# Josephson junction detectors for Majorana modes and Dirac fermions

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We demonstrate that the current-voltage ($I$-$V$) characteristics of resistively and capacitively shunted Josephson junctions (RCSJs) hosting localized subgap Majorana states provide a phase-sensitive method for their detection. The $I$-$V$ characteristics of such RCSJs, in contrast to their resistively shunted counterparts, exhibit subharmonic odd Shapiro steps. These steps, owing to their subharmonic nature, exhibit qualitatively different properties compared to harmonic odd steps of conventional junctions. In addition, the RCSJs hosting Majorana bound states also display an additional sequence of steps in the devil's staircase structure seen in their $I$-$V$ characteristics; such a sequence of steps makes their $I$-$V$ characteristics qualitatively distinct from that of their conventional counterparts. A similar study for RCSJs with graphene superconducting junctions hosting Dirac-like quasiparticles reveals that the Shapiro step width in their $I$-$V$ curves bears a signature of the transmission resonance phenomenon of their underlying Dirac quasiparticles; consequently, these step widths exhibit a $\pi$ periodic oscillatory behavior with variation of the junction barrier potential. We discuss experiments which can test our theory.

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## I. INTRODUCTION

The possibility of realization of Majorana zero modes, particles with anyonic statistics described by real wave functions, has attracted tremendous interest in recent years [1]. Several suggestions regarding condensed-matter systems which can host such fermions have recently been put forth [2–9]. Out of these, the most promising ones for experimental realization turn out to be those which host Majorana modes as localized subgap states in their superconducting ground state [5–9]. Typically, the occurrence of such states requires unconventional superconducting pairing symmetry such as $p$- or $d$-wave pairing [10,11]. However, recent proposals have circumvented this requirement; it was shown that such bound states can occur either at the end of a one-dimensional (1D) wire in a magnetic field with spin-orbit coupling and in the presence of a proximate $s$-wave superconductor [6,7] or in superconducting junctions atop a topological insulator surface hosting Dirac fermions on the surface of a topological insulator [8]. Such Majorana fermions leave their signature as a midgap peak in tunneling conductance measurement [12] as well through the fractional Josephson effect [13].

Another interesting phenomenon in recent years has been the discovery of materials whose low-energy quasiparticles obey Dirac-like equations. These materials are commonly dubbed Dirac materials; graphene and topological insulators are common examples of such materials [14,15]. These materials can exhibit superconductivity via the proximity effect with Cooper pairing occurring between Dirac electrons with opposite momentum [16,17]; it is well known that transport properties of such superconductors differ from their conventional counterparts and can serve as experimental signatures of the Dirac nature of their constituent quasiparticles [16,17].

The experimental detection of Majorana modes has mainly relied on either measurement of a midgap peak [18] or detection of even Shapiro steps in a Josephson junction of superconductors hosting Majorana modes [19]. The effectiveness of the former set of experiments in the detection of Majorana modes has been questioned since the midgap peak did not lead to the expected $2e^2/h$ value of the tunneling conductance and could have also occurred because of several other effects such as the presence of magnetic impurities leading to the Kondo effect [20] and impurity-induced subgap states [21]. In this sense, the presence of even Shapiro steps at $V = n\hbar\omega_J/e$ [and the absence of odd ones at $(2n+1)\hbar\omega_J/2e$] in Josephson current measurement, where $\omega_J$ is the Josephson frequency, $n$ is an integer, and $V$ is the applied external voltage, provides a more definite detection of such fermions since they constitute a phase-sensitive signature which is free of effects of disorder [13]. Consequently, theoretical studies of the ac Josephson effect for unconventional superconductors which hosts Majorana modes has received a lot of attention lately [22,23]. Theoretical studies of the Josephson effect in graphene Josephson junctions has also been carried out [24,25]; it was shown that the critical current of such junctions shows a novel oscillatory dependence on the barrier potential of the junctions. However, the features of ac Josephson effects in either of these systems for a resistively and capacitively shunted Josephson junction (RCSJ) in the presence of external radiation has not been studied previously. In this context, we mention that the analysis presented in this paper for junctions hosting Majorana subgap states is somewhat idealized in the sense that it does not provide a full treatment of Landau-Zener tunneling and other related quantum effects; however, we do provide a qualitative discussion and identify a regime of junction parameters where our analysis is expected to hold qualitatively.

In this work, we study the Josephson junction described by a RCSJ model where the superconductors forming the junction either host Majorana modes at the interface or constitute Dirac-like quasiparticles. In the former case, we show that in contrast to their counterpart in conventional junctions, the $I$-$V$ characteristics display *subharmonic odd Shapiro steps* whose width vanishes for resistive junctions (in the limit where the junction capacitance approaches zero) [22,23]. We provide an analytical formula for the step width of both even and odd steps for such junctions, show that the analytic result matches exact numerics closely, and demonstrate, on the basis of this analytical result, that the behavior of this ratio is qualitatively different for Josephson junctions with and without Majorana bound states. In particular we demonstrate that the ratio of the odd and even step widths decreases exponentially with the junction capacitance for junctions with Majorana modes; in contrast, this ratio does not vary appreciably for conventional $s$-wave junctions. Thus it serves as a robust indicator for bound Majorana states in a Josephson junction. Our analytical results, supported by numerical analysis, reproduces the phenomenon of the absence of odd Shapiro steps in Josephson junctions with Majorana bound states as a special limiting property of resistive Josephson junctions; thus our work indicates that the absence of odd Shapiro steps, while sufficient, is not a necessary characteristic of Majorana bound states in a Josephson junction. We also find that the $I$-$V$ characteristics of junctions with Majorana bound states show a qualitatively different devil's staircase structure which is distinct from their $s$-wave counterparts. In particular, they display an additional sequence of steps which follows Farey's sum rule [26]; such sequences are absent in $I$-$V$ characteristics of conventional $s$-wave junctions. In the latter case, for junctions of superconductors hosting Dirac quasiparticles, we show that the width of the Shapiro steps displays $\pi$-periodic oscillatory dependence on the barrier potential of the junction. Such a behavior is a direct consequence of the transmission resonance phenomenon of the Dirac-like quasiparticles of the superconductors forming the junctions and is qualitatively distinct from conventional junctions hosting quasiparticles which obey Schrodinger's equation. We note that our work shows that a RCSJ can act as phase-sensitive detection device for both Majorana bound states in topological superconductors and Dirac-like quasiparticles in a superconductor; it is therefore expected to be of interest to theorists and experimentalists working on both Majorana modes and Dirac materials.

The plan of the rest of this work is as follows. In Sec. II, we provide our analytical and numerical results for junctions which host subgap Majorana bound states. In Sec. III, we present our results on junctions which host Dirac-like quasiparticles. Finally, we summarize our main results, provide a discussion of experiments that can test our theory, and conclude in Sec. IV.

## II. JUNCTIONS WITH MAJORANA MODES

The basic design of the circuit which we propose to serve as the detector is shown in Fig. 1. To analyze the property of this circuit, we first consider the Josephson junction. In our proposal, this comprises two superconductors with order parameters $\Delta_R$ (for $x > d/2$) and $\Delta_L$ (for $x < -d/2$) separated by a barrier region of width $d$ ($-d/2 \leqslant x \leqslant d/2$) characterized by a barrier potential $V_0$, as shown in Fig 1. The superconductors can be either topological superconductors with (effective) $p$-wave pairing [6,7] or superconductors with Dirac-like quasiparticles which have $s$-wave pair potential [16,17,24,25]. In this section, we analyze the former case in detail. We note at the outset that the present analysis will hold for topological superconductors in 1D wire geometry [6,7] provided that the transverse dimension $L$ is set to zero.

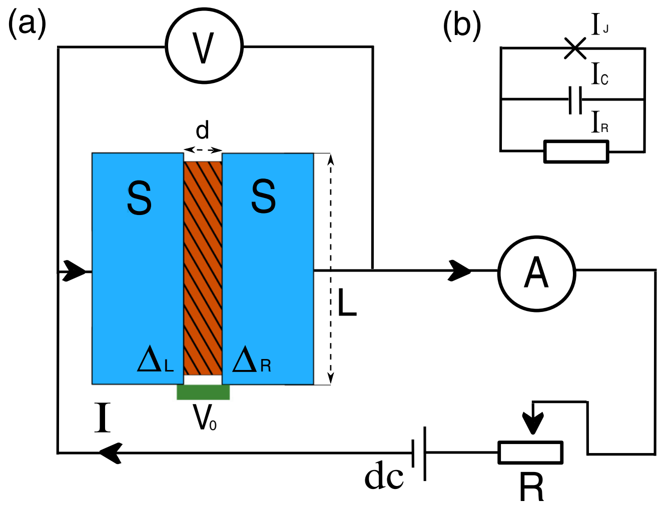


FIG. 1. (Color online) Schematic representation of the Josephson junction in the RCSJ circuit (see inset). The junction has width $L$ in the transverse direction, and the barrier region separating the two superconductors has a thickness $d$ and is modeled by a potential $V_0$. $I_J$, $I_C$, and $I_R$ are the Josephson current, displacement current, and quasiparticle currents, respectively.

Josephson junctions, shown in Fig. 1, are known to support localized subgap Andreev bound states, which can be obtained as the solution of the Bogoliubov–de Gennes (BdG) equation. For topological superconductors which support $p$-wave pairing, the BdG equation reads

$$\{[H_\beta + V(x)]\tau_z + [\Delta_\beta(x)\tau_+ + \text{H.c.}]\}\,\psi_\beta = E\psi_\beta, \quad (1)$$

where $\beta = R, L$ for the right and the left superconductors, $\psi_\beta = [\psi_{\beta\uparrow}(x,k_\parallel), \psi^\dagger_{\beta\downarrow}(x,k_\parallel)]$ is the two-component BdG wave function, $V(x) = V_0\delta(x)$ is the barrier potential (we take the limit of a thin barrier for which $d \to 0$), and $H_\beta = \hbar^2k^2/(2m) - \mu$ denotes the dispersion of the left and right superconductors, with $\mu$ being the chemical potential, $m$ being the electron mass, and $k^2 = -\partial_x^2 + k_\parallel^2$. In what follows, we are going to assume $p_x$-wave pairing and write $\Delta_L(x) = \Delta_0 k_{xF}/k_F$ and $\Delta_R = \Delta_0 k_{xF}/k_F \exp(i\phi)$, where $k_F$ are the Fermi momenta of the two superconductors, $\phi$ is the phase difference across the junction, $\Delta_0$ is the amplitude of the superconducting gap, and $k_{xF}$ is the $x$ component of the Fermi momentum. The wave functions $\psi_\beta$ satisfy the boundary condition $\psi_L(x=0) = \psi_R(x=0)$ and $\partial_x\psi_L(x=0) - \partial_x\psi_R(x=0) = k_F\chi_1(k_\parallel)\psi_L(x=0)$, where $\chi_1(k_\parallel) \equiv \chi_1 = 2U_0/\hbar v_F(k_\parallel)$ is the dimensionless barrier potential for a given transverse momentum of the quasiparticles. The localized subgap solutions of Eq. (1) are given by [13,27]

$$E_1 = -\Delta_0\cos(\phi/2)/\sqrt{1+\chi_1^2/4}. \quad (2)$$

Note that for $\phi = \pi$, $E = 0$, and it has been shown that this state constitutes a realization of Majorana modes [8]. Since these subgap states are the only ones with $\phi$-dependent dispersion, one find the zero-temperature Josephson current as

$$I_1(\phi) = \frac{2e}{\hbar}\partial E_1/\partial\phi = \int_{-k_F}^{k_F} \frac{dk_y}{2\pi} \frac{e\Delta_0 \sin(\phi/2)}{2\hbar\sqrt{1+\chi_1^2/4}}. \quad (3)$$

As noted in Ref. [13], the current is $4\pi$ periodic, and a substitution $\phi \to 2eVt/\hbar$ in the presence of a bias voltage $V$ leads to the fractional ac Josephson effect [13,27].

We now use Eqs. (3) to obtain the response of an RCSJ circuit constructed out of the superconducting junctions discussed above. The RCSJ model, shown in the inset of Fig. 1, includes a resistive component to take into account the dissipative process, which may occur, for example, due to quasiparticle tunneling, and a shunting capacitance $C$, which takes into account the displacement currents due to possible charge accumulation in the leads [28]. The current phase relationship for this model, in the presence of an external radiation, is given by [28,29]

$$\ddot{\phi} + \beta\dot{\phi} + I_J(\phi)/I_c = I/I_c + A\sin(\omega t)/I_c, \quad (4)$$

where $I_c$ is the critical current of the junction, $A$ and $\omega$ are the amplitude and frequency of the external radiation, $I_J(\phi) = I_1$ for superconducting junctions with Majorana fermions, $\beta = \sqrt{\hbar/(2eI_cR^2C_0)}$, $R$ and $C_0$ denote the resistance and capacitance of the junction, and we have scaled $t \to t/\tau$, where $\tau = \sqrt{\hbar C_0/(2eI_c)}$.

Before proceeding further, we note that Eq. (2) holds in the ideal limit where the two subgap Majorana modes at the two ends of each wire do not interact [23]. Such an interaction leads to hybridization of amplitude $\delta$ between the two Majorana branches: $E_{1\text{hyb}} = \pm\sqrt{\delta^2 + E_1^2}$ [23]. The hybridization amplitude $\delta$ arising from such interaction is exponentially suppressed for long wires $\delta \sim \exp[-L/(2\xi)]$, where the coherence length $\xi$, for 1D nanowires, depends on the spin-orbit-coupling strength of the wire. In the presence of an external voltage, this leads to a Landau-Zener tunneling probability $P_t = \exp[-2\pi\delta^2/(E_J\hbar\dot{\phi})]$, where $E_J \sim \Delta_0$ is the maximal Josephson energy. The manifestation of $4\pi$ periodicity is evident when $P_t \simeq 1$. This occurs when $\delta^2/(E_J\hbar\dot{\phi}) \ll 1$; a complete determination of the frequency and voltage range where $P_t \simeq 1$ requires the self-consistent solution of Eq. (4) and $I_J = \partial E_{1\text{hyb}}/\partial\phi$. We have not attempted that in this work; however, we note that such a regime can always be obtained for long enough wire since $\delta$ is exponentially suppressed in this regime, leading to $P_t \simeq 1$. An analysis of these conditions for resistive junctions can be found in Ref. [23]; in the rest of this work, we shall assume $P_t \simeq 1$ and work with Eq. (2).

In what follows, we first obtain an approximate analytical solution of Eq. (4) in Sec. II A, which demonstrates the existence of subharmonic odd Shapiro steps for junctions hosting Majorana bound states. Then in Sec. II B we carry out a detailed numerical study of Eq. (4) where the analytical results are verified and the devil's staircase structure of the Shapiro steps is studied.

### A. Perturbative analytical solution

In this section we consider perturbative analysis of Eq. (4) for an unconventional Josephson junction composed of superconductors hosting subgap Majorana states with $E = E_J\cos(\phi/2)$. We begin from the equation of such a junction given by Eq. (4) and analyze this equation for $\omega,\beta\omega,A \gg 1$. The key point regarding this analysis is the observation in the regime mentioned above that it is possible to expand $\phi$ as [30]

$$\phi = \sum_n \epsilon^n\phi_n, \quad I = \sum_{n=0}^{\infty}\epsilon^n I_n, \quad (5)$$

where $I_0$ is the applied current and $I_n$, $\epsilon \ll 1$, and $I_n$ for $n > 0$ are determined self-consistently from the condition of the absence of additional dc voltage: $\lim_{T\to\infty}\int_0^T \dot{\phi}_n = 0$ [28].

The equations for $\phi_n$ can be obtained by equating terms in the same order of $\epsilon$. The procedure is standard and yields

$$\ddot{\phi}_n + \beta\dot{\phi}_n = f_n(t) + I_n, \quad f_0 = A\sin(\omega t),$$
$$f_1 = -\sin[\phi_0/2], \quad f_2 = \phi_1\cos[\phi_0/2]/2. \quad (6)$$

Note that the $n = 0$ equation represents the autonomous $I$-$V$ curve of the junction and is independent of the nonlinear sinusoidal term.

To solve these equations, we note that this represents a linear first-order differential equation in $\dot{\phi}_n$; consequently, we define $y_n = \dot{\phi}_n$ and write

$$\dot{y}_n + \beta y_n = I_n + f_n(t). \quad (7)$$

These equations admit the solution

$$y_n(t) = \beta^{-1}I_n + e^{-\beta t}\int_0^t e^{\beta t'} f_n(t')dt'$$
$$\phi_n(t) = \int_0^t y_n(t')dt' + \phi_n(0). \quad (8)$$

For $n = 0$, this yields

$$\phi_0(t) = \phi' + I_0 t/\beta + \frac{A}{\omega\gamma}\sin(\omega t + \alpha_0), \quad (9)$$

where $\alpha_0 = \arccos(\omega/\gamma)$, $\gamma = \sqrt{\beta^2 + \omega^2}$, and $\phi'$ is the dc phase of the junction. The supercurrent at this order is given by

$$I_s^{(0)} \sim \sin[\phi_0(t)/2] = \mathrm{Im}(e^{i\phi_0(t)/2}) \quad (10)$$

$$= \mathrm{Im}\sum_{n=-\infty}^{\infty} J_n(x)e^{i([I_0/(2\beta)+n\omega]t+n\alpha_0+\phi'/2)} \quad (11)$$

where $x = A/(2\gamma\omega)$. Thus the Shapiro steps occur when the ac component of the supercurrent vanishes: $I_0 = 2|n|\omega$ (even steps). The width of the $n$th even step can be read off from Eq. (11): $W_{\text{even}} = \Delta I_s^{\text{even}} = 2J_n(x)$. Thus Eqs. (10) and (11) lead to expression for width of harmonic steps. Note that in contrast to the conventional junctions where $I_s \sim \sin[\phi_0]$, only even harmonic steps occur for junctions which host subgap Majorana steps.

Next, we obtain the solution for $\phi_1$. Substituting Eq. (11) in Eq. (7), we find, after some straightforward algebra,

$$\phi_1 = \sum_{n=-\infty}^{\infty} J_n(x)(\gamma_n\omega_n)^{-1}\cos(\omega_n t + n\alpha_0 + \delta_0 + n\phi'/2), \quad (12)$$

where $\omega_n = I_0/2\beta + n\omega$, $\delta_n = \arccos(\omega_n/\gamma_n)$, and $\gamma_n = \sqrt{\omega_n^2 + \beta^2}$. At this order, the supercurrent is given by

$$\begin{aligned} I_s^{(1)} &\sim \frac{1}{2}\phi_1(t)\cos[\phi_0(t)/2] \\ &= \sum_{n_1,n_2=-\infty}^{\infty} J_{n_1}(x)J_{n_2}(x)(2\gamma_{n_1}\omega_{n_1})^{-1} \\ &\quad \times \sin[\omega_{n_1}t + n_1(\alpha_0 + \phi'/2) + \delta_{n_1}] \\ &\quad \times \cos[\omega_{n_2}t + n_2(\alpha_0 + \phi'/2) + \delta_{n_2}] \\ &= \sum_{n_1,n_2} J_{n_1}(x)J_{n_2}(x)(4\gamma_{n_1}\omega_{n_1})^{-1} \\ &\quad \times (\sin\{[\omega_{n_1} + \omega_{n_2}]t + [n_1 + n_2](\alpha_0 + \phi'/2) + \delta_{n_1}\} \\ &\quad + \sin\{[\omega_{n_1} - \omega_{n_2}]t + [n_1 - n_2](\alpha_0 + \phi'/2) + \delta_{n_1}\}). \end{aligned} \tag{13}$$

At this order, we find that there are additional steps in the dc component of the supercurrent; these steps occur at $|n_1 + n_2|\omega = I_0$, for which the first of the two terms on the right side of Eq. (14) becomes independent of time. A set of these steps occurs at $(n_1 + n_2) = 2n - 1$ for integers $n = 1,2...$ and constitutes the odd Shapiro steps. Thus we find that the odd steps for a junction of superconductors hosting Majorana ground states are necessarily of subharmonic nature. The width of these steps can be read off from Eq. (14) to be

$$W_{\text{odd}} = \Delta I_{sn}^{\text{odd}} = \sum_{n_1} \frac{J_{n_1}(x)J_{2n-1-n_1}(x)}{2[(\{2n-1-2n_1\}\omega)^2/4 + \beta^2]} \tag{14}$$

We note that when $C_0 \to 0$, $\beta \to \infty$, and the subharmonic steps vanish, leading to the result that only even harmonic Shapiro steps exist for resistive Josephson junctions hosting subgap Majorana steps. Thus our analysis reproduces the absence of odd Shapiro steps in Josephson junctions with Majorana bound states as a special case [13,22,23]. We also note that these odd steps have a completely different origin than the analogous steps discussed in Ref. [23] since they occur without any $\sim \sin(\phi)$ dependence of $I_J$. Finally, we note that the ratio of the $n$th even and the adjacent odd Shapiro steps for these junctions are given by

$$\eta_n = \frac{W_{\text{even}}}{W_{\text{odd}}} = \frac{2J_n(x)}{\sum_{n_1} \frac{J_{n_1}(x)J_{2n+1-n_1}(x)}{2[(\{2n-1-2n_1\}\omega)^2/4+\beta^2]}} \tag{15}$$

In the next section, we shall compare the analytical expression [Eq. (15)] with numerical results obtained by exact numerical solution of Eq. (4).

### B. Exact numerical results

To compute the $I$-$V$ characteristics, we study the temporal dependence of $V = \hbar\dot{\phi}/(2e)$ obtained by numerical solution of Eq. (4) as a function of time for a fixed bias current $I$. The dc component of the voltage is obtained by standard procedure [29,31] from $V$ and plotted as a function of $I$ to generate the $I$-$V$ characteristics.

The central results that we obtain from this analysis are as follows. First, for topological superconductors hosting Majorana subgap states, we find that for a significant range

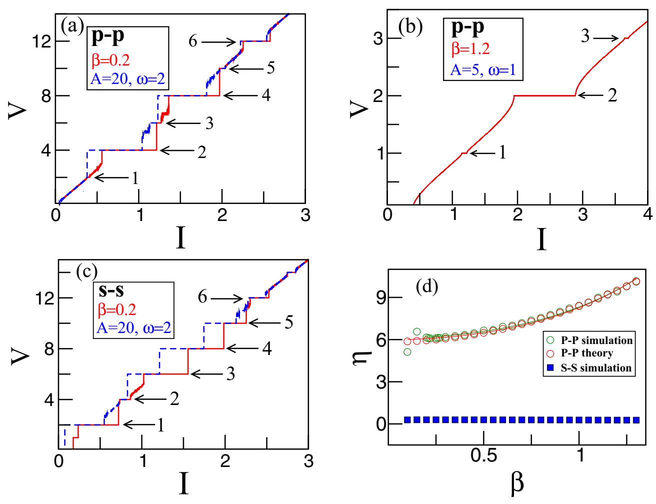


FIG. 2. (Color online) (a) and (b) CVC of the $p$-wave Josephson junction. (a) shows the CVC in the underdamped region ($\beta = 0.2$) and for $A = 20$ and $\omega = 2$, while (b) represents the overdamped region ($\beta = 1.2$) with $A = 5$ and $\omega = 1$ (in appropriate dimensionless units; see text). (c) The CVC of the $s$-wave junction [all parameters are the same as in (a)]. (d) The ratio of the widths of the even and odd Shapiro steps, $\eta = W_{\text{even}}(2\omega)/W_{\text{odd}}(\omega)$, for the $p$-wave and the $s$-wave junctions (inset) as a function of $\beta$ for $A = 10$ and $\omega = 3$. For the $p$ wave, $\eta \sim \exp(0.31\beta^2)$, while for the $s$ wave, $\eta$ does not vary with $\beta$. (a), (b), and (c) have $V$ and $I$ scaled in units of $\hbar/(\tau e)$ and $I_c$, respectively. The red solid (blue dashed) curves in (a) and (c) correspond to data for increasing (decreasing) current sweeps; these data coincide in the overdamped region, as shown in (b).

of external coupling and in the underdamped region $\beta < 1$, both even and odd Shapiro steps appear as expected from the analytical results obtained in Sec. II A. The even steps at $V = 2n\hbar\omega/e$ are enhanced compared to their odd counterparts at $V = (2n+1)\hbar\omega/e$, as shown in Figs. 2(a) and 2(b) for underdamped ($\beta < 1$) and overdamped ($\beta > 1$) regions, respectively. Figure 2(c) shows an analogous plot for the $s$-wave superconductors. The dominance of the even steps over the odd ones is characterized by $\eta_1 \equiv \eta$ [Eq. (15)]. A plot of $\eta$ as a function of $\beta$ shown in Fig. 2(d) demonstrates that $\eta \sim \exp(0.3\beta^2)$ for junctions with Majorana modes. We also note that the theoretical result for $\eta$ obtained from Eq. (15) provides a near-perfect match with the exact numerics, demonstrating the accuracy of the analytical solution over a wide range of $\beta$. We further note that the behavior of $\eta$ as a function of $\beta$ is in complete contrast to its counterpart for $s$-wave superconductors where $\eta$ does not vary appreciably with $\beta$, as shown in Fig. 2(d). Thus the exponential dependence of $\eta$ on the junction capacitance $C$ constitutes a phase-sensitive signature of the presence of the Majorana modes. Note that it is generally expected that only even Shapiro steps occur in $I$-$V$ characteristics of Josephson junctions, which supports Majorana modes due to $4\pi$ periodicity of the Josephson current [13,22,23]; this is a consequence of analysis of the problem in the limit of zero junction capacitance $C \to 0$ [13,22,23], where $\beta \to \infty$, leading to vanishing of the odd steps. However, our study shows that an underdamped RCSJ of such an unconventional Josephson junctions with finite $C$

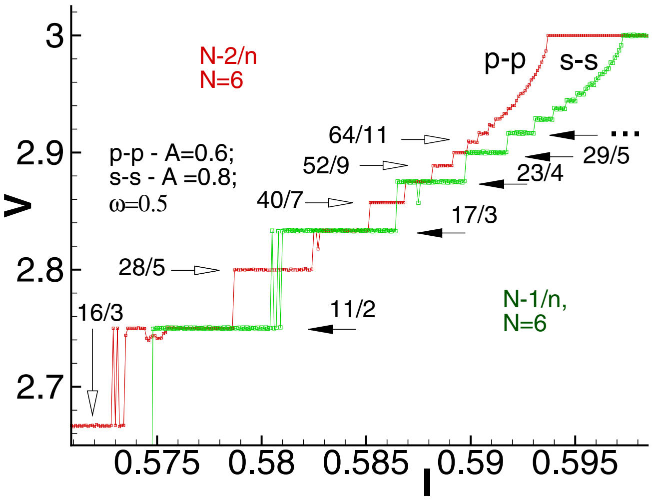


FIG. 3. (Color online) Plots of the self-similar structure for $p$-wave ($A = 0.6$) and $s$-wave ($A = 0.8$) Josephson junctions for $D = 0.7$ and $\omega = 0.5$. The additional fractions marked with arrows pointing to the right belong to the additional sequence characteristics of Josephson junctions with Majorana subgap states and obey the Farey sum rule. $V$ and $I$ are scaled in units of $\hbar/(e\tau)$ and $I_c$, respectively; see text for details.

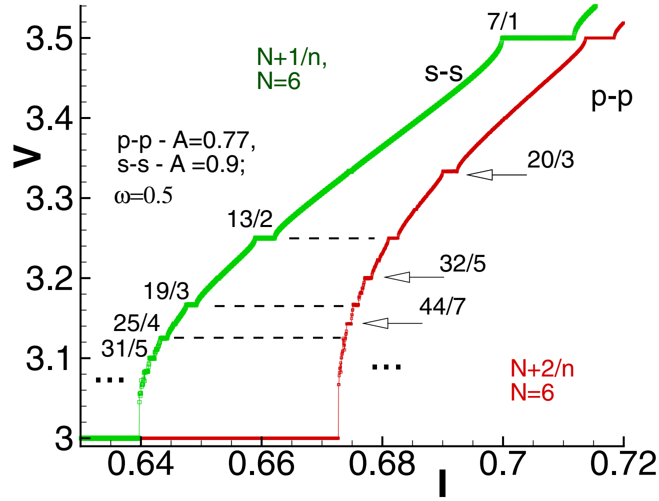


FIG. 4. (Color online) Same as Fig. 3, but with different amplitudes of external radiation (for $p$ waves, $A = 0.77$; for $s$ waves, $A = 0.9$). Note that the additional sequence of steps for $p$ waves persists.

can display both even and odd steps. Thus the presence of odd Shapiro steps does not necessarily signify the absence of Majorana modes, especially if the RCSJ is underdamped.

Another qualitative difference between the $I$-$V$ characteristics of Josephson junctions hosting Majorana states and their conventional counterparts occurs in the devil's staircase structures of the Shapiro steps occurring within fixed-bias current intervals in these junctions. Such steps are known to occur for conventional $s$-wave junctions, and their voltage-frequency relation can be represented by a continued fraction as

$$V = (N \pm 1/\{n \pm [1/m \pm 1/(p \pm 1)]\})\omega. \tag{16}$$

The fractions involving $N$ are termed as first level fractions, those with $N$ and $n$ are second level fractions, and so on. The steps obey Farey sum rule [26] and their structure for both $s$- and $p$- wave junctions is shown in Figs. 3 and 4. We find that for conventional junctions, the steps corresponding to second level fractions occur at $V = (N \pm 1/n)\omega$. In contrast, for junctions hosting Majorana subgap states, the steps correspond to $V = (N \pm 2/n)\omega$ (Figs. 3 and 4) leading to several additional steps within a given range of $A$. This difference in structure lead us to hypothesize that in contrast to conventional Josephson junctions, the steps for the junctions with Majorana bound states show additional fractions. We therefore suggest that the presence of these additional specific continued fractions may be considered as a signature of Majorana fermions. Two specific examples of this phenomenon is presented in Figs. 3 and 4. In Fig. 3, we find that the continued fraction $V = (N - 1/n)\omega$ with $N = 6$ which appears in $s$-wave Josephson junctions at $\beta = 0.2$, $\omega = 0.5$ and $A = 0.8$. In contrast, as clearly demonstrated in Fig. 3, the steps for the $p$-wave junction occur at $V = (N - 2/n)\omega$ with $N = 6$, and several $n$. Further, as shown in Ref. [29], the continued fraction $V = (N + 1/n)\omega$ with $N = 6$ appears for conventional Josephson junctions at $A = 0.9$. In contrast, for Josephson junctions with Majorana subgap states, as shown in Fig. 4, the steps occur at $V = (N + 2/n)\omega$ with $N = 6$. Thus for both these cases, for junctions with Majorana fermions, the steps occur at $V = (N \pm 2/n)\omega$ leading to the additional sequence of steps. Our numerics therefore suggests that the difference between junctions without and with Majorana subgap states lies in the manifestation of the subharmonics in their I-V characteristics; the steps follow continued fractions characterized by $N \pm 1/n$ for the former and $N \pm 2/n$ for the latter as $A$ is increased. We conjecture that the additional factor of 2 leading to the extra steps is a manifestation of the $4\pi$ periodicity of the junctions. Finally, we point out that the difference between these two type of junctions can also be stressed by comparing the largest width of the first subharmonic in these continued fractions. We find that with the increase of $A$, the largest subharmonic is $V = 20\omega/3$ (the series is $8/1, 7/1, 20/3, \ldots$) for junctions with Majorana fermions; in contrast, for junctions without Majorana subgap states, the largest width correspond to $V = 13\omega/2 (7/1, 13/2, \ldots)$.

## III. JUNCTIONS WITH DIRAC FERMIONS

For superconducting junctions hosting Dirac quasiparticles such as in graphene [12,16], the pair potential has $s$-wave symmetry: $\Delta_L = \Delta_0$ and $\Delta_R = \Delta_0 \exp(i\phi)$. Here pairing occurs between electrons with opposite spin and momenta; in graphene this necessitates pairing between electrons of $K$ and $K'$ valleys [14]. The BdG equations are described in terms of four-component wave functions $\psi = (\psi^K_{A\uparrow}, \psi^K_{B\uparrow}, \psi^{K'\dagger}_{A\downarrow}, \psi^{K'\dagger}_{B\downarrow})$ and are given by

$$\{[H' - V(x)]\tau_3 + [\Delta(x)\tau_+ + \text{H.c.}]\}\psi = E\psi, \tag{17}$$

where $A,B$ denote sublattice indices $H'_\beta = \hbar v_F(-i\sigma_x\partial_x \pm \sigma_y k_y) - \mu$, $\vec{\tau}$ and $\vec{\sigma}$ denote Pauli matrices in valley and pseudospin (sublattice) spaces, respectively, $\mu$ is the Fermi energy, $v_F$ is the Fermi velocity of the Dirac quasiparticles described by $H'$, and a plus (minus) sign corresponds to electrons in the $K$ ($K'$) valley. The pair potential takes the form $\Delta(x) = \Delta_L\theta(d/2 + x) + \Delta_R\theta(x - d/2)$ and $V(x) = V_0\theta(d/2 - x)\theta(d/2 + x)$. The localized subgap Andreev states are obtained by demanding the continuity of $\psi$ at $x = \pm d/2$ and are given, in the thin-barrier limit ($V_0 \to \infty$, $d \to 0$, and $V_0 d/\hbar v_F = \chi$), by [25]

$$E_2 = \pm\Delta_0\sqrt{1 - T(k_y,\chi)\sin^2(\phi/2)}, \tag{18}$$

where $T(k_y,\chi) = \cos^2(\gamma)/[1 - \cos^2(\gamma)\sin^2(\chi)]$ is a measure of transparency of the junction and $\sin(\gamma) = \hbar v_F k_y/\mu$. Note that Eq. (18), in contrast to Eq. (2), is $2\pi$ periodic in $\phi$. The corresponding Josephson current at zero temperature is given by $I_2(\phi) = \frac{2e}{\hbar}\int_{-\pi/2}^{\pi/2} d\gamma\, \cos(\gamma)\partial E_2/\partial\phi$ and leads to

$$I_2 = I_0\Delta_0 \int_{-\pi/2}^{\pi/2} d\gamma\, \cos(\gamma)\sin(\phi)T(\gamma,\chi)/|E_2|, \tag{19}$$

where $I_0 = e\Delta_0 E_F L/(2\hbar^2\pi v_F)$. Equation (19) shows that the Josephson current is an oscillatory function of the dimensionless barrier strength for such junctions. We note that whereas the forms of Eqs. (18) and (19) are generic for $s$-wave conventional tunnel junctions, the oscillatory dependence of $T$ on $\chi$ is a consequence of the Dirac nature of graphene quasiparticles and is not observed in junctions made of conventional superconductors [25].

To chart out the $I$-$V$ characteristics of the superconducting junctions which host such Dirac quasiparticles, we analyze Eq. (4) numerically with $I_J = I_2$ and obtain the corresponding Shapiro step structure. The procedure followed here is identical to the one charted out in Sec. II B. We find that the Shapiro step structure in the $I$-$V$ characteristics is same as the conventional $s$-wave superconductor displaying harmonic odd and even steps, as shown in Fig. 5(a). However, the width of these steps $W$ varies with the dimensionless barrier potential $\chi$ in an oscillatory manner, as shown in Fig. 5(b). This is in complete contrast to the dependence of $W$ in the conventional junctions, where the step widths are a monotonically decreasing function of the barrier potential. This behavior of $W$ can be qualitatively understood as follows. In a RCSJ, $W$ can be related to the magnitude of $I_J$, which, in turn, depends on the transparency of the junction: $W \sim (1 + \chi^2/4)^{-1/2}$ for conventional junctions. In a conventional junction, the increase of the barrier potential $\chi$ leads to a monotonic decrease of the transparency; consequently, $W$ decreases monotonically with increasing $\chi$. However, for a RCSJ made out of Dirac materials, the transparency of the junction $T(k_y,\chi)$ is a $\pi$-periodic oscillatory function of the dimensionless barrier strength $\chi$ with maxima at $\chi = n\pi$ due to the transmission resonance condition of the Dirac quasiparticles [17]. Consequently, one expects $I_J$ and hence $W$ to oscillate with $\chi$. This expectation is corroborated in Fig. 5(b), where the $\pi$-periodic oscillation of the step width is plotted as a function of $\chi$. We note that such an oscillatory behavior is a direct manifestation of the transmission resonance condition of the Dirac quasiparticles; it thus provides a qualitative distinction between Josephson-junction-hosting Schrodinger and Dirac quasiparticles.

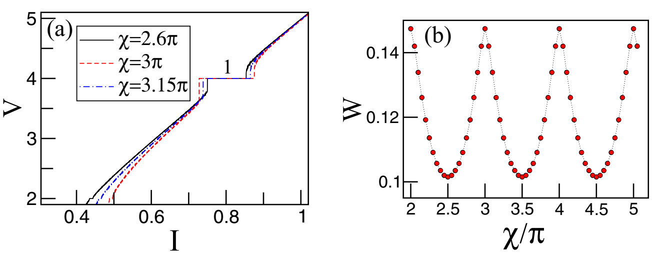


FIG. 5. (Color online) (a) CVC of the graphene Josephson junction with $A = 1$ and $\omega = 4$ for several values of $\chi$, indicating the variation in the width of the main Shapiro step at $\omega$. (b) Plot of the width of the Shapiro step at $\omega$ as a function of $\chi$ showing $\pi$ periodic oscillatory behavior.

## IV. DISCUSSION

In this work we have studied the $I$-$V$ characteristics of a RCSJ where the superconductors making up the junction either host subgap Majorana bound states or have Dirac-like character of the Bogoliubov quasiparticles. The former set of junctions occurs for $p$-wave superconductors [12] or 1D nanowires [6,7] with strong spin-orbit coupling and transverse magnetic field, while graphene superconduction junctions provide an example of the latter class. We find that the $I$-$V$ characteristics of RCSJs for each of these classes of junctions are qualitatively different from their conventional counterparts. Thus such junctions may serve as phase-sensitive detectors of Majorana and Dirac fermions realized using superconducting platforms.

For junctions hosting subgap Majorana states, we find two essential characteristics which are qualitatively different from their $s$-wave counterparts. First, the odd Shapiro steps are subharmonic in nature; the ratio of their width to that of adjacent even Shapiro steps is a decreasing function of the junction capacitance. This is in contrast to the conventional $s$-wave junctions, where the ratio is largely independent of $C$. We note that our result in this regard shows that the absence of odd Shapiro steps is a sufficient condition for having subgap Majorana modes; however, it is not necessary since such an absence requires, in addition to the presence of the Majorana modes, resistive Josephson junctions. Our result thus constitutes a generalization of the detection criteria for Majorana modes realized using a superconducting platform. Second, we find that the devil's staircase structure of the Shapiro steps in Josephson junctions with Majorana subgap states involves additional sequences which satisfy the Farey sum rule. This feature, as shown in our work, constitutes a qualitative difference between Josephson junctions with and without Majorana subgap states.

For junctions with Dirac quasiparticles, we find that even with $s$-wave symmetry, the Shapiro step width is a $\pi$-periodic oscillatory function of the barrier potential of the junction. We trace the origin of this phenomenon to the transmission resonance of the Dirac-Bogoliubov quasiparticles in such superconductors and demonstrate that the oscillatory behavior is a qualitatively distinct signature of the Dirac nature of the superconducting quasiparticles.

The numerical estimate of typical frequencies at which the devil's staircase structure can be obtained is as follows. For standard experiments $I_c \sim 1$ nA and $C_0 \simeq 1$ pF. Using these numbers, one can estimate $\omega_p = \sqrt{2eI_c/(\hbar C_0)} \simeq 1$ GHz. In all the figures, we have used $\omega$ ranges between $0.5\omega_p \simeq 0.5$ GHz and $2\omega_p \simeq 2$ GHz. In particular, the devil's staircase structure is seen at an external radiation frequency of 0.5 GHz. The self-similar structure is seen at energy range of $5\omega$–$6\omega$, which is around 2–3 GHz. In this context, we note that the required frequency range is small enough to avoid possible smearing due to $2\pi$ periodicity arising due to multimode effects [23]. In addition, this estimate holds for zero-temperature analysis; however, it is expected to be qualitatively accurate for $k_B T \ll \Delta_0$ when quasiparticle poisoning and thermal decoherence rates do not play a significant role. Also, we note that it is possible to model the dissipation and noise in the junction by assuming it to be coupled through a thermal bath using the standard Caldeira-Leggett formalism [32]; the Langevin or saddle-point equation corresponding to that analysis at low temperature, where the effects of white noise can be ignored, reduces to Eq. (4) of our work with the resistive term being renormalized by the coefficient of dissipation. This formalism also allows for the study of the effect of quantum fluctuations and noise in such junctions beyond saddle-point approximation, which is left as a possible subject of future study.

The experiments to test our theory involve measurement on the RCSJ under applied radiation with definite amplitude $A$ and frequency $\omega$. Such experiments are rather standard for $s$-wave junctions [33]; more recently, such experiments have been performed for a 1D Majorana nanowire setup with resistive junctions [19]. Our specific suggestion involves measurement of $\eta$ as a function of the effective junction capacitance for Josephson junctions with Majorana bound states in a circuit with finite capacitance which can be modeled by a RCSJ; we predict the presence of subharmonic odd Shapiro steps for such junctions, whose width depends on the junction capacitance and leads to an exponential dependence of $\eta$ on the junction capacitance [see Fig. 2(d)]. In addition, we suggest the presence of an additional sequence in the devil's staircase structure of the Shapiro steps. For junctions with Dirac quasiparticles, which can be made with graphene [34], we predict that the width of the Shapiro steps will display $\pi$-periodic oscillatory dependence on the junction barrier potential.

We note here that it is possible to have additional odd steps in a realistic experimental system due to presence of small but additional $2\pi$ periodic terms ($I_J \sim \sin[\phi]$) in the Josephson current due to a variety of reasons [19,23]. However, such steps are harmonic and for them $\eta$ would not show appreciable variation with $\beta$ [bottom curve of Fig. 2(d)]; in contrast, the subharmonic odd steps that we focus on here has $\eta \sim \exp[0.3\beta^2]$ [upper curves of Fig. 2(d)]. The size of the steps vary with the junction capacitance and thus displays qualitatively different behavior.

In conclusion, we have studied RCSJ Josephson junction circuits and have shown that they can serve as phase-sensitive detectors for both Majorana and Dirac quasiparticles in such junctions. We have charted out the properties of such junctions, which are qualitatively distinct from their $s$-wave counterparts, and have suggested experiments which can test our theory.

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