Exchange interaction of two spin qubits mediated by a superconductor

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Entangling two quantum bits by letting them interact is the crucial requirement for building a quantum processor. For qubits based on the spin of the electron, these two qubit gates are typically performed by exchange interaction of the electrons captured in two nearby quantum dots. Since the exchange interaction relies on tunneling of the electrons, the range of interaction for conventional approaches is severely limited as the tunneling amplitude decays exponentially with the length of the tunneling barrier. Here, we present an approach to couple two spin qubits via a superconducting coupler. In essence, the superconducting coupler provides a tunneling barrier for the electrons which can be tuned with exquisite precision. We show that as a result exchange couplings over a distance of several microns become realistic, thus enabling flexible designs of multiqubit systems.

I. INTRODUCTION

Semiconductor based electron-spin qubits have made significant progress towards scalability. Single qubit gate fidelities demonstrated in some devices meet the requirements for quantum error correction [1,2], with other approaches not being far behind [3–5]. Two qubit gates have also been realized [6,7], and adequate fidelities seem within reach [2]. However, all these gates act over a very limited range of typically less than 1 μm, which severely constrains scalability as the resulting small qubit spacing leaves little room for wiring and local control electronics. For example, a surface code architecture [8], which is the currently most promising mainstream approach to error correction, requires a two-dimensional (2D) lattice of qubits with nearest-neighbor coupling. One possible solution is to use charge coupling. Simulations indicate that the coupling range can be extended to at least 10 μm with floating electrostatic couplers [9] while maintaining a strength comparable to that demonstrated in experiments with immediately adjacent qubits [6]. With this coupling strength, the currently achievable level of charge noise [10] still leads to coherence times that are two orders of magnitude too short to reach the required gate fidelities. Another possibility is to transfer electrons between qubits, e.g., using surface acoustic waves [11,12] or electrostatic gates [13]. First evidence indicates that the spin projection can be preserved during such a transfer [13,14]. While one may hope that one can also achieve spin coherent transfer, this remains to be shown experimentally. Furthermore, these approaches entail a rather cumbersome complexity of the device and its operation.

A very attractive remedy would be to directly extend the range of tunnel or exchange coupling between localized electron spins representing a qubit. In principle, this could be achieved with a very long and shallow tunnel barrier, but in practice disorder would lead to localization on a scale of a few hundred nm. Here, we theoretically analyze the possibility to use a superconductor that is tunnel coupled to the qubit electrons (Fig. 1) to mediate such coupling over extended distances without being limited by localization. Qualitatively, the main idea is to use extended quasiparticle states for coupling while relying on the gap to freeze out all the low-energy degrees of freedom in the coupler, thus suppressing decoherence.

A key question is what coupling range and strength can be achieved with this approach. To address this question for a simple model system, we compute the exchange coupling between two distant electrons (e.g., localized in semiconductor quantum dots) that are tunnel coupled to a 2D superconducting ground plane and can be detuned electrostatically with respect to the latter. We derive an expression relating the mediated coupling strength to that achievable with direct coupling via the Green's function of the superconductor, considering both the ballistic and disordered case. Using realistic estimates of the relevant parameters, we find that a micron-scale coupling with a practically useful strength of 10 to 100 MHz is achievable.

An important qualitative result is that the decay length of the coupling is determined by the detuning between the quantum dot levels and the upper edge of the superconducting gap. This detuning can be controlled with high precision via gate voltages. Therefore, the decay length can be substantially larger than the superconducting coherence length. This result is in contrast to alternative proposals [15,16] considering crossed Andreev reflection as coupling mechanism in a similar setting and finding a decay on the scale of the coherence length.

The remainder of the paper is organized as follows. In Sec. II, we discuss the general setup and present its Hamiltonian. Section III discusses the results for the case of a clean superconductor. These results are contrasted in Sec. IV with the results for the disordered case. In Sec. V, we present realistic experimental parameters that allow an exchange coupling over a distance of several microns. Some technical details about the disorder average are moved to the Appendix.

II. SETUP

We want to study the exchange-coupling strength in a setup where two semiconducting spin qubits are coupled via a thin superconducting film [see Fig. 1(a)]. The total Hamiltonian $H = H_D + H_{\mathrm{BCS}} + H_T$ consists of three parts which we will discuss individually in the following. The dots can be modeled by the Hamiltonian $H_D = H_1 + H_2$ with

$$H_j = \epsilon_j n_j + \tfrac{1}{2} U_j n_j (n_j - 1). \qquad (1)$$

It involves the operator $n_j = \sum_\sigma n_{j\sigma}$ counting the number of electrons in dot j where $n_{j\sigma} = d_{j\sigma}^\dagger d_{j\sigma}$ and $d_{j\sigma}$ are fermionic operators. The two dots are tunnel coupled via a

(a)

(b)

$|0,1; \bm{k} \uparrow\rangle$

H_T

H_T

$|1,1_S;0\rangle$

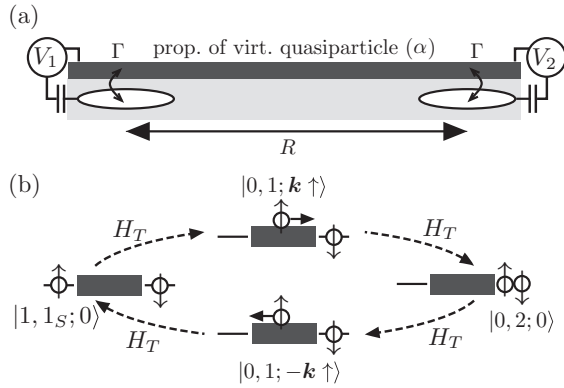$|0,2;0\rangle$

H_T

H_T

$|0,1; -\bm{k} \uparrow\rangle$

FIG. 1. (a) Sketch of the setup analyzed in the paper. Two semiconducting quantum dots (white ellipses) are tunnel coupled with strength Γ to a two-dimensional superconducting film (dark gray). The levels of the dot are tunable by the gate voltages $V_{1,2}$. (b) Steps in the virtual process leading to the exchange interaction. The initial conditions with one electron in each quantum dot are shown to the left. Given the fact that the electrons are in a singlet state, the following virtual process which is fourth order in H_T may take place: the electron in the left dot tunnels into the superconductor, where it propagates as a quasiparticle above the gap. The electron enters the second dot, which becomes doubly occupied. The two remaining tunneling processes reverse the path and take the electron back to the initial state. Note that the reverse path could be also taken by the other electron, which leads to a spin flip and explains the factor of 2 in Eq. (4).

Hamiltonian H_T to a thin 2D superconducting film of size $L \times L$. We measure the energies relative to the chemical potential of the superconductor and assume the tunability of the level position ϵ_j by nearby gates V_j. The parameter $U_j > 0$ describes the Coulomb interaction due to the repulsion of multiple electrons on a single dot. We note that the eigenenergies $\varepsilon_{n_1,n_2} = \epsilon_1 n_1 + \frac{1}{2}U_1 n_1(n_1 - 1) + \epsilon_2 n_2 + \frac{1}{2}U_2 n_2(n_2 - 1)$ of the dot Hamiltonian H_D only depend on the occupation n_1, n_2 and not on the spin state of the electrons. This is due to the absence of a magnetic field in our description of the system. Note that, if needed, the application of a (weak) magnetic field can be included perturbatively in the end. We seek a situation where the states $|11\rangle$ at energy $\varepsilon_{1,1} = \epsilon_1 + \epsilon_2$ and $|02\rangle$ at energy $\varepsilon_{0,2} = 2\epsilon_2 + U_2$ are almost degenerate with $\delta\varepsilon = \varepsilon_{0,2} - \varepsilon_{1,1} > 0$ much smaller than the typical energy spacing in the dots. In this situation, we only have to take the states $|1,1\rangle$ and $|0,2\rangle$ into account. The near degeneracy can be achieved by setting $\epsilon_1 = \epsilon_2 + U_2 - \delta\varepsilon$.

We model the thin film of the superconductor by the conventional BCS Hamiltonian $H_{\mathrm{BCS}} = \sum_{\bm{k}\sigma} E_k \beta_{\bm{k}\sigma}^\dagger \beta_{\bm{k}\sigma}$. The spectrum of the superconductor is given by $E_k = (\xi_k^2 + \Delta^2)^{1/2}$ with $\Delta > 0$ the energy gap of the superconductor and $\xi_k = \hbar^2 k^2 / 2m - \mu$; here, m is the electron mass and μ is the chemical potential of the superconductor. The fermionic Bogoliubov operators $\beta_{\bm{k}\sigma}$ are related to conventional electronic degrees of freedom via the unitary transformation

$$c_{\bm{k}\uparrow} = u_k \beta_{\bm{k}\uparrow} + v_k \beta_{-\bm{k}\downarrow}^\dagger, \quad c_{-\bm{k}\downarrow}^\dagger = -v_k \beta_{\bm{k}\uparrow} + u_k \beta_{-\bm{k}\downarrow}^\dagger \quad (2)$$

with the parameters $u_k, v_k \geqslant 0$ determined by $u_k^2 = 1 - v_k^2 = \frac{1}{2}(1 + \xi_k/E_k)$. In the following, we denote with $|0\rangle$ the ground state of the superconductor and correspondingly $|\bm{k}\sigma\rangle = \beta_{\bm{k}\sigma}^\dagger |0\rangle$ denote the (single-particle) excitations.

Coupling between the superconductor and the dot is provided by the tunneling Hamiltonian

$$H_T = -t \sum_\sigma [c_\sigma^\dagger(0)d_{1\sigma} + c_\sigma^\dagger(R)d_{2\sigma}] + \mathrm{H.c.}$$

$$= -\frac{t}{L} \sum_{\bm{k}\sigma} [c_{\bm{k}\sigma}^\dagger d_{1\sigma} + e^{-ik_x R} c_{\bm{k}\sigma}^\dagger d_{2\sigma}] + \mathrm{H.c.}, \quad (3)$$

where we have taken into account that the two dots are separated by a distance R (along the x axis). For simplicity, we have assumed t to be the same in the two dots. In the following, it will be useful to parametrize t by the tunneling rate $\Gamma/\hbar$ in the normal state with $\Gamma = 2\pi t^2 \rho_0$; here, $\rho_0 = m/2\pi\hbar^2$ denotes the density of states of the normal state (per spin). Note that t and thus Γ depend exponentially on the distance from the quantum dot to the superconducting layer which sets a limit on the depth of the quantum well in which the dots are formed; we will comment on this requirement at the end of Sec. V.

As we are considering the exchange effect mediated by the superconductor, an important parameter will be the energy difference between the initial state $\varepsilon_{1,1} = 2\epsilon_2 + U_2 - \delta\varepsilon$ and the intermediate state $\varepsilon_{0,1} + E_k = \epsilon_2 + E_k$ with the electron in the superconducting wire. For the latter to be an excited state, we demand that $|\varepsilon_{1,1} - \varepsilon_{0,1}| < \Delta$. In particular, we are interested in a situation where $\varepsilon_{1,1} - \varepsilon_{0,1} = \epsilon_2 + U_2 - \delta\varepsilon$ is smaller but not much smaller than Δ (i.e., the level of dot 1 is tuned close to the gap edge). We parametrize this by the energy offset $M = \Delta - (\epsilon_2 + U_2 - \delta\varepsilon) > 0$ between the initial state with one electron in each dot and the intermediate state where the electron of dot 1 is transferred to the superconductor. Note that M can be tuned independently of $\delta\varepsilon$ by ϵ_2.

In this situation, the dominant contribution for the exchange comes from a virtual process where we start from $|1,1\rangle$ going over to a state $|0,1\rangle$ plus a low-energy excitation in the superconductor, then we reach $|0,2\rangle$ before we retrace the steps [see Fig. 1(b)]. In order that this exchange interaction can lead to a reduction of the ground-state energy, the spins of the electrons in the initial $|1,1\rangle$ state have to be in a singlet as otherwise the $|0,2\rangle$ state is forbidden by Pauli exclusion. The interaction thus assumes the form $H_{\mathrm{ex}} = \frac{1}{4}J\bm{\sigma}_1 \cdot \bm{\sigma}_2$ with $J > 0$ the energy difference between the singlet (which is lowered in energy) and the triplet state (which is unchanged).

Assuming that the spins are in the singlet $|1,1_S\rangle = 2^{-1/2}(d_{1\uparrow}^\dagger d_{2\downarrow}^\dagger - d_{1\downarrow}^\dagger d_{2\uparrow}^\dagger)|0\rangle$, we compute the lowering of the ground-state energy in fourth-order perturbation theory in the tunneling Hamiltonian H_T (the triplet energy does not change as discussed above). We obtain $J = \alpha\Gamma^2/\delta\varepsilon$ with the dimensionless coupling constant

$$\alpha = \frac{1}{\Gamma^2} \left| \sum_{\bm{k}\sigma} \frac{\langle 0,2;0|H_T|0,1;\bm{k}\sigma\rangle\langle 0,1;\bm{k}\sigma|H_T|1,1_S;0\rangle}{\varepsilon_{1,1} - \varepsilon_{0,1} - E_k} \right|^2$$

$$= \frac{1}{2\pi^2 \rho_0^2} |g(R; \Delta - M)|^2, \quad (4)$$

where we have introduced the Green's function of the superconductor

$$g(r; E) = -i \int_0^\infty dt \, \langle 0 | c_\sigma(\boldsymbol{r}) e^{i(E - H_{\mathrm{BCS}})t/\hbar} c_\sigma^\dagger(0) | 0 \rangle$$

$$= \int \frac{d^2 k}{(2\pi)^2} \frac{u_k^2 e^{i\boldsymbol{k} \cdot \boldsymbol{r}}}{E - E_k} \tag{5}$$

that we will compute in the following. Note that because we are interested in values $E = \Delta - M < \Delta$, we do not need to distinguish between advanced and retarded Green's functions. The exchange interaction $J = \alpha J_0$ is thus given by the product of the bare result $J_0 = \Gamma^2/\delta\varepsilon$ which would be achievable in the case the dots were in direct contact and a distance dependent renormalization factor $\alpha < 1$ describing the reduction due to the finite spatial separation. Note that in Eqs. (4) and (5) we have assumed that the superconductor is in the ground state without quasiparticle excitations present, which requires that the electron temperature is much smaller than the superconducting gap Δ.

III. CLEAN SC

In the case of a clean superconductor, it is straightforward to evaluate Eq. (5). Going over to polar coordinates and assuming $M \ll \Delta \ll \mu$ and $k_F r \gg 1$ yields the semiclassical expression

$$g(r; E) = \frac{\rho_0}{2} \int_{-\infty}^\infty d\xi_k \int_0^{2\pi} \frac{d\phi}{2\pi} \frac{e^{i(k_F + \xi_k/\hbar v_F) r \cos \phi}}{E - \Delta - \xi_k^2/2\Delta}$$

$$= -\frac{\rho_0}{\sqrt{2\pi k_F r}} \,\mathrm{Re} \int_{-\infty}^\infty d\xi_k \frac{e^{i k_F r - i\pi/4 + i\xi_k r/\hbar v_F}}{M + \xi_k^2/2\Delta}$$

$$= \rho_0 \left(\frac{\pi \Delta}{M k_F r} \right)^{1/2} \cos(k_F r + 3\pi/4) e^{-r/2\xi}, \tag{6}$$

with the effective coherence length $\xi = \hbar v_F/\sqrt{8\Delta M}$ that is a factor $(\Delta/M)^{1/2} \gg 1$ longer than the bare coherence length $\xi_0 = \hbar v_F/\pi\Delta$. In Eq. (6), we have taken into account that for the relevant part of the integral we have $\xi_k \approx \hbar v_F(k - k_F) \ll \Delta$ such that $u_k^2 \approx \frac{1}{2}$ and $E_k \approx \Delta + \xi_k^2/2\Delta$.

Having evaluated the Green's function in the semiclassical limit, we are in the position to evaluate the dimensionless coupling constant $\alpha = 2\cos^2(k_F R + 3\pi/4)\alpha_0$ with

$$\alpha_0 = \frac{\Delta}{4\pi M k_F R} e^{-R/\xi}. \tag{7}$$

The $\cos^2$ dependence of α originates in our model from the tunneling at a point (i.e., momentum-independent tunneling amplitudes). In a realistic situation, the diameter d of the dot is large such that $k_F d \gg 1$. In this case, the result will be modified. In particular, the tunneling amplitude will depend on the momentum mismatch between the dot and the superconductor. A careful consideration of these effects turns out to be rather subtle and beyond the scope of the present work (see, e.g., Ref. [17]). When comparing the results for a clean to a disordered superconductor in Sec. V, we use α_0 to ease comparison between the clean and the disordered case. This corresponds to replacing $\cos^2$ by its typical value $1/2$. In this

way, we avoid the dependence of the results on microscopic details that are relevant only in the clean case.

IV. DISORDERED SC

In order to treat the case of a disordered superconductor it is useful to go over from the Green's function $g(r; E)$ to the Gorkov Green's function

$$G(r; E) = \int \frac{d^2 k}{(2\pi)^2} \frac{(E + \xi_k) e^{i\boldsymbol{k} \cdot \boldsymbol{r}}}{E^2 - E_k^2} \tag{8}$$

as the latter has better analytical properties allowing us to set up perturbation theory in terms of Feynman diagrams [18]. In the limit $M \ll \Delta \ll \mu$ that we are interested in, we have

$$\frac{E + \xi_k}{E^2 - E_k^2} \approx \frac{u_k^2}{E - E_k} \tag{9}$$

such that the two Green's functions G and g can be used interchangeably. It is a well-known result [18] that under an impurity average the Gorkov Green's function is simply given by $\overline{G(r; E)} = G(r; E)e^{-r/2\ell}$ with ℓ the mean free path in the disordered system.

In order to obtain the exchange coupling through a disordered superconductor, we should average α over disorder. It is important to note that the impurity average (denoted by the overline) cannot simply be performed separately on the two Green's functions constituting α. The reason is that this neglects interference effects which are relevant for a disordered sample with long phase-coherence length. In fact, impurity scattering that involves both Green's functions even becomes dominant at large distances due to the emergence of a new length scale, which in the diagrammatic language is subsumed by the ladder diagrams forming the diffuson.

Following ideas of Refs. [19,20], we calculate the diffuson approximation of the product of the Green's function entering α in the Appendix. We obtain the result ($E < \Delta$)

$$\int \frac{d^2 k}{(2\pi)^2} \overline{G(\boldsymbol{k}; E)G(\boldsymbol{k} - \boldsymbol{q}; E)}$$

$$\approx \frac{\pi \rho_0}{2} \frac{\Delta^2}{(\Delta^2 - E^2)[(\Delta^2 - E^2)^{1/2} + \hbar D q^2/2]} \tag{10}$$

with the diffusion constant $D = v_F \ell/2$; here, $G(\boldsymbol{k}; E)$ denotes the Fourier transform of $G(\boldsymbol{r}; E)$. The expression (10) is valid for weak disorder with $k_F \ell \gg 1$ and for $q \ll k_F$. Using this expression, we can obtain the result for the disorder-averaged exchange coupling (q, ϕ are the polar coordinates of $\boldsymbol{q}$):

$$\overline{\alpha} = \frac{\Delta}{8\pi \rho_0 M} \int \frac{d^2 q}{(2\pi)^2} \frac{e^{iqR \cos \phi}}{\sqrt{2\Delta M} + \hbar D q^2/2}$$

$$= \frac{\Delta K_0(R/\xi_D)}{2\pi M k_F \ell} \approx \frac{\Delta \xi_D^{1/2}}{2(2\pi)^{1/2} M k_F \ell R^{1/2}} e^{-R/\xi_D} \tag{11}$$

with $\xi_D = \sqrt{\ell \xi/2}$.

When comparing the ballistic result (7) to the diffusive result (11) there are two major differences.

(i) The exponential decay is controlled by different length scales ξ versus ξ_D.

(ii) The algebraic decay of the former is given by R^{-1} whereas the latter decays more slowly with the power $R^{-1/2}$ [21].

V. ESTIMATE OF THE COUPLING STRENGTH

We would like to end by discussing realistic length scales R over which the superconducting film can be employed in order to exchange couple two spin qubits. Starting from realistic values $J_0 \simeq h(1-10)\,\mathrm{GHz}$ for the direct exchange coupling of two qubits, we aim at achieving a dimensionless coupling constant $\alpha \simeq 10^{-3}$ in order to end up with useful exchange-coupling constants $J \simeq h(1-10)\,\mathrm{MHz}$. For the superconductor we propose aluminum with a gap parameter $\Delta/k_B = 2.2\,\mathrm{K}$, which corresponds to a coherence length $\xi_0 = 2.3\,\mu\mathrm{m}$ for a clean sample. Aluminum has a Fermi velocity $v_F = 2.0 \times 10^6\,\mathrm{m/s}$ that implies a Fermi wave vector $k_F = 17\,\mathrm{nm}^{-1}$. The most crucial parameter which makes it possible to increase α is the detuning M. Some of the requirements bounding M from below are that the detuning should be held stable over the time of exchange interaction and that the smearing of the superconducting gap γ (the so-called Dynes parameter) should be much smaller than M. Recent experiments have shown that γ can be as small as $10^{-6}\Delta$ [22]. Given this input, we take a conservative choice of $M = 1\,\mu\mathrm{eV}$ which corresponds to $M/\Delta = 5 \times 10^{-3}$. As a result, we obtain the clean effective coherence length $\xi = 36\,\mu\mathrm{m}$ and the dimensionless coupling constant α_0 of Eq. (7) (see Fig. 2)

$$\alpha_0 = \frac{0.94\,\mathrm{nm}}{R} e^{-R/36\,\mu\mathrm{m}} \tag{12}$$

in a clean system as it is, for example, obtained by epitaxial growth [23]. For a distance of $R = 1\,\mu\mathrm{m}$ this evaluates to $\alpha_0 = 9.1 \times 10^{-4}$. We note that the value of α_0 is completely dominated by the prefactor. Thus, we expect that the analysis becomes more favorable for the case of disordered aluminum as is obtained, for example, by sputtering.
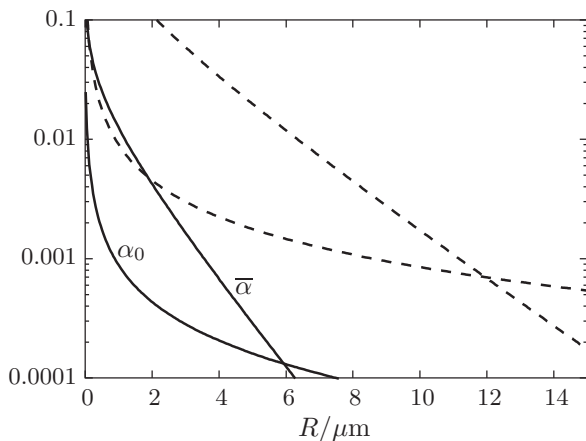

FIG. 2. Plot of the dimensionless coupling constant for clean aluminum (α_0) and disordered aluminum ($\overline{\alpha}$) as a function of the distance R of the spin qubits that are coupled. The plot shows both the results for $M/\Delta = 5 \times 10^{-3}$ (solid lines) and the results for $M/\Delta = 5 \times 10^{-4}$ (dashed lines).

Extrapolating from Refs. [24–26], a realistic value of the mean free path in aluminum is $\ell = 100\,\mathrm{nm}$, which translates to a diffusive coherence length $\xi_D = 1.3\,\mu\mathrm{m}$. This yields

$$\overline{\alpha} = \frac{0.027\,\mu\mathrm{m}^{1/2}}{R^{1/2}} e^{-R/1.3\,\mu\mathrm{m}} \tag{13}$$

and evaluates for $R = 1\,\mu\mathrm{m}$ to the more favorable result $\overline{\alpha} = 1.3 \times 10^{-2}$.

Figure 2 shows that for distances R smaller than a characteristic distance R^* the coupling via a diffusive superconductor is larger than the one through a clean system [27]. For $R > R^*$ this reverses. For the typical case $\ell \ll \xi$, the characteristic distance R^* is approximately given by

$$R^* \approx \frac{\xi_D}{2} \ln(\pi \xi/2\ell). \tag{14}$$

For $M/\Delta = 5 \times 10^{-3}$, we find numerically that $R^* = 5.9\,\mu\mathrm{m}$ and that $\overline{\alpha}$ is larger than 10^{-3} up to $R = 3.6\,\mu\mathrm{m}$.

Provided accurate control over the gate voltages, the range of the exchange coupling can be extended even further. For example, lowering M by a factor of 10, i.e., for $M = 0.1\,\mu\mathrm{eV}$, we obtain the dashed lines in Fig. 2. We see that for distances smaller than $R^* = 12\,\mu\mathrm{m}$ the diffusive material leads to a stronger exchange interaction. Furthermore, $\overline{\alpha}$ is larger than 10^{-3} for distances up to $R = 11\,\mu\mathrm{m}$.

Note that the operation of spin qubits normally requires a magnetic field to obtain a Zeeman splitting, which must be much smaller than the critical field of the superconducting electrode (about 100 mT for bulk aluminum). Using a thin film and applying the field in plane can enhance the critical field; for thin films of thickness below 10 nm, critical fields of the order of 1 T can be achieved [28]. While for GaAs quantum dots fields on the order of 0.5 T are desirable to suppress nuclear spin dephasing [29], a few mT should be sufficient for Si quantum dots due to the absence of nuclear spins. Hence one may expect that at least for the latter the suppression of superconductivity due to the applied field will not be a limiting factor.

To achieve a strong interaction with $J_0 \simeq \hbar \times 100\,\mathrm{GHz}$, it is clearly advantageous to utilize a large tunnel coupling t. This requirement is not compatible with conventional GaAs heterostructures, where electrons are typically located about 100 nm below the surface with surface states leading to a large energy barrier. We envision solving this problem by shallow undoped structures with a low Al content in the barrier, thus reducing the band offset, and high-quality interfaces [23]. In Si, the possibility to use metal-oxide-semiconductor quantum dots [2] with a very thin oxide layer or other barriers to passivate the surface states should more readily enable a sizable t.

VI. CONCLUSION

We have derived the strength of the exchange interaction between two spin qubits which are connected via a 2D film of superconducting material acting as a coupler. We have shown that the bare exchange interaction for two neighboring dots is reduced by a dimensionless coupling constant α incorporating all effects of the finite distance R. We have presented results for the case of both a clean and a disordered superconductor. We have shown that for distances R smaller than a characteristic

distance R^* a diffusive superconductor outperforms the clean one due to the prefactor in α for the former being a factor $(R/\ell)^{1/2}$ larger than for the latter. We have shown that a diffusive superconductor with a moderate mean free path of $\ell = 100\,\text{nm}$ enables useful exchange coupling strengths over a distance of more than $10\,\mu\text{m}$. In practice, it would be convenient to use superconducting wires rather than an extended film to mediate the coupling. In this case the further confinement would eliminate the $R^{-1/2}$ decay of the prefactor in Eq. (11), leaving only the exponential decay and allowing substantially stronger coupling. This suggests that superconducting wires might be suitable for mediating an exchange coupling between quantum dots with a separation large enough to implement a 2D lattice. An additional advantage is that the coupling strength can be varied easily and rapidly by changing the electrostatic potential of the superconductor.

APPENDIX: CALCULATION OF THE DIFFUSON

In this Appendix, we apply the ideas of Refs. [19,20] to the specific case of disordered 2D superconducting film. The calculation of the impurity average has to be performed in Nambu space, because the Green's function has anomalous components due to the superconducting condensate. All the Green's functions can be combined in the matrix Green's function

$$\hat{G}(k; E) = \frac{E + \xi_k \tau_3 + \Delta \tau_1}{E^2 - E_k^2} \quad (A1)$$

acting via the Pauli matrices τ_i on the Nambu space. Note that the connection to the Gorkov Green's function introduced in the main text is given by $G(k; E) = \hat{G}(k, E)_{11}$. We assume throughout this section that $0 < E < \Delta$ such that the excitations are virtual and similar to the Matsubara formalism, and we do not have to worry about retarded and advanced Green's functions. If required, the results for $E > \Delta$ can simply be obtained by analytical continuation.

We are interested in performing an impurity average in the potential $\hat{V}(\boldsymbol{r}) = v \sum_i \delta^{(2)}(\boldsymbol{r} - \boldsymbol{r}_i)\tau_3$ where $\boldsymbol{r}_i$ denotes the position of the ith impurity. The connection with the mean free path is given by the Born result $\hbar v_F/\ell = 2\pi n_i v^2 \rho_0$ where n_i is the density of impurities. The impurity averaged Green's function is given by (see Ref. [18])

$$\overline{\hat{G}(k; E)} = \frac{\bar{E} + \xi_k \tau_3 + \bar{\Delta}\tau_1}{\bar{E}^2 - \xi_k^2 - \bar{\Delta}^2} \quad (A2)$$

where $\bar{E} = \eta E$, $\bar{\Delta} = \eta \Delta$, and

$$\eta = 1 + \frac{\hbar v_F}{2\ell(\Delta^2 - E^2)^{1/2}}. \quad (A3)$$

In order to calculate the disorder average of a product of Green's functions, it is useful to introduce the four matrix diffusons (τ_0 is the identity matrix):

$$D_i(q) = nv^2 \int \frac{d^2k}{(2\pi)^2} \tau_3 \overline{\hat{G}(\boldsymbol{k}; E)\tau_i \hat{G}(\boldsymbol{k} - \boldsymbol{q}; E)}\tau_3. \quad (A4)$$

In terms of these, the combination of Green's functions in Eq. (10) is given by

$$\int \frac{d^2k}{(2\pi)^2} \overline{G(\boldsymbol{k}; E)G(\boldsymbol{k} - \boldsymbol{q}; E)} = \frac{1}{2nv^2}[D_0(q) + D_3(q)]_{11}. \quad (A5)$$

The dominant contribution to D_i in the weak disorder limit originates from ladder diagrams which keep the relative momentum $\boldsymbol{q}$ conserved. The zeroth-order term is given by

$$D_i^{(0)}(q) = nv^2 \int \frac{d^2k}{(2\pi)^2} \tau_3 \overline{\hat{G}(\boldsymbol{k}; E)} \tau_i \overline{\hat{G}(\boldsymbol{k} - \boldsymbol{q}; E)}\tau_3. \quad (A6)$$

As we are interested in small relative momenta q, we can expand and obtain $D_i^{(0)}(q) = a_i + b_i q^2$ with

$$a_0 = nv^2 \rho_0 \int d\xi_k \, \tau_3 \overline{\hat{G}(k; E)}^2 \tau_3 = \frac{\hbar v_F}{2\ell} \frac{\bar{\Delta}^2 - \bar{\Delta}\bar{E}\tau_1}{(\bar{\Delta}^2 - \bar{E}^2)^{3/2}}. \quad (A7)$$

The expansion in q is obtained by the replacement $\xi_{k-q} = \xi_k - \hbar v_F q \cos \phi$ with ϕ the angle between $\boldsymbol{k}$ and $\boldsymbol{q}$. The first order in q vanishes due to the integration over ϕ. The first nonvanishing contribution reads

$$b_i = \frac{\hbar^2 v_F^2 a_i}{8(\bar{E}^2 - \bar{\Delta}^2)}. \quad (A8)$$

As $D_0^{(0)}$ has a term proportional to τ_1, we need to calculate additionally $D_1^{(0)}$ in order to obtain a closed set of equations. We obtain the expression

$$a_1 = \frac{\hbar v_F}{2\ell} \frac{\bar{\Delta}\bar{E} - \bar{E}^2 \tau_1}{(\bar{\Delta}^2 - \bar{E}^2)^{3/2}}. \quad (A9)$$

For completeness, we give also the other expressions

$$a_2 = \frac{\hbar v_F}{2\ell} \frac{\tau_2}{(\bar{\Delta}^2 - \bar{E}^2)^{1/2}}, \quad a_3 = 0. \quad (A10)$$

For further convenience, we denote by $D_{i,j}^{(0)}$ the term in $D_i^{(0)}$ proportional to τ_j such that $D_i^{(0)} = \sum_j D_{i,j}^{(0)}\tau_j$ by definition.

Summing the ladder diagrams is equivalent to the Dyson type equations [30]:

$$D_i(q) = D_i^{(0)}(q) + \sum_j D_{i,j}^{(0)}(q)D_j(q). \quad (A11)$$

The system is closed in the subspace $i, j \in \{0, 1\}$. Solving the linear system of equations, we obtain the result

$$D_i(q) = \frac{D_i^{(0)} + \left(D_{0,1}^{(0)}D_{1,0}^{(0)} - D_{0,0}^{(0)}D_{1,1}^{(0)}\right)\tau_i}{1 - D_{0,0}^{(0)} - D_{1,1}^{(0)} - D_{0,1}^{(0)}D_{1,0}^{(0)} + D_{0,0}^{(0)}D_{1,1}^{(0)}}. \quad (A12)$$

Inserting the expressions for $D^{(0)}$ yields the final result

$$D_0(q) = \frac{\hbar v_F}{2\ell} \frac{\Delta^2 - \Delta E \tau_1}{(\Delta^2 - E^2)[(\Delta^2 - E^2)^{1/2} + \hbar D q^2/2]} \quad (A13)$$

valid for small q. For reference, we also give the result

$$D_1(q) = \frac{\hbar v_F}{2\ell} \frac{\Delta E - E^2 \tau_1}{(\Delta^2 - E^2)[(\Delta^2 - E^2)^{1/2} + \hbar D q^2/2]}. \quad (A14)$$

The diffuson D_2 only couples to itself, thus the solution of Eq. (A11) is simply

$$D_2(q) = \frac{D_2^{(0)}}{1 - D_{2,2}^{(0)}} = \frac{\hbar v_F}{2\ell} \frac{\tau_2}{(\Delta^2 - E^2)^{1/2} + \hbar D q^2/2}. \quad (A15)$$

The D_3 diffuson vanishes such that only D_0 enters the expression (A5). From Eq. (A5), we obtain the result quoted in the main text.

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