Hall effect, edge states, and Haldane exclusion statistics in two-dimensional space

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We clarify the relation between two kinds of statistics for particle excitations in planar systems: the braid statistics of anyons and the Haldane exclusion statistics (HES). It is shown nonperturbatively that the HES exists for incompressible anyon liquid in the presence of a Hall response. We also study the statistical properties of a specific quantum anomalous Hall model with Chern-Simons term by perturbation in both compressible and incompressible regimes, where the crucial role of edge states to the HES is shown.

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I. INTRODUCTION

Anyons are particle excitations obeying the *braid statistics*, which interpolates continuously between fermion and boson statistics in two dimensions(2D) [1–4]. Abelian braid statistics can be characterized by the phase factor $e^{i(\alpha-1)\pi}$ of the many-body wave-function when two anyons are exchanged, with "any" $\alpha \in [0,2)$, thus explaining their name. Fermions(bosons) correspond to $\alpha = 0(1)$. In real world, anyons seem to appear in the fractional quantum Hall systems [5,6], and presumably on the magnetized surface of topological insulators [7]. Their relation to high-temperature superconductivity, initiated decades ago [8,9], is still being actively pursued nowadays in a renewed form [10–12], requiring the knowledge of anyon thermodynamics, which, however, is barely known even for free anyons with $0 < \alpha < 1$.

Another interpolation between fermion and boson statistics was proposed by Haldane. It is known as *Haldane's exclusion statistics* (HES)[13] or Haldane-Wu statistics (HWS)[14], based upon the following state-counting ideas; for a many-body system in finite volume, the dimension $d(N)$ of the Hilbert space of the $(N+1)$-th particle depends on N, which leads to the definition of statistical interaction g via $\Delta d = (g-1)\Delta N$ [13]. Obviously, $g = 0$ for fermions, $g = 1$ for bosons, and other intermediate values of g define fractional HES. Unlike the aforementioned braid statistics, the HES is not limited to planar systems and one can calculate their thermodynamic properties explicitly. The representative quantity is the energy distribution function $n_H(\epsilon,g)$, which was derived by Wu [14]. It turns out that particles with HES (nonmutual) have a well-defined Fermi energy ϵ_F at zero temperature if $g \neq 1$. When $\epsilon < \epsilon_F$, $n_H(\epsilon,g) = 1/(1-g)$, otherwise $n_H(\epsilon,g) = 0$, thus interpolating between Fermi-Dirac (FD) distribution with $g = 0$ and Bose-Einstein (BE) distribution with $g = 1$ at zero temperature. At finite temperatures, n_H is much more complicated than FD and BE distributions. In analogy with Landau-Fermi liquid theory, a theory of interacting HES particles called Haldane liquid was developed in Ref. [15]. An important result is the generalization of Luttinger theorem: the interaction does not change the volume enclosed by the Fermi surface for HES.

In one-dimensional (1D) systems a deep connection between Luttinger liquids, braid statistics and HES was established in Refs. [16,17], but for planar systems, braid statistics and HES are not always equivalent. In fact the free nonrelativistic anyons do not show the evidence of HES. On the other hand, the nonrelativistic anyon model in the strong magnetic field is exactly solvable after projection to the lowest Landau level (LLL) [18–20] and it is described by an equation of state consistent with HES [20]. It was shown that the filling factor of the LLL at $T = 0$ equals to $1/(1-\alpha)$ [20,21] indicating $g = \alpha$. This is so far the only known exactly solvable model obeying HES in 2D and it has a flat dispersion. Therefore the clarification of the relation between HES and anyons' braid statistics appears to be a step of great interest for providing a class of planar models obeying HES with nontrivial (nonflat) dispersion.

In this paper, we show that the HES exists quite generically in 2D systems as long as there is a Hall response. A general relation between g and α is established *nonperturbatively* by examining the total anyon number in the ground state. Note that determining the energy distribution function $n_H(\epsilon,g)$ requires a Fock structure in terms of single anyon states, not found in anyonic systems where the N-body wave function is a multivalued section depending nontrivially on the variables of all particles [22,23]. On the other hand, the total anyon number is easy to calculate. In fact, according to the Haldane's definition of the statistical interaction g, one immediately obtains $d(N) - d(0) = (g-1)N$. If the entire band is filled, N reaches its maximum denoted by $N(g)$, and $d(N) = 0$. Note that $d(0)$ is also the maximum of fermion filling number $N(0)$, hence we find the integral form of Haldane's statistical interaction

$$\frac{N(g)}{N(0)} = \frac{1}{1-g} \quad \text{or} \quad N(g) - N(0) = \frac{g}{1-g} N(0), \qquad (1)$$

which is consistent with Haldane's energy distribution function $n_H(\epsilon, g)$, being more general, valid also for Haldane liquids [15]. The above description can be easily generalized to the case of multiple species with *mutual* HES. Labelling the species with a latin subindex $i, j, \ldots$, with obvious meaning of symbols, one finds $d_i(\{N_l\}) - d_i(\{0\}) = \sum_j (g_{ij} - \delta_{ij}) N_j$, with $g_{ij}(j \neq i)$ defining the *mutual* HES.

In the following, we give two proofs of the relation between anyons' braid statistics and HES, the first one involving an adiabatic argument given in Sec. II, while the second one given in Sec. III is based upon a specific model in a random phase approximation (RPA). Section IV is the conclusion.

II. ADIABATIC ARGUMENT

According to Wilczek [2], one can map an anyon into a fermion bound to a flux tube by a singular gauge transformation, so that the multivalueness of the anyons' wave function, allowed by the nontrivial topology of the configuration space of 2D identical particles, is removed in the new fermion system. In this sense, anyons can be viewed as charged fermions with long range gauge interaction in which the braid statistics is encoded. The flux binding can be achieved in the Lagrangian formalism by introducing the Chern-Simons(CS) term. The resulting Lagrangian in the Minkowski space-time reads

$$\mathcal{L} = \mathcal{L}_M - a_\mu J^\mu + \frac{1}{4\pi\alpha} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda, \qquad (2)$$

where J_μ is the current and a_μ is the statistical CS gauge field. By "charge" we mean the statistical charge coupled to a_μ's field, which is taken as unit here. At present, the precise form of the fermion Lagrangian $\mathcal{L}_M$ is not important, and it is only assumed to yield the Hall conductance σ_h. We could take $\mathcal{L}_M$ as describing the conventional quantum Hall system, the magnetized surface state of a topological insulator, or other quantized anomalous Hall insulators without magnetic field [24–29].

By integrating out the a^0 field, one obtains the following constraint on the charge density and the statistical flux:

$$\vec{\nabla} \times \vec{a} = -2\pi\alpha J^0. \qquad (3)$$

The Chern-Simons parameter α characterizes the braid statistics, which measures the Aharonov-Bohm phase when one fermion circles around another or equivalently two fermions are interchanged.

To illustrate the relation between the braid statistics and the HES, we isolate a thermodynamically large region S as shown in Fig. 1(a) from the rest. Inside S we turn on the statistical parameter α adiabatically, generating the statistical flux $\Phi \equiv \int_{\partial S} \vec{a} \cdot d\vec{x}$ through the region S. This in turn induces a current $J^i = -\sigma_h \epsilon^{il} E^l$ with $E^l = -\partial_t a^l$ for a *Hall insulator* in the scaling limit. Then, according to the continuity equation and Eq. (3), we have

$$\partial_t N = -\int_S d^2\vec{x}\, \partial_i J^i = 2\pi\sigma_h \partial_t(\alpha N), \qquad (4)$$

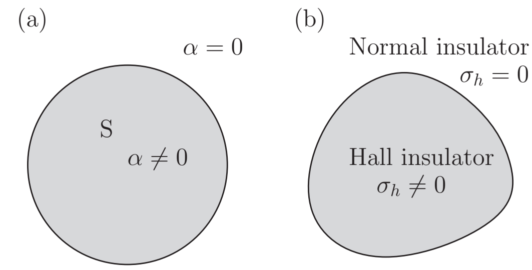

FIG. 1. (a) The Chern-Simons coupling parameter α varies adiabatically with time inside the shadowed region S, being zero outside. (b) A Hall insulator connected with a normal insulator.

where $N \equiv \int_S d^2\vec{x} J^0$ is the total particle number in S. Equation (4) implies that $N(1 - 2\pi\sigma_h\alpha)$ is time independent, therefore the particle number for a general value of α is given by $N(\alpha)/N(0) = 1/(1 - 2\pi\sigma_h\alpha)$ in the scaling limit. Comparing with Eq. (1), we obtain the condition

$$g = 2\pi\sigma_h\alpha, \qquad (5)$$

between the statistical parameters g for HES and α for the braid statistics. As an example, let's revisit the nonrelativistic anyons in a strong magnetic field. If we take only the LLLs into account, $\sigma_h = 1/(2\pi)$, one finds $g = \alpha$ according to Eq. (5), which recovers the result given in Refs. [20,21].

In this adiabatic argument, it is crucial for $\mathcal{L}_M$ to describe a Hall insulator, while a normal insulator with $\sigma_h = 0$ does not respond to the Chern-Simons flux in the required way. Therefore anyons in a normal insulator can not obey HES. This proof can be generalized to *mutual* statistics obeyed by multiple species of particles labeled by a latin subscript i as after Eq. (1). Let us consider anyons with statistical charges q_i and Hall conductivity $\sigma_{h,i}$ coupled to a common Chern-Simon field as in Eq. (2), then the mutual exchange statistics between species i and j is given by $\alpha_{ij} \equiv \alpha q_i q_j$. The argument above generalizes Eq. (5) to $g_{ij} = 2\pi\sigma_{h,i}\alpha_{ij}$, thus relating also a mutual HES [14] to anyons' braid statistics. The details on the mutual statistics are given in the Appendix.

Next, by invoking the edge states, inevitable in a quantum Hall system with boundaries, we give another proof of Eq. (5) admitting some generalization beyond the insulating case. In fact, even in Eq. (2) a boundary term would be required to ensure gauge invariance (see, e.g., Ref. [30]), but it has not been written explicitly because it is irrelevant in the above adiabatic argument, based only on the continuity equations in the bulk.

III. RPA CALCULATION

We just showed that the Hall response is crucial for anyons obeying HES. In practice, a Hall insulator can be realized with or without magnetic field. Since anyons in a strong magnetic field have been studied before [18–21], in this article we consider the quantum anomalous Hall(QAH) system without Landau levels, which has been proposed theoretically [24,26] and realized in recent experiments [27]. To be specific we take the fermion Lagrangian $\mathcal{L}_M$ as a 2D massive Dirac fermion with $\mathcal{N}_f$ flavors, which can be written conveniently in the

Euclidean space-time as

$$\mathcal{L}_M = \sum_{s=1}^{\mathcal{N}_f} \bar{\psi}_s \left(\frac{1}{i} \not{\partial} + i\gamma^0 \mu + m_s \right) \psi_s, \tag{6}$$

where $\gamma^\mu = \{\sigma_3, i\sigma_1, i\sigma_2\}$, $\not{\partial} \equiv \partial_\mu \gamma^\mu$, $\bar{\psi} = \psi^\dagger \gamma^0$, and m_s is the fermion mass of flavor s. In realistic systems, there might be impurities and conventional Coulomb interactions between fermions, but for simplicity we ignore them here. In the following, we present a perturbation calculation of the total anyon number in the ground state.

The chemical potential in $\mathcal{L}_M$ breaks the Lorentz invariance down to the spatial rotation invariance. Combined with the gauge invariance of Eq. (2), one finds that the fermion polarization bubble $\Pi^{(2)}_{\mu\nu}(q|\mu)$ takes the following form [31]:

$$\Pi^{(2)}_{\mu\nu}(q|\mu) = (q^2\delta_{\mu\nu} - q_\mu q_\nu) h_1(q|\mu) + \delta_{\mu k}(\vec{q}^2 \delta^{kl} - q^k q^l) \times \delta_{l\nu} h_2(q|\mu) + \epsilon^{\mu\nu\lambda} q_\lambda h_3(q|\mu), \tag{7}$$

where we denote the space-time indices with Greek symbols, while Latin characters denote the spatial coordinates. The three h_i's functions can be calculated straightforwardly. The first two terms are the conventional ones of the three-dimensional relativistic fermions with finite density. It is the antisymmetric h_3 term that is peculiar and the relevant one for 2D fractional exclusion statistics.

Without chemical potential or for $|\mu| \leqslant \min(|m_s|)$, one finds $\lim_{q\to 0} h_3 = \mathcal{N}_f/(4\pi)\sum_{s=1}^{\mathcal{N}_f} \mathrm{sgn}(m_s)$ [32–35], which provides the quantized Hall conductance σ_h of the system. According to the Nielsen-Ninomiya theorem [36], the flavor number $\mathcal{N}_f$ should be even except for some cases with topological reasons. If the signs of m_s's annihilate in pairs, $\mathcal{L}_M$ describes a normal insulator with $\sigma_h = 0$. Once these signs do not cancel completely, we obtain a Hall insulator with integer Hall conductance. As an example we note that the Haldane's QAH model [24] has its low-energy physics described by a two-flavor Dirac fermion with the same mass term. For simplicity, we assume all masses m_s have the same positive value $m > 0$, then $\sigma_h = \mathcal{N}_f/(4\pi)$. Unlike other polarization terms (h_1 and h_2), this induced Chern-Simons term is a topological effect, which is independent of the specific characteristics of the model and nonperturbative in nature. Indeed, it depends only on the flavor number $\mathcal{N}_f$. We then adopt the random phase approximation by taking only this topological term into account and obtain the following propagator of the gauge field,

$$\tilde{D}_{\mu\nu}(x,x') = i2\pi\tilde{\alpha}\partial^{-2}\epsilon^{\mu\nu\lambda}\partial_\lambda\delta^{(3)}(x-x'), \tag{8}$$

where $\tilde{\alpha} = \alpha/(1 - 2\pi\sigma_h\alpha)$ and the Landau gauge is assumed for convenience.

To calculate the anyon number, we first evaluate the free energy $\mathcal{F}(\mu,\alpha)$ to the lowest perturbation order with all gauge boson lines being replaced by $\tilde{D}_{\mu\nu}$ of Eq. (8), which includes the direct and exchange terms $\mathcal{F}_d$ and $\mathcal{F}_e$:

$$\mathcal{F}_d = -\frac{1}{2\beta}\int d^3x d^3x' J^\mu(x)\tilde{D}_{\mu\nu}(x-x')J^\nu(x'), \tag{9}$$

$$\mathcal{F}_e = \frac{\mathcal{N}_f}{2\beta}\int d^3x d^3x' \tilde{D}_{\mu\nu}(x-x') \mathrm{tr}[S_0(x'-x|\mu)\gamma^\mu S_0(x-x'|\mu)\gamma^\nu], \tag{10}$$

where β is the inverse temperature and $S_0(x|\mu)$ is the free fermion propagator of a single flavor. Although it is almost the simplest approximation, it can reproduce the *nonperturbative* results for the anyon insulator given earlier, since it already incorporates the important topological effect in the propagator $\tilde{D}_{\mu\nu}$.

We first consider the insulating case with $\mu = 0$. To investigate the anyon number in the filled band (or vacuum), we require that the statistical charges of particles in the valence band are not screened at all, so that the vacuum expectation value of anyon density $J^0(x) \neq 0$. However, $J^i(x)$ still vanishes if one calculates the fermion loop directly since a filled band usually can not carry any persistent current. Note that the RPA propagator $\tilde{D}_{\mu\nu}$ is antisymmetric and one might conclude $\mathcal{F}_d = 0$ at first glance. This is true for the normal insulators, but it is not correct for the Hall insulator where a persistent chiral current does exist at the sample edges [30,37], as required by gauge invariance, though it vanishes in the bulk. Since $\tilde{D}_{\mu\nu}$ contains an infrared singularity, one can kill two birds with one stone by putting the system in a finite area Ω in contact with normal insulator as shown in Fig. 1(b). For the static state, the current densities are time independent and, if the characteristic function of Ω is denoted by $f_\Omega(\vec{x})$, the chiral current density can be written as $\vec{J}(\vec{x}) = I(\mu)(\hat{z}\times\vec{\nabla} f_\Omega)$ with I the edge current. Then the direct term can be written as

$$\mathcal{F}_d(\mu,\alpha) = -i2\pi\tilde{\alpha}\int d^2\vec{x} J^0(\vec{x})\vec{\nabla}^{-2}\epsilon^{kj}\partial_j J^k(\vec{x}) = -2\pi\tilde{\alpha} I(\mu) N(\mu,0), \tag{11}$$

which leads to the modification of anyon number

$$\Delta N_d(\mu,\alpha) = 2\pi\tilde{\alpha}\sigma_h N(\mu,0) + 2\pi\tilde{\alpha} I(\mu)\frac{\partial N}{\partial\mu}. \tag{12}$$

In the presence of the Chern-Simons term, the chiral current also induces an electric field perpendicular to the boundary, which is proportional to $2\pi\alpha I(\mu)$ and leads to an additional inner pressure. This is the origin of the second term in the right-hand side (r.h.s.) of Eq. (12). Indeed, $\partial N/\partial\mu$ is proportional to the compressibility κ_T for a fixed area. For the insulating state, where κ_T is zero, we then recover the results given earlier and we can extend it to the more general case of incompressible, but not necessarily insulating systems. It is also noted that in this case, there is no contribution to the particle number from the exchange term $\mathcal{F}_e$ and from other possible interactions, since they do not depend on μ as long as the bulk gap is not spoiled.

We now consider a compressible anyon gas with $\mu > m$, where the bulk excitation gap vanishes. In this case, we are only interested in the anyon gas in the conduction band, and assume the valence band is screened by a static background with opposite charges. The finite Fermi sea modifies the polarization bubble Eq. (7) [38,39]. The Hall conductance σ_h is no longer quantized and the compressibility κ_T is also nonzero accordingly. In this case not only the second term in the r.h.s. of Eq. (12) but also the exchange term $\mathcal{F}_e$ contributes.

The calculation of $\mathcal{F}_e$ follows the standard procedure for the QED with finite density given in Refs. [31,40]:

$$\mathcal{F}_e(\mu,\alpha) = -\frac{\tilde{\alpha} m \mathcal{N}_f \Omega}{4\pi}(\mu - m)^2. \tag{13}$$

Differentiating $\mathcal{F}_e$ with respect to μ leads to the modification of particle number from the exchange term,

$$\Delta N_e(\mu,\alpha) = \frac{2\tilde{\alpha} m}{\mu + m} N(\mu,0) \tag{14}$$

with $N(\mu,0) \equiv \mathcal{N}_f \Omega(\mu^2 - m^2)/(4\pi)$. The total particle number change is the sum of Eqs. (12) and (14).

Finally, we consider a charge neutral situation in which all statistical charges in both valence and conduction bands are screened completely, so that all fermion loops vanish and only the exchange term survives. Further assuming $\mu - m \ll m$, we obtain $N(\mu,\alpha) \approx (1+\tilde{\alpha})N(\mu,0)$. If $\mathcal{N}_f = 2$, we obtain $g = \alpha$, a result similar to the insulating case is obtained. The relation between HES and the exchange diagram has also been discussed in Ref. [41] where they set $\mathcal{N}_f = 1$ and did not consider the RPA.

IV. DISCUSSION AND CONCLUSION

We have discussed the exclusion statistics of anyons in a generic quantum Hall system. Instead of analyzing the distribution of single-particle states requiring Fock structure, we study the total particle number in the many body ground state in a finite volume, which has been shown to characterize the crucial aspect of the statistics. Our result proves that the total particle number of the ground state of anyons satisfies HES if the bulk excitations are fully gapped or more generally for an incompressible system. The crucial role played by edge currents for this result is clarified. Anyons in normal insulators without Hall conductance and edge states do not satisfy the HES.

The compressible anyon gas does not show the HES as clean as the incompressible liquid. Actually, it is noticed that the braiding statistics is not well defined at all in the 2D compressible superfluids [42]. Anyhow, for compressible systems if the parameter α is well defined the relation between g and α is much more complicated than Eq. (5), involving also the system-dependent parameters like fermion mass. Nonetheless, at fixed chemical potential the particle number in finite volume strongly depends on the statistical parameter α, and it even diverges at $2\pi\sigma_h\alpha = 1$ [see, e.g., Eq. (12)] as a Bose liquid. Since anyons can be viewed as fermions with gauge interaction, this indicates the violation of the original Luttinger theorem for fermions [43] by the Chern-Simons interaction with noninteger α.

Our approach provides specific microscopic models satisfying HES with nontrivial dispersion, it allows easily judging whether a system can obey HES or not, and points a direction for searching such an exotic statistics in real materials. We hope that our work may shed light on a large class of models in 2D, as the relation with HES did in 1D. In particular, it allows a cleaner treatment of the semionic ($\alpha = 1/2$) holons in the 2D t-J model, relevant to the cuprate high-T_c superconductors [11,12,44], in analogy with the semionic holons of the 1D t-J model [45,46].

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APPENDIX: ADIABATIC ARGUMENT FOR MUTUAL STATISTICS

In this Appendix, we give the details of the adiabatic argument for the mutual statistics. We first consider s kinds of particles satisfying the mutual Haldane's exclusion statistics (HES) [13,14] characterized by the following equation:

$$\Delta d_i = \sum_j (g_{ij} - \delta_{ij})\Delta N_j, \tag{A1}$$

where d_i and N_i are the dimensionality of the new adding particle and the total particle number of kind-i, respectively. g_{ij} for $i \neq j$ is the *mutual HES* [14].

Integrating Eq. (A1), one finds the dimensionality d_i as a function of the particle numbers $\{N_j\}$,

$$d_i(\{N_j\}) = \sum_j (g_{ij} - \delta_{ij})N_j + N_{i,0}, \tag{A2}$$

where $N_{i,0} \equiv d_i(0)$ is the dimension of the available Hilbert space for kind-i particle when there are no particles, which is also the maximal filling number for fermions when $g_{ij} = 0$(namely, no exotic HES). If the system is insulating, the dimensionality $d_i(\{N_j\}) = 0$, and the particle number $N_{j,g}$ can be solved from Eq. (A2) with the following form:

$$N_{i,0} = \sum_j (\delta_{ij} - g_{ij})N_{j,g}. \tag{A3}$$

Next, let us consider the *braid* statistics. Suppose we have a model involving s kinds of anyons which can be described by fermions coupled to a common Chern-Simons field with the following Lagrangian:

$$\mathcal{L} = \mathcal{L}_M[\psi_j, \psi_j^\dagger] - \sum_{j=1}^{s} q_j J_\mu^{(j)} a^\mu + \frac{1}{4\pi\alpha}\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda, \tag{A4}$$

where $J^{(j)}$ is the current of kind-j particle, and q_j is the corresponding statistical charge. As in Sec. II, we also assume $\mathcal{L}_M$ providing a nontrivial Hall response $\sigma_{h,j} q_j^2$ for kind-j particle. Each particle is now bound with flux ϕ_j depending on its statistical charge q_j:

$$\phi_j = -2\pi\alpha q_j. \tag{A5}$$

Following the adiabatic argument given in Sec. II, we isolate a thermodynamically large area S, through which the total flux reads

$$\Phi = \sum_{j=1}^{s} N_j \phi_j = -2\pi\alpha \sum_{j=1}^{s} q_j N_j. \tag{A6}$$

As we turn on α adiabatically, according to the continuity equation, we obtain

$$\partial_t(q_i N_i) = -(\sigma_{h,i} q_i^2)\partial_t \Phi = 2\pi(\sigma_{h,i} q_i^2)\partial_t \left(\alpha \sum_{j=1}^{s} q_j N_j\right). \tag{A7}$$

For convenience, we can introduce the *exchange matrix* $\alpha_{ij} \equiv \alpha q_i q_j$, which reflects the anyons' mutual *braid* statistics between kind-i and kind-j particles. Then, Eq. (A7) can be rewritten as

$$\partial_t(N_i - 2\pi\sigma_{h,i} \sum_j \alpha_{ij} N_j) = 0, \tag{A8}$$

which implies the term in the bracket is time-independent when α is changed from 0 to a finite value adiabatically, therefore

$$N_i(\alpha) - 2\pi\sigma_{h,i} \sum_j \alpha_{ij} N_j(\alpha) = N_i(0), \tag{A9}$$

where $N_i(0)$ is the total fermion number of kind-i when $\alpha = 0$. Identifying $N_i(\alpha)$ $[N_i(0)]$ in Eq. (A9) with $N_{i,g}(N_{i,0})$ in Eq. (A3), we establish a relation between mutual HES g_{ij} and the anyons' mutual *braid* statistics α_{ij},

$$g_{ij} = 2\pi\sigma_{h,i}\alpha_{ij}. \tag{A10}$$

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