+, S_ε^+, and V_ε^+ are given, respectively, by Eqs. (B11), (B10), and (B14) from Appendix B. For the MFPT τ_- from the well around the energy minimum at $u_X = -1$, S_ε^-, V_ε^-, and f_ε^- can be obtained simply by replacing h by $-h$ and J by $-J$ in all the equations for S_ε^+, V_ε^+, and f_ε^+. Thus, we have $\tau_- = \tau_+(-h, -J)$. Now recalling that τ_+ and τ_- are related to the corresponding escape rates from the individual wells via $\Gamma_+ = (2\tau_+)^{-1}$ and $\Gamma_- = (2\tau_-)^{-1}$ [38] so that the VLD reversal time is given by $\tau^{\rm VLD} = (\Gamma_+ + \Gamma_-)^{-1}$, we have

$$\tau^{\rm VLD} = \frac{2\tau_+\tau_-}{\tau_+ + \tau_-}. \tag{44}$$

In the high-barrier limit, we have from Eqs. (35), (B11), (B10), and (B14) a simple asymptotic equation,

$$\tau_\pm \sim \frac{1}{\alpha f_A^\pm S_{\varepsilon_C}^\pm} e^{\sigma(1\pm h)^2 - \frac{\sigma}{\alpha}\int_{\varepsilon_A^\pm}^{\varepsilon_C} \frac{V_{\varepsilon'}^\pm}{S_{\varepsilon'}^\pm} d\varepsilon'}, \tag{45}$$

where

$$f_A^\pm = \frac{1}{2\pi\tau_0}\sqrt{(1 \pm h + \delta)(1 \pm h)}, \tag{46}$$

$$S_{\varepsilon_C}^\pm = 8\delta\sigma\left(1 - \frac{h^2}{1+\delta}\right)\left\{\sqrt{\frac{1-h^2}{\delta}} + \frac{h}{\sqrt{1+\delta}} \arctan\left[\frac{h}{\sqrt{(1-h^2)(1+\delta^{-1})}}\right] \pm \frac{h\pi}{2}\right\}. \tag{47}$$

In the absence of a dc bias field, $h \to 0$, where

$$S_\varepsilon^\pm|_{h\to 0} = 8\sigma\sqrt{\delta - \varepsilon}[\varepsilon K(m_\varepsilon) + E(m_\varepsilon)],$$
$$V_\varepsilon^\pm|_{h\to 0} = \pm 2\pi J \frac{1+\varepsilon}{\sqrt{1+\delta}},$$
$$f_\varepsilon^\pm|_{h\to 0} = \frac{\sqrt{\delta - \varepsilon}}{4\tau_0 K(m_\varepsilon)},$$
$$m_\varepsilon = \frac{\delta(1+\varepsilon)}{\delta - \varepsilon}.$$

Equation (43) coincides with corresponding equations of Taniguchi *et al.* [21a,b]. Here $K(m)$ and $E(m)$ are the complete elliptic integrals of the first and second kind, respectively [45].

For zero STT, i.e., $J = 0$, and nonzero dc bias field, the high-barrier asymptotic equation (45) yields the known results [32]

$$\tau_\pm^{as} \sim \frac{e^{\sigma(1\pm h)^2}}{\alpha f_A^\pm S_{\varepsilon_C}^\pm}, \quad \sigma(1-h)^2 \gg 1. \tag{48}$$

Furthermore, for zero STT ($J = 0$) and zero dc bias field ($h = 0$), the free energy Eq. (37) is a double-well potential with two *equivalent* wells. Therefore, $\tau_+ = \tau_- = \tau$, so that the overall reversal time is then $\tau^{\rm VLD} = \tau$, which can be written as the analytic equation

$$\tau^{\rm VLD} = \frac{\tau_0\sigma}{2\alpha}\int_{-1}^{0} \frac{e^{\sigma\varepsilon}\int_{-1}^{\varepsilon} \frac{K(m_{\varepsilon'})e^{-\sigma\varepsilon'}}{\sqrt{\delta - \varepsilon'}} d\varepsilon'}{\sqrt{\delta - \varepsilon}[E(m_\varepsilon) + \varepsilon K(m_\varepsilon)]} d\varepsilon. \tag{49}$$

For $\delta = 0$ (uniaxial anisotropy), we have from Eq. (49),

$$\tau^{\rm VLD} = \frac{\tau_0\sigma}{2\alpha}\int_{-1}^{0} \frac{e^{\sigma\varepsilon}}{\sqrt{-\varepsilon}(1+\varepsilon)} \int_{-1}^{\varepsilon} \frac{e^{-\sigma\varepsilon'} d\varepsilon'}{\sqrt{-\varepsilon'}} d\varepsilon = \tau_0 \frac{\sqrt{\pi\sigma}}{\alpha}\int_{-1}^{0} \frac{[\mathrm{erfi}(\sqrt{\sigma}) - \mathrm{erfi}(z\sqrt{\sigma})]e^{-\sigma z^2}}{1 - z^2} dz. \tag{50}$$

Equation (50) agrees in all respects with the MFPT τ^{VLD} from Eq. (12). For high barriers, $\sigma \gg 1$, τ^{VLD} from Eqs. (49) and (50) are closely approximated by the known formula [32,36]

$$\tau_{\mathrm{as}}^{\mathrm{VLD}} \sim \frac{\tau_0 \pi e^{\sigma}}{4\alpha\sigma\sqrt{\delta(1+\delta)}} \quad (\delta\sigma > 0.5), \tag{51}$$

and Brown's asymptotic equation (10), respectively.

VI. MATRIX CONTINUED FRACTION SOLUTION

Alternatively, the magnetization reversal time can be calculated numerically by using the matrix continued fraction approach [13,46] developed for nonzero STT in Ref. [30a]. The solution of the Langevin equation (1) can be reduced to the solution of an infinite hierarchy of differential-recurrence equations for the after effect functions of the statistical moments (averaged spherical harmonics $Y_{l,m}(\vartheta,\varphi)$ [44]) $c_{l,m}(t) = \langle Y_{l,m}\rangle(t) - \langle Y_{l,m}\rangle_0$ (where $\langle Y_{l,m}\rangle_0$ are the equilibrium averages of the statistical moments) governing the magnetization relaxation of a nanomagnet [30a],

$$\frac{d}{dt}c_{l,m}(t) = \sum_{l',m'} e_{l',m',l,m}c_{l,m}(t), \tag{52}$$

where $e_{l',m',l,m}$ are time-independent coefficients; the explicit formulas for $e_{l',m',l,m}$ for the problem at hand are given in Appendix C. A method of derivation of the moment system Eq. (52) for arbitrary anisotropy and nonzero STT is given in Ref. [30a]. We remark that Eq. (52) can also be derived from the Fokker-Planck equation (6) [30a].

The differential-recurrence Eq. (52) for the statistical moments can always be transformed into the tridiagonal vector differential-recurrence equation [13,43]

$$\dot{\mathbf{C}}_n(t) = \mathbf{Q}_n^{-}\mathbf{C}_{n-1}(t) + \mathbf{Q}_n\mathbf{C}_n(t) + \mathbf{Q}_n^{+}\mathbf{C}_{n+1}(t), \tag{53}$$

where $\mathbf{C}_n(t)$ are the column vectors arranged in an appropriate manner from $c_{l,m}(t)$, and $\mathbf{Q}_n^{\pm},\mathbf{Q}_n$ are matrices with elements defined via $e_{l',m',l,m}$ (these column vectors and matrices are given in Appendix C). The general method of evaluating the smallest nonvanishing eigenvalue λ_1 of the Fokker-Planck operator from the tridiagonal vector differential-recurrence equation (53) is described in Chap. 10 of Ref. [43]. Furthermore, as shown in Ref. [46] (see also [13], Chap. 2, Sec. 2.11.2), one can calculate λ_1 from the secular or characteristic equation

$$\det(\lambda\mathbf{I} - \mathbf{S}) = 0, \tag{54}$$

where $\mathbf{I}$ is the unit matrix and the matrix $\mathbf{S}$ is defined in terms of the matrices $\mathbf{Q}_n^{\pm},\mathbf{Q}_n$ as

$$\mathbf{S} = -[\mathbf{Q}_1 + \mathbf{Q}_1^{+}\,\boldsymbol{\Delta}_2(0)\mathbf{Q}_2^{-}][\mathbf{I} - \mathbf{Q}_1^{+}\,\boldsymbol{\Delta}'_2(0)\mathbf{Q}_2^{-}]^{-1}. \tag{55}$$

Here $\boldsymbol{\Delta}_n(s)$ is the matrix continued fraction defined by the recurrence equation

$$\boldsymbol{\Delta}_n(s) = [s\mathbf{I} - \mathbf{Q}_n - \mathbf{Q}_n^{+}\,\boldsymbol{\Delta}_{n+1}(s)\mathbf{Q}_{n+1}^{-}]^{-1},$$

and the prime designates the derivative of $\boldsymbol{\Delta}_2(s)$ with respect to s. Thus λ_1 is the smallest nonvanishing eigenvalue of $\mathbf{S}$ yielding the reversal time of the magnetization $\tau = 1/\lambda_1$. Matrix continued fractions provide an easy method of computation of the reversal time via λ_1 due to the relatively small dimension of all the matrices involved in Eq. (55). Here, the primary purpose of the matrix continued fraction method is to determine the accuracy of the MFPT equations.

VII. COMPARISON OF ANALYTICAL AND NUMERICAL RESULTS

We can now compare the reversal time τ^{VLD} from the exact integral solutions of Eqs. (43) and (44) both with the asymptotic VLD escape rate $\tau_{\mathrm{as}}^{\mathrm{VLD}}$, Eqs. (45)–(47), and with the inverse of the smallest nonvanishing eigenvalue λ_1 calculated numerically by the matrix continued fraction method. All the calculations have been done for damping parameters corresponding to the true VLD limit, $\alpha S_{E_C} \ll 1$ for all values of the barriers. The spin-polarization factor P selected is $P = 0.3$ yielding $b_P = 4P^{3/2}/[3(1+P)^3 - 16P^{3/2}] = 0.166$ ($P \approx 0.3 - 0.4$ are typical values for ferromagnetic metals [6]). Furthermore, for $D_{\parallel} = 0.034$, $\gamma = 2.2 \times 10^5\ \mathrm{m\,A^{-1}\,s^{-1}}$, $M_{\mathrm{S}} \approx 1.4 \times 10^6\ \mathrm{A\,m^{-1}}$ (cobalt), we have $\tau_0 \approx 4.8 \times 10^{-11}\ \mathrm{s}$.

The magnetization reversal time is shown in Fig. 3 as a function of the anisotropy (inverse temperature) parameter σ for various spin-polarized current parameters J, damping parameters α, and dc bias field parameters h and typical values of the other model parameters. Apparently, τ^{VLD} and ${\lambda_1}^{-1}$ lie very close to each other for virtually all σ. Furthermore, in the high-barrier limit (large σ), $\tau_{\mathrm{as}}^{\mathrm{VLD}}$ from the asymptotic Eqs. (45)–(47) provides an accurate approximation to both ${\lambda_1}^{-1}$ and τ^{VLD}. However, for $\sigma < 5$, $\tau_{\mathrm{as}}^{\mathrm{VLD}}$ deviates considerably from both of these so that it cannot be used to calculate the reversal time. Clearly, the STT effects are governed by the ratio J/α so that by altering J/α the ensuing variation of τ may be as much as several orders of magnitude [Fig. 3(a)]. Furthermore, τ may greatly exceed or, on the other hand, be much less than the value pertaining to zero STT $J = 0$. Moreover, the increase or decrease in τ is entirely governed by the *direction* of the current, i.e., by the sign of J as indeed expected. The temperature dependence of τ can be understood via the *effective potential barriers* $\Delta E^{\pm}$ in Eq. (45), namely,

$$\Delta E^{\pm} = \sigma(1 \pm h)^2 - \frac{\sigma}{\alpha}\int_{\varepsilon_A^{\pm}}^{\varepsilon_C} \frac{V_{\varepsilon'}^{\pm}}{S_{\varepsilon'}^{\pm}}d\varepsilon'. \tag{56}$$

Evidently, for large anisotropy parameters, $\sigma > 5$, the *temperature dependence* of τ has the customary Arrhenius behavior $\tau \sim e^{\Delta E^{\pm}}$, i.e., exponential increase with decreasing temperature. Here τ is markedly dependent on the ratio J/α and the dc bias field parameter h because the barrier height of the shallow well is strongly influenced by both J/α and h.

VIII. CONCLUSION

We have derived analytic formulas for the magnetization reversal time of nanomagnets driven by spin-polarized currents in the VLD range. Our principal result is the general equation (31) yielding the reversal time via quadratures, which can, in principle, be evaluated for *any* anisotropy potential. Yet another merit of Eq. (31) is that it is valid in parameter ranges where escape rate equations such as Eqs. (45)–(47)

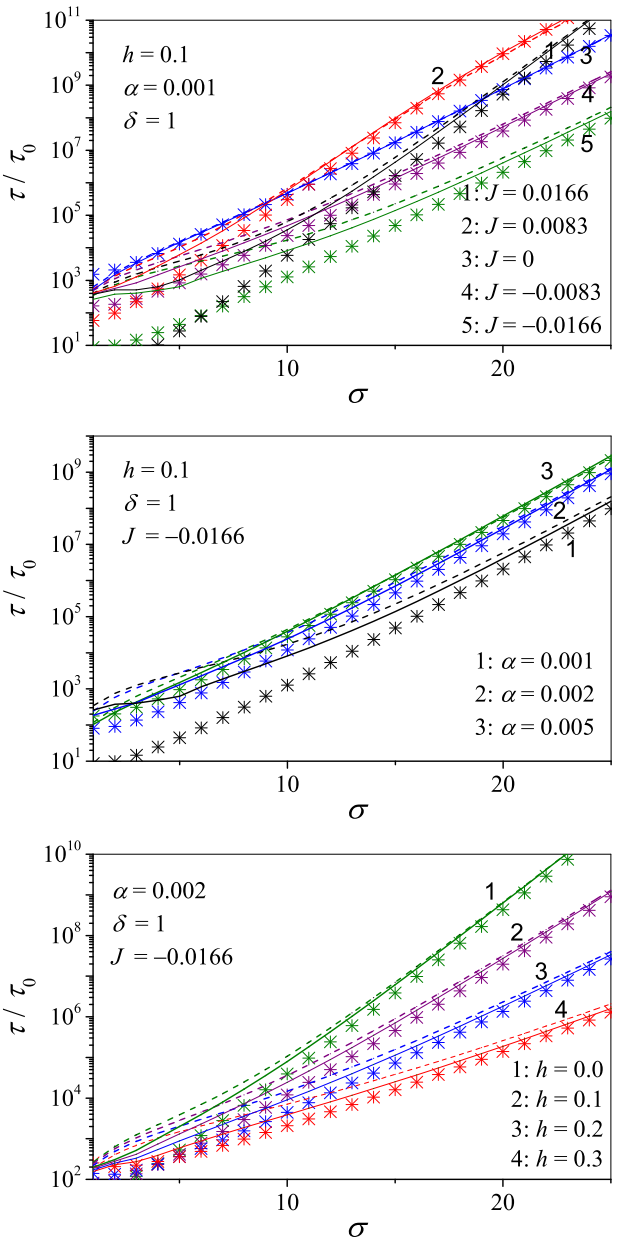

FIG. 3. Normalized times τ/τ_0, τ_{as}/τ_0, and $1/(\tau_0\lambda_1)$ versus the anisotropy (inverse temperature) parameter σ for various spin-polarized currents J (a), damping coefficients α (b), and dc field parameters h (c). Solid lines: numerical solution for the inverse of the smallest nonvanishing eigenvalue $1/(\tau_0\lambda_1)$ of the Fokker-Planck operator. Dashed lines: τ/τ_0 from the MFPT Eqs. (43) and (44). Asterisks: the VLD asymptotic, Eqs. (45)–(47).

do not apply at all, e.g., for *low barriers*. Equation (31) is also valuable as a benchmark solution with which numerical calculations of the reversal time from the magnetic Langevin and/or Fokker-Planck equation in the VLD limit must agree. As already mentioned, the range of the validity of the results obtained is defined by the inequality, $\alpha S_{E_C} \ll 1$. However, the asymptotic formula Eq. (35) for the VLD reversal time may be generalized to a wider damping range as is so for nanomagnets for zero STT. Indeed, for values of damping up to intermediate values, $\alpha \leqslant 1$, Coffey *et al.* [47] have shown that the Mel'nikov Meshkov formalism [48] for bridging the VLD and TST escape rates as a function of the dissipation parameter for mechanical particles can be extended to estimate the relaxation time of the magnetization of nanomagnets in the absence of STT. According to Coffey *et al.* [47], an asymptotic equation for bridging the magnetization reversal time from a single well for $\alpha \leqslant 1$ and $\Delta E \gg 1$ is given by

$$\tau = \frac{\tau^{\mathrm{TST}}}{A(\alpha S_{E_C})}, \tag{57}$$

where $\tau^{\mathrm{TST}} = f_A^{-1} e^{\Delta E}$ is the TST reversal time, S_{E_C} is the dimensionless action variable defined by Eq. (14), and $A(\alpha S)$ is the so-called depopulation factor given by [48]

$$A(\alpha S) = e^{\frac{1}{\pi}\int_0^\infty \frac{\ln\{1-\exp[-\alpha S(\lambda^2+1/4)]\}}{\lambda^2+1/4} d\lambda}. \tag{58}$$

Because $A(\alpha S) \to 1$ as $\alpha S \gg 1$ and $A(\alpha S)/\alpha \to S$ as $\alpha \to 0$ [48], Eq. (57) transparently reduces to Eqs. (7) and (35) in the TST and VLD limits, respectively. Furthermore, for bistable potentials with nonequivalent wells (as treated here), τ can be evaluated as [13,36]

$$\tau \approx \frac{\tau_+^{\mathrm{TST}}\tau_-^{\mathrm{TST}} A\left(\alpha S_{E_C}^+ + \alpha S_{E_C}^-\right)}{(\tau_+^{\mathrm{TST}} + \tau_-^{\mathrm{TST}}) A\left(\alpha S_{E_C}^+\right) A\left(\alpha S_{E_C}^-\right)}. \tag{59}$$

Equations (57) and (59) are also valid for nonzero STT. In particular, using the formal definition of the TST reversal times for two wells, viz.,

$$\tau_\pm^{\mathrm{TST}} = \frac{1}{f_A^\pm} e^{\sigma(1\pm h)^2 - \frac{\sigma}{\alpha}\int_{\varepsilon_A^\pm}^{\varepsilon_C} \frac{V_{\varepsilon'}^\pm}{S_{\varepsilon'}^\pm} d\varepsilon'}, \tag{60}$$

and Eqs. (46) and (47), Eq. (57) yields STT and dc bias field effects on the magnetization reversal time for the anisotropy potential equation (37) for $\alpha \leqslant 1$ and $\Delta E \gg 1$.

ACKNOWLEDGMENTS

We thank P.-M. Déjardin, D. Jordan, and J.-E. Wegrowe for helpful conversations. W.T.C. also acknowledges the French Embassy in Ireland for enabling a research visit to Perpignan. D.J.B. acknowledges the SimSci Structured Ph.D. Program at University College Dublin for financial support. This research is supported by the Programme for Research in Third Level Institutions (PRTLI) Cycle 5 and is cofunded by the European Regional Development Fund.

APPENDIX A: DERIVATION OF THE ENERGY-CONTROLLED EQUATION (22)

In order to transform the magnetic Langevin equation (1) to the Langevin equations (15) and (16) for the energy E and phase ϕ variables, we first notice that the parametrization $\mathbf{M} = \mathbf{M}(E,\phi)$ is specified by the following relations [7,39]:

$$\frac{\partial \mathbf{M}}{\partial \phi} = \frac{\gamma}{\Omega_E}[\mathbf{H} \times \mathbf{M}], \tag{A1}$$

$$\frac{v\mu_0}{kT}\frac{\partial \mathbf{M}}{\partial E} = -\frac{[\mathbf{M} \times [\mathbf{H} \times \mathbf{M}]]}{|[\mathbf{H} \times \mathbf{M}]|^2}, \tag{A2}$$

and the equation of motion for $E(t)$ and $\phi(t)$ are given by

$$\frac{dE}{dt} = \left(\frac{\partial E}{\partial \mathbf{M}} \cdot \dot{\mathbf{M}}\right) = -\frac{v\mu_0}{kT}(\mathbf{H} \cdot \dot{\mathbf{M}}), \tag{A3}$$

$$\frac{d\phi}{dt} = \frac{\Omega_E}{\gamma}\frac{([\mathbf{H} \times \mathbf{M}] \cdot \dot{\mathbf{M}})}{|[\mathbf{H} \times \mathbf{M}]|^2}, \tag{A4}$$

where $\Omega_E = 2\pi f_E$. Now, substituting $\dot{\mathbf{M}}$ from Eq. (1) into Eqs. (A3) and (A4), we have

$$\begin{aligned}\frac{dE}{dt} = {} & \frac{v\mu_0\gamma}{kT}(\mathbf{H} \cdot [\mathbf{M} \times \mathbf{H}]) - \frac{v\alpha\mu_0}{kTM_S}(\mathbf{H} \cdot [\mathbf{M} \times \dot{\mathbf{M}}]) \\ & - \frac{v\mu_0\gamma}{kTM_S}(\mathbf{H} \cdot [\mathbf{M} \times [\mathbf{M} \times \mathbf{I}_S]]) + \frac{\gamma v\mu_0}{kT}(\mathbf{H} \cdot [\mathbf{M} \times \mathbf{h}]),\end{aligned} \tag{A5}$$

$$\begin{aligned}\frac{d\phi}{dt} = {} & \Omega_E\left[1 + \frac{\alpha}{M_S}\frac{([\mathbf{H} \times \mathbf{M}] \cdot [\mathbf{M} \times \dot{\mathbf{M}}])}{\gamma|[\mathbf{H} \times \mathbf{M}]|^2}\right. \\ & + \frac{([\mathbf{H} \times \mathbf{M}] \cdot [\mathbf{M} \times [\mathbf{M} \times \mathbf{I}_S]])}{M_S|[\mathbf{H} \times \mathbf{M}]|^2} \\ & \left. + \frac{([\mathbf{H} \times \mathbf{M}] \cdot [\mathbf{h} \times \mathbf{M}])}{|[\mathbf{H} \times \mathbf{M}]|^2}\right],\end{aligned} \tag{A6}$$

Noticing that the first term in the right-hand side of Eq. (A5) is zero due to the scalar triple-product definition; noting that in the VLD regime and for small spin-polarized currents, all terms of order α^2, $J\alpha$, J^2, etc., in Eqs. (A5) and (A6) may be neglected; and using Eqs. (A1) and (A2), we obtain

$$\begin{aligned}\frac{dE}{dt} = {} & -\frac{v\alpha\mu_0}{\gamma kTM_S}|\dot{\mathbf{M}}_{\rm pr}|^2 + \frac{v\mu_0}{kTM_S}(\mathbf{I}_S \cdot [\mathbf{M} \times \dot{\mathbf{M}}_{\rm pr}]) \\ & + \frac{v\mu_0\Omega_E}{kT}\left(\frac{\partial \mathbf{M}}{\partial \phi} \cdot \mathbf{h}\right),\end{aligned} \tag{A7}$$

$$\frac{d\phi}{dt} = \Omega_E\left[1 - \frac{\gamma M_S}{|\dot{\mathbf{M}}_{\rm pr}|^2}(\mathbf{I}_S \cdot \dot{\mathbf{M}}_{\rm pr}) - \frac{v\mu_0}{kT}\left(\frac{\partial \mathbf{M}}{\partial E} \cdot \mathbf{h}\right)\right]. \tag{A8}$$

The two terms in Eq. (A7) for the energy originate, respectively, from Gilbert damping and STT terms while the second term in Eq. (A8) originates from the fact that STT may have a component along the Stoner-Wohlfarth orbit. The Larmor term obviously does not contribute to Eq. (A7) while Gilbert damping drops from Eq. (A8) for ϕ whose dynamics in the VLD limit are mostly governed by the uniform precession. Obviously, Eqs. (A7) and (A8) can be written in the form of the Langevin equations (15) and (16).

Now the Fokker-Planck equation for the probability density function $W(E,\phi,t)$ corresponding to the Langevin equations (15) and (16) is formally given in energy and phase variables, $\xi_1 = E$ and $\xi_2 = \phi$, by [43]

$$\begin{aligned}\dot{W} = {} & -\frac{\partial}{\partial E}\left[D_1^{(1)}W - \sum_j \frac{\partial}{\partial \xi_j}\left(D_{1j}^{(2)}W\right)\right] \\ & - \frac{\partial}{\partial \phi}\left[D_2^{(1)}W - \sum_j \frac{\partial}{\partial \xi_j}\left(D_{2j}^{(2)}W\right)\right],\end{aligned} \tag{A9}$$

where $D_i^{(1)}$ and $D_{ij}^{(2)}$ are, respectively, the *drift* and *diffusion* coefficients defined as

$$D_i^{(1)} = F_i + D\left(\mathbf{g}_1 \cdot \frac{\partial \mathbf{g}_i}{\partial E}\right) + D\left(\mathbf{g}_2 \cdot \frac{\partial \mathbf{g}_i}{\partial \phi}\right), \tag{A10}$$

$$D_{ij}^{(2)} = D(\mathbf{g}_i \cdot \mathbf{g}_j). \tag{A11}$$

F_i and $\mathbf{g}_i$ are defined by Eqs. (17)–(20), and the diffusion coefficient D is defined in Eq. (2). Noting that [7,39]

$$\left(\mathbf{g}_2 \cdot \frac{\partial \mathbf{g}_1}{\partial \xi_i}\right) = -\left(\mathbf{g}_1 \cdot \frac{\partial \mathbf{g}_2}{\partial \xi_i}\right),$$

and

$$\frac{\partial \mathbf{g}_2}{\partial \phi} = -\frac{\partial \mathbf{g}_1}{\partial E} + \mathbf{g}_1\frac{\partial \ln \Omega_E}{\partial E},$$

we can write the drift and diffusion coefficients Eqs. (A10) and (A11) as

$$D_1^{(1)} = F_1 + \frac{\partial |\mathbf{g}_1|^2}{\partial E} - \frac{\partial \ln \Omega_E}{\partial E}|\mathbf{g}_1|^2, \tag{A12}$$

$$D_2^{(1)} = F_2 + \frac{\partial}{\partial \phi}|\mathbf{g}_2|^2, \tag{A13}$$

$$D_{12}^{(2)} = D_{21}^{(2)} = 0, \tag{A14}$$

$$D_{11}^{(2)} = D|\mathbf{g}_1|^2, \tag{A15}$$

$$D_{22}^{(2)} = D|\mathbf{g}_2|^2. \tag{A16}$$

Substituting (A12)–(A16) into Eq. (A9) and noting Eqs. (17)–(20), we obtain

$$\begin{aligned}\frac{\partial W}{\partial t} = {} & \alpha\frac{\partial}{\partial E}\left[\frac{v\mu_0}{\gamma M_S kT\Omega_E}|\dot{\mathbf{M}}_{\rm pr}|^2\left(1 + \frac{\partial}{\partial E}\right)\Omega_E W\right. \\ & \left. - \frac{v\mu_0}{\alpha M_S kT}(\mathbf{I}_S \cdot [\mathbf{M} \times \dot{\mathbf{M}}_{\rm pr}])W\right] \\ & + \frac{\partial}{\partial \phi}\left[-\Omega_E W + \Omega_E\frac{\gamma M_S}{|\dot{\mathbf{M}}_{\rm pr}|^2}(\mathbf{I}_S \cdot \dot{\mathbf{M}}_{\rm pr})W\right. \\ & \left. + \frac{v\mu_0\alpha}{\gamma M_S kT}\Omega_E^2\left|\frac{\partial \mathbf{M}}{\partial E}\right|^2\frac{\partial W}{\partial \phi}\right].\end{aligned} \tag{A17}$$

The Fokker-Planck equation (A17), because the two state variables E and ϕ are again involved, is difficult to treat. However, since in the VLD limit on long time scales, $f_E t \gg 1$, ϕ is a fast variable and E is slow and is almost conserved (so that $W(E,\phi,t)$ nearly equilibrates in ϕ and slowly evolves in E), the ϕ dependence in Eq. (A17) can be eliminated by exploiting the periodicity of W in ϕ along a precessional orbit that formally corresponds to averaging $W(E,\phi,t)$ over ϕ, namely [7,39],

$$W(E,t) = \frac{1}{2\pi}\int_0^{2\pi} W(E,\phi,t)d\phi.$$

Thus noting that

$$\frac{v\mu_0}{2\pi f_E \gamma kT M_S}\int_0^{2\pi}|\dot{\mathbf{M}}_{\rm pr}|^2 d\phi = \frac{v\mu_0}{\gamma kT M_S}\int_0^{1/f_E}|\dot{\mathbf{M}}_{\rm pr}|^2 dt$$
$$= \frac{v\mu_0}{M_S kT}\oint_E([\mathbf{H}\times\mathbf{M}]\cdot d\mathbf{M}) = S_E,$$

$$\frac{v\mu_0}{2\pi f_E M_S kT}\int_0^{2\pi}(\mathbf{I}_S\cdot[\mathbf{M}\times\dot{\mathbf{M}}_{\rm pr}])d\phi$$
$$= \frac{v\mu_0}{M_S kT}\int_0^{1/f_E}(\mathbf{I}_S\cdot[\mathbf{M}\times\dot{\mathbf{M}}_{\rm pr}])dt = V_E,$$

we find that after averaging, Eq. (A17) becomes the energy-controlled diffusion equation (22) for $W(E,t)$. Here we have used that $d\phi = 2\pi f_E dt$ [7].

APPENDIX B: CALCULATIONS OF S_ε^+, V_ε^+, AND f_ε^+

Because Eqs. (41) and (42) lead to the equality

$$\left(u_X + \frac{h}{\delta+1}\right)^2 + \frac{\delta}{\delta+1}u_Y^2 = \text{const} = p_\varepsilon^2,$$

we can formally introduce a new function $u(t)$ related to $u_X(t)$, $u_Y(t)$, and $u_Z(t)$ via

$$u_X(t) = p_\varepsilon u(t) - h(\delta+1)^{-1}, \tag{B1}$$

$$u_Y(t) = p_\varepsilon\sqrt{(1+\delta^{-1})[1-u^2(t)]}, \tag{B2}$$

$$u_Z(t) = p_\varepsilon\sqrt{\delta^{-1}[u(t)-e_+][u(t)-e_-]}, \tag{B3}$$

where

$$p_\varepsilon^2 = \frac{\delta-\varepsilon}{\delta+1} + \frac{h^2}{(\delta+1)^2},$$
$$e_\pm = -\frac{h\delta}{p_\varepsilon(\delta+1)} \pm \frac{\sqrt{h^2-\varepsilon}}{p_\varepsilon}.$$

By substitution of $u_X(t)$, $u_Y(t)$, $u_Z(t)$ from Eqs. (B1)–(B3) into Eq. (38), we see that $u(t)$ satisfies the following differential equation:

$$\frac{du}{dt} = \frac{p_\varepsilon}{\tau_0}\sqrt{(1+\delta)(1-u^2)(u-e_+)(u-e_-)}. \tag{B4}$$

The solution of the differential equation (B4) within the energy region between the bottom of the well and the saddle point $-1-2h \leqslant \varepsilon \leqslant h^2$, $0 \leqslant m_\varepsilon \leqslant 1$ is [49]

$$u(t) = \frac{a_\varepsilon - \mathrm{sn}^2(\omega_\varepsilon t + w|m_\varepsilon)}{a_\varepsilon + \mathrm{sn}^2(\omega_\varepsilon t + w|m_\varepsilon)}, \tag{B5}$$

where $\mathrm{sn}(u|m)$ is Jacobi's doubly periodic elliptic function [45,49], w is an integration constant (initial phase), and

$$a_\varepsilon = \frac{1+e_+}{1-e_+},\quad m_\varepsilon = \frac{(1+e_-)(1-e_+)}{(1+e_+)(1-e_-)}$$
$$\omega_\varepsilon = \frac{p_\varepsilon}{2\tau_0}\sqrt{(\delta+1)(1+e_+)(1-e_-)}.$$

Now noticing Eqs. (38)–(42) and (B1)–(B3), we can write the dimensionless action S_ε^+ from Eq. (23) as

$$S_\varepsilon^+ = 2\tau_0\sigma\int_0^{1/f_\varepsilon^+}\left[\dot{u}_X^2(t)+\dot{u}_Y^2(t)+\dot{u}_Z^2(t)\right]dt$$
$$= \frac{2\sigma p_\varepsilon^2}{\tau_0 f_\varepsilon^+}\left[\varepsilon(\delta+1) - h^2 + 2hp_\varepsilon(\delta+1)f_\varepsilon^+\int_0^{1/f_\varepsilon^+}u(t)dt + (1+\delta-h^2)f_\varepsilon^+\int_0^{1/f_\varepsilon^+}u^2(t)dt\right], \tag{B6}$$

where $u(t)$ is given by Eq. (B5). The integrals in Eq. (B6), namely, $\int_0^{1/f_\varepsilon^+}u(t)dt$ and $\int_0^{1/f_\varepsilon^+}u^2(t)dt$, can be evaluated as

$$f_\varepsilon^+\int_0^{1/f_\varepsilon^+}u(t)dt = -1 + 2a_\varepsilon I(1), \tag{B7}$$

$$f_\varepsilon^+\int_0^{1/f_\varepsilon^+}u^2(t)dt = 1 - 4a_\varepsilon I(1) + 4a_\varepsilon^2 I(2), \tag{B8}$$

where the integrals $I(n)$ defined as

$$I(n) = f_\varepsilon^+\int_0^{1/f_\varepsilon^+}\frac{dt}{[a_\varepsilon + \mathrm{sn}^2(\omega_\varepsilon t + w|m_\varepsilon)]^n} \tag{B9}$$

can be expressed using the table of integrals from Sec. 1.17.1 of Ref. [50] as

$$I(1) = \frac{\Pi(-a_\varepsilon^{-1}|m_\varepsilon)}{a_\varepsilon K(m_\varepsilon)},$$

$$I(2) = \frac{1}{2a_\varepsilon(1+a_\varepsilon)}\left[-1 + \frac{E(m_\varepsilon)}{(1+a_\varepsilon m_\varepsilon)K(m_\varepsilon)} + \frac{1+3a_\varepsilon^2 m_\varepsilon + 2a_\varepsilon m_\varepsilon + 2a_\varepsilon}{(1+a_\varepsilon m_\varepsilon)a_\varepsilon K(m_\varepsilon)}\Pi\left(-a_\varepsilon^{-1}|m_\varepsilon\right)\right].$$

Here $K(m)$, $E(m)$, and $\Pi(n|m)$ are the complete elliptic integrals of the first, second, and third kind, respectively [45]. By substituting Eqs. (B7) and (B8) into Eq. (B6), we ultimately have the action S_ε^+ in terms of known functions, viz.,

$$S_\varepsilon^+ = \frac{2\sigma p_\varepsilon^2(1+\delta)}{\tau_0 f_\varepsilon^+}\left\{\varepsilon + 2hp_\varepsilon - \frac{h^2}{1+\delta} + \frac{1+\delta-h^2}{(1+\delta)(1+a_\varepsilon)}\left[1 - a_\varepsilon + \frac{2a_\varepsilon E(m_\varepsilon)}{(1+a_\varepsilon m_\varepsilon)K(m_\varepsilon)}\right]\right.$$
$$\left. + \left[4hp_\varepsilon - 2\frac{(1+\delta-h^2)\left(1-a_\varepsilon^2 m_\varepsilon\right)}{(1+\delta)(1+a_\varepsilon)(1+a_\varepsilon m_\varepsilon)}\right]\frac{\Pi\left(-a_\varepsilon^{-1}|m_\varepsilon\right)}{K(m_\varepsilon)}\right\}, \tag{B10}$$

where the precession frequency f_ε^+ is given by

$$f_\varepsilon^+ = \frac{\omega_\varepsilon}{4K(m_\varepsilon)} = \frac{p_\varepsilon\sqrt{(\delta+1)(1+e_+)(1-e_-)}}{8\tau_0 K(m_\varepsilon)}. \tag{B11}$$

(Jacobi's elliptic function $\mathrm{sn}(u|m)$ has the period $4K(m)$ [45]).

Finally, we can also find V_ε^+ from Eq. (24) yielding

$$V_\varepsilon^+ = J\int_0^{1/f_\varepsilon^\pm}[\dot{u}_Z(t)u_Y(t) - \dot{u}_Y(t)u_Z(t)]dt, \tag{B12}$$

which can be rearranged using Eqs. (38)–(42) and (B1) as

$$V_\varepsilon^+ = \frac{J}{\tau_0}\int_0^{1/f_\varepsilon^+}\left[h + (\varepsilon+1)u_X(t) + hu_X^2(t)\right]dt$$
$$= \frac{J}{\tau_0 f_\varepsilon^+}\left[hp_\varepsilon^2 + p_\varepsilon\left(\varepsilon + 1 - \frac{2h^2}{\delta+1}\right)f_\varepsilon^+\int_0^{1/f_\varepsilon^+}u(t)dt + hp_\varepsilon^2 f_\varepsilon^+\int_0^{1/f_\varepsilon^+}u^2(t)dt\right]. \tag{B13}$$

By substituting Eqs. (B7) and (B8) into Eq. (B13), we ultimately have V_ε^+ in terms of known functions, namely,

$$V_\varepsilon^+ = \frac{J}{\tau_0 f_\varepsilon^+}\left\{h + \frac{h^3}{(\delta+1)^2} - \frac{h(\varepsilon+1)}{\delta+1} + p_\varepsilon\left(\varepsilon + 1 - \frac{2h^2}{\delta+1}\right)\left[2\frac{\Pi\left(-a_\varepsilon^{-1}|m_\varepsilon\right)}{K(m_\varepsilon)} - 1\right]\right.$$
$$\left. + \frac{hp_\varepsilon^2}{1+a_\varepsilon^{-1}}\left[a_\varepsilon^{-1} - 1 + 2\frac{a_\varepsilon^{-1}E(m_\varepsilon) - \left(a_\varepsilon^{-2} - m_\varepsilon\right)\Pi\left(-a_\varepsilon^{-1}|m_\varepsilon\right)}{\left(a_\varepsilon^{-1} + m_\varepsilon\right)K(m_\varepsilon)}\right]\right\}. \tag{B14}$$

APPENDIX C: EXPLICIT FORM OF THE DIFFERENTIAL-RECURRENCE EQUATIONS

For the free-energy density given by Eq. (37) and the nonconservative potential given by Eq. (5) where the magnetization direction of the fixed layer $\mathbf{e}_P$ is along the easy (X) axis, a 21-term differential-recurrence relation can be found from Eq. (52) as described in details in Ref. [30a] and is given by

$$\frac{d}{dt}c_{n,m}(t) = v_{n,m}^{--}c_{n-2,m-2}(t) + v_{n,m}c_{n-2,m}(t) + v_{n,m}^{++}c_{n-2,m+2}(t)$$
$$+ w_{n,m}^{--}c_{n-1,m-2}(t) + w_{n,m}^{-}c_{n-1,m-1}(t) + w_{n,m}c_{n-1,m}(t) + w_{n,m}^{+}c_{n-1,m+1}(t) + w_{n,m}^{++}c_{n-1,m+2}(t)$$
$$+ x_{n,m}^{--}c_{n,m-2}(t) + x_{n,m}^{-}c_{n,m-1}(t) + x_{n,m}c_{n,m}(t) + x_{n,m}^{+}c_{n,m+1}(t) + x_{n,m}^{++}c_{n,m+2}(t)$$
$$+ y_{n,m}^{--}c_{n+1,m-2} + y_{n,m}^{-}c_{n+1,m-1}(t) + y_{n,m}c_{n+1,m}(t) + y_{n,m}^{+}c_{n+1,m+1}(t) + y_{n,m}^{++}c_{n+1,m+2}(t)$$
$$+ z_{n,m}^{--}c_{n+2,m-2}(t) + z_{n,m}c_{n+2,m}(t) + z_{n,m}^{++}c_{n+2,m+2}(t), \tag{C1}$$

where

$$v_{n,m} = -\frac{n+1}{2n-1}\sqrt{\frac{[(n-1)^2-m^2](n^2-m^2)}{(2n+1)(2n-3)}}\frac{\sigma}{\tau_N}\left(\frac{1}{2}+\delta\right),$$

$$v_{n,m}^{\pm\pm} = \frac{\sigma}{4\tau_N}\frac{n+1}{2n-1}\sqrt{\frac{(n\mp m-3)(n\mp m-2)(n\mp m-1)(n\mp m)}{(2n+1)(2n-3)}},$$

$$w_{n,m} = -\frac{im\sigma}{\alpha\tau_N}\sqrt{\frac{n^2-m^2}{4n^2-1}}\left(\frac{1}{2}+\delta\right),$$

$$w_{n,m}^{\pm} = \pm\frac{1}{\tau_N}\sqrt{\frac{(n\mp m)(n\mp m-1)}{4n^2-1}}\left\{\frac{n+1}{2}\sigma h - \frac{J(n+1)}{4\alpha}\right\},$$

$$
\begin{aligned}
w_{n,m}^{\pm\pm} &= \pm i\frac{\sigma}{4\tau_{\mathrm{N}}\alpha}\sqrt{\frac{(n\mp m-2)(n\mp m-1)(1+n\pm m)(n\mp m)}{4n^2-1}},\\
x_{n,m} &= -\frac{n(n+1)}{2\tau_{\mathrm{N}}} - \frac{n(n+1)-3m^2}{(2n-1)(2n+3)}\frac{\sigma}{\tau_{\mathrm{N}}}\left(\frac{1}{2}+\delta\right),\\
x_{n,m}^{\pm} &= \frac{i}{\tau_{\mathrm{N}}}\sqrt{(n\pm m+1)(n\mp m)}\left\{\frac{\sigma h}{2\alpha}+\frac{J}{4}\right\},\\
x_{n,m}^{\pm\pm} &= -\frac{\sqrt{(n\pm m+1)(n\pm m+2)(n\mp m-1)(n\mp m)}}{(2n-1)(2n+3)}\frac{3\sigma}{4\tau_{\mathrm{N}}},\\
y_{n,m} &= -i\frac{m\sigma}{\alpha\tau_{\mathrm{N}}}\sqrt{\frac{(n+1)^2-m^2}{(2n+1)(2n+3)}}\left(\frac{1}{2}+\delta\right),\\
y_{n,m}^{\pm} &= \pm\frac{1}{\tau_N}\sqrt{\frac{(1+n\pm m)(2+n\pm m)}{(1+2n)(3+2n)}}\left\{\frac{n\sigma h}{2}-\frac{nJ}{4\alpha}\right\},\\
y_{n,m}^{\pm\pm} &= \mp i\frac{\sigma}{4\alpha\tau_{\mathrm{N}}}\sqrt{\frac{(1+n\pm m)(2+n\pm m)(3+n\pm m)(n\mp m)}{(1+2n)(3+2n)}},\\
z_{n,m} &= \frac{n}{2n+3}\sqrt{\frac{[(n+1)^2-m^2][(n+2)^2-m^2]}{(2n+1)(2n+5)}}\frac{\sigma}{\tau_{\mathrm{N}}}\left(\frac{1}{2}+\delta\right),\\
z_{n,m}^{\pm\pm} &= -\frac{\sigma}{4\tau_{\mathrm{N}}}\frac{n}{2n+3}\sqrt{\frac{(n+1\pm m)(2+n\pm m)(3+n\pm m)(4+n\pm m)}{(2n+1)(2n+5)}}.
\end{aligned}
$$

Equation (C1) is a particular case of a 25-term differential-recurrence relation for a general form of the nonconservative potential derived in Ref. [30a] with $v_{n,m}^{\pm}=0$ and $z_{n,m}^{\pm}=0$.

In order to rewrite Eq. (C1) in the form of the tridiagonal vector differential-recurrence equation (53), we define the column vectors $\mathbf{C}_n(t)$ via $c_{n,m}(t)$ as follows:

$$
\mathbf{C}_0(t)=\mathbf{0},\quad \mathbf{C}_n(t)=\begin{pmatrix} c_{2n,-2n}(t)\\ c_{2n,-2n+1}(t)\\ \vdots\\ c_{2n,2n}(t)\\ c_{2n-1,-2n+1}(t)\\ c_{2n-1,-2n+2}(t)\\ \vdots\\ c_{2n-1,2n-1}(t)\end{pmatrix},\quad n\geqslant 1,
$$

while the matrices $\mathbf{Q}_n,\mathbf{Q}_n^{+},\mathbf{Q}_n^{-}$ are defined as

$$
\mathbf{Q}_n=\begin{pmatrix}\mathbf{X}_{2n} & \mathbf{W}_{2n}\\ \mathbf{Y}_{2n-1} & \mathbf{X}_{2n-1}\end{pmatrix},\quad \mathbf{Q}_n^{+}=\begin{pmatrix}\mathbf{Z}_{2n} & \mathbf{Y}_{2n}\\ \mathbf{0} & \mathbf{Z}_{2n-1}\end{pmatrix},\quad \mathbf{Q}_n^{-}=\begin{pmatrix}\mathbf{V}_{2n} & \mathbf{0}\\ \mathbf{W}_{2n-1} & \mathbf{V}_{2n-1}\end{pmatrix}.
$$

In turn, the matrices $\mathbf{Q}_n,\mathbf{Q}_n^{+},\mathbf{Q}_n^{-}$ themselves consist of five submatrices $\mathbf{V}_l$, $\mathbf{W}_l$, $\mathbf{X}_l$, $\mathbf{Y}_l$, and $\mathbf{Z}_l$ of dimensions $(2l+1)\times(2l-3)$, $(2l+1)\times(2l-1)$, $(2l+1)\times(2l+1)$, $(2l+1)\times(2l+3)$, and $(2l+1)\times(2l+5)$, respectively. The elements of these five- and three-diagonal submatrices, which are formed from the coefficients occurring in Eq. (C1), are given by

$$
\begin{aligned}
(\mathbf{V}_l)_{n,m} &= \delta_{n-4m}v_{l,-l+m+3}^{--}+\delta_{n-2m}v_{l,-l+m+1}+\delta_{nm}v_{l,-l+m-1}^{++},\\
(\mathbf{W}_l)_{n,m} &= \delta_{n-3m}w_{l,-l+m+2}^{--}+\delta_{n-2m}w_{l,-l+m+1}^{-}+\delta_{n-1m}w_{l,-l+m}+\delta_{nm}w_{l,-l+m-1}^{+}+\delta_{n+1m}w_{l,-l+m-2}^{++},\\
(\mathbf{X}_l)_{n,m} &= \delta_{n-2m}x_{l,-l+m+1}^{--}+\delta_{n-1m}x_{l,-l+m}^{-}+\delta_{nm}x_{l,-l+m-1}+\delta_{n+1m}x_{l,-l+m-2}^{+}+\delta_{n+2m}x_{l,-l+m-3}^{++},\\
(\mathbf{Y}_l)_{n,m} &= \delta_{n-1m}y_{l,-l+m}^{--}+\delta_{nm}y_{l,-l+m-1}^{-}+\delta_{n+1m}y_{l,-l+m-2}+\delta_{n+2m}y_{l,-l+m-3}^{+}+\delta_{n+3m}y_{l,-l+m-4}^{++},\\
(\mathbf{Z}_l)_{n,m} &= \delta_{nm}z_{l,-l+m-1}^{--}+\delta_{n+2m}z_{l,-l+m-3}+\delta_{n+4m}z_{l,-l+m-5}^{++},
\end{aligned}
$$

where δ_{ij} is Kronecker's delta.

[1] J. C. Slonczewski, Current-driven excitation of magnetic multilayers, J. Magn. Magn. Mater. **159**, L1 (1996).

[2] L. Berger, Emission of spin waves by a magnetic multilayer traversed by a current, Phys. Rev. B **54**, 9353 (1996).

[3] M. D. Stiles and J. Miltat, Spin-transfer torque and dynamics, in *Spin Dynamics in Confined Magnetic Structures III*, edited by B. Hillebrands and A. Thiaville (Springer-Verlag, Berlin, 2006), p. 225.

[4] D. C. Ralph and M. D. Stiles, Spin transfer torques, J. Magn. Magn. Mater. **320**, 1190 (2008).

[5] Y. Suzuki, A. A. Tulapurkar, and C. Chappert, Spin-injection phenomena and applications, in *Nanomagnetism and Spintronics*, edited by T. Shinjo (Elsevier, Amsterdam, 2009), Chap. 3, p. 94.

[6] G. Bertotti, C. Serpico, and I. D. Mayergoyz, *Nonlinear Magnetization Dynamics in Nanosystems* (Elsevier, Amsterdam, 2009).

[7] T. Dunn, A. L. Chudnovskiy, and A. Kamenev, Dynamics of nano-magnetic oscillators, in *Fluctuating Nonlinear Oscillators*, edited by M. Dykman (Oxford University Press, London, 2012).

[8] Yu. V. Gulyaev, P. E. Zilberman, A. I. Panas, and E. M. Epshtein, Spintronics: exchange switching of ferromagnetic metallic junctions at a low current density, Usp. Fiz. Nauk **179**, 359 (2009) [Phys. Usp. **52**, 335 (2009)].

[9] R. Heindl, W. H. Rippard, S. E. Russek, M. R. Pufall, and A. B. Kos, Validity of the thermal activation model for spin-transfer torque switching in magnetic tunnel junctions, J. Appl. Phys. **109**, 073910 (2011); W. H. Rippard, R. Heindl, M. R. Pufall, S. E. Russek, and A. B. Kos, Thermal relaxation rates of magnetic nanoparticles in the presence of magnetic fields and spin-transfer effects, Phys. Rev. B **84**, 064439 (2011).

[10] J. Swiebodzinski, A. Chudnovskiy, T. Dunn, and A. Kamenev, Spin torque dynamics with noise in magnetic nanosystems, Phys. Rev. B **82**, 144404 (2010).

[11] E. B. Myers, F. J. Albert, J. C. Sankey, E. Bonet, R. A. Buhrman, and D. C. Ralph, Thermally Activated Magnetic Reversal Induced by a Spin-Polarized Current, Phys. Rev. Lett. **89**, 196801 (2002); A. Fabian, C. Terrier, S. Serrano Guisan, X. Hoffer, M. Dubey, L. Gravier, J.-Ph. Ansermet, and J.-E. Wegrowe, Current-Induced Two-Level Fluctuations in Pseudo-Spin-Valve (Co/Cu/Co) Nanostructures, *ibid.* **91**, 257209 (2003); F. B. Mancoff, R. W. Dave, N. D. Rizzo, T. C. Eschrich, B. N. Engel, and S. Tehrani, Angular dependence of spin-transfer switching in a magnetic nanostructure, Appl. Phys. Lett. **83**, 1596 (2003); I. N. Krivorotov, N. C. Emley, A. G. F. Garcia, J. C. Sankey, S. I. Kiselev, D. C. Ralph, and R. A. Buhrman, Temperature Dependence of Spin-Transfer-Induced Switching of Nanomagnets, Phys. Rev. Lett. **93**, 166603 (2004); R. H. Koch, J. A. Katine, and J. Z. Sun, Time-Resolved Reversal of Spin-Transfer Switching in a Nanomagnet, *ibid.* **92**, 088302 (2004); M. L. Schneider, M. R. Pufall, W. H. Rippard, S. E. Russek, and J. A. Katine, Thermal effects on the critical current of spin torque switching in spin valve nanopillars, Appl. Phys. Lett. **90**, 092504 (2007).

[12] T. L. Gilbert, A Lagrangian formulation of the gyromagnetic equation of the magnetic field, Phys. Rev. **100**, 1243 (1955); abstract only; full report in Armour Research Foundation Project No. A059, Supplementary Report (1956). Reprinted in T. L. Gilbert, A phenomenological theory of damping in ferromagnetic materials, IEEE Trans. Magn. **40**, 3443 (2004).

[13] W. T. Coffey and Y. P. Kalmykov, *The Langevin Equation*, 3rd ed. (World Scientific, Singapore, 2012).

[14] Z. Li and S. Zhang, Thermally assisted magnetization reversal in the presence of a spin-transfer torque, Phys. Rev. B **69**, 134416 (2004).

[15] W. Wernsdorfer, E. Bonet Orozco, K. Hasselbach, A. Benoit, D. Mailly, O. Kubo, H. Nakano, and B. Barbara, Macroscopic Quantum Tunneling of Magnetization of Single Ferrimagnetic Nanoparticles of Barium Ferrite, Phys. Rev. Lett. **79**, 4014 (1997); W. T. Coffey, D. S. F. Crothers, J. L. Dormann, Yu. P. Kalmykov, E. C. Kennedy, and W. Wernsdorfer, Thermally Activated Relaxation Time of a Single Domain Ferromagnetic Particle Subjected to a Uniform Field at an Oblique Angle to the Easy Axis: Comparison with Experimental Observations, *ibid.* **80**, 5655 (1998).

[16] M. Oogane, T. Wakitani, S. Yakata, R. Yilgin, Y. Ando, A. Sakuma, and T. Miyazaki, Magnetic damping in ferromagnetic thin films, Jpn. J. Appl. Phys. **45**, 3889 (2006).

[17] M. C. Hickey and J. S. Moodera, Origin of Intrinsic Gilbert Damping, Phys. Rev. Lett. **102**, 137601 (2009).

[18] D. M. Apalkov and P. B. Visscher, Spin-torque switching: Fokker-Planck rate calculation, Phys. Rev. B **72**, 180405(R) (2005).

[19] G. Bertotti, I. D. Mayergoyz, and C. Serpico, Analysis of random Landau-Lifshitz dynamics by using stochastic processes on graphs, J. Appl. Phys. **99**, 08F301 (2006).

[20] T. Taniguchi and H. Imamura, Thermally assisted spin transfer torque switching in synthetic free layers, Phys. Rev. B **83**, 054432 (2011); Thermal switching rate of a ferromagnetic material with uniaxial anisotropy, *ibid.* **85**, 184403 (2012).

[21] T. Taniguchi, Y. Utsumi, and H. Imamura, Thermally activated switching rate of a nanomagnet in the presence of spin torque, Phys. Rev. B **88**, 214414 (2013); T. Taniguchi, Y. Utsumi, M. Marthaler, D. S. Golubev, and H. Imamura, Spin torque switching of an in-plane magnetized system in a thermally activated region, *ibid.* **87**, 054406 (2013); T. Taniguchi and H. Imamura, Current dependence of spin torque switching rate based on Fokker-Planck approach, J. Appl. Phys. **115**, 17C708 (2014).

[22] W. H. Butler, T. Mewes, C. K. A. Mewes, P. B. Visscher, W. H. Rippard, S. E. Russek, and R. Heindl, Switching distributions for perpendicular spin-torque devices within the macrospin approximation, IEEE Trans. Magn. **48**, 4684 (2012).

[23] K. A. Newhall and E. Vanden-Eijnden, Averaged equation for energy diffusion on a graph reveals bifurcation diagram and thermally assisted reversal times in spin-torque driven nanomagnets. J. Appl. Phys. **113**, 184105 (2013).

[24] J. M. Lee and S. H. Lim, Thermally activated magnetization switching in a nanostructured synthetic ferrimagnet, J. Appl. Phys. **113**, 063914 (2013).

[25] D. V. Berkov and N. L. Gorn, Magnetization precession due to a spin-polarized current in a thin nanoelement: Numerical simulation study, Phys. Rev. B **72**, 094401 (2005); Non-linear magnetization dynamics in nanodevices induced by a spin-polarized current: micromagnetic simulation, J. Phys. D: Appl. Phys. **41**, 164013 (2008).

[26] K. Ito, Micromagnetic simulation on dynamics of spin transfer torque magnetization reversal, IEEE Trans. Magn. **41**, 2630 (2005).

[27] T. Taniguchi, M. Shibata, M. Marthaler, Y. Utsumi, and H. Imamura, Numerical study on spin torque switching in thermally activated region, Appl. Phys. Express **5**, 063009 (2012).

[28] D. Pinna, A. D. Kent, and D. L. Stein, Spin-transfer torque magnetization reversal in uniaxial nanomagnets with thermal noise, J. Appl. Phys. **114**, 033901 (2013).

[29] M. d'Aquino, C. Serpico, R. Bonin, G. Bertotti, and I. D. Mayergoyz, Stochastic resonance in noise-induced transitions between self-oscillations and equilibria in spin-valve nanomagnets, Phys. Rev. B **84**, 214415 (2011).

[30] Y. P. Kalmykov, W. T. Coffey, S. V. Titov, J. E. Wegrowe, and D. Byrne, Spin-torque effects in thermally assisted magnetization reversal: Method of statistical moments, Phys. Rev. B **88**, 144406 (2013); D. Byrne, W. T. Coffey, Y. P. Kalmykov, S. V. Titov, and J. E. Wegrowe, Spin-transfer torque effects in the dynamic forced response of the magnetization of nanoscale ferromagnets in superimposed ac and dc bias fields in the presence of thermal agitation, *ibid.* **91**, 174406 (2015).

[31] L. Neel, Théorie du traînage magnétique des ferromagnétiques en grains fins avec applications aux terres cuites, Ann. Geophys. **5**, 99 (1949).

[32] W. F. Brown, Jr., Thermal fluctuations of a single-domain particle, Phys. Rev. **130**, 1677 (1963).

[33] W. F. Brown, Jr., Thermal fluctuations of fine ferromagnetic particles, IEEE Trans. Magn. **15**, 1196 (1979).

[34] I. Klik and L. Gunther, First-passage-time approach to overbarrier relaxation of magnetization, J. Stat. Phys. **60**, 473 (1990); Thermal relaxation over a barrier in single domain ferromagnetic particles, J. Appl. Phys. **67**, 4505 (1990).

[35] W. T. Coffey, Finite integral representation of characteristic times of orientation relaxation processes: Application to the uniform bias force effect in relaxation in bistable potentials, Adv. Chem. Phys. **103**, 259 (1998).

[36] W. T. Coffey and Y. P. Kalmykov, Thermal fluctuations of magnetic nanoparticles, J. Appl. Phys. **112**, 121301 (2012).

[37] H. A. Kramers, Brownian motion in a field of force and the diffusion model of chemical reactions, Physica **7**, 284 (1940).

[38] P. Hänggi, P. Talkner, and M. Borkovec, Reaction-rate theory: Fifty years after Kramers, Rev. Mod. Phys. **62**, 251 (1990).

[39] W. T. Coffey, Y. P. Kalmykov, and S. V. Titov, Magnetization reversal time of magnetic nanoparticles at very low damping, Phys. Rev. B **89**, 054408 (2014); D. Byrne, W. T. Coffey, W. J. Dowling, Y. P. Kalmykov, and S. V. Titov, On the Kramers very low damping escape rate for point particles and classical spins, Adv. Chem. Phys. **156**, 393 (2015).

[40] J. Z. Sun, Spin-current interaction with a monodomain magnetic body: A model study, Phys. Rev. B **62**, 570 (2000).

[41] B. J. Matkowsky, Z. Schuss, and C. Tier, Uniform expansion of the transition rate in Kramers' problem, J. Stat. Phys. **35**, 443 (1984).

[42] A. Barone and G. Paterno, *Physics and Applications of the Josephson Effect* (Wiley, New York, 1982).

[43] H. Risken, *The Fokker-Planck Equation*, 2nd ed. (Springer, Berlin, 1989).

[44] D. A. Varshalovitch, A. N. Moskalev, and V. K. Khersonskii, *Quantum Theory of Angular Momentum* (World Scientific, Singapore, 1988).

[45] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. Stegun (Dover, New York, 1964).

[46] Y. P. Kalmykov, Evaluation of the smallest nonvanishing eigenvalue of the Fokker-Planck equation for the Brownian motion in a potential. II. The matrix continued fraction approach, Phys. Rev. E **62**, 227 (2000).

[47] W. T. Coffey, D. A. Garanin, and D. J. McCarthy, Crossover formulas in the Kramers theory of thermally activated escape rates—application to spin systems, Adv. Chem. Phys. **117**, 483 (2001); P. M. Déjardin, D. S. F. Crothers, W. T. Coffey, and D. J. McCarthy, Interpolation formula between very low and intermediate-to-high damping Kramers escape rates for single-domain ferromagnetic particles, Phys. Rev. E **63**, 021102 (2001).

[48] V. I. Mel'nikov and S. V. Meshkov, Theory of activated rate processes: exact solution of the Kramers problem, J. Chem. Phys. **85**, 1018 (1986); V. I. Mel'nikov, The Kramers problem: Fifty years of development, Phys. Rep. **209**, 1 (1991).

[49] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th ed. (Cambridge University Press, London, 1927).

[50] A. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series, Vol. 3: More Special Functions* (Gordon and Breach, New York, 1990), Sec. 1.17.1.