Collective field theory for quantum Hall states

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We develop a collective field theory for fractional quantum Hall (FQH) states. We show that in the leading approximation for a large number of particles, the properties of Laughlin states are captured by a Gaussian free field theory with a background charge. Gradient corrections to the Gaussian field theory arise from the covariant ultraviolet regularization of the theory, which produces the gravitational anomaly. These corrections are described by a theory closely related to the Liouville theory of quantum gravity. The field theory simplifies the computation of correlation functions in FQH states and makes manifest the effect of quantum anomalies.

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I. INTRODUCTION

Since the work of Laughlin [1], a common approach to analyzing the physics of the fractional quantum Hall effect (FQHE) starts with a trial ground-state wave function for N electrons. Despite its success, this approach is an impractical framework for studying the collective behavior of a large number of electrons ($N \sim 10^6$, in samples exhibiting the QHE). As a result, some subtle properties of QHE states, such as the gravitational anomaly [2–10], were computed only recently.

The effects of *quantum anomalies* are essential in the physics of the QHE. Although anomalies originate at short distances on the order of the magnetic length, they control the large-scale properties of the state, such as transport. It was recently shown in Ref. [10] that, like the Hall conductance, transport coefficients determined by the gravitational anomaly are expected to be quantized on QH plateaus. For this reason, it is important to formulate the theory of the QH effect in a fashion that makes the quantum anomalies manifest. The field theory approach seems the most appropriate for this purpose.

In this paper, we develop a field theory for Laughlin states. This approach naturally captures universal features of the QHE, and it emphasizes the geometric aspects of QH states. We demonstrate how the field theory encompasses recent developments in the field [2–10], and we obtain some properties of quasihole excitations. A preliminary treatment of this approach appears in Ref. [3].

The field theory framework uncovers a connection between the QHE and random geometry, specifically two-dimensional (2D) Liouville quantum gravity. Since its introduction, the Laughlin wave function has been a practical model wave function mainly because of the plasma analogy. This analogy to a 2D statistical mechanical system allowed the most salient features of the state—uniform density and fractional quasihole charge—to be easily captured by a saddle-point approach to the partition function of the equivalent plasma.

Every analysis to date has stopped at the saddle point. As a result, subtle features of the theory such as the gravitational anomaly were missed. We show how the Laughlin wave-function maps to a full quantum field theory. This approach allows us to go beyond the saddle point and includes quantum fluctuations previously inaccessible by the plasma picture.

We present an analysis of the path-integral measure based on quantum anomalies. In the process, we find that accounting for the anomalies gives rise to the Liouville action. Thus, the correction to the plasma mapping involves a quantum theory of gravity, or random geometry.

The universal properties of the QHE are encoded in the dependence of the ground-state wave function on electromagnetic and gravitational backgrounds (see, e.g., [2]). For that reason, we study QH states on a Riemann surface, and for simplicity we focus on genus-0 surfaces.

We restrict our analysis to the Laughlin states. Our approach is closely connected to the hydrodynamic theory of QH states of Ref. [11] and the collective field theory approach of Gervais, Sakita, and Jevicki developed in Ref. [12] and extended in Refs. [13,14]. The action of the field theory for Laughlin states is written in Sec. III. The leading part, Eq. (10), is equivalent to the classical energy of a 2D neutralized Coulomb plasma when the discreteness of particles is not taken into account. This is used in the familiar plasma analogy of Ref. [1] to deduce the equilibrium density, as well as properties of the quasihole state such as charge and statistics. The other terms in the action are more subtle but equally significant, and they give rise to important effects including the gravitational anomaly.

II. COLLECTIVE FIELD THEORY

We start with some general remarks about the collective field theoretical approach. To compute the expectation value of an observable $O(z_1,\dots,z_N)$ within the ground state $\Psi(z_1,\dots,z_N)$, one has to evaluate a multiple integral over the individual particle coordinates,

$$\langle O\rangle = \int \Psi^* O \Psi \, dV_1 \cdots dV_N\,, \quad dV_i = \sqrt{g(z_i)}d^2 z_i, \quad (1)$$

and then proceed with the large-N limit. The field theory approach assumes instead that the appropriate variables are collective modes. In the QH systems, the ground state at a fixed background gauge potential is a holomorphic function of coordinates. On a Riemann surface, this means that the wave function is holomorphic in complex (or isothermal) coordinates where the metric is $ds^2 = \sqrt{g}\, dz\, d\bar{z}$. Therefore, holomorphic collective modes suffice for a complete field theory of the QHE. On genus-0 surfaces they are power sums,

$$a_{-k} = \sum_{i=1}^{N} z_i^k, \quad k \geqslant 1, \quad D\varphi = \prod_{k>0} da_{-k} d\bar{a}_{-k}.$$

The sum is taken in the $N \to \infty$ limit, and the measure of integration $D\varphi$ represents a functional integration over the real *collective field* $\varphi(\xi)$, where we denote $\xi = (z,\bar{z})$. For further discussion of the measure, see Sec. VI. The field is defined such that its current, the holomorphic derivative $\partial_z\varphi$, is a generating function of the modes a_{-k},

$$i\partial_z\varphi \equiv -i\sum_{k\geqslant 1} a_{-k}z^{-k-1}. \tag{2}$$

In this definition we assume that the field has no zero modes $\int \varphi\, dV = 0$ and is therefore globally defined on the Riemann surface. Expectation values are obtained by a functional integral over the field with the appropriate action

$$\langle O\rangle = \frac{\int O[\varphi]e^{-\Gamma[\varphi]}D\varphi}{\int e^{-\Gamma[\varphi]}D\varphi} \tag{3}$$

as opposed to the multiple integral in Eq. (1). The collective field φ defined by its expansion at infinity (2) can be extended to the finite part of the plane excluding the positions of particles where the current has poles $\partial\varphi|_{z\to z_i} \sim -1/(z-z_i)$. This field is defined as

$$\varphi(\xi) = 4\pi\sum_i G(\xi,\xi_i), \tag{4}$$

where G is the Green function of the Laplace-Beltrami operator Δ with the zero mode removed, and which satisfies

$$-\Delta G(\xi,\xi') = \delta^{(2)}(\xi-\xi') - \frac{1}{V}.$$

By definition, the collective field is a solution of the Poisson equation

$$-\Delta\varphi = 4\pi\left(\rho - \frac{N}{V}\right), \tag{5}$$

where $\rho(\xi)$ is the particle density.

We now specialize our discussion to the Laughlin state on genus-0 surfaces, but the final results hold for any genus. The Laughlin wave function reads

$$\Psi = \tfrac{1}{\sqrt{\mathcal{Z}}}\prod_{i<j}(z_i-z_j)^m e^{\frac{1}{2}\sum_i Q(\xi_i)}, \tag{6}$$

$$\hbar\Delta Q = -2eB, \tag{7}$$

where $m = 1/\nu$ is an integer, ν is the filling fraction, and Q is the "magnetic" potential of a slowly varying magnetic field B. Below we set $e = \hbar = 1$.

The normalization $\mathcal{Z}$, known as the generating functional, was studied in Refs. [2,3]. The generating functional is independent of the choice of coordinates and depends only on the geometry of the surface through functionals of the metric.

At a given magnetic field, the state is normalizable if the maximal number of particles is

$$N = \nu N_\phi + \tfrac{1}{2}\chi, \tag{8}$$

where χ is the Euler characteristic of the surface ($\chi = 2$ for a sphere) and $N_\phi = \frac{1}{2\pi}\int B\, dV$ is the total number of magnetic flux quanta. We assume that the state contains a maximal number of particles so the surface is completely filled and the particle density has no boundary.

Our goal is to represent the probability density $dP = |\Psi|^2\prod_i dV_i$ as a functional integral over the collective field Eq. (4) such that $dP \to e^{-\Gamma[\varphi]}D\varphi$.

III. MAIN RESULTS

Now we can formulate some results for the Laughlin state. We compute the action $\Gamma[\varphi]$ in Eq. (3) in the leading $1/N$ approximation. The action consists of three parts,

$$\Gamma[\varphi] = \Gamma_G[\varphi] + \Gamma_B[\varphi] + \Gamma_L[\varphi], \tag{9}$$

which are conveniently written in terms of the field φ and related field $\sigma = \log\sqrt{\rho/(N/V)}$,

$$\Gamma_G[\varphi] = \frac{1}{8\pi\nu}\int[(\nabla\varphi)^2 - R\varphi - 4\nu B\varphi]dV, \tag{10}$$

$$\Gamma_B[\varphi] = \frac{2}{\nu}\left(\nu - \frac{1}{2}\right)\frac{N}{V}\int e^{2\sigma}\sigma\, dV, \tag{11}$$

$$\Gamma_L[\varphi] = \frac{1}{24\pi}\int[(\nabla\sigma)^2 + R\sigma]dV, \tag{12}$$

where R is a scalar curvature of the surface. The actions (10)–(12) are derived in Secs. IV–VII. We remind the reader that the field φ is defined such that $\int\varphi\, dV = 0$, so the coupling with the curvature R and the magnetic field B in Eq. (10) occurs only if the curvature and magnetic field are not uniform. If they are uniform, the magnetic field enters only through relation (8).

The action is nonlinear since σ and φ are connected by Eq. (5). It consists of three distinct terms at different orders in $1/N$, in descending order. This can be seen by noticing that φ defined by (4) is of the order N, while σ is of the order 1.

The leading term (10) of the action is the Gaussian free field with a *background charge* that describes the coupling to curvature; cf. [15–17]. The background charge is directly related to the shift $\chi/2$ in Eq. (8). Perturbatively, the action (10) is equivalent to the Liouville theory of gravity (see, e.g., [18]) in the sense that the background charge increases the central charge of the Gaussian field from 1 to $1+3\nu^{-1}$. As a consequence, the conformal dimension of the vertex operator $e^{-a\varphi}$ is

$$h_a = \tfrac{1}{2}a(1-a\nu). \tag{13}$$

The conformal dimension is equal to the spin of the quasihole. This result refines the erroneous notion that the spin of a quasihole matches its mutual statistics and the charge deficit; both equal the filling fraction ν at $a = 1$.[1]

Formally the action (10) is that of a Gaussian free field and possesses conformal invariance. This invariance breaks at the next order of the action (11), except in the case of the bosonic Laughlin state $\nu = 1/2$ at which (11) vanishes.

Finally, the Polyakov-Liouville action (12) manifests the gravitational anomaly. This part of the action alone is identical to the action of the Liouville theory of gravity if the density $\rho = (N/V)e^{2\sigma}$ is identified as a random metric [from this point of view, the field φ plays the role of a random Kähler potential

[1]To the best of our knowledge, the spin of the quasihole was correctly computed in Ref. [30]; see also [19,20].

(cf. [21])]. The action does not possess the cosmological term since the number of particles is fixed and $\int e^{2\sigma} dV = V$. We can check the consistency of the action against some known results.

Minimizing the action, we find the first three leading terms of the $1/N$ expansion of the ground-state value of the particle density previously obtained in Ref. [2]. If the magnetic field is uniform, it is also a gradient expansion in curvature,

$$\langle\rho\rangle = \bar\rho + \left[\frac{1}{2\nu}\left(\nu - \frac{1}{2}\right) + \frac{1}{12}\right](l^2\Delta)\frac{R}{8\pi}, \quad \bar\rho = \frac{\nu B}{2\pi} + \frac{R}{8\pi}, \tag{14}$$

where $l = \sqrt{\hbar/eB}$ is the magnetic length.

The $\bar\rho$ term in Eq. (14) comes from (10). Integrating over the density yields the particle number (8), where the $R/(8\pi)$ term yields the background charge of $\chi/2$ due to the Gauss-Bonnet theorem $\int R\, dV = 4\pi\chi$. The order l^2 term in Eq. (14), which receives contributions from both (11) and (12), does not contribute to the particle number.

Linearizing the action on a flat space yields the propagator of density modes,

$$\Gamma[\varphi] \approx \frac{V}{2N}\sum_k S^{-1}(k)|\rho_k|^2, \tag{15}$$

where $S(k)$ is the static structure factor expanded to order k^6, first computed in Ref. [22] (see also [2]),

$$S^{-1}(k) = \frac{2}{(kl)^2}\left[1 + \frac{1}{\nu}\left(\nu - \frac{1}{2}\right)(kl)^2 + \frac{1}{48\nu}(kl)^4 \cdots\right].$$

Other results are described below.

IV. BOLTZMANN WEIGHT

The first step in constructing the collective field theory is expressing the wave function (6) as a functional of the collective field. The amplitude of (6) is interpreted as the Boltzmann weight of the neutralized Coulomb plasma $|\Psi|^2 \sim e^{-E}$, with temperature set to unity. We express the energy in terms of the Green function and the Kähler potential K defined by the conditions $\partial_z\partial_{\bar z}K = (\pi/V)\sqrt{g}$ and $K \sim \log|z|^2 + O(1/|z|)$ at infinity. Note that for constant B, the potential becomes $Q = -N_\phi K$. The energy reads

$$E = -2\iint \rho(\xi)G(\xi,\xi')B(\xi')dV_\xi dV_{\xi'} - N\int Q\frac{dV}{V} - \frac{1}{2}NN_\phi\int K\frac{dV}{V} + \frac{2\pi}{\nu}\sum_{i\neq j}G(\xi_i,\xi_j). \tag{16}$$

The last term in Eq. (16) takes into account the discreteness of particles.

In the continuum limit, we have to replace the sums over particle positions $\sum_{i\neq j} G(\xi_i,\xi_j)$ by integrals over the density taking into account the excluded self-interaction at $i = j$. We must therefore regularize the Green function $G(\xi_i,\xi_j)$ at coinciding points. The regularized Green function is defined by subtracting the logarithm of the geodesic distance $|\xi - \xi'|g^{1/4}$ between the points in units of the typical separation between particles, which is of the order of $\rho^{-1/2}$,

$$G_R(\xi) = \lim_{\xi\to\xi'}\left(G(\xi,\xi') + \frac{1}{4\pi}\log[|\xi - \xi'|^2\rho\sqrt{g}]\right). \tag{17}$$

Thus $\sum_{i\neq j} G(\xi_i,\xi_j)$ must be replaced by

$$\int\left[\int G(\xi,\xi')\rho(\xi')dV_{\xi'} - G_R(\xi)\right]\rho(\xi)dV_\xi.$$

Bringing all pieces together and integrating by parts,

$$E = E_0 + \Gamma_G[\varphi] - \frac{1}{2\nu}\int \rho\log\rho\, dV, \tag{18}$$

where $\Gamma_G[\varphi]$ is given by (10), and

$$E_0 = \frac{N}{\nu V}\iint \log|\xi - \xi'|^2\left(\bar\rho(\xi') - \frac{1}{2}\frac{N}{V}\right)dV_\xi dV_{\xi'},$$

where $\bar\rho$ is defined in Eq. (14). This gives the field-theoretical representation of the wave function. We comment that the short-distance regularization is determined by the density ρ and for that reason depends on the state of the plasma. A similar regularization scheme was employed for a 1D plasma in Ref. [23].

V. ENTROPY

The next step is to pass from integration over coordinates of individual particles to integration over the macroscopic density. This is a standard method in statistical mechanics (used in a setting similar to ours in Ref. [23]). The transformation defines the Boltzmann entropy $S_B[\rho] = -\int \rho\log(\rho/\bar\rho)\, dV$,

$$\prod_i \sqrt{g(\xi_i)}d^2\xi_i \to e^{S_B}D\rho.$$

Combining the Boltzmann weight and the entropy, we obtain the probability density

$$dP \to e^{-E[\rho]+S_B[\rho]}D\rho.$$

Here, the free energy of local equilibrium is

$$E - S_B = E_0 + \Gamma_G + \Gamma_B.$$

We observe that the Boltzmann entropy and the short-distance regularization of the Coulomb energy (18) combine to form Γ_B.

VI. GHOSTS

The next step is to determine the measure $D\rho$. Passing from $\rho \to \varphi$ comes at the price of a Jacobian, which is given by the spectral determinant of the Laplace-Beltrami operator,

$$D\rho \sim \det(-\Delta)D\varphi. \tag{19}$$

The determinant can be represented by (1,0) Faddeev-Popov ghosts as $\det(-\Delta) = \int e^{-\int \bar\eta(-\Delta)\eta\, dV} D\eta\, D\bar\eta$, where η are complex fermionic modes.

VII. GRAVITATIONAL ANOMALY

The last step involves the functional measure in Eq. (19). The procedure we outline below is commonly used in the theory of quantum gravity. Let us denote by X a field φ or ghosts $\eta,\bar\eta$ and consider the deviation δX from a given value of the field, say its mean. We define the norm of the deviation as

$$||\delta X||^2 = \sum_{i=1}^{N}[\delta X(\xi_i)]^2 = \int (\delta X)^2 \rho \, dV \tag{20}$$

and assume that the measure is normalized as $\int DX \exp[-||\delta X||^2] = 1$. Such normalization is supported by calculations based on the Ward identity for Laughlin states [3]. Thus the measure for both φ and the ghost fields depends in a nontrivial fashion on the density, and thus on φ itself. So although the ghosts appear decoupled from the rest of the action, in fact they are not.

The density ρ appearing in Eq. (20) can be treated as a conformal factor of the metric and thus removed from the measure by a conformal transformation of coordinates $dV \to \rho^{-1}dV$. It is known, however, that under conformal transformation, the measure transforms anomalously as

$$DX \to e^{c_X \Gamma_L[\sigma]} DX,$$

where c_X is the central charge of the field X, where $\Gamma_L[\rho]$ is the Polyakov-Liouville action (5) [24]; see also [25]. This is the *Weyl* or *gravitational anomaly*, which appears here in a similar fashion as in the quantum theory of gravity. Applying this to the collective field φ with the central charge $+1$ and ghost with the central charge -2, we obtain the measure

$$e^{-\Gamma_L[\rho]} D\varphi \, D\eta \, D\bar\eta.$$

After the Polyakov-Liouville action is taken into account, the short-distance regularization of the field φ and ghosts does not depend on density. Since the ghosts are decoupled, their contribution is the spectral determinant of the Laplace operator. Summing up, the probability distribution is

$$dP = \mathcal{Z}^{-1}\det(-\Delta) e^{-E_0-\Gamma[\varphi]} D\varphi. \tag{21}$$

The ghosts determinant contributes to the finite-size correction to the free energy of the Coulomb plasma [3,26].

Now we turn to some applications.

VIII. DENSITY AND GENERATING FUNCTIONAL

We start by computing the generating functional—the normalization factor of the Laughlin wave function or (21). The integral on the left-hand side of (21) is 1. The relevant contribution to the integral on the right-hand side of (21) comes from the Gaussian approximation. It consists of the on-shell action $\Gamma[\varphi_c]$ computed on the "classical" solution φ_c, which minimizes the action. Computing Gaussian fluctuations, it is sufficient to take into account only the leading part of the action (10),

$$\int e^{-\Gamma[\varphi]} D\varphi = [\det(-\Delta)]^{-\frac{1}{2}} e^{-\Gamma[\varphi_c]}.$$

Thus integrating (21) gives

$$\mathcal{Z} = [\det(-\Delta)]^{\frac{1}{2}} e^{-\Gamma_0}, \quad \Gamma_0 = E_0 + \Gamma[\varphi_c]. \tag{22}$$

In the three first leading orders in the $1/N$ solution of $\delta\Gamma[\varphi]/\delta\varphi = 0$ is the ground-state value of the field $\varphi_c = \langle\varphi\rangle$, which, through (5), determines the ground-state value of the density. Solving in the leading order in $1/N$, we obtain Eq. (14).

Inserting (14) back into (9), we find

$$\Gamma[\varphi_c] = -\frac{2\pi}{\nu}\iint \bar\rho(\xi')G(\xi,\xi')\bar\rho(\xi')dV_\xi dV_{\xi'}.$$

The final result for the functional Γ_0 in Eq. (22) is best expressed in terms of the gauge potential and spin connection. Their complex components are defined by

$$2i(\partial_{\bar z}A_z - \partial_z A_{\bar z}) = B\sqrt{g}, \quad 2i(\partial_{\bar z}\omega_z - \partial_z\omega_{\bar z}) = \tfrac{1}{2}R\sqrt{g}.$$

In the transverse gauge $\partial_{\bar z}A_z = -\partial_z A_{\bar z}$, $\partial_{\bar z}\omega_z = -\partial_z\omega_{\bar z}$, the functional Γ_0 has a compact form,

$$\Gamma_0 = -\frac{2}{\pi\nu}\int \left|\left(\nu A_z + \frac{1}{2}\omega_z\right)\right|^2 dz\, d\bar z.$$

It remains to recall the value of the spectral determinant of the Laplace operator in Eq. (22). Up to a metric-independent term, it is given by the Polyakov formula [27]

$$\log\det(-\Delta) = -\frac{1}{3\pi}\int |\omega_z|^2 dz\, d\bar z.$$

As a result (cf. [3]),

$$\log\mathcal{Z} = \int\left[\frac{2}{\pi\nu}\left|\left(\nu A_z + \frac{1}{2}\omega_z\right)\right|^2 - \frac{1}{6\pi}|\omega_z|^2\right]dz\, d\bar z. \tag{23}$$

In the form (23) it is valid on a surface with any genus.

The authors of Ref. [10] argued that the elements of the Hessian matrix of the generating functional,

$$\sigma_H = \frac{\pi}{2}\frac{\delta^2\log\mathcal{Z}}{\delta A_z \delta A_{\bar z}}, \quad 2\varsigma_H = \frac{\pi}{2}\frac{\delta^2\log\mathcal{Z}}{\delta\omega_z\delta A_{\bar z}}, \quad -\frac{c_H}{12} = \frac{\pi}{2}\frac{\delta^2\log\mathcal{Z}}{\delta\omega_z\delta\omega_{\bar z}},$$

are universal transport coefficients precisely quantized on QH plateaus. Here σ_H is the Hall conductance, ς_H determines the current caused by changing the metric, and the third coefficient, c_H, describes forces exerted on the fluid as a result of changing the metric. We refer to [10] for further details. For Laughlin states, these coefficients are encoded in Eq. (23),

$$\sigma_H = \nu, \quad \varsigma_H = 1/4, \quad c_H = 1 - 3/\nu. \tag{24}$$

IX. QUASIHOLES—GAUGE ANOMALY

Introduced by Laughlin [1], a quasihole state with charge a on a compact surface reads

$$\Psi_a = \frac{e^{\frac{1}{2}\nu a[Q(w)-aK(w)]}}{\sqrt{\mathcal{Z}_a[w,\bar w]}}\left[\prod_{i=1}^{N}(z_i - w)^a e^{-\frac{a}{2}K(z_i,\bar z_i)}\right]\Psi, \tag{25}$$

where w is a holomorphic coordinate of the quasihole, Ψ is the ground state (6) with N particles subject to the condition (8), a is a positive integer less than $m = 1/\nu$, and K is defined above (16). The factor of $\exp[-\frac{a}{2}K(\xi_i)]$ neutralizes the insertion of the quasihole. This state covers the entire surface. The exponential factor of $\frac{a\nu}{2}[Q - aK]$ in Eq. (25) is added for convenience.

A quasihole is represented by the vertex operator $V_a(w,\bar w) = e^{-a\varphi(w,\bar w)}$. In particular, the normalization factor

$\mathcal{Z}_a$, the generating functional for a quasihole state, reads up to constants

$$\mathcal{Z}_a[w,\bar{w}] \sim \langle V_a(w,\bar{w})\rangle,$$

where the average is taken over the ground state (6) without the quasihole. As such, the quasihole may be seen as a source for the action (10) $\Gamma \to \Gamma + a\varphi(w)$. However, there is a caveat. The quasihole disturbs the electronic density around itself in a vicinity of the size of the magnetic length. At the limit of a vanishing magnetic length, the density becomes singular. At the same time, the derivation of the action was based on the assumption that the density is smooth. Therefore, the derivation must be reexamined to take into account the feedback of the singularity.

The leading $1/N$ value of (23) is given by the Gaussian part of the action (10),

$$\mathcal{Z}_a \approx \exp\left(-a\langle\varphi\rangle + \frac{a^2}{2}\langle\varphi^2\rangle_c\right). \tag{26}$$

The mean of the field φ determined by (10) is

$$\langle\varphi(\xi)\rangle \approx 4\pi\int G(\xi,\xi')\bar{\rho}(\xi')dV_{\xi'} = \nu Q + \frac{1}{2}\log\sqrt{g(\xi)}.$$

The variance is $\langle\varphi^2\rangle_c \equiv \langle\varphi^2\rangle - \langle\varphi\rangle^2 = 4\pi\nu G_R$, where the regularized Green function is given by (17). But the G_R depends on the density itself, and in the leading approximation one replaces the density by its mean such that $\langle\varphi^2\rangle_c = \nu\log(\langle\rho\rangle\sqrt{g})$. Putting this together, we obtain

$$\mathcal{Z}_a \approx (\sqrt{\langle\rho\rangle})^{\nu a^2}(\sqrt{g})^{-h_a}, \tag{27}$$

where $h_a = \frac{a}{2}(1-\nu a)$ is the conformal dimension as in Eq. (13).

In the leading approximation, the factor $\langle\rho\rangle$ in Eq. (27) can be treated as a constant. Then (21) suggests that h_a is the conformal dimension of the quasihole state: the quasihole state transforms as a primary field under a holomorphic transformation. Symbolically,

$$w \to f(w), \quad V_a \to [f'(w)]^{h_a}V_a.$$

Because the state is holomorphic [up to the normalization factors in Eq. (25)], the holomorphic dimension h_a is also the spin of the state. Later we show this in a more direct manner.

In the next-to-leading approximation, we cannot assume that the density (27) is a constant. As with the gravitational anomaly above, the field transforms as $\varphi \to \varphi - a\nu\log\sqrt{\rho}$, which modifies the vertex operator,

$$V_a = (\sqrt{\rho})^{\nu a^2}e^{-a\varphi},$$

such that the regularization of the two-point correlation function at coincident points is independent of the state density. Alternatively, we may say that the quasihole contributes to the action as a source $\Gamma \to \Gamma + a\varphi - a^2\nu\log\sqrt{\rho}$. Thus the stationary point of the action reads

$$\frac{\delta\Gamma}{\delta\varphi(\xi)} = -a\left(1 + \frac{\nu a}{8\pi\rho}\Delta\right)\delta(w-\xi). \tag{28}$$

In the linear approximation, we treat ρ in Eq. (28) as a constant $\approx\nu/(2\pi l^2)$ and use (15). As a result, we obtain the first two terms of the expansion in $(kl)^2$,

$$\rho_k \approx \frac{2\nu a}{(kl)^2}\left(-1 + \frac{a}{4}(kl)^2\right)S(k) \approx -\nu a + \frac{(kl)^2}{2}(a\nu - h_a).$$

Equivalently, the first two moments of the density $\delta\rho = \langle\rho\rangle - \frac{N}{V}$ are

$$\int \delta\rho\, dV = -\nu a, \tag{29}$$

$$\frac{1}{2l^2}\int r^2\delta\rho\, dV = -\nu a + h_a. \tag{30}$$

The first is the fractional charge deficit $-\nu a$. This result goes back to [1]. The second moment is more involved [28,29]. It shows the dimension of the state [3]. Curiously, the second moment vanishes at $\nu = \frac{1}{3}$ and $a = 1$.

Having determined the generating functional, we compute the adiabatic phase $\gamma_{\mathcal{C}}$ acquired by the quasiholes by transporting one around a closed path $\mathcal{C}$. For simplicity, we compute the adiabatic phase when one hole with coordinate w_1 moves around a closed path $\mathcal{C}$ enclosing another quasihole with coordinate w_2. The extension of (26) and (27) to the case of two quasiholes is

$$\mathcal{Z}_{a_1a_2}(w_1,w_2) = \mathcal{Z}_{a_1}(w_1)\mathcal{Z}_{a_2}(w_2)e^{4\pi\nu a_2a_1G(w_1,w_2)}, \tag{31}$$

where we used $\langle\varphi(w_2)\varphi(w_2)\rangle_c = 4\pi\nu G(w_1,w_2)$ and (26).

The adiabatic phase reads

$$\gamma_{\mathcal{C}} = 2i\int\left[\oint_{\mathcal{C}}\overline{\Psi}\partial_{w_1}\Psi dw_1\right]dV_1\cdots dV_N.$$

Since the state is a holomorphic function of the position of the quasiholes, only the normalization factor in Eq. (25) contributes to the phase,

$$\gamma_{\mathcal{C}} = -2\pi a_1\nu\,\Phi_{\mathcal{C}} + i\oint_{\mathcal{C}}\partial_{w_1}\log\mathcal{Z}_{a_1a_2}dw_1.$$

The first term is the Aharonov-Bohm phase picked up by a particle with charge $-a_1\nu$ enclosing the magnetic flux $\Phi_{\mathcal{C}} = (N_\Phi + a_1 + a_2)\text{area}(\mathcal{C})/V$ in units of the flux quantum. The contribution of the second term follows from (31),

$$i\oint_{\mathcal{C}}\partial_{w_1}\log\mathcal{Z}_{a_1a_2}dw_1 = -h_{a_1}\Omega_{\mathcal{C}} + 2\pi\nu a_1a_2. \tag{32}$$

It contains the solid angle $\Omega_{\mathcal{C}} = i\oint d\log\sqrt{g} = \frac{1}{2}\int_{\mathcal{C}} R dV$. The coefficient in front of it is the spin of the quasihole, equal to the holomorphic dimension (13). This formula extends the result of Ref. [30], which was for the adiabatic phase of a single quasihole ($a = 1$) on a sphere.

The last term in Eq. (32), $4\pi i\nu a_2a_1\oint dG(w_1,w_2)$, which vanishes if the contour $\mathcal{C}$ does not enclose w_2, is commonly referred to as the mutual statistics of the quasiholes. When the quasiholes are identical, it is equal to νa^2, and it differs from the spin.

X. EFFECT OF SPIN

Lastly, we comment on the effect of spin of quantum Hall states. The spin, yet another characterization of the QH state was introduced in Ref. [3]. The inclusion of spin comes as a generalization of the lowest Landau level (LLL). We recall that the LLL are defined as zero modes of the anti-holomorphic component of the kinetic momentum operator $\bar{\pi} = -i\hbar\bar{\partial} + \hbar s\bar{\omega} - e\bar{A}$ where $\bar{\omega} = -(i/2)\bar{\partial}\log\sqrt{g}$, where parameter s is the spin. Throughout, the paper we set the spin to zero. Inclusion of spin shifts the magnetic potential Q in (7) by $-s\log\sqrt{g}$, such that the modified Q now satisfies the Poisson equation $\Delta Q = -\frac{2e}{\hbar}B + sR$. As a result, the action acquires an additional term $\frac{s}{4\pi}\int\varphi R dV$, which shifts the background charge in the Gaussian action

$$\Gamma_G[\varphi] = \frac{1}{8\pi\nu}\int[(\nabla\varphi)^2 - (1-2\nu s)R\varphi - 4\nu B\varphi]dV.$$

The Boltzmann entropy (11) and the Polyakov-Liouville action (12) remain the same. Below we list some effects of spin.

Spin does not appear in local properties evaluated at distances where change of curvature is negligible, for example in a flat space. In particular the structure factor $S(k)$ of Sec. III, the charge of the quasi-hole (29) and its moment (30) are independent of spin.

However, global characteristics depend on spin. As such, the relation (8) between the total number of particles and magnetic flux becomes

$$N = \nu N_\phi + \frac{1}{2}(1-2\nu s)\chi. \tag{33}$$

The spin modifies the conformal dimension (13) defined in (27) and appearing in the adiabatic phase (32)

$$h_a = \frac{1}{2}a(1-2\nu s - a\nu).$$

However, the second moment (30) will not acquire any spin dependence, and will maintain its relation to the conformal dimension at $s = 0$.

Spin also enters the generating functional (23)

$$\log\mathcal{Z} = \int\left[\frac{2}{\pi\nu}\left|\left(\nu A_z + \frac{1}{2}(1-2\nu s)\omega_z\right)\right|^2 - \frac{1}{6\pi}|\omega_z|^2\right]dzd\bar{z}.$$

Consequently, the Hall conductance does not depend on spin, but the geometric transport coefficients in (24) do

$$\varsigma_H = \frac{1}{4}(1-2\nu s),\ c_H = 1 - 3\nu^{-1}(1-2\nu s)^2$$

For more details regarding the inclusion of spin into the FQHE on a curved space, see Ref. [3].

XI. CONCLUSION

In summary, we formulated the theory of the Laughlin QH states as a field theory of a scalar Bose field. The field theory consists of the Gaussian action with the background charge and the subleading corrections representing the gravitational anomaly. We demonstrated that this theory captures conformal properties of quasiholes, the adiabatic transport, and it clarifies the effect of the gravitational anomaly.

Finally, we comment that the action similar to (9) has been considered in Ref. [21] as an admissible action for a random metric. The actions become analogous upon identifying the fluctuating density as a random metric and the field φ as a fluctuating Kähler potential.

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