Superconducting transition temperature: Interacting Fermi gas and phonon mechanisms in the nonadiabatic regime

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We analyze the mathematical structure of equations for temperature T_C of the superconductivity transition in a gas of interacting Fermi particles or at the phonon-mediated pairing in a metal in the case of nonadiabatic conditions $\omega_0 \geqslant E_F$, i.e., when the characteristic phonon frequency ω_0 is comparable or larger than the Fermi energy E_F. As the methods of calculating T_C in common superconductors are not applicable in the nonadiabatic regime, the integral equations for T_C are derived in the logarithmic approximation. The new equations contain no divergent terms in the antiadiabatic limit. The results can be immediately generalized to anisotropic band superconductors.

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I. INTRODUCTION

The record $T_C \approx 40\,\mathrm{K}$ for "ordinary" phonon-mediated superconductivity in MgB_2 [1] has recently been overrun by the discovery of superconductivity with a critical temperature as high as $T_C \geqslant 100\,\mathrm{K}$ in the single-layer FeSe deposited on the $SrTiO_3$ substrate [2,3]. The angle-resolved photoemission spectroscopy [4] confirms that the band FeSe electrons interact with a surface phonon mode with the frequency ω_0 between 80 and 100 meV. The bottom of the electron pocket at the M point lies only 60 meV below the chemical potential so that the Fermi energy E_F is *smaller* than the frequency ω_0 of the active phone. This is in sharp contrast to ordinary superconductors in which θ_D the Debye temperature on the order 100–300 K is small $\theta_D \ll E_F$.

In metals the extension of the weak-coupling BCS model to the electron-phonon interactions of the arbitrary strength is provided by the set of Migdal-Eliashberg equations [5,6]. However the applicability of the latter is *subject to the condition of small* adiabatic parameter $\omega_0/E_F \ll 1$ [5]. In the single-layer FeSe on the strontium titanate substrate $\omega_0/E_F \approx 1.3$–1.7, and this criterion is not fulfilled.

Another case of the violated adiabatic Migdal provision is superconductivity in bulk $SrTiO_3$ in which at low doping the Fermi energy is surprisingly small ($E_F \geqslant 1\,\mathrm{meV}$; $T_C \approx (0.07$–$0.2\mathrm{K})$ [7,8]). As to the phonon's spectrum in $SrTiO_3$, the latter stretches from the acoustic branches to the high-frequency optical mode with $\omega_0 \geqslant 100\,\mathrm{meV}$ [9]. This case, thereby presents the instance of the phonon-mediated superconductivity in the *extreme antiadiabatic* limit $\omega_0 \gg E_F$.

Note that by varying the ratio between ω_0 and E_F one also somewhat changes the very concept of the phonon pairing. From condition $\omega_0/E_F \ll 1$ in the adiabatic regime follows the celebrated retardation effect of the BCS theory. Indeed, each of the two electrons comprising the Cooper pair for the characteristic phonon time $1/\omega_0$ shifts by $d \approx v_F/\omega_0$ on the order of $(1/p_F)(E_F/\omega_0)$. Thereby the effect of the Coulomb repulsion is reduced because the latter is screened on the atomic scale; meanwhile the electrons of the pair stay apart from each other on distances d larger than the typical interatomic distance $a \approx 1/p_F$ by the factor $E_F/\omega_0 \gg 1$.

In the opposite limit $\omega_0/E_F \gg 1$ electrons of the pair sense instantaneously and at the same time both the direct Coulomb repulsion and the potential created by the local lattice distortion. Therefore, for the Cooper pairs to form the strength of the phonon attraction must *prevail* over the direct electron-electron Coulomb repulsive interaction. At ω_0 on the same order as E_F the electron-electron potential interaction must be treated on an equal basis with attraction via the virtual exchange by a phonon. Therefore, in the nonadiabatic limit, from a mathematical viewpoint, there is no difference between the direct Coulomb potential and that of the instantaneous electron-phonon attraction.

Changes in the underlying physical picture must find reflection in the *mathematical apparatus* of the theory. In conventional metals contributing to the superconducting gap are only the electrons within a narrow vicinity of the Fermi energy $\Delta\varepsilon \approx \omega_0 \ll E_F$. Convergence of the logarithmic integrals in the Cooper channel in this case is guaranteed by the phonon Green's function [6,10]. However, in the opposite limit $\omega_0 \gg E_F$ the question arises as to what replaces the phonon frequency as a cutoff parameter. For instance, it is common in the literature at calculating T_C using the bandwidth as such a cutoff, although, as is shown below, such a choice is not substantiated.

The problem is pertinent, in particular, to the physics of cold gases and for the first time was studied for the gas of the neutral Fermi particles with weak attractive interaction [11]. In that follows, we extend the method [11] to the case of the nonadiabatic phonon-mediated superconductivity pairing.

The validity of the Migdal-Eliashberg equations [5,6] has been repeatedly questioned in the theoretical literature on various mechanisms as an alternative to the phonon-mediated pairing, such as the so-called plasmon and excitonic mechanisms expected to provide higher T_C or in discussions with regard to the nature of high T_C superconductivity in cuprates. The inevitable methodic difficulty is that with $\omega_0/E_F \geqslant 1$ the key advantage of the Migdal theory, namely, the possibility of neglecting all contributions from the so-called "crossing" diagrams is lost. The authors of the theoretical publications on the theme are often preoccupied analyzing contributions from particular classes of crossing diagrams such as, for instance, the so-called vertex corrections [12].

In our opinion, in the absence of small parameters such incremental improvements lead nowhere. Instead, we suggest the analysis in the frameworks of the weak-coupling approximation. We argue that such analysis allows studying every qualitative feature pertinent to a particular nonadiabatic mechanism. At the time, the weak-coupling BCS theory turned out extremely successful at the interpretation of the experimental data (see, e.g., the review in Ref. [13]).

We start with the isotropic parabolic model for the electronic spectrum.

II. THE WEAK-COUPLING LIMIT

The canonic BCS weak-coupling expression for the temperature T_C of the superconductivity transition has the form

$$T_C = W\ \exp(-1/\lambda). \tag{1}$$

Here in (1) W is the order of magnitude cutoff parameter in the Cooper channel. At $\omega_0 \ll E_F$ the phoning-mediated attraction is effective in the narrow vicinity of the Fermi energy. So, $W \approx \omega_0$ where ω_0 is on the order of the Debye temperature or a typical phonon frequency. (The proportionality $\omega_0 \propto M^{-1/2}$ where M is an atomic mass lies at the core of the isotope effect.) In Eq. (1) λ is proportional to the product of the electron-phonon interaction constant and the density of states at the Fermi level ($\lambda \ll 1$).

In the limit $\omega_0 \gg E_F$ the only cutoff in the Cooper channel can be the Fermi energy itself. One of the consequences is that such a discriminating signature of a phonon mechanism as the isotope effect is absent in this limit.

Below the integral equations for the temperature of the superconductivity transition are analyzed in the general case of ω_0 the same order of magnitude as E_F.

III. PHONON-MEDIATED SUPERCONDUCTIVITY AT $\omega_0 \sim E_F$ IN GENERAL

As mentioned, the key element of the Migdal [5] theory of the electron- phonon interaction in metals is the possibility to discard in the diagrammatic expansion contributions coming from the so-called crossing diagrams. This greatly advantageous feature at $\omega_0/E_F \geqslant 1$ is lost already in the normal phase. Performing the analytical calculations reduces itself to the analysis of the perturbation expansion. Correspondingly, below all results related to the Cooper instability in nonadiabatic conditions are obtained in the so-called logarithmic approximation.

In particular, in the limit $\omega_0 \gg E_F$ the expression $T_C = \text{const} \times E_F \exp(-1/\lambda)$ contains a numerical factor const on the order of unity. Formally, for the validity of the result must be large $\ln(E_F/T_C) \gg 1$. In practice, due to the exponential dependence in the expression (1) for the approach to work it is sufficient to have $\ln(E_F/T_C) > 1$. In each concrete case this can be verified if the Fermi energy E_F is also known experimentally.

The signature of the superconducting instability is the pole appearing in the scattering amplitude for two electrons $\Gamma(p,q-p|p',q-p')$ at the transition temperature T_C [10]. The amplitude is the sum of all diagrams in the Cooper channel. Commonly one analyzes the diagrammatic series at zero summary momentum and frequency $q = 0$. The resulting equation in the notations $\Gamma(p,q-p|p',q-p') \equiv \Gamma(p|p')$ reads

$$\Gamma(p|p') = \tilde{\Gamma}(p|p') - \frac{T}{(2\pi)^3} \times \sum_{n'} \int d\vec{k}\ \tilde{\Gamma}(p|k)G(k)G(-k)\Gamma(k|p'). \tag{2}$$

In Eq. (2) $\tilde{\Gamma}(p|p')$ represents the block of the so-called irreducible diagrams, i.e., the diagrams that cannot be cut into the two parts by crossing only two parallel electronic lines. From a mathematic viewpoint the very instability in the Cooper channel owes its origin to the logarithmic singularity and divergent contributions at the summation and integrations inside the blocks represented in Eq. (2) by the product of two Green's functions $G(k)G(-k)$. The electron Green's function in the thermodynamic technique has the form [10]

$$G(k) = \left[i\nu_n - \left(\vec{k}^2 - p_F^2\right)/2m\right]^{-1}. \tag{3}$$

Generally, the matrix element for the electron-electron scattering has the following form:

$$M(p|p') = V(\vec{p}-\vec{p}') - \gamma^2(\vec{p}-\vec{p}') \times D_0(p-p'). \tag{4}$$

Here $D_0(p-p')$ is the phonon Green's function,

$$D_0(p-p') = -\omega_0^2(\vec{p}-\vec{p}')/[(\varepsilon_n-\varepsilon_m)^2 + \omega_0^2(\vec{p}-\vec{p}')], \tag{5}$$

$V(\vec{p}-\vec{p}')$ is the direct Coulomb term; $\gamma(\vec{p}-\vec{p}')$ is the electron-phonon coupling constant. $M(p|p')$ is the first term in the perturbation expansion for the block $\tilde{\Gamma}(p|p')$ in Eq. (2). The Coulomb potential $V(\vec{p}-\vec{p}')$ in Eq. (4) will be omitted. Although formally $V(\vec{p}-\vec{p}')$ could be treated below on equal footing with the phonons, however, because of its singular role of the Coulomb repulsion in the theory of superconductivity it needs special consideration (see Ref. [14]). Below we focus on the interaction mediated by phonons in nonadiabatic conditions. From now on,

$$M(p|p') = -\gamma^2(\vec{p}-\vec{p}')D_0(p-p'). \tag{4a}$$

IV. POSING THE QUESTION

To make what follows below more transparent, the equations are simplified further by temporarily omitting dispersion in the spectrum of phonons. In (4a) and (5) now $\omega_0(\vec{p}-\vec{p}') \equiv \omega_0$,

$$M(p|p') \Rightarrow M(\varepsilon_n|\varepsilon_m) = -\gamma^2\omega_0^2/\left[(\varepsilon_n-\varepsilon_m)^2 + \omega_0^2\right], \tag{6}$$

and $\Gamma(p|p')$ depends only on the energy variables. Changing the notations again into $\Gamma(p|p') \Rightarrow \Gamma(\varepsilon_n|\varepsilon_m)$ one writes

$$\Gamma(\varepsilon_n|\varepsilon_{n'}) = \tilde{\Gamma}(\varepsilon_n|\varepsilon_{n'}) - T\sum_m \tilde{\Gamma}(\varepsilon_n|\varepsilon_m)\Pi^{(d)}(\varepsilon_m)\Gamma(\varepsilon_m|\varepsilon_{n'}). \tag{7}$$

The notation $\Pi^{(d)}(\varepsilon_m)$ in Eq. (7) stands for the integral,

$$\Pi^{(d)}(\varepsilon_m) = \int \frac{d\vec{k}}{(2\pi)^d} G(k)G(-k) = \int \frac{d\vec{k}}{(2\pi)^d} \frac{1}{\nu_m^2 + \left[(\vec{k}^2 - p_F^2)/2m\right]^2}. \tag{8}$$

[In (7) and (8) and in the equations below index d signifies dimensionality of the problem: $d = 2,3$].$M(\varepsilon_n|\varepsilon_m)$ (6) and, hence, $\Gamma(\varepsilon_n|\varepsilon_m)$ decrease at large $\varepsilon_n > \omega_0$; therefore *formally*, in Eq. (7) the summation over ε_m at a given ε_n converges at ε_m on the order of ω_0, the phonon frequency. In reality, such a cutoff makes sense only in a common metal where $\omega_0 \ll E_F$ and both the integration and the summation in (2) and (7) are limited to the narrow vicinity of the Fermi energy; in that case one returns to the familiar BCS prefactor $W \sim \omega_0$ in the expression (1) for T_C. In a general case when ω_0 is on the order of the Fermi energy E_F the diagrammatic expansion in Eq. (2) contains nonphysical terms that come about with the use of the perturbation theory.

In fact, in the opposite limit of $\omega_0 \gg E_F$, from Eqs. (6) and (7) one concludes that at $\varepsilon_n < \omega_0$, $\Gamma(\varepsilon_n|\varepsilon_{n'})$ is a constant $\Gamma(\varepsilon_n|\varepsilon_{n'}) \equiv \bar{\Gamma}$. However, substituting a constant $\Gamma(\varepsilon_n|\varepsilon_{n'})$ into the right-hand side of (7) one obtains the diverging expression.

The difficulty has the obvious origin. At high energies, that is, at $\vec{k}^2 \gg p_F^2$, electrons in the Fermi gas are indistinguishable from free electrons. Far from the Fermi surface the product $G(k)G(-k)$ goes over into the product $G^{(0)}(k)G^{(0)}(-k)$ of the two Green's functions for the two free Fermi particles,

$$G^{(0)}(k)G^{(0)}(-k) = \frac{1}{\varepsilon_m^2 + [\vec{k}^2/2m]^2}. \tag{9}$$

That is, the contribution from $\Pi^{(d)}(\varepsilon_m)$ (8) at large ε_m coincides with the expression of the second-order Born correction to the scattering amplitude for two *free* electrons, and the divergence in (7) must be removed by properly renormalizing the interaction [11].

As shown below, integrations in (7) actually converge at $\varepsilon_m \approx \mu$, and in the antiadiabatic regime the role of the cutoff parameter in Eq. (1) belongs to the Fermi energy itself $W \Rightarrow E_F$.

V. LOGARITHMIC APPROXIMATION IN THE GENERAL CASE

Let us focus on the treatment of the interactions mediated by photons in *nonadiabatic* conditions.

In general terms, the "logarithmic accuracy" signifies the approximation where the smallness of the product $\nu(E_F)M(p|p') \ll 1$ of the density of states and the matrix element is compensated by a large logarithmic factor $\nu(E_F)M(p|p')\ln(W/T) \approx 1$.

Return to the general case of ω_0 on the same order magnitude as E_F and rewrite Eq. (2) in the form ready for use both in the three dimensions (3D) and two dimensions (2D) ($d = 2,3$):

$$\Gamma(p|p') = \tilde{\Gamma}(p|p') - \frac{T}{(2\pi)^d} \times \sum_{n'} \int d\vec{k}\ \tilde{\Gamma}(p|k)G(k)G(-k)\Gamma(k|p'). \tag{2a}$$

Restoring dispersion in the phonons spectrum in Eq. (4a), for the bare vertex $\Gamma^{(1)}(p|p')$ in all equations we substitute the expression $\Gamma^{(1)}(p|p') \equiv M(p|p')$ from Eq. (4a). Define the quantum-mechanical scattering amplitude $\Gamma^{(0)}(p|p')$ for two electrons or any two band Fermi excitations by the equation,

$$\Gamma^{(0)}(p|p') = \Gamma^{(1)}(p|p') - \frac{T}{(2\pi)^d} \sum_{n'} \int d\vec{k}\ \Gamma^{(1)}(p|k)G^{(0)}(k) \times G^{(0)}(-k)\Gamma^{(0)}(k|p'). \tag{10}$$

Inverting Eq. (10) introduces the operator $\hat{L}$,

$$\hat{L}\Gamma^{(0)}(p|p') \equiv \Gamma^{(0)}(p|p') + \frac{T}{(2\pi)^d} \sum_{n'} \int d\vec{k}\ \Gamma^{(1)}(p|k)G^{(0)}(k) \times G^{(0)}(-k)\Gamma^{(0)}(k|p'). \tag{11}$$

The operator $\hat{L}$ in Eq. (11) provides the relation between the scattering matrix element in the Born approximation $\Gamma^{(1)}(p|p')$ and the exact scattering amplitude $\Gamma^{(0)}(p|p')$. The latter can be defined independently by solving the Schrödinger equation. In principle, the result would be equivalent to the *exact* sum of all terms in the diagrammatic expansion. (The inverse operator $\hat{L}^{-1}$ below deals with the well-known mathematical problem of reconstructing the potential from the data on the angular dependence of the scattering cross section.)

Rewrite Eq. (2a) in the form

$$\Gamma(p|p') + \frac{T}{(2\pi)^d} \sum_{n'} \int d\vec{k}\ \Gamma^{(1)}(p|k)G^{(0)}(k)G^{(0)}(-k)\Gamma^{(0)}(k|p') = \Gamma^{(1)}(p|p') + \Gamma^{(2)}(p|p') - \frac{T}{(2\pi)^d} \sum_{n'} \int d\vec{k}\ \Gamma^{(1)}(p|k) \times [G(k)G(-k) - G^{(0)}(k)G^{(0)}(-k)]\Gamma(k|p') - \frac{T}{(2\pi)^d} \sum_{n'} \int d\vec{k}\ \Gamma^{(2)}(p|k)G(k)G(-k)\Gamma(k|p'). \tag{12}$$

In Eq. (12) $\Gamma^{(1)}(p|p') \equiv M(p|p')$ and $\Gamma^{(2)}(p|p')$ are the irreducible diagrams of the second and higher orders in the value of $M(p|p')$.

From (10) and (11) follows:

$$\hat{L}\Gamma(p|p') = \hat{L}\Gamma^{(0)}(p|p') + \Gamma^{(2)}(p|p') - \frac{T}{(2\pi)^d} \sum_{n'} \int d\vec{k} \times \hat{L}\Gamma^{(0)}(p|k)[G(k)G(-k) - G^{(0)}(k)G^{(0)}(-k)] - \frac{T}{(2\pi)^d} \sum_{n'} \int d\vec{k}\ \Gamma^{(2)}(p|k)G(k)G(-k)\Gamma(k|p'). \tag{13}$$

Applying the inverse operator $\hat{L}^{-1}$ to the both sides of Eq. (13) one finally arrives at the following equation:

$$\Gamma(p|p') = \Gamma^{(0)}(p|p') + \Gamma^{(2)}(p|p') - \frac{T}{(2\pi)^d} \sum_{n'} \int d\vec{k}\ \Gamma^{(0)}(p|k)\tilde{\Pi}^{(d)}(k)\Gamma(k|p') - \frac{T}{(2\pi)^d} \sum_{n'} \int d\vec{k}\ \Gamma^{(2)}(p|k)G(k)G(-k)\Gamma(k|p'). \tag{14}$$

[Restricting by the first Born approximation in Eq. (10) one has $\Gamma^{(0)}(p|p') \simeq M(p|p')$; $\hat{L}^{-1}\Gamma^{(2)}(p|p') \simeq \Gamma^{(2)}(p|p')$].

In the upper line of Eq. (14) $\tilde{\Pi}^{(d)}(k)$ is the difference,

$$\tilde{\Pi}^{(d)}(k) = [G(k)G(-k) - G^{(0)}(k)G^{(0)}(-k)]. \quad (15)$$

Convergence at the summation and integration over the momentum in the upper line of Eq. (14) is now guaranteed either by the presence of the phonon Green's function in Eq. (4) or by the fact that the block $\tilde{\Pi}^{(d)}(k)$ in (15) decreases at large $k = (\varepsilon_m, \vec{k})$. As to the second term in Eq. (14), recall that by the very definition [see in Eq. (2)], $\tilde{\Gamma}(p|p')$ represents the block of the so-called irreducible diagrams, i.e., the diagrams that cannot be cut into the two parts by crossing only two parallel electronic lines. Therefore the second term $\Gamma^{(2)}(p|p')$ comes from interactions between particles and is nonzero for (p, p') only in the vicinity of the Fermi surface. Correspondingly, the summation and integration in the second term Eq. (11) take place near the chemical potential.

To determine the temperature of the superconductivity onset one must find the *eigenvalue* of the homogeneous equation (14). Rewriting $\Gamma(p|p') \Rightarrow \psi(p) \times \psi(p')$ from (14) follows the equation for the function $\psi(p)$:

$$\psi(p) = -\frac{T}{(2\pi)^d} \sum_m \int d\vec{k}\; M(p|p')\tilde{\Pi}^{(d)}(k)\psi(k) - \frac{T}{(2\pi)^d} \times \sum_{n'} \int d\vec{k}\; \Gamma^{(2)}(p|k)G(k)G(-k)\psi(k). \quad (14a)$$

VI. MATRIX EQUATION IN THE CASE OF DISPERSIONLESS PHONONS

It is instructive to solve Eq. (14a) in the case of $M(\varepsilon_n|\varepsilon_m) \equiv -\gamma^2\omega_0^2/[(\varepsilon_n - \varepsilon_m)^2 + \omega_0^2]$ Eq. (6). In this example, most calculations can be performed at the end.

Denote the solution of the homogeneous Eq. (14) for dispersionless phonons as $\psi(k) \Rightarrow \psi(\varepsilon_n)$. One has

$$\psi(\varepsilon_n) = -T \sum_m M(\varepsilon_n|\varepsilon_m)\tilde{\Pi}^{(d)}(\varepsilon_m)\psi(\varepsilon_m) - \frac{T}{(2\pi)^d} \times \sum_{n'} \int d\vec{k}\; \Gamma^{(2)}(p|k)G(k)G(-k)\psi(\varepsilon_m). \quad (16)$$

[Compare with Eq. (14a). For the derivation of the expressions for the kernel $\tilde{\Pi}^{(d)}(\varepsilon_m)$, see Appendix A]. In two dimensions,

$$\tilde{\Pi}^{(2)}(\varepsilon_n) = \left(\frac{m}{2\pi}\right)\left(\frac{1}{\varepsilon_n}\arctan\frac{\mu}{\varepsilon_n}\right). \quad (17a)$$

For three dimensions,

$$\tilde{\Pi}^{(3)}(\varepsilon_n) = \frac{(2m)^{3/2}}{2\pi}\left(\frac{1}{2\varepsilon_n}\right)[\sqrt{\mu + i\varepsilon_n} + \sqrt{\mu - i\varepsilon_n} - \sqrt{i\varepsilon_n} - \sqrt{-i\varepsilon_n}] \quad (17b)$$

($\mu \equiv E_F$). Without terms quadratic in $M(\varepsilon_n|\varepsilon_m)$ Eq. (16) acquires the transparent form of the matrix equation,

$$\psi(\varepsilon_n) = -T\sum_m M(\varepsilon_n|\varepsilon_m)\tilde{\Pi}^{(d)}(\varepsilon_m)\psi(\varepsilon_n). \quad (18)$$

To emphasize this once again, $\tilde{\Pi}^{(d)}(\varepsilon_m)$ in the expressions (17a) and (17b) decreases at $\varepsilon_m > E_F$, thereby convergence of the summation in (18) is guaranteed at *any ratio* between ω_0 and E_F. [At the arbitrary ω_0 and E_F, Eq. (18) will be solved numerically elsewhere.]

In the limit $\omega_0 \ll E_F$ one is returned to the BCS result. In the opposite limit $\omega_0 \gg E_F = \mu$ the main contribution in (18) comes from $\varepsilon_m \ll \omega_0$. At $\varepsilon_n, \varepsilon_m \ll \omega_0$, $M(\varepsilon_n|\varepsilon_m)$ is a constant [$M(\varepsilon_n|\varepsilon_m) \simeq -\gamma^2$], and in the antiadiabatic limit the temperature T_C is defined by the algebraic equation,

$$1 = \gamma^2 T \sum_m \tilde{\Pi}(\varepsilon_m). \quad (19)$$

[The summation over ε_m in $T\sum_m \tilde{\Pi}(\varepsilon_m)$ is carried out in Appendix B].

Before proceeding further, return however to the contribution from the last term in Eqs. (14), (14a), and (16) containing $\Gamma^{(2)}(p|k)$. $\Gamma^{(2)}(p|k)$ is of the second order in $M(p|p')$. One can verify that all second-order diagrams constituting $\Gamma^{(2)}(p|k)$ belong to the class of crossing diagrams [11]. Therefore, in the adiabatic limit $\omega_0 \ll E_F$ this term is negligibly small.

At $\omega_0 \gg E_F$, the logarithmic contribution that comes about from the summation and the integrations of the product of the two Green's functions $G(k)G(-k)$ in (16) is multiplied by a factor that must be calculated by performing the internal integrations in the diagrammatic expression for $\Gamma^{(2)}(p|k)$. Since $\Gamma^{(2)}(p|k)$ is *quadratic* in $M(p|p')$, the result contributes only into the numeric coefficient in front of the expression for T_C. After simple, but somewhat tedious calculations, one obtains

$$1 = \gamma^2 \frac{m}{\pi\hbar^2} \ln\left(\frac{2\mu\gamma}{\pi e^2 T}\right), \quad (20a)$$

in 2D; in the 3D case [11],

$$1 = \gamma^2 \frac{m p_F}{2\pi^2\hbar^3} \ln\left[\left(\frac{2}{e}\right)^{7/3} \frac{\mu\gamma}{\pi T}\right]. \quad (20b)$$

Substituting $\mu \equiv E_F$ for the temperature of transition follows:

$$T_C^{(2D)} = \left(\frac{2\gamma\mu}{\pi e^2}\right) \exp\left(-\frac{\pi\hbar^2}{m\gamma^2}\right) \simeq 0.15 E_F \exp\left(-\frac{\pi\hbar^2}{m\gamma^2}\right), \quad (21a)$$

and

$$T_c^{(3D)} = \frac{\gamma}{\pi}\left(\frac{2}{e}\right)^{7/3} E_F \exp\left[-\frac{2\pi^2\hbar^3}{m p_F \gamma^2}\right] \approx 0.27 E_F \exp\left[-\frac{2\pi^2\hbar^3}{m p_F \gamma^2}\right]. \quad (21b)$$

Depending on the value of the parameters in the exponent and the value of E_F itself, T_C may be higher than one may expect in the adiabatic limit at the same value of γ.

At constant $M(\varepsilon_n|\varepsilon_m)$ the equations, as was just shown, could be solved to the very end. If the momentum dependence were included in $M(\varepsilon_n|\varepsilon_m)$, instead of the algebraic Eq. (19), one is to solve the integral equation (14). The details will vary depending on the problem at hand. Thus, for instance, the account of the Coulomb potential makes it necessary to consider the screening, etc. Solving Eq. (14) is beyond the scope of this paper.

VII. TEMPERATURE OF TRANSITION WITH LOGARITHMIC ACCURACY

In its new form the integral equation (14) is encumbered by the presence of the additional contribution with $\Gamma^{(2)}(p|k)$. Singling out the logarithmic factor of the block of $G(k)G(-k)$, this contribution is proportional to $\propto \Gamma^{(2)}(p|0)\nu(E_F)\ln(W/T)\Gamma(0|p')$ [here $(\cdots|0)$ signifies the choice of p or k at the Fermi level]. Equation (14a) acquires the form

$$\psi(p) = -\frac{T}{(2\pi)^d}\sum_m \int d\vec{k}\ \Gamma^{(0)}(p|k)\tilde{\Pi}^{(d)}(k)\psi(k) + F(p)\psi(0). \tag{14b}$$

In principle, both $\Gamma^{(2)}(p|0) \propto M^2$ [and $F(p)$] can be calculated for the arbitrary form of the interaction in (4) and (5), but calculations are tedious, and the resulting expressions are not useful.

Meanwhile, in the limiting case of a constant $\Gamma^{(1)}(\varepsilon_n|\varepsilon_{n'}) \approx -\gamma^2$*the exact solution* was obtained in Sec. VI, and it was shown that taking the $\Gamma^{(2)}(p|k)$ terms into account changes only the *numeric factor* of the order of unity in the expression (1) for T_C. With such accuracy the homogeneous weak-coupling Eq. (14) can be solved *directly*.

In fact, in Eq. (2) write

$$-\frac{T}{(2\pi)^3}\sum_{n'}\int d\vec{k}\ \tilde{\Gamma}(p|k)\tilde{\Pi}^{(d)}(k)\Gamma(k|p')$$

$$\Rightarrow \quad -T\sum_{n'}\int \tilde{\Gamma}(p|k)[m p_F \sin\theta\, d\theta)/(2\pi)^2] \times d\varsigma\ \tilde{\Pi}^{(d)}(k)\Gamma(k|p'). \tag{22}$$

[In (22) $\varsigma = (\vec{k}^2 - p_F^2)/2m \approx v_F(p - p_F)$; θ is the angle between two vectors $\vec{p}$ and $\vec{k}$. $\tilde{\Pi}^{(d)}(k)$ is from Eq. (15)].

Let $\varepsilon_n = \varepsilon_m$ and the vectors $\vec{p}$ and $\vec{k}$ be *on the Fermi surface*. The expression for $\tilde{\Gamma}(p|k) \equiv \tilde{\Gamma}(\theta)|_{FS}$ defines the factor in front of the logarithmic singularity in Eq. (22),

$$\left[\int_0^\pi \sin\theta\, d\theta\ \tilde{\Gamma}(\theta)\big|_{FS}\right]\frac{m p_F}{(2\pi)^2}\int_0^W \frac{d\varsigma}{\varsigma} th\frac{\varsigma}{2T}$$

$$\Rightarrow \quad \lambda\ \ln\left(\frac{2W\gamma}{\pi T}\right). \tag{23}$$

Equation (23) provides a definition of λ in $T_C = \text{const}\ W\ \exp(-1/\lambda)$.

For the problem at hand, $\tilde{\Gamma}(p|k)|_{FS} = -\gamma^2[2p_F(1-\cos\theta)]$. As to the prefactor, $W \approx \omega_0$ in the adiabatic limit, and $W \approx E_F$ in the extreme case of $\omega_0 \gg E_F$.

In detail the variation of the cutoff parameter W from $W \approx \omega_0$ to $W \approx E_F$ can be explored [omitting for simplicity the last term with $\Gamma^{(2)}(p|k)$] from the homogeneous integral equation (14),

$$\psi(p) = -\frac{T}{(2\pi)^d}\sum_m \int d\vec{k}\ \Gamma^{(0)}(p|k)\tilde{\Pi}^{(d)}(k)\psi(k), \tag{24}$$

where $\Gamma^{(0)}(p|k) = M(p|k)$ from Eq. (4a).

VIII. CONCLUSION AND SUMMARY

The Migdal theory of the electron-phonon interaction in metals is not applicable in the nonadiabatic regime, and the consistent analysis of equations for the temperature of the superconductivity transition is possible only in the weak-coupling approximation. We concentrated on studying the Cooper instability for the phonon-mediated attraction in the general case.

It was shown that in Eqs. (2) and (2a) in case of a nonadiabatic regime there appear nonphysical contributions that must be removed. In the diagrammatic Eqs. (2) and (2a) expansion is in powers of the *bare* matrix elements for scattering between excitations $\Gamma^{(1)}(p|p')$, not in terms of their *exact* quantum mechanical expressions of the latter $\Gamma^{(0)}(p|p')$. In the exact form the relation between the two is given by Eq. (10). In the framework of the weak-coupling BCS-like approximation this can be realized by subtracting the second-order Born corrections to the scattering amplitudes of the free Fermi particles.

The equations for temperature T_C of the superconductivity transition are given in the new form by the integral equations (14a) and (14b).

Several exact results could be obtained in the antiadiabatic limit for *dispersionless* phonons. The method can be applied to the arbitrary short-range interactions between Fermi particles [11]. In the general case and at an arbitrary value of the Migdal parameter ω_0/E_F the transition temperature T_C is found with logarithmic accuracy.

Most of the methods and the results discussed in the above can be extended to anisotropic superconductivity as well. The perturbation expansion can be developed on the arbitrary basis, and therefore the basic structure of Eq. (2) remains the same. In the limit $\omega_0 \ll E_F, \bar{W}$, where $\bar{W}$ is the bandwidth, one returns to a commonplace anisotropic BCS superconductivity in which case the interactions only in a vicinity of the Fermi surface are important. One must be more cautious at the so-called "realistic" calculations of the transition temperature assuming a specific band structure for the electronic spectrum in the case of a nonadiabatic regime. When the phonon frequency ω_0 is on the order or larger $E_F, \bar{W}$, one comes across the same divergences emerging as above via the (Born) perturbation corrections to the expressions of the matrix elements for *noninteracting* band excitations. One can easily verify, however, that the necessary procedure subtracting in Eq. (14) the product of the free (band) Green's functions $G^{(0)}(k)G^{(0)}(-k)$ in Eq. (15) remains exactly the same in its main features.

Note in conclusion that Eq. (2) for a *multiband* superconductor can obviously be rewritten in the matrix form.

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APPENDIX A

For the isotropic spectrum the integration over momentum $\vec{k}$ in

$$\tilde{\Pi}^{(d)}(\varepsilon_m) = \int d\vec{k}(2\pi)^{-d}[G(k)G(-k) - G^{(0)}(k)G^{(0)}(-k)]$$

is performed analytically by integrating over the variable $u = 2mk^2$. In the 2D calculation of $\tilde{\Pi}^{(2)}(\varepsilon_n)$ in Eq. (15) reduces to the standard integrals.

In 3D the expression (17b) for $\tilde{\Pi}^{(3)}(\varepsilon_n)$ is presented in the form

$$\tilde{\Pi}^{(3)}(\varepsilon_n) = \frac{(2m)^{3/2}}{(2\pi)^2}\int_o^\infty u^{1/2}du\left\{\frac{1}{\varepsilon_n^2 + (u-\mu)^2} - \frac{1}{\varepsilon_n^2 + u^2}\right\}.$$

Rotating the *contour*$\{0;\infty\}$ in the complex plane by 2π and calculating the residues one arrives at the expression Eq. (17b).

APPENDIX B

Transform the sum over ε_m in $T\sum_m \tilde{\Pi}(\varepsilon_m)$ into the integrals; in 2D,

$$T\sum_m \tilde{\Pi}^{(2)}(\varepsilon_m) = \frac{1}{2}\int_0^\infty du\left\{\frac{th(u-\mu)/2T}{u-\mu} - \frac{th(u/2T}{u}\right\},$$

and in 3D,

$$T\sum_m \tilde{\Pi}^{(3)}(\varepsilon_m) = \frac{1}{2}\int_0^\infty u^{1/2}du\left\{\frac{th(u-\mu)/2T}{u-\mu} - \frac{th(u/2T}{u}\right\}.$$

After integrating by parts and making use of the relation $T_C \ll \mu$ one arrives at the results in Eqs. (20) and (21) of the main text.

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