Edge-state transport in Floquet topological insulators

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Floquet topological insulators are systems in which the topology emerges only when a time-periodic perturbation is applied. In these systems one can define quasienergy states which replace the equilibrium stationary states. The system exhibits its nontrivial topology by developing edge-localized quasienergy states which lie in a gap of the quasienergy spectrum. These states represent a nonequilibrium analog of the topologically protected edge states in equilibrium topological insulators which exhibit an edge conductance of $2e^2/h$. Here we explore the transport properties of the edge states in a Floquet topological insulator. In stark contrast to the equilibrium result, we find that the two-terminal conductivity of these edge states is significantly different from $2e^2/h$. This fact notwithstanding, we find that for certain external potential strengths the conductivity is smaller than $2e^2/h$ and robust to the effects of disorder and smooth changes to the Hamiltonian's parameters. This robustness is reminiscent of the robustness found in equilibrium topological insulators. We provide an intuitive understanding of the reduction of the conductivity in terms of a picture where electrons in edge states are scattered by photons. We also consider the Floquet sum rule [A. Kundu and B. Seradjeh, Phys. Rev. Lett. **111**, 136402 (2013)], which was proposed in a different context. The summed conductivity recovers the equilibrium value of $2e^2/h$ whenever edge states are present. We show that this sum rule holds in our system using both numerical and analytic techniques.

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I. INTRODUCTION

Over the past decade topological insulators have become well known for their novel transport properties. The hallmark of these systems is their linearly dispersing, in-gap states. These states correspond to counterpropagating, helical edge modes. In a two-dimensional geometry these edge modes represent one-dimensional channels and lead to specific transport properties.

One example is a two-terminal device, where a source and drain are attached to the left and right of a sample and a bias voltage is applied across these terminals. The conductivity when these Fermi energies are placed in the gap (where the edge states live) is $\sigma = 2e^2/h$ [1–3]. In a six-terminal, or Hall-bar, geometry specific values of multiterminal resistances are expected [1–3] and these resistances are unique to counterpropagating, helical edge modes.

Although the number of confirmed topological insulators is ever increasing, materials with the correct physical parameters to support this state of matter are hard to come by. This has led many authors to consider ways in which to drive a material without any topological properties into a topological state. When a time-periodic potential is used to accomplish this task the resulting nonequilibrium topological state is called a Floquet topological insulator.

The field of Floquet topological insulators (and Floquet topological superconductors) has produced many interesting results of late [4–22]. The introduction of a time-periodic potential into the system breaks continuous time-translational invariance and so one must dispense with the notion of an energy spectrum. A time-periodic field does have discrete time-translational invariance and therefore one has the ability to define an analogous concept called the "quasienergy" spectrum [23]. In Floquet topological insulators one uses an externally applied time-periodic field of carefully chosen parameters to drive the system into its topological phase. The topology is manifest in in-gap, edge modes which are created in the quasienergy spectrum. Such a system then represents a nonequilibrium analog of topological insulators, but with the added flexibility of an external periodic potential.

In this work we study the transport properties of Floquet edge states. Our goal is to test whether transport through the edge modes of a two-dimensional Floquet topological insulator is quantized and robust as in the case of equilibrium topological insulators. We initially focus on a two-terminal geometry and then move on to consider a six-terminal setup. We expect the results and intuition developed here to be readily generalizable to other geometries. In general we find that the two-terminal conductivity of the Floquet edge states is significantly different from the typical equilibrium value of $2e^2/h$ and can be either larger or smaller than this distinctive value depending on how the strength of the external field is tuned. The same holds true for the resistance measurements typical of a topological insulator in a six-terminal setup.

The main results of this paper may be summarized as follows. The existence of quasienergy edge states in the Floquet topological insulator is accompanied by a conductivity of $\sigma < 2e^2/h$, when the chemical potential lies in the quasienergy gap. In addition, the value of $2e^2/h$ is obtained as a sum rule when the conductivity is summed over all "sidebands," i.e., over all energies which differ from a particular energy in the gap by an integer number of photon energies. Physically, the result $\sigma < 2e^2/h$ for nonequilibrium edge states corresponds to the presence of photons inhibiting access to the topologically protected edge states of the system [24].

Moreover, in regions where the conductivity is smaller than $2e^2/h$, we find that the calculated values are robust to the effects of disorder, system size, and changes to the Hamiltonian that maintain the energy gap. Such behavior is reminiscent of topologically protected edge states in equilibrium topological insulators and we indeed find that for the external potential strengths where we see this robustness there exist linearly dispersing, in-gap edge states in the quasienergy spectrum. In regions where the conductivity is larger than $2e^2/h$ no robustness exists and the gap is closed; hence we are probing bulk effects.

The reduction of the topologically protected conductivity away from $2e^2/h$ can be intuitively understood by borrowing some machinery from the field of photon-assisted tunneling (PAT) [25]. Namely, an electron that would normally tunnel into the edge states of the system has a finite probability of absorbing/emitting a photon and being scattered out of the edge state. From the viewpoint of quasienergy states this comes from understanding that the definite energy states of the leads do not perfectly overlap with the quasienergy states of the Floquet topological insulator [15]. The heuristic description in terms of scattering of electrons by photons can be applied to observe a so-called "Floquet sum rule" [4]. In short, the sum rule recovers all of the conductivity lost from PAT by summing over lead energies separated by photon energies $\hbar\Omega$, Ω being the frequency of the driving field. We have confirmed this sum rule using both numerics and an approximate analytic approach.

II. MODEL

Our model Hamiltonian is that of a quantum well heterostructure [1] irradiated by linearly polarized light and subjected to a disorder potential. It is given as follows

$$H_S = \sum_{\mathbf{k}} \psi_{\mathbf{k}}^{\dagger} \begin{pmatrix} \hat{H}(\mathbf{k},t) & 0 \\ 0 & \hat{H}^*(-\mathbf{k},t) \end{pmatrix} \psi_{\mathbf{k}} - \sum_{i,\alpha} w_i \psi_{i,\alpha}^{\dagger} \psi_{i,\alpha}, \tag{1}$$

where $\psi_{\mathbf{k}}^{\dagger}$ is a four-component creation operator for electrons at momenta $\mathbf{k}$ in angular momentum state $m_J = (1/2,3/2,-1/2,-3/2)$ and $\psi_i^{\dagger}$ is its Fourier transform. The first term above is the Hamiltonian of the clean, irradiated heterostructure and we have used $\hat{H}(\mathbf{k}) = \epsilon_{\mathbf{k}}\sigma_0 + \mathbf{d}(\mathbf{k})\cdot\sigma + 2(\mathbf{V}\cdot\sigma)\cos\Omega t$. The second term takes into account disorder. We have used the standard definitions $\mathbf{d}(\mathbf{k}) = (A\sin k_x, A\sin k_y, M - 4B + 2B(\cos k_x + \cos k_y))$ and $\epsilon_{\mathbf{k}} = C - 2D(2 - \cos k_x - \cos k_y)$ and draw the disorder parameters, $\{w_i\}$, randomly from an evenly distributed sample between $-W/2$ and $W/2$.

Following Lindner and coworkers [5], we set $C = D = 0$, $A = B = 0.2|M|$ and set $|M| = 1$ throughout (i.e., all energies are in units of $|M|$). To simulate a trivial system we set $M = -1$ so that $\mathrm{sgn}(M/B) = -1$ [1,5]. We take $\mathbf{V} = V_{\mathrm{ext}}\hat{z}$ for concreteness. Note that the field we consider here, and our subsequent observations in this paper, is assumed to always be "on." We consider a field that was turned on in the distant past and is not switched off throughout the duration of our calculations. Furthermore, from this point forward we fix $\hbar\Omega = 2.3|M|$.

The goal of this paper will be to understand the transport properties of the nonequilibrium system described above. In order to accomplish this we must couple the system to leads/electrodes. We leave the specifics of this process to the appendices of this paper and here only discuss the matter at a high level. We model the leads as being static (in time); i.e., the time-dependent field abruptly turns off at the leads. Ultimately the leads result in the system experiencing a self-energy proportional to $\Gamma(t)/2 = \sum_\lambda \Gamma_\lambda(t)/2$, where $\Gamma_\lambda(t)$ is the contribution of lead λ. In this paper we will work to simplify our discussion by employing the "wide-band" approximation. This phenomenological approach assumes that the density of states of the leads is constant over the energy scales in which we are interested. This amounts to assuming that the Fourier transform of the lead operators, $\Gamma_\lambda(\epsilon)$, is independent of ϵ; i.e., $\Gamma_\lambda(\epsilon) \simeq \Gamma_\lambda$.

In the work in Ref. [24] we have studied a system with $\mathrm{sgn}(M/B) > 0$. In other words, the system we were concerned with was a topological insulator, more specifically a quantum spin-Hall insulator, *before* any periodic perturbation was applied. Our work was interested in observing the behavior of the topological edge-states in this system in the presence of a time-periodic drive. In contrast, our work here is focused on a system with trivial topology in equilibrium; there are no edge states without the time dependence. The system is a true Floquet topological insulator in the sense that its edge states only develop after a time-periodic perturbation is applied. These edge states rely crucially on band mixing that comes from the periodic perturbation being *on-resonance* [5]; i.e., the quantity $\hbar\Omega$ connects different parts of the band structure. The work in Ref. [24] only considers off-resonant light, where $\hbar\Omega$ does not join any existing eigenstates. This is in contrast to other systems, for example graphene [6–10,26], where the Floquet topological insulator can be driven using an off-resonant perturbation [26].

Our understanding relies primarily on Floquet states [23]. Floquet states are the extension of stationary states to time-periodic systems. In a time-periodic system one deals with (Floquet) states that solve the Schrödinger equation and are characterized by a definite quasienergy. These states are traditionally written as $|\psi_{\tilde{\eta}}(t)\rangle = e^{-i\tilde{\eta}t/\hbar}|\phi_{\tilde{\eta}}(t)\rangle$, which leads to the eigenvalue equation $[H(t) - i\hbar\partial_t]|\phi_{\tilde{\eta}}(t)\rangle = \tilde{\eta}|\phi_{\tilde{\eta}}(t)\rangle$, where $H(t)$ is the full Hamiltonian of the system, $\tilde{\eta}$ are the quasienergies, and $|\phi_{\tilde{\eta}}(t+T)\rangle = |\phi_{\tilde{\eta}}(t)\rangle$. We note that if $|\phi_{\tilde{\eta}}(t)\rangle$ is an eigenstate with quasienergy $\tilde{\eta}$, then $e^{i\Omega t}|\phi_{\tilde{\eta}}(t)\rangle$ is also an eigenstate but with quasienergy $\tilde{\eta} + \hbar\Omega$. Therefore the quasienergy spectrum is only unique up to integer multiples of $\hbar\Omega$. This allows us to define a "Brillouin zone" for the quasienergies; we will call this the Floquet zone. For this work we consider $0 \leqslant \eta < \hbar\Omega$; we will use the convention η to denote quasienergies confined to this zone while $\tilde{\eta}$ above is unconfined. This reflects the fact that energy in a time-periodic system is only conserved modulo $\hbar\Omega$; an electron in a quasienergy state $|\phi_\eta(t)\rangle$ can always absorb or emit a photon.

III. TWO-TERMINAL CONDUCTIVITY

Let us begin with our results for the two-terminal conductivity of this system. We calculate the conductivity numerically using Floquet-Landauer theory [10,26]. In this two-terminal setup, shown in Fig. 1(a), we consider the leads to be kept at a voltage such that the Fermi level of both leads, which we will refer to as the lead energies, takes a value E. We then apply a (vanishingly) small bias voltage V/e so that the Fermi energies of the two leads are $\epsilon_L = E + V$ and $\epsilon_R = E$. We study the differential conductivity at a lead energy of $E = \Omega/2$ which is where the edge states are expected to be found [5]. Referring to our results in Fig. 2(a) we see that, with the exception of a small area near $V_{\mathrm{ext}} = 0.3|M|$, the two-terminal conductivity generally decreases with V_{ext} in

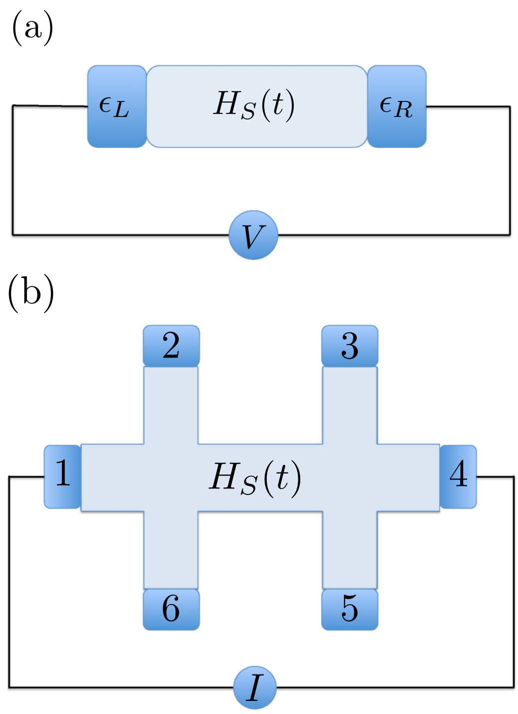

FIG. 1. Two geometries considered in this work. On the top we have the two-terminal setup with a bias voltage V/e offsetting the two Fermi energies ϵ_L and ϵ_R. On the bottom we have the six-terminal setup with a current I being driven between leads 1 and 4.

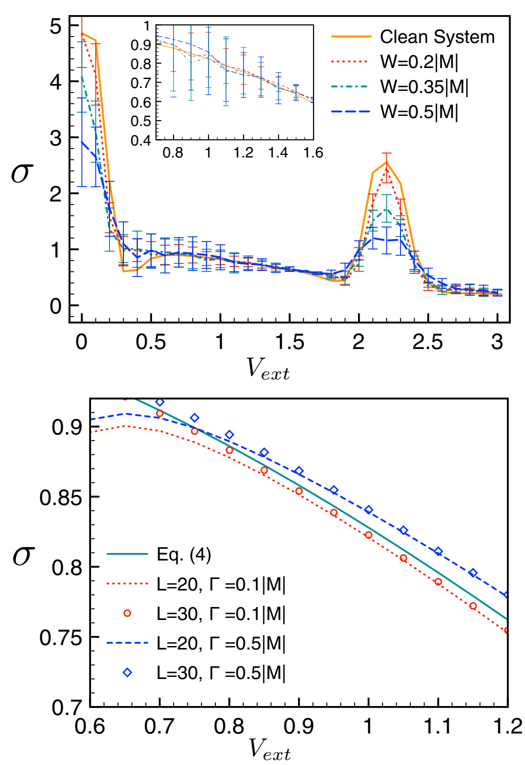

FIG. 2. Two-terminal conductivity in units of e^2/h. The top shows the conductivity at $E = \Omega/2$ in a two-terminal setup as a function of external potential strength for various disorder strengths W and system size $L = 20$. The bottom is the conductivity for various values of the system size, L, and the lead coupling parameter Γ over a region where edge states are present.

the range of parameters considered. We note that nowhere do we see a saturation to a value of $\sigma = 2e^2/h$, nor any other constant value. This fact notwithstanding, our results do have the striking feature that after a certain value of V_{ext} the conductivity becomes insensitive to the effects of disorder; in that region all of the curves overlap. In Fig. 2(b) we see that in this same region our results are insensitive to system length L and to the parameter Γ which describes the strength of the coupling to the leads. Thus we note our second result: for some values of V_{ext} the calculated conductivity is robust in the same way as for an equilibrium topological insulator.

The robustness in the conductivity coexists with the presence of edge states in the quasienergy spectrum. To show this we consider the system in the absence of leads and in a semi-infinite cylindrical geometry. By semi-infinite geometry we mean a system with open boundary conditions in the y direction and periodic boundary conditions in the x direction The quasienergy for our model appears in Fig. 3 for several values of the external potential strength V_{ext}. For small driving strength the gap remains closed, but as the strength is increased the gap opens up leaving linearly dispersing states. Further inspection of these states reveals that they reside on the edge of the system [5]. In general we have found that when this gap is open and large enough to withstand the effects of disorder or coupling to the leads, the value calculated for the conductivity is robust in the same sense as edge states in a topological insulator.

IV. PHOTON-INHIBITED TRANSPORT AND FLOQUET SUM RULE

Thus far we have shown that when this system plays host to edge states the conductivity that we find appears to be topologically robust. We now address the question of why it does not have the hallmark value of $2e^2/h$. For this we further generalize a technique inspired by photon-assisted tunneling [25,27] and used in Ref. [24]. In this work it was shown that for a topological heterostructure the presence of an external time-periodic field reduces the conductivity away from $2e^2/h$. This reduction and other subsequent results can be accounted for by understanding that the external potential not only "dresses" the quantum well Hamiltonian but also splits this dressed system into sidebands [25,27]. The splitting means that the edge states of the system are only populated probabilistically, accounting for the reduction in the standard transport quantities. The specific application of Ref. [24] relied crucially on the driving potential being *off-resonance*, i.e., that it did not mix portions of the equilibrium band structure. The current system requires *on-resonance* light in order to drive the system into a topological state. In spite of this, our results are conducive to a similar interpretation in that we see topologically robust results in Fig. 2 that are different from $2e^2/h$.

To put this discussion on more general grounds we appeal to Floquet theory. As discussed previously, in a time-periodic system the states of interest are the steady-state solutions $|\psi_\eta(t)\rangle = e^{-i\eta t/\hbar}|\phi_\eta(t)\rangle$, where η is the quasienergy and $[H(t) - i\hbar\partial_t]|\phi_\eta(t)\rangle = \eta|\phi_\eta(t)\rangle$. For periodic dependence in

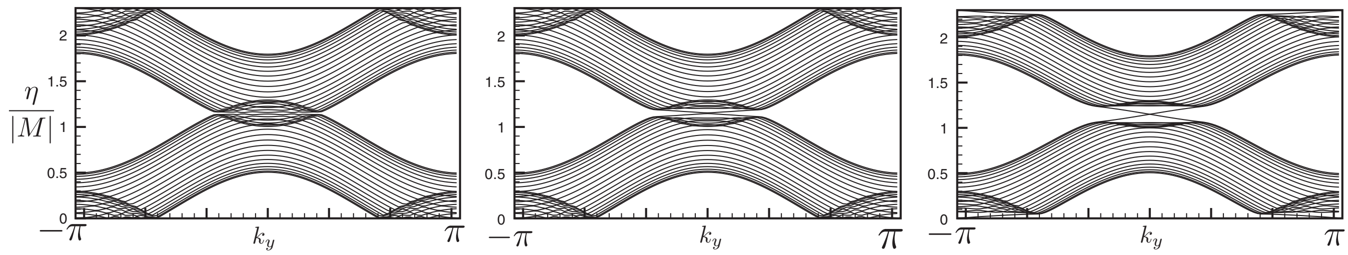

FIG. 3. Quasienergy spectrum of a topologically trivial sample at different driving amplitudes in a semi-infinite cylindrical geometry. The left plot is for $V_{\text{ext}} = 0.1|M|$, the middle $V_{\text{ext}} = 0.3|M|$, and the right $V_{\text{ext}} = 0.9|M|$.

t we are free to define the following decomposition:

$$|\phi_\eta(t)\rangle = \sum_n e^{-in\Omega t}|\phi_n\rangle. \tag{2}$$

In the literature the states $|\phi_n\rangle$ are commonly called sideband states [28] and are determined as solutions to the eigenvalue equation $\sum_n \bar{H}_{n,m}|\phi_m\rangle = (\eta + n\hbar\Omega)|\phi_n\rangle$ where $\bar{H}_{n,m} = \frac{1}{T}\int_0^T dt e^{i(n-m)\Omega t} H(t)$.

We now calculate the time-averaged expectation value of the energy in the steady state $|\psi_\eta(t)\rangle$ which we define as $\bar{E} = \frac{1}{T}\int_0^T dt \langle\psi_\eta(t)|H(t)|\psi_\eta(t)\rangle$. Using the sideband decomposition in Eq. (2) and the fact that $|\psi_\eta(t)\rangle$ solves the Schrödingier equation immediately gives

$$\bar{E} = \sum_n (\eta + \hbar\Omega n)\langle\phi_n|\phi_n\rangle. \tag{3}$$

Noting that $\langle\phi_n|\phi_n\rangle \geqslant 0$ and $\sum_n \langle\phi_n|\phi_n\rangle = 1$ (the latter property follows from the normalization of $|\phi_\eta(t)\rangle$) allows us to interpret the above average as follows. In the quantum state $|\psi_\eta(t)\rangle$ the energies $\eta + \hbar\Omega n$ occur with probability $\langle\phi_n|\phi_n\rangle$.

We now tie the above statistical interpretation to our observations of the transport in the Floquet topological insulators. For the system of interest one can calculate the appropriate quasienergies $0 \leqslant \eta < \hbar\Omega$ and their corresponding wave functions $|\phi_\eta(t)\rangle$; these are the steady states of our sample. Now, when electrons from the lead are injected into the system at some definite energy E, as opposed to an equilibrium case, only a portion of the sample state overlaps with the definite energy lead state [15]. Physically, we envision this in terms of electrons being able to absorb or emit photons once they enter the sample. For lead electrons at an energy $E = \eta + N\hbar\Omega$ there is only a probability $\langle\phi_N|\phi_N\rangle$ that the electron will absorb/emit enough photons to access the sample state with quasienergy η. This quasienergy spectrum may contain topologically protected edge states [12–16]. Now, when we try to access these states from a charge transport point of view we can only access the state at a certain probability, because of the possibility to absorb/emit photons. Therefore expected signatures of these edge states, e.g., $\sigma = 2e^2/h$ conductance, are probabilistically suppressed.

Note that this argument does not rely on the periodic perturbation being on or off resonance; it is simply a consequence of the discrete time-translational invariance. Therefore, when one is dealing with Floquet edge states it should be kept in mind that the weight of these edge states is distributed into sidebands as discussed above. Indeed in the current system one can approximately obtain a description of the conductivity at specific lead energies $E + n\hbar\Omega$ (where the edge states live). Quoting only the result here (for a detailed derivation please see the appendices),

$$\sigma(E + N\hbar\Omega) \simeq \frac{1}{2}\left[J_N^2\left(\frac{2V_{\text{ext}}}{\hbar\Omega}\right) + J_{N+1}^2\left(\frac{2V_{\text{ext}}}{\hbar\Omega}\right)\right] \times\tilde{\sigma}(E, V_{\text{ext}}), \tag{4}$$

where the relevant energy E is chosen to be in the vicinity of $\hbar\Omega/2$, where the localized quasienergy states appear. $\tilde{\sigma}(E, V_{\text{ext}})$ is a complicated function of the model parameters and, interestingly, cannot be thought of as the conductivity of some effective static system. We find numerically that $\tilde{\sigma}(E, V_{\text{ext}}) \simeq 2e^2/h$ when edge states are present in the quasienergy spectrum. The important implication of the above formula is that the conductivity can be thought of as an overall probabilistic factor times a conductivity expression. The above approximate result compares very well to our numerical calculations. A plot of this function appears in Fig. 2(b).

With the intuition for why the conductivity is suppressed in Floquet topological edge states, let us move on to present results for how the value of $2e^2/h$ can be recovered. In short, by setting lead energies at $\hbar\Omega/2 + n\hbar\Omega$ and summing over all n we should be able to recollect the lost statistical weight from the photon scattering. Towards this end we consider the quantity [4]

$$\bar{\sigma}(E, V_{\text{ext}}) = \sum_n \sigma(E + n\hbar\Omega) = \tilde{\sigma}(E, V_{\text{ext}}). \tag{5}$$

We calculate $\bar{\sigma}(E \simeq \Omega/2)$ for various different values of V_{ext} and also at different disorder strengths. Our results are presented in Fig. 4. What we see is quite satisfying: for a window of V_{ext} values we see that $\bar{\sigma} = 2e^2/h$. Moreover, this window corresponds to the same parameter regime where there are in-gap quasienergy edge states, and insensitivity of the system to disorder, system size, and other parameters in Figs. 2 and 3.

We understand the plot in Fig. 4 as follows. For smaller V_{ext} the external field is not strong enough to open a gap and "expose" the edge states. Therefore the conduction σ is a result of bulk processes and thus sensitive to disorder strength. As V_{ext} gets large enough to open a sufficiently stable gap the edge states appear in this gap and are unobscured by bulk states. Here we see $\bar{\sigma} = 2e^2/h$ and an *insensitivity* to disorder strength. Eventually V_{ext} becomes so strong that the gap closes

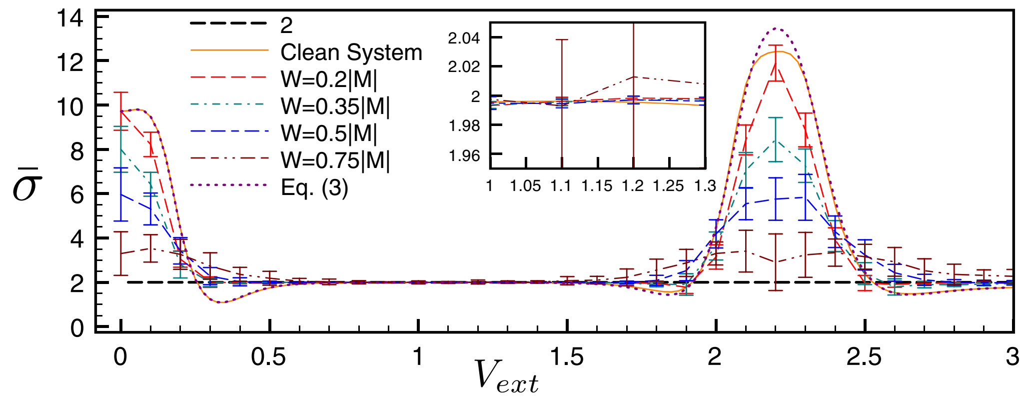

FIG. 4. Results for disorder-averaged summed conductivity, i.e., Eq. (5) in the text, with $M = -1$ and $E = \Omega/2$ and in units of e^2/h for various disorder strengths W. The inset shows a zoomed in picture of the first area of conductivity quantization. The disorder plots are constructed by averaging over 40 randomly drawn collections of disorder potentials while the error bars represent one standard deviation of these data. Note that some error bars in the insets are too small to see. These data have been obtained from a calculation on a 20×20 lattice.

again and bulk states dominate. In this case $\bar{\sigma}(\hbar\Omega/2) > 2e^2/h$ representing bulk conductivity. When the gap opens again at a larger external potential we see a reversion back to $\bar{\sigma} = 2e^2/h$.

V. HALL-BAR GEOMETRY AND EDGE STATES

Up to now we have presented our findings in a two-terminal device geometry. We now move on to study a six-terminal device in an effort to further illustrate that the conductivity discussed above is indeed a result of conduction along the edge of the sample, and not some coincidental edge effect. Our setup is motivated by experiments on Hall-bar systems [2,10,29]. An illustration of the setup that we have in mind appears in Fig. 1(b).

In the absence of a periodic driving potential the six-terminal geometry is used as follows. Assuming that all voltages are close to the Fermi level and that all leads are identical so that we can drop the pumped current [10], one approximates Eq. (A5) as

$$\bar{I}_\lambda = -\frac{e^2}{h}\sum_{\lambda'}[T_{\lambda,\lambda'}(E_f)V_{\lambda'} - T_{\lambda',\lambda}(E_f)V_\lambda], \tag{6}$$

where λ labels each of the side terminals. For a spin-Hall insulator dissipationless edge states exist and so one expects $T_{i,i+1} = T_{i+1,i} = 1$, where we periodically identify $6+1 \rightarrow 1$. Now one imagines driving a current I from contact 1 to contact 4. We then have $\mathbf{I} = (I,0,0,-I,0,0)^T$ where we have defined $\mathbf{I}_i = \bar{I}_i$. Inverting Eq. (6) one can find the voltages required to drive such a current. Doing so gives $V_1 - V_4 = \frac{3h}{2e^2}I$ and $V_2 - V_3 = \frac{h}{2e^2}I$. Defining $R_{i,j} = (V_i - V_j)/I$ as the resistance between terminals i and j we find $R_{1,4} = \frac{3}{2}h/e^2$ and $R_{2,3} = \frac{1}{2}h/e^2$. These values are unique to transport from dissipationless, helical edge states.

Here we will discuss a generalization of this concept to the effect which we have discussed so far. For our insulator in the topological regime we find that $T_{i,j} = 0$ except for the off-diagonal elements $T_{i,i+1}$ and $T_{i+1,i}$, where again we periodically identify $6+1 \rightarrow 1$. In contrast to the equilibrium observation we find that $T_{i,i+1} = T_{i+1,i} = T^{(n)} \neq 1$ where $T^{(n)}$ is the tunneling value for the lead Fermi energies near the gap in the nth Floquet zone. We now again imagine driving a current I from contact 1 to contact 4 with the bias voltages set near the middle of the gap in the nth Floquet zone. Defining $R^{(n)}_{i,j} = (V_i - V_j)/I$ as the resistance between terminals i and j in this case we find $R^{(n)}_{1,4} = \frac{3}{2T^{(n)}}h/e^2$ and $R^{(n)}_{2,3} = \frac{1}{2T^{(n)}}h/e^2$. Note that the signature results in terms of rational fractions of h/e^2 are lost; they have been reduced by a factor of $T^{(n)}$.

Using the above result we can realize two interesting properties of these Floquet devices. The first is the following:

$$\frac{R^{(n)}_{1,4}}{R^{(n)}_{2,3}} \simeq 3 \;\; \forall n. \tag{7}$$

That is, taking the ratio of these two resistances gives 3 regardless of which Floquet zone the Fermi energies are set in. Finally, in analogy with how we can retain the quantized value of σ in the two-terminal device considered above we can retain the equilibrium result here as follows:

$$\bar{R}^F_{1,4} = \left(\sum_n \frac{1}{R^{(n)}_{1,4}}\right)^{-1} = \frac{3}{2}h/e^2 \tag{8}$$

and

$$\bar{R}^F_{2,3} = \left(\sum_n \frac{1}{R^{(n)}_{2,3}}\right)^{-1} = \frac{1}{2}h/e^2. \tag{9}$$

These results are consistent with the picture developed above of edge states that are only occupied in a probabilistic way. The fact that $T_{i,j\neq i\pm 1} = 0$ reflects the edge conductance. The fact that $T_{i,i\pm 1} < 1$ reflects the fact that in a periodically driven system electrons entering lead i have a probability to absorb or emit a photon before reaching terminal $i \pm 1$. Thus $T_{i,j}$ is reduced. By summing over all Floquet zones we again effectively sum over all of these probabilities and retain the expected equilibrium result.

As a numerical test of the above we calculate T_{ij} for lead energies $(n+1/2)\hbar\Omega$ for $V_{\text{ext}} = 0.675|M|$, i.e., where we expect to see in-gap edge states. In our calculation we find $T_{ij} = 0$ for $i \neq j$ except when $j = i \pm 1$. We observe that $T_{i,i\pm 1}(n=0) = T_{i,i\pm 1}(n=-1) \simeq 0.46$,

$T_{i,i\pm1}(n=1) = T_{i,i\pm1}(n=-2) \simeq 0.04$, and zero for all other n's. These numerical values satisfy $\sum_n T_{i,i+1}(n) = 1$ and therefore satisfy the results established in this section.

VI. CONCLUSIONS

We have explored the transport properties of Floquet topological edge states in a quantum well heterostructure. At first we took a numerical approach which showed that in the presence of Floquet edge states in the quasienergy spectrum the two-terminal conductivity is topologically robust, albeit not quantized to $2e^2/h$.

To explain the reduction of the two-terminal conductivity compared to the equilibrium value of $2e^2/h$ we appealed to an intuitive description in terms of electrons being scattered by photons. This picture consists of viewing the Floquet edge states in the quasienergy spectrum as having their weight distributed into sidebands of energies $\eta + n\hbar\Omega$. The result of this sideband distribution is that as we attempt to inject an electron from a lead at some energy E there is a certain probability that it will absorb/emit enough photons to find the Floquet edge state.

The heuristic picture in terms of scattering by photons motivated us to propose a means to salvage the equilibrium conductivity of $2e^2/h$. This can be done using a recently proposed Floquet sum rule [4], which in our formalism has a natural interpretation. In our picture the topological Floquet states represent a superposition of states in various sidebands. The different coefficients in the superposition $\langle\phi_n|\phi_\eta\rangle$ determine the overlap. Our Floquet edge states nicely obey this sum rule showing that a summed conductivity value of $\bar{\sigma} = 2e^2/h$ is found when the external field is such that edge states in the quasienergy spectrum exist. Moreover, the result $\bar{\sigma} = 2e^2/h$ is robust to disorder even up to very large disorder strengths.

Finally, we have extended our results to study a six-terminal, or Hall-bar, setup. Here the equilibrium signatures of the quantum spin-Hall effect are several characteristic resistance measurements. We have shown that these resistances are increased relative to the equilibrium case. This fact notwithstanding, following our intuition from the two-terminal results we have suggested a sum rule for the six-terminal resistance measurements that recovers the equilibrium result. This sum rule is also intuitively explained in terms of photon inhibition of edge states.

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APPENDIX A: FLOQUET-LANDAUER FORMALISM FOR TRANSPORT

Here we present only the essential aspects of the Floquet-Landauer formalism. A more detailed description of our specific approach to this problem can be found in the Supplemental Material of [24]. Moreover, an excellent review can be found in Ref. [28].

We begin with a generic Hamiltonian which is given by

$$H = H_S(t) + H_L + H_C, \tag{A1}$$

where

$$H_S(t) = c^\dagger \mathcal{H}_S(t) c \tag{A2}$$

is the Hamiltonian of the sample where $c^\dagger = (c_1^\dagger, \ldots, c_{N_s}^\dagger)$ is a vector containing creation operators for each of the N_s degrees of freedom in the sample and $\mathcal{H}_S(t)$ is a $N_s \times N_s$ matrix coupling these degrees of freedom. This matrix contains both the static properties of the sample as well as the time-dependent effects of the periodic field. Next,

$$H_L = \sum_\lambda b_\lambda^\dagger \mathcal{H}_{L,\lambda} b_\lambda \tag{A3}$$

is the Hamiltonian of all of the leads where $b_\lambda^\dagger = (b_{1,\lambda}^\dagger, \ldots, b_{N_{l,\lambda},\lambda}^\dagger)$ is a vector containing creation operators for each of the N_l degrees of freedom in the lead λ and $\mathcal{H}_{l,\lambda}$ is an $N_{l,\lambda} \times N_{l,\lambda}$ matrix coupling these degrees of freedom. Finally,

$$H_C = \sum_\lambda (b_\lambda^\dagger \mathcal{K}_\lambda c + \text{H.c.}) \tag{A4}$$

is the Hamiltonian coupling the sample to each of the leads. $\mathcal{K}_\lambda$ is the $N_{l,\lambda} \times N_s$ matrix that describes these coupling strengths. The above model is completely general and makes no specification of band structure or dimension of the system.

The time-averaged current flowing through the sample out of lead λ can be shown to read as follows:

$$\bar{I}_\lambda = \frac{e}{h} \sum_{\lambda'} \int d\epsilon [T_{\lambda,\lambda'}(\epsilon) f_{\lambda'}(\epsilon) - T_{\lambda',\lambda}(\epsilon) f_\lambda(\epsilon)], \tag{A5}$$

where $f_\lambda(\epsilon) = \frac{1}{1+\exp[\beta_\lambda(\epsilon - eV_\lambda)]}$ and we have defined the transmission matrices

$$T_{\lambda,\lambda'}(\epsilon) = \sum_n \text{Tr}[\Gamma_\lambda(\epsilon + \hbar n\Omega) G^{(n)}(\epsilon) \Gamma_{\lambda'}(\epsilon) [G^{(n)}(\epsilon)]^\dagger], \tag{A6}$$

where

$$G^{(n)}(\epsilon) = \frac{1}{T} \int_0^T e^{in\Omega t} G(t,\epsilon) dt \tag{A7}$$

and $G(t,\epsilon)$ solves

$$\left(i\hbar \frac{d}{dt} + \epsilon - \mathcal{H}_S(t)\right) G(t,\epsilon) + i \int_0^\infty e^{i\epsilon\tau/\hbar} \Gamma(\tau) G(t-\tau,\epsilon) d\tau = I_{N_s \times N_s}. \tag{A8}$$

In the above $\Gamma(t)$ can be thought of as the self-energy obtained from integrating out the leads in the system. From our microscopic Hamiltonian above it is given by

$$\Gamma_\lambda(t) = \frac{1}{\hbar} \mathcal{K}_{L,\lambda}^\dagger g_{\lambda,L}(t) \mathcal{K}_{L,\lambda} \tag{A9}$$

and $\Gamma(t) = \sum_\lambda \Gamma_\lambda(t)$. In the above we have also defined $g_{\lambda,L}(t-t') = \exp[-\frac{i}{\hbar}\mathcal{H}_{L,\lambda}(t-t')]$, which is the temporal Green's function of the isolated leads. In this paper we will work to simplify our discussion by employing the "wide-band" approximation. This phenomenological approach assumes that the density of states of the leads is constant over the energy scales in which we are interested. This amounts to assuming that the lead operators, $\Gamma_\lambda(\epsilon)$, are independent of ϵ, $\Gamma_\lambda(\epsilon) \simeq \Gamma_\lambda$. Moreover, we make the assumption of identical leads so that $(\Gamma_\lambda)_{i,j} = \Gamma\delta_{i,j}\delta_{i,\mathbf{x}_\lambda}$, where $\mathbf{x}_\lambda$ are the set of all sample degrees of freedom connected to lead λ. We note that as we are dealing with topological transport properties none of these details should change our results.

In a two-terminal device the sample has leads attached to its left and right edges. For this type of device we label these leads as L for left and R for right. By conservation of current we must have $\bar{I}_R = -\bar{I}_L$, since the current entering the right lead must be equal to the current leaving the left lead. It is then sufficient to think only of $\bar{I}_R$. We now imagine biasing our sample so that we have a voltage $E/e + V/e$ on the left lead and a voltage $E/e + 0$ on the right lead, where E is the Fermi level of both leads. We then define the (differential) conductance as

$$\sigma(E) = \frac{d\bar{I}_R}{dV}\bigg|_{V=0} = \frac{e^2}{h}T_{R,L}(E). \tag{A10}$$

Thus for this geometry we have the simple result that the conductivity is simply given by the total transmission coefficient from the left lead to the right lead.

APPENDIX B: APPROXIMATE RESULT FOR THE CONDUCTIVITY

Here we detail the steps taken to derive the approximate result for the conductivity presented in Eq. (4). We begin (in the wide-band approximation) with the equation of motion for $G(t,\epsilon)$:

$$\left(i\hbar\frac{d}{dt} + \epsilon - \mathcal{H}_S(t) + \frac{i}{2}\Gamma\right)G(t,\epsilon) = I. \tag{B1}$$

We note the fact that $G(t,\epsilon)$ explicitly depends on time. This is in contrast to equilibrium where $G(t,\epsilon) \to G(\epsilon)$ has no time dependence [as $\mathcal{H}_S(t) = \mathcal{H}_S$] and the above simplifies to $[\epsilon - \mathcal{H}_S(t) + \frac{i}{2}\Gamma]G(\epsilon) = I$. The remaining time index can be shown to be periodic in time [28] and therefore we are free to define

$$G^{(n)}(\epsilon) = \frac{1}{T}\int_0^T dt e^{in\Omega t}G(t,\epsilon). \tag{B2}$$

It is the above object that we ultimately need to find.

This task is simplified by considering instead the auxiliary equation

$$\left(i\hbar\frac{d}{dt} + \epsilon - \mathcal{H}_S(t) + \frac{i}{2}\Gamma\right)\mathcal{G}(t,t',\epsilon) = \delta(t-t') \tag{B3}$$

and then noting that

$$G(t,\epsilon) = \int dt'\mathcal{G}(t,t',\epsilon). \tag{B4}$$

We will now focus on Eq. (B3). By writing $\mathcal{H}_S(t) - \frac{i}{2}\Gamma = \hat{\mathcal{H}}_S + H_{\text{ext}}(t)$ and introducing the rotating-frame picture

$$\mathcal{G}(t,t',\epsilon) = U_V(t)\check{\mathcal{G}}(t,t',\epsilon)U_V^\dagger(t'), \tag{B5}$$

where $i\hbar\frac{d}{dt}U_V(t) = H_{\text{ext}}(t)U_V(t)$. Then it follows that $\check{\mathcal{G}}(t,t',\epsilon)$ is a solution to

$$\left(i\hbar\frac{d}{dt} + \epsilon - \hat{\mathcal{H}}_S(t)\right)\check{\mathcal{G}}(t,t',\epsilon) = \delta(t-t') \tag{B6}$$

with the "rotating" version of $\hat{\mathcal{H}}_S$ being $\hat{\mathcal{H}}_S(t) = U_V(t)^\dagger\hat{\mathcal{H}}_S U_V(t)$. If the external, periodic potential were some potential with no internal structure coupling the degrees of freedom of the system, then we would have $\hat{\mathcal{H}}_S(t) = \hat{\mathcal{H}}_S$ as $H_{\text{ext}}(t)$ would commute with all other terms. This would immediately lead to a static system and a direct analog of photon-assisted tunneling.

For this particular problem $H_{\text{ext}}(t)$ commutes with itself at different times. As a result $U_V(t+T) = U_V(t)$ and it is useful to define

$$U_V(n) = \frac{1}{T}\int_0^T dt e^{in\Omega t}U_V(t),$$
$$\hat{\mathcal{H}}_S(n) = \frac{1}{T}\int_0^T dt e^{in\Omega t}\hat{\mathcal{H}}_S(t). \tag{B7}$$

Using all of these ingredients we then have

$$G^{(n)}(\epsilon) = \frac{1}{T}\int_0^T dt e^{in\Omega t}\int dt' U_V(t)\check{\mathcal{G}}(t,t',\epsilon)U_V^\dagger(t'). \tag{B8}$$

Our task becomes to solve Eq. (B6). We do so by defining

$$\check{\mathcal{G}}(t,t',\epsilon) = \frac{1}{T}\sum_{n,m} e^{-in\Omega t}e^{im\Omega t'}\check{\mathcal{G}}_{n,m}(\epsilon), \tag{B9}$$

which reduces Eq. (B6) to the difference equation

$$(\ell\hbar\Omega + \epsilon - \hat{\mathcal{H}}_S(0))\check{\mathcal{G}}_{\ell,m}(\epsilon) - \sum_{n\neq\ell}\hat{\mathcal{H}}_S(\ell-n)\check{\mathcal{G}}_{n,m}(\epsilon) = \delta_{\ell,m}, \tag{B10}$$

where $\hat{\mathcal{H}}_S(\ell)$ is defined in Eq. (B7). All of the Fourier expansions above when used with Eqs. (B2) and (B4) then give

$$G^{(n)}(\epsilon) = \sum_{\ell,m} U_V(n-\ell)\check{\mathcal{G}}_{\ell,m}(\epsilon)U_V^\dagger(-m). \tag{B11}$$

We now make note of a symmetry in the difference equation for $\check{\mathcal{G}}_{\ell,m}(\epsilon)$. Namely we note that simultaneously shifting $\ell \to \ell - k$, $m \to m - k$, and $\epsilon \to \epsilon + k\hbar\Omega$ for any integer k in Eq. (B10) shows that if $\check{\mathcal{G}}_{\ell,m}(\epsilon)$ is a solution then so is $\check{\mathcal{G}}_{\ell-k,m-k}(\epsilon + k\hbar\Omega)$; we thus identify [30]

$$\check{\mathcal{G}}_{\ell,m}(\epsilon) = \check{\mathcal{G}}_{\ell-k,m-k}(\epsilon + k\hbar\Omega). \tag{B12}$$

From the above one can conclude that all of the relevant information is contained in $\check{\mathcal{G}}_{\ell,0}(\epsilon) \equiv G_V^{(\ell)}(\epsilon)$. The operator $G_V^{(\ell)}(\epsilon)$ solves the equation

$$(\ell\hbar\Omega + \epsilon - \hat{\mathcal{H}}_S(0))G_V^{(\ell)}(\epsilon) - \sum_n \hat{\mathcal{H}}_S(\ell-n)G_V^{(n)}(\epsilon) = \delta_{\ell,0}. \tag{B13}$$

We can then write $\check{\mathcal{G}}_{\ell,m}(\epsilon) = G_V^{(\ell-m)}(\epsilon + m\hbar\Omega)$. Plugging this in above yields

$$G^{(n)}(\epsilon) = \sum_{\ell,m} U_V(n-\ell)G_V^{(\ell-m)}(\epsilon + m\hbar\Omega)U_V^\dagger(-m) \tag{B14}$$

sending $m \to -m$ and then $\ell \to \ell - m$ gives the main result of this discussion:

$$G^{(n)}(\epsilon) = \sum_{\ell,m} U_V(n+m-\ell)G_V^{(\ell)}(\epsilon - m\hbar\Omega)U_V^\dagger(m). \quad \text{(B15)}$$

We will use the above to find an approximate formula for the conductivity. We we will develop a few other necessary relations

So far our discussion has been general with the only assumption being that $[H_{\text{ext}}(t), H_{\text{ext}}(t')] = 0, \forall (t,t')$ (a more general treatment in the absence of this restriction is currently a work in progress [31]). At this point we specialize to the quantum well system which has been discussed in the main text. Our first task is to define the operators $U_V(m)$. We are interested in $\mathcal{H}_{\text{ext}}(t) = 2V_{\text{ext}}\cos(\Omega t)$ where $V_{\text{ext}} = V_e \sigma_z \otimes I_L$, I_L is the identity operator on the lattice and σ_z acts on spin. This operator commutes with itself at different times and is diagonal in "spin-lattice" space. This allows us to easily define the time evolution operator as follows:

$$U_V(t) = e^{-i\int_0^t dt' \mathcal{H}_{\text{ext}}(t')} = \exp\left[-i\frac{2V_e}{\hbar\Omega}\sin(\Omega t)\sigma_z \otimes I_L\right]. \quad \text{(B16)}$$

Note that no time ordering is required in the exponential because $[H_{\text{ext}}(t), H_{\text{ext}}(t')] = 0$. Finding the Fourier series of the above periodic function is made possible by the identity $e^{-ix\sin(\Omega t)} = \sum_m J_m(x)e^{-i\Omega m t}$ where $J_m(\cdot)$ is the Bessel function of the first kind of order m. We thus have

$$U_V(t) = \sum_m J_m\left(\frac{2V_e}{\hbar\Omega}\sigma_z \otimes I_L\right)e^{-i\Omega m t}. \quad \text{(B17)}$$

We can then read off

$$U_V(m) = J_m\left(\frac{2V_e}{\hbar\Omega}\sigma_z \otimes I_L\right) = J_m\left(\frac{2V_e}{\hbar\Omega}\right)S_m \otimes I_L = J_m\left(\frac{2V_{\text{ext}}}{\hbar\Omega}\right), \quad \text{(B18)}$$

where $S_m = \sigma_0$ if m is even and σ_z if m is odd and the second and third equality can be established using a series expansion of the Bessel function. The second relation above comes from the Bessel function property $J_m(-x) = (-1)^m J_m(x)$. The third relation, $J_m(\frac{2V_{\text{ext}}}{\hbar\Omega})$, is a compact form which will be useful in a derivation of the transmission elements.

With an explicit formula of $U_V(m)$ in hand we proceed to plug Eq. (B15) for the Green's function into the formula for $T_{\lambda,\lambda'}(\epsilon)$ which gives

$$T_{\lambda,\lambda'}(\epsilon) = \sum_{n,\ell,m}\sum_{\ell',m'} \text{Tr}\left[J_{n+m'-\ell'}\left(\frac{2V_{\text{ext}}}{\hbar\Omega}\right)J_{n+m-\ell}\left(\frac{2V_{\text{ext}}}{\hbar\Omega}\right) \times \Gamma_\lambda G_V^{(\ell)}(\epsilon - m\hbar\Omega)J_m\left(\frac{2V_{\text{ext}}}{\hbar\Omega}\right)\Gamma_{\lambda'}J_{m'}\left(\frac{2V_{\text{ext}}}{\hbar\Omega}\right) \times \left[G_V^{(\ell')}(\epsilon - m'\hbar\Omega)\right]^\dagger\right], \quad \text{(B19)}$$

where we have used the fact that the lead self-energies commute with V_{ext} (the leads make no distinction between different spins of particles). The sum over Bessel functions gives a delta function. After some index relabeling we are left with

$$T_{\lambda,\lambda'}(\epsilon) = \sum_{\ell,\ell',m}\text{Tr}\left[\Gamma_\lambda G_V^{(\ell)}(\epsilon - (m+\ell)\hbar\Omega)\Gamma_{\lambda'}J_{m+\ell}\left(\frac{2V_{\text{ext}}}{\hbar\Omega}\right) \times J_{m+\ell'}\left(\frac{2V_{\text{ext}}}{\hbar\Omega}\right)\left[G_V^{(\ell')}(\epsilon - (m+\ell')\hbar\Omega)\right]^\dagger\right]. \quad \text{(B20)}$$

We see that knowledge of $G_V^{(\ell)}(\epsilon - (m+\ell)\hbar\Omega)$ will allow us to find the tunneling matrices and hence the two-terminal conductivity. Towards this end we now write a formal solution of the difference equation for $G_V^{(\ell)}(\epsilon)$.

To make the notation more compact let us define

$$\bar{H}_\ell = \begin{cases}\hat{\mathcal{H}}_S(\ell): & \ell \neq 0,\\ 0: & \ell = 0.\end{cases} \quad \text{(B21)}$$

Besides being less cumbersome, this convention allows us to drop the restrictions on the sums and eventually employ Einstein summation convention. Starting from Eq. (B10) and acting on both sides with $[\ell\hbar\Omega + \epsilon - \hat{\mathcal{H}}_S(0)]^{-1}$ then gives

$$G_V^{(\ell)}(\epsilon) = (\ell\hbar\Omega + \epsilon - \hat{\mathcal{H}}_S(0))^{-1}\delta_{\ell,0} + \sum_n (\ell\hbar\Omega + \epsilon - \hat{\mathcal{H}}_S(0))^{-1}\bar{H}_{\ell-n}G_V^{(n)}(\epsilon). \quad \text{(B22)}$$

Let us note that $g_F(\epsilon) = [\epsilon - \hat{\mathcal{H}}_S(0)]^{-1}$ is the Green's function of the static system with Hamiltonian $\hat{\mathcal{H}}_S(0)$. Therefore

$$G_V^{(\ell)}(\epsilon) = g_F(\epsilon)\delta_{\ell,0} + \sum_j g_F(\epsilon + \ell\hbar\Omega)\bar{H}_{\ell-j}G_V^{(j)}(\epsilon). \quad \text{(B23)}$$

We now iterate this equation. To do this let us introduce some notation to extract the useful part of the above. First, we use implied summation over repeated indices that are not ℓ. Second, we define $g_i = g_F(\epsilon + i\hbar\Omega)$. This gives

$$G_V^{(\ell)}(\epsilon) = g_0\delta_{\ell,0} + g_\ell\bar{H}_\ell g_0 + g_\ell\bar{H}_{\ell-j}g_j\bar{H}_{j-j'}G_V^{(j')}(\epsilon). \quad \text{(B24)}$$

Repeated iteration of the above difference equation allows us to write

$$G_V^{(\ell)}(\epsilon) = g_0\delta_{\ell,0} + g_\ell(\bar{H}_\ell + \bar{H}_{\ell-j}g_j\bar{H}_j + \bar{H}_{\ell-j}g_j\bar{H}_{j-j'}g_{j'}\bar{H}_{j'} + \bar{H}_{\ell-j}g_j\bar{H}_{j-j'}g_{j'}\bar{H}_{j'-\alpha}g_\alpha\bar{H}_\alpha + \bar{H}_{\ell-j}g_j\bar{H}_{j-j'}g_{j'} \times \bar{H}_{j'-\alpha}g_\alpha\bar{H}_{\alpha-\beta}g_\beta\bar{H}_{\beta-\sigma}g_\sigma\bar{H}_\sigma + \cdots)g_0 \quad \text{(B25)}$$

or

$$G_V^{(\ell)}(\epsilon) = g_F(\epsilon)\delta_{\ell,0} + g_F(\epsilon + \ell\hbar\Omega)\check{D}_\ell(\epsilon)g_F(\epsilon), \quad \text{(B26)}$$

where

$$\check{D}_\ell(\epsilon) = \bar{H}_\ell + \bar{H}_{\ell-j}g_j\bar{H}_j + \bar{H}_{\ell-j}g_j\bar{H}_{j-j'}g_{j'}\bar{H}_{j'} + \bar{H}_{\ell-j}g_j\bar{H}_{j-j'}g_{j'}\bar{H}_{j'-\alpha}g_\alpha\bar{H}_\alpha + \bar{H}_{\ell-j}g_j\bar{H}_{j-j'}g_{j'} \times \bar{H}_{j'-\alpha}g_\alpha\bar{H}_{\alpha-\beta}g_\beta\bar{H}_{\beta-\sigma}g_\sigma\bar{H}_\sigma + \cdots. \quad \text{(B27)}$$

The above series can in turn be generated by (restoring the summation symbols)

$$\check{D}_\ell(\epsilon) = \bar{H}_\ell + \sum_j \bar{H}_{\ell-j} g_j \bar{H}_j + \sum_{j,j'} \bar{H}_{\ell-j} g_j \bar{H}_{j-j'} g_{j'} \check{D}_{j'}(\epsilon). \tag{B28}$$

Physically, the Green's function $G_V^{(\ell)}(\epsilon)$ is described by all possible processes starting at energy ϵ [indicated by amplitude $g_F(\epsilon)$] where the electrons absorb/emit a *net* number of photons ℓ [indicated by amplitude $\check{D}_\ell(\epsilon)$] and then end up at an energy eigenstate $\epsilon + \ell\hbar\Omega$ [hence $g_F(\epsilon + \ell\hbar\Omega)$].

We are interested in energies $E + N\hbar\Omega$ where $E \simeq \hbar\Omega/2$. Near such energies the Green's functions we need are then given by

$$\begin{aligned} G_V^{(\ell)}&(E + (N - m - \ell)\hbar\Omega) \\ &= g_F(E + (N - m)\hbar\Omega)\delta_{\ell,0} + g_F(E + (N - m)\hbar\Omega) \\ &\quad \times \check{D}_\ell(E + (N - m - \ell)\hbar\Omega) g_F(E + (N - m - \ell)\hbar\Omega). \end{aligned} \tag{B29}$$

We now assume that for the parameters we are interested in the unit of energy $\hbar\Omega$ connects two (and not more) points on the spectrum of $\hat{\mathcal{H}}_S(0)$. Namely an energy $-\Omega\hbar$ can move us from $\Omega/2$ to $-\Omega/2$. Any other photon processes are not possible though. Plotting the spectrum of $\hat{\mathcal{H}}_S(0)$ reveals this to be true for the parameters we have considered in the main text.

The above discussion leads us to make the approximation

$$g_F(E + K\hbar\Omega) = \delta_{K,0} g_F(E) + \delta_{K,-1} g_F(-E). \tag{B30}$$

The motivation for the above approximation is as follows. $g_F(\epsilon)$ is the Green's function for the system with Hamiltonian $\hat{\mathcal{H}}_S(0)$. We are interested in energies $\epsilon = E \simeq \hbar\Omega/2$, where Ω is the driving frequency. Now, owing to the finite bandwidth of the dressed Hamiltonian $\hat{\mathcal{H}}_S(0)$, there are no energy eigenstates at $E + N\hbar\Omega$ for $N > 0$. There are states at $E - \hbar\Omega \simeq -\Omega/2$ but none for any $\hbar\Omega$ below this. Thus the above approximation ignores values of K for which $\hat{\mathcal{H}}_S(0)$ has no states at $E + K\hbar\Omega$.

Let us now define

$$\tilde{g}_F(E) = g_F(E) + g_F(E)\check{D}_0(E) g_F(E) \tag{B31}$$

and use the fact that $E - \hbar\Omega \simeq -E$. We immediately find

$$\begin{aligned} G_V^{(\ell)}&(E + (N - m - \ell)\hbar\Omega) \\ &= \delta_{N,m}(\tilde{g}_F(E)\delta_{\ell,0} + g_F(E)\check{D}_1(-E) g_F(-E)\delta_{\ell,1}) \\ &\quad + \delta_{N+1,m}(\tilde{g}_F(-E)\delta_{\ell,0} + g_F(-E)\check{D}_{-1}(E) g_F(E)\delta_{\ell,-1}). \end{aligned} \tag{B32}$$

Plugging this into the transmission matrix formula gives

$$\begin{aligned} T_{\lambda,\lambda'}&(E + N\hbar\Omega) \\ &= J_N^2\left(\frac{2V_{\rm ext}}{\hbar\Omega}\right)\{\mathrm{Tr}[\Gamma_\lambda \tilde{g}_F(E)\Gamma_{\lambda'}\tilde{g}_F^\dagger(E)] + \mathrm{Tr}[\Gamma_\lambda g_F(-E)\check{D}_{-1}(E) g_F(E)\Gamma_{\lambda'}(g_F(-E)\check{D}_{-1}(E) g_F(E))^\dagger]\} \\ &\quad + J_{N+1}^2\left(\frac{2V_{\rm ext}}{\hbar\Omega}\right)\{\mathrm{Tr}[\Gamma_\lambda \tilde{g}_F(-E)\Gamma_{\lambda'}\tilde{g}_F^\dagger(-E)] + \mathrm{Tr}[\Gamma_\lambda g_F(E)\check{D}_1(-E) g_F(-E)\Gamma_{\lambda'}(g_F(E)\check{D}_1(-E) g_F(-E))^\dagger]\} \\ &\quad + J_{N+1}\left(\frac{2V_{\rm ext}}{\hbar\Omega}\right) J_N\left(\frac{2V_{\rm ext}}{\hbar\Omega}\right)\{\mathrm{Tr}[\Gamma_\lambda \tilde{g}_F(E)\Gamma_{\lambda'} S_1 (g_F(E)\check{D}_1(-E) g_F(-E))^\dagger] + \mathrm{Tr}[\Gamma_\lambda g_F(E)\check{D}_1(-E) g_F(-E)\Gamma_{\lambda'} S_1 \tilde{g}_F^\dagger(E)]\} \\ &\quad + J_{N+1}\left(\frac{2V_{\rm ext}}{\hbar\Omega}\right) J_N\left(\frac{2V_{\rm ext}}{\hbar\Omega}\right)\{\mathrm{Tr}[\Gamma_\lambda \tilde{g}_F(-E)\Gamma_{\lambda'} S_1 (g_F(-E)\check{D}_{-1}(E) g_F(E))^\dagger] + \mathrm{Tr}[\Gamma_\lambda g_F(-E)\check{D}_{-1}(E) g_F(E)\Gamma_{\lambda'} S_1 \tilde{g}_F^\dagger(-E)]\}, \end{aligned} \tag{B33}$$

where $S_1 = \sigma_z \otimes I_L$. Let us now consider $D_{-1}(E)$. Using our generating function we have

$$\check{D}_\ell(E) = \bar{H}_\ell + \bar{H}_{\ell-j} g_j \bar{H}_j + \bar{H}_{\ell-j} g_j \bar{H}_{j-j'} g'_j \check{D}_{j'}(E). \tag{B34}$$

This equation can be solved for both $D_{-1}(E)$ and $D_1(-E)$, and the results are

$$D_{-1}(E) = -[I + \bar{H}_1 g_F(E)\bar{H}_1 g_F(-E)]^{-1}\bar{H}_1 \tag{B35}$$

and

$$D_1(-E) = [I + \bar{H}_1 g_F(-E)\bar{H}_1 g_F(E)]^{-1}\bar{H}_1. \tag{B36}$$

Further let us note that

$$D_0(E) = [I - \bar{H}_1 g_F(-E)\bar{H}_{-1} g_F(E)]^{-1}\bar{H}_1 g_F(-E)\bar{H}_{-1}. \tag{B37}$$

Using this one can show that

$$\tilde{g}_F(E) = g_F(E) + g_F(E)\check{D}_0(E) g_F(E) = \frac{1}{E - [\bar{H}_0 + \bar{H}_1 g_F(-E)\bar{H}_{-1}]}. \tag{B38}$$

So $\tilde{g}_F(E)$ is the Green's function of a system with an effective Hamiltonian $\bar{H}_0 + \bar{H}_1 g_F(-E)\bar{H}_{-1}$. Now let us define

$$F_{\rm up}(E) = g_F(E)\check{D}_1(-E) g_F(-E) = g_F(E) H_1 \tilde{g}_F(-E) \tag{B39}$$

and

$$F_{\rm down}(E) = g_F(-E)\check{D}_{-1}(E)g_F(E) = -F_{\rm up}(-E). \tag{B40}$$

Using these definitions the tunneling elements can be written

$$\begin{aligned} T_{\lambda,\lambda'}(E+N\hbar\Omega) =& J_N^2\left(\frac{2V_{\rm ext}}{\hbar\Omega}\right)\{{\rm Tr}[\Gamma_\lambda\tilde{g}_F(E)\Gamma_{\lambda'}\tilde{g}_F^\dagger(E)] + {\rm Tr}[\Gamma_\lambda F_{\rm up}(-E)\Gamma_{\lambda'}F_{\rm up}^\dagger(-E)]\} \\ &+ J_{N+1}^2\left(\frac{2V_{\rm ext}}{\hbar\Omega}\right)\{{\rm Tr}[\Gamma_\lambda\tilde{g}_F(-E)\Gamma_{\lambda'}\tilde{g}_F^\dagger(-E)] + {\rm Tr}[\Gamma_\lambda F_{\rm up}(E)\Gamma_{\lambda'}F_{\rm up}^\dagger(E)]\} \\ &+ J_{N+1}\left(\frac{2V_{\rm ext}}{\hbar\Omega}\right)J_N\left(\frac{2V_{\rm ext}}{\hbar\Omega}\right)\{{\rm Tr}[\Gamma_\lambda\tilde{g}_F(E)\Gamma_{\lambda'}S_1F_{\rm up}^\dagger(E)] + {\rm Tr}[\Gamma_\lambda F_{\rm up}(E)\Gamma_{\lambda'}S_1\tilde{g}_F^\dagger(E)]\} \\ &-, J_{N+1}\left(\frac{2V_{\rm ext}}{\hbar\Omega}\right)J_N\left(\frac{2V_{\rm ext}}{\hbar\Omega}\right)\{{\rm Tr}[\Gamma_\lambda\tilde{g}_F(-E)\Gamma_{\lambda'}S_1F_{\rm up}^\dagger(-E)] + {\rm Tr}[\Gamma_\lambda F_{\rm up}(-E)\Gamma_{\lambda'}S_1\tilde{g}_F^\dagger(-E)]\}. \end{aligned} \tag{B41}$$

We now note that in our numerical calculations our system has energy eigenstates distributed symmetrically around $E = 0$ (as we have taken $C = D = 0$). In such a system we must have $WH_SW^\dagger = -H_S$ where W is some operator. From this it follows that $W\tilde{g}_F(-E)W^\dagger = -\tilde{g}_F(E)$ and also $WF_{\rm up}(-E)W^\dagger = -F_{\rm up}(-E)$. Using the fact that the leads (and $\Gamma_{\lambda'}S_1$) also obey this symmetry and inserting the identity in the form $W^\dagger W = I$ in strategic places above leaves

$$\begin{aligned} T_{\lambda,\lambda'}(E+N\hbar\Omega) =& \left[J_N^2\left(\frac{2V_{\rm ext}}{\hbar\Omega}\right) + J_{N+1}^2\left(\frac{2V_{\rm ext}}{\hbar\Omega}\right)\right] \\ &\times\{{\rm Tr}[\Gamma_\lambda\tilde{g}_F(E)\Gamma_{\lambda'}\tilde{g}_F^\dagger(E)] \\ &+ {\rm Tr}[\Gamma_\lambda F_{\rm up}(E)\Gamma_{\lambda'}F_{\rm up}^\dagger(E)]\}, \end{aligned} \tag{B42}$$

which we write as

$$\begin{aligned} T_{\lambda,\lambda'}(E+N\hbar\Omega) =& \frac{1}{2}\left[J_N^2\left(\frac{2V_{\rm ext}}{\hbar\Omega}\right) + J_{N+1}^2\left(\frac{2V_{\rm ext}}{\hbar\Omega}\right)\right] \\ &\times\hat{T}_{\lambda,\lambda'}(E,V_{\rm ext}), \end{aligned} \tag{B43}$$

where

$$\begin{aligned} &\hat{T}_{\lambda,\lambda'}(E,V_{\rm ext}) \\ &\quad = 2\{{\rm Tr}[\Gamma_\lambda\tilde{g}_F(E)\Gamma_{\lambda'}\tilde{g}_F^\dagger(E)] + {\rm Tr}[\Gamma_\lambda F_{\rm up}(E)\Gamma_{\lambda'}F_{\rm up}^\dagger(E)]\}. \end{aligned} \tag{B44}$$

We note that in a two-terminal geometry when there are edge states in the quasienergy spectrum we find that $\hat{T}_{\lambda,\lambda'}(E,V_{\rm ext}) = 2$. Each of the two terms above looks reminiscent of a conductivity. The first term looks like a contribution coming from edge states at an energy E, while the second term looks like a contribution coming from the edge states at $-E$ transitioning to E.

1. Expression for the effective Hamiltonian

We close with an expression for a derivation of the effective Hamiltonian. Recall that $\hat{\mathcal{H}}_S(t) = U_V^\dagger(t)\hat{\mathcal{H}}_SU_V(t)$ and that $\hat{\mathcal{H}}_S(n) = \frac{1}{T}\int_0^T e^{in\Omega t}\bar{H}(t)$; then by using the Fourier decomposition of $U_V(t)$ we can write

$$\hat{\mathcal{H}}_S(n) = \sum_m U_V^\dagger(m)\hat{\mathcal{H}}_SU_V(m+n). \tag{B45}$$

If we now recall the relation for the U_V operators derived above, then it immediately follows that

$$\hat{\mathcal{H}}_S(n) = \sum_m J_m\left(\frac{2V_e}{\hbar\Omega}\right)J_{m+n}\left(\frac{2V_e}{\hbar\Omega}\right)S_m\otimes I_L\hat{\mathcal{H}}_SS_{m+n}\otimes I_L. \tag{B46}$$

Now let us write

$$\hat{\mathcal{H}}_S = \sum_{\alpha,\beta}\bar{\mathcal{H}}_{\alpha,\beta}\sigma_\alpha\otimes R_\beta, \tag{B47}$$

where R_β is a complete set of operators in the space of the lattice. Then, defining $\zeta = \frac{2V_e}{\hbar\Omega}$, noting that for $\alpha = 1$ or 2 one can show that $S_m\sigma_\alpha S_{n+m} = (-1)^m\sigma_\alpha S_n$ whereas for $\alpha = 0$ or $\alpha = 3$ they do nothing, and using the Bessel function identities $\sum_m[J_m(\zeta)J_{m+n}(\zeta)] = \delta_{n,0}$ and $\sum_m[J_m(\zeta)(-1)^mJ_{m+n}(\zeta)] = \sum_m[J_m(\zeta)J_{-m+n}(\zeta)] = J_n(2\zeta)$, we have

$$\begin{aligned} \hat{\mathcal{H}}_S(m) =& \delta_{n,0}\sum_{\alpha=0,3,\beta}\bar{\mathcal{H}}_{\alpha,\beta}(\sigma_\alpha)\otimes R_\beta \\ &+ J_n(2\zeta)\sum_{\alpha=1,2,\beta}\bar{\mathcal{H}}_{\alpha,\beta}(\sigma_\alpha S_n)\otimes R_\beta. \end{aligned} \tag{B48}$$

It is most convenient to write this as

$$\hat{\mathcal{H}}_S(m) = \delta_{n,0}\bar{\mathcal{H}}_{03} + J_n(2\zeta)\bar{\mathcal{H}}_{12}(n). \tag{B49}$$

Thus the field does not touch terms in $\hat{\mathcal{H}}_S$ proportional to σ_0 or σ_3 and "dresses" the σ_1 and σ_2 terms.

The Green's function g_F and the transmission matrix elements $T_{\lambda,\lambda'}^F$ are the result of calculating the Green's function and the transport properties of a system described by a static Hamiltonian $\hat{\mathcal{H}}_S(0)$. Such a Hamiltonian looks similar to our original static Hamiltonian (before periodic perturbation) but with $\bar{\mathcal{H}}_{12}$ renormalized by the Bessel function $J_0(2\zeta)$. Moreover, σ is not simply the static conductivity of $\bar{\mathcal{H}}_{03}$.

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