Magnon-phonon interconversion in a dynamically reconfigurable magnetic material

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The ferrimagnetic insulator yttrium iron garnet (YIG) is an important material in the field of magnon spintronics, mainly because of its low magnetic losses. YIG also has very low acoustic losses, and for this reason the conversion of a state of magnetic excitation (magnons) into a state of lattice vibration (phonons), or vice versa, broadens its possible applications in spintronics. Since the magnetic parameters can be varied by some external action, the magnon-phonon interconversion can be tuned to perform a desired function. We present a quantum theory of the interaction between magnons and phonons in a ferromagnetic material subject to a dynamic variation of the applied magnetic field. It is shown that when the field gradient at the magnetoelastic crossover region is much smaller than a critical value, an initial elastic excitation can be completely converted into a magnetic excitation, or vice versa. This occurs with conservation of linear momentum and spin angular momentum, implying that phonons created by the conversion of magnons have spin angular momentum and carry spin current. It is shown further that if the system is initially in a quantum coherent state, its coherence properties are maintained regardless of the time dependence of the field.

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I. INTRODUCTION

Phenomena involving spin waves, the collective excitations of spins in magnetic materials, are becoming increasingly more promising in device applications for signal processing in the microwave gigahertz frequency range. This fact, together with the continuing developments in magnetic hybrid structures and observations of novel physical effects, has made magnon spintronics an active and exciting field of research. The use of magnons, which are the quanta of spin waves, to carry, transport, and process information in devices made with insulating ferrimagnetic materials is very attractive, among other reasons, because it is free from energy dissipation due to Ohmic losses [1,2].

A key material for magnon spintronics is the ferrimagnetic insulator yttrium iron garnet ($Y_3Fe_5O_{12}$–YIG), thanks to its unique magnetic properties. For several decades YIG has been the prototype material for investigating the physics of a variety of spin-wave phenomena. In the 1960s and 1970s the main interest was in the study of the excitation and propagation properties of spin waves in bulk YIG crystals motivated by their interesting physical properties and potential applications in microwave signal processing devices [3–8]. In the 1980s and 1990s the interest turned to nonlinear dynamics, bifurcation, and chaos in bulk YIG crystals and soliton phenomena in YIG films [9–13]. In the 2000s parametrically pumped magnons in YIG revealed unique Bose-Einstein condensation phenomena at room temperature [14–18]. Recently there has been an upsurge of interest in YIG motivated by the discovery of spintronic effects in hybrid structures containing this material [19], such as the spin pumping produced by ferromagnetic resonance (FMR) and spin waves [20–26], the spin-Seebeck effect [27–32], the control of spin-wave damping by thermal gradients [33–35] or by the spin-Hall effect [36], the spin-Hall magnetoresistance, and related effects [37–39].

Owing to the magnetostrictive properties of materials, elastic deformations in crystals change the energy of the spins. Conversely, changes in the spin configuration modify the elastic energy [40]. As a consequence, if a spin wave in a crystal lattice has frequency and wave vector close to those of an elastic wave, they can become strongly coupled and form a magnetoelastic wave, or a hybridized magnon-phonon excitation. Magnetoelastic waves with frequency of a few gigahertz were extensively studied in bulk YIG crystals in the 1960s, motivated by scientific interest and potential application in controllable delay lines [41–44]. Very recently, with the development of novel material structures for spintronics, there has been a revival of interest in the coupling between spin excitations and elastic waves. Uchida *et al.* [45] have used low-frequency elastic waves to generate spin currents in YIG by means of the magnon-phonon interaction, which are then detected by the voltage produced in attached Pt strips, demonstrating an acoustic spin-pumping effect. The magnon-phonon interaction has also been exploited by Weiler *et al.* [46] to excite FMR in a cobalt film by surface acoustic waves, by Rückriegel *et al.* to calculate the magnon-phonon damping in YIG films [47], and by Kamra *et al.* [48] to study theoretically the magnetization dynamics in a ferromagnet excited by means of elastic waves.

In this paper we present a study of the magnon-phonon conversion in a magnetic medium whose parameters vary in time using a quantum formulation. We show that magnons can be efficiently converted into phonons, or vice versa, by an appropriate variation in time of the magnetic parameters, such as the applied magnetic field or the magnetization. This is of interest in the context of reconfigurable magnon spintronics, whereby the function of a device can be tunable by an external action during operation [1,49–51]. The theory of magnetoelastic waves in a magnetic field that varies either in space or in time was developed a long time ago only in semiclassical terms [40,42]. The quantum theory presented here reveals important aspects of the phenomena involved, such as the spin angular momentum and the spin current associated with phonons.

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The paper is organized as follows. Section II presents the background material for the following sections, namely, the quantization of the magnetic and elastic variables by means of transformations that lead to magnon and phonon operators for a static applied field and the magnetoelastic interaction Hamiltonian in terms of these operators. In Sec. III**,** the total magnon-phonon Hamiltonian is diagonalized and the possible states of the system are discussed. In Sec. IV we consider the coupled magnon-phonon equations of motion for a time-dependent applied field and calculate the momentum magnon-phonon conversion efficiencies. Finally, in Sec. V, we present some additional remarks on the quantum states involved and discuss the relations between the results of the quantum and semiclassical treatments.

II. MAGNONS AND PHONONS IN A FERROMAGNETIC MEDIUM

The analysis presented in this paper applies to a simple Heisenberg ferromagnetic cubic crystal, magnetized to saturation by an external magnetic field that can vary in time. Although YIG is ferrimagnetic and has several magnon branches, the properties of its acoustic magnons can be well described by the model. In this section the field is assumed to be static and uniform. The total Hamiltonian of the system can be expressed in terms of the spin operator and the elastic displacement operator at each lattice site. In a first approximation, the Hamiltonian can be written as the sum of three parts: a magnetic component depending only on the spins, a pure elastic part, and a magnetoelastic term depending on both the spin and the elastic displacement.

A. Quantization of the spin excitations: Magnons

We consider a magnetic Hamiltonian consisting of the interactions between individual spins with the magnetic field (Zeeman interaction) and the exchange interactions between neighboring spins. For simplicity we do not consider at this point the dipolar interaction because it complicates the treatment of the magnetic Hamiltonian. Later we will introduce the effect of the dipolar interaction in the spin-wave frequency. We also neglect the magnetic anisotropy interaction, which is small in YIG. The magnetic Hamiltonian is then

$$\mathrm{H}_m = -\gamma\hbar \sum_i \vec{S}_i \cdot \vec{H} - \sum_{i\neq j} J_{ij}\, \vec{S}_i \cdot \vec{S}_j, \tag{1}$$

where $\gamma = g\mu_B/\hbar$ is the gyromagnetic ratio, g is the spectroscopic splitting factor, μ_B the Bohr magneton, $\hbar$ the reduced Plank constant, $\vec{S}_i$ is the spin (in units of $\hbar$) at the lattice site i with position vector $\vec{r}_i$, J_{ij} is the exchange constant of the interaction between spins $\vec{S}_i$ and $\vec{S}_j$, and $\vec{H}$ is the magnetic field, considered to be uniform and lying in the z direction of a Cartesian coordinate system. We treat the quantized excitations of the magnetic system with the approach of Holstein-Primakoff [3,4,52,53], which consists of transformations that express the spin operators in terms of boson operators that create or destroy magnons. In the first transformation the components of the local spin operator are related to the creation and annihilation operators of spin deviation at site i, denoted respectively by a_i^+ and a_i, which satisfy the boson commutation rules $[a_i, a_j^+] = \delta_{ij}$ and $[a_i, a_j] = 0$. Using a coordinate system with $\hat{z}$ along the equilibrium direction of the spins, defining $S_i^+ = S_i^x + iS_i^y$ and $S_i^- = S_i^x - iS_i^y$, it can be shown that the relations that satisfy the commutation rules for the spin components and the boson operators are [3,4,52,53]

$$S_i^+ = (2S)^{1/2}(1 - a_i^+ a_i/2S)^{1/2} a_i, \tag{2a}$$

$$S_i^- = (2S)^{1/2} a_i^+ (1 - a_i^+ a_i/2S)^{1/2}, \tag{2b}$$

$$S_i^z = S - a_i^+ a_i, \tag{2c}$$

where S is the spin and $a_i^+ a_i \equiv n_i$ is the operator for the number of spin deviations at site i. Since we will not consider here interactions between magnons, we use the linear approximation, whereby only the first terms in Eqs. (2) are kept, $S_i^+ \cong (2S)^{1/2} a_i$, $S_i^- \cong (2S)^{1/2} a_i^+$, and $S_i^z \cong S - a_i^+ a_i$. With these transformations one can express the magnetic Hamiltonian in a quadratic form containing only lattice sums of products of two boson operators:

$$\mathrm{H}_m = \sum_{i\neq j} (\gamma\,\hbar\, H + 2S\, J_{ij}) a_i^+ a_j. \tag{3}$$

The next step consists in introducing a transformation from the localized field operators to collective boson operators a_k^+ and a_k, which satisfy the boson commutation rules $[a_k, a_{k'}^+] = \delta_{kk}$ and $[a_k, a_{k'}] = 0$,

$$a_i = N^{-1/2} \sum_k e^{i\vec{k}.\vec{r}_i} a_k, \tag{4}$$

where N is the number of spins in the system and $\vec{k}$ is a wave vector. Using Eq. (4) in Eq. (3) the magnetic Hamiltonian becomes

$$\mathrm{H}_m = \sum_{\mathrm{k}} \hbar\,\omega_m(k)\, a_k^+ a_k, \tag{5}$$

where $\hbar\omega_m(k)$ is the energy of a magnon with frequency ω_k and wave vector $\vec{k}$, while a_k and a_k^+ are, respectively, destruction and creation operators for magnons. Considering the exchange interaction only between nearest neighbors, from Eq. (3) we obtain $\omega_m(k) = \gamma H + 2zJS(1-\gamma_k)/\hbar$, where z is the number of nearest-neighbor spins with exchange interaction constant J, and $\gamma_k = z^{-1}\sum_\delta e^{i\vec{k}\cdot\vec{\delta}}$ is a geometrical factor, determined by the nearest-neighbors' position vector $\vec{\delta} = \vec{r}_j - \vec{r}_i$. For a cubic crystal with coordinate axes along the [100] directions, $\gamma_k = (\cos k_x a + \cos k_y a + \cos k_z a)/3$, where a is the lattice parameter. For $ka \ll 1$, $\gamma_k \approx 1 - k^2 a/z^2$, and the magnon frequency is given by the quadratic dispersion relation

$$\omega_m(k) = \gamma(H + Dk^2), \tag{6}$$

where $D = 2JSa^2/\gamma\hbar$ is the exchange stiffness parameter. If the dipolar interaction between the spins is introduced in the Hamiltonian (3), another transformation involving the magnon operators is necessary [3,4,52,53] to diagonalize the Hamiltonian and one obtains the magnon dispersion relation

$$\omega_m(k) = \gamma[(H + Dk^2)(H + Dk^2 + 4\pi M \sin^2\theta_k)]^{1/2}, \tag{7}$$

where θ_k is the angle between the wave vector and the direction of the magnetic field and M is the magnetization. Notice that Eq. (7) is valid only for wave number k such that $ka \ll 1$ and $kd \gg 1$, where d is a typical dimension of the sample. For smaller k the magnetostatic boundary conditions in the sample surfaces become important and the dispersion relation depends on the sample geometry [5–7]. In experiments one excites and detects spin excitations by means of the magnetization dynamics, and thus we shall work with the magnetization operator. Assuming a continuous description of the crystal, we use the relation $\vec{M}(\vec{r}) = g\mu_B(N/V)\sum_i \vec{S}_i$. Considering that $\vec{M}$ in equilibrium lies along $\hat{z}$, one can write $\vec{M} = \hat{z}\,M_z + \vec{m}$ and introduce the transverse circularly polarized components $m^{\pm} = m_x \pm i m_y$. With the transformations (2) and (4) one can express the transverse components of the magnetization in terms of the magnon operators,

$$m^+(\vec{r}) = [m^-(\vec{r})]^+ = \frac{M}{(2NS)^{1/2}} \sum_k e^{i\vec{k}.\vec{r}} a_k. \quad (8)$$

In the continuum approximation the magnetic Hamiltonian can also be written as

$$\begin{aligned} \mathrm{H}_m &= \int d^3r \left(H m_z + \frac{D}{2\gamma\hbar M} \frac{\partial m_i}{\partial x_j} \frac{\partial m_i}{\partial x_j} \right) \\ &= \sum_k \hbar\omega_m(k)\, a_k^+ a_k, \end{aligned} \quad (9)$$

where the repeated indices indicate summation. Note that $n_k = a_k^+ a_k$ is the operator for the magnon number, so that the Hamiltonian (9) corresponds to the total magnon energy operator. Another operator of interest in the continuum approach of spin waves is the linear momentum density, for which the i component is [53,54],

$$g_m^i = \frac{1}{2\gamma M}\left(\vec{m} \times \frac{\partial \vec{m}}{\partial x_i}\right) \cdot \hat{z}. \quad (10)$$

Using Eq. (8) one can show that the total magnon linear momentum given by the integration of Eq. (10) in the volume is [53]

$$\vec{p}_m = \sum_k \hbar\vec{k}\, a_k^+ a_k, \quad (11)$$

where $\hbar\vec{k}$ is the linear momentum of one magnon. Finally, another operator of interest is the angular momentum flow carried by spin waves. The magnon spin current density (per unit area) with polarization $\hat{z}$ is given by [20,55–57]

$$\vec{J}_{Sm}^{z} = \frac{D}{M}(\vec{m} \times \nabla\vec{m}) \cdot \hat{z}. \quad (12)$$

Using Eqs. (8) and (12) one can show that the spin current density carried by magnons is given by [20,57]

$$\vec{J}_{Sm}^{z} = -\frac{\hbar}{V}\sum_k \vec{v}_{mk}\, a_k^+ a_k, \quad (13)$$

where $\vec{v}_{mk} = 2\gamma D\vec{k}$ is the k-magnon group velocity. Since one magnon corresponds to one spin deviation, Eq. (13) is consistent with the view that each magnon carries an angular momentum $-\hbar\hat{z}$. To conclude this section, we recall a few properties of magnon states. The eigenstates $|n_k\rangle$ of the Hamiltonian in Eq. (5), which are also eigenstates of the number operator $n_k = a_k^+ a_k$, can be obtained by applying integral powers of the creation operator to the vacuum,

$$|n_k\rangle = [(a_k^+)^{n_k}/(n_k\,!)^{1/2}]\,|0\rangle, \quad (14)$$

where the vacuum state is defined by the condition $a_k\,|0\rangle = 0$. These stationary states describe systems with a precisely defined number of magnons n_k and an uncertain phase. They form a complete orthonormal set which can be used as a basis for the expansion of any state of spin excitation. They are used in nearly all quantum treatments of thermodynamic properties, relaxation mechanisms, and other phenomena involving magnons. However, since $\langle n_k|a_k\,|n_k\rangle = 0$, they have zero expectation value for the small-signal transverse magnetization operators m_x and m_y and thus do not have a macroscopic wave function. In order to establish a correspondence between classical and quantum spin waves, one should use the concept of coherent magnon states [53,58], defined in analogy to the coherent photon states introduced by Glauber [59]. A coherent magnon state is the eigenket of the circularly polarized magnetization operator in Eq. (8). It can be written as the direct product of single-mode coherent states, defined as the eigenstates of the annihilation operator,

$$a_k\,|\alpha_k\rangle = \alpha_k|\alpha_k\rangle, \quad (15)$$

where the eigenvalue α_k is a complex number. Although the coherent states are not eigenstates of the unperturbed Hamiltonian and as such do not have a well-defined number of magnons, they have nonzero expectation values for the magnetization m^+ with a well-defined phase. It can be shown that the coherent states can be expanded in terms of the eigenstates of the unperturbed Hamiltonian [58,59],

$$|\alpha_k\rangle = e^{-|\alpha_k|^2/2}\sum_{n_k}(\alpha_k)^{n_k}/(n_k\,!)^{1/2}\,|n_k\rangle. \quad (16)$$

The number of magnons in the coherent state $|\alpha_k\rangle$ obtained directly from Eq. (16) is $\langle\alpha_k|n_k\,|\alpha_k\rangle = |\alpha_k|^2$. Further, the expectation values of the components of the magnetization operators for a single coherent state with eigenvalue $\alpha_k = |\alpha_k|\,\exp(i\phi_k)$ can be obtained using the definition (15) in Eq. (8). They are

$$\langle m_x(\vec{r},t)\rangle = \frac{M}{(NS/2)^{1/2}}|\alpha_k|\cos(\vec{k}\cdot\vec{r} - \omega_m t + \phi_k), \quad (17)$$

$$\langle m_y(\vec{r},t)\rangle = \frac{M}{(NS/2)^{1/2}}|\alpha_k|\sin(\vec{k}\cdot\vec{r} - \omega_m t + \phi_k). \quad (18)$$

The transverse components of the magnetization in Eqs. (17) and (18), together with $\hat{z}\,M_z$, correspond to the classical view of a spin wave, namely, the magnetization precesses around the equilibrium direction with a phase that varies along the direction of propagation.

B. Quantization of elastic waves: Phonons

Let us consider that the ferromagnetic crystal is a continuous solid, elastically isotropic, with average mass density ρ. We also assume that it is a cubic crystal so that, within the linear approximation, the relation between the stress tensor and the strain tensor involves only two different elastic constants, c_{12}

and c_{44}. The elastic deformations of the solid are expressed in terms of the vector displacement $\vec{u} = \vec{r} - \vec{r}'$, where $\vec{r}$ is the initial position of an atom or of a volume element, and $\vec{r}'$ is the position after deformation. The contributions of the elastic system to the Hamiltonian arise from the kinetic and potential energies. Introducing the momentum density conjugate to the displacement, $\rho\,\partial u_i/\partial t$, in the linear approximation, the elastic Hamiltonian can be written as [60]

$$\mathrm{H}_e = \int d^3r \left(\frac{\rho}{2}\frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t} + \frac{\alpha}{2}\frac{\partial u_i}{\partial x_i}\frac{\partial u_j}{\partial x_j} + \frac{\beta}{2}\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} \right), \quad (19)$$

where the elastic constants are written as $\alpha = c_{12} + c_{44}$, $\beta = c_{44}$, for the Cartesian coordinate system chosen with axes lying along the [100] crystallographic directions. In order to obtain the collective excitation operators for the elastic system, we use the canonical transformation [60]

$$u_i(\vec{r},t) = \left(\frac{\hbar}{V}\right)^{1/2} \sum_{k,\mu} \varepsilon_{i\mu}(\vec{k})\, Q_k^{\mu}(t) e^{i\,\vec{k}\cdot\vec{r}}, \quad (20)$$

$$\rho\,\dot{u}_i(\vec{r},t) = \left(\frac{\hbar}{V}\right)^{1/2} \sum_{k,\mu} \varepsilon_{i\mu}(\vec{k})\, P_k^{\mu}(t) e^{-i\,\vec{k}\cdot\vec{r}}, \quad (21)$$

where $\varepsilon_{i\mu} = \hat{x}_i \cdot \hat{\varepsilon}(\vec{k},\mu)$ and $\hat{\varepsilon}(\vec{k},\mu)$ are unitary polarization vectors. We denote by $\mu = 1,2$ the two polarizations transverse to the wave vector $\vec{k}$ and $\mu = 3$ the longitudinal polarization. Notice that from Hermiticity it follows that $Q_k^i = Q_{-k}^{i\,+}$ and $P_k^i = P_{-k}^{i\,+}$. The quantization of the elastic vibrations is made through the commutation relations involving $u_i(\vec{r})$ and its conjugate momentum density $\rho\,\partial\vec{u}/\partial t$. The only noncommuting pair is such that

$$[u_i(\vec{r}), \rho\,\dot{u}_j(\vec{r}')] = i\hbar\,\delta_{ij}\delta(\vec{r} - \vec{r}'), \quad (22)$$

which leads to

$$\left[Q_k^{\mu}, P_{k'}^{\nu}\right] = i\hbar\,\delta_{kk'}\delta_{\mu\nu}. \quad (23)$$

In order to diagonalize the elastic Hamiltonian it is necessary to introduce the canonical transformation

$$Q_k^{\mu} = \left[\frac{\hbar}{2\rho\omega_{p\mu}(k)}\right]^{1/2} (b_{\mu-k}^{+} + b_{\mu k}), \quad (24)$$

$$P_k^{\mu} = i\left[\frac{\rho\,\hbar\omega_{p\mu}(k)}{2}\right]^{1/2} (b_{\mu k}^{+} - b_{\mu-k}), \quad (25)$$

where

$$\omega_{p\mu}(k) = k\,[(\beta + \alpha\delta_{\mu 3})/\rho]^{1/2} \quad (26)$$

is the phonon frequency. With this transformation, the Hamiltonian (19) becomes

$$\mathrm{H}_e = \sum_{k,\mu} \hbar\omega_{p\mu}(k)\,(b_{\mu k}^{+} b_{\mu k} + 1/2). \quad (27)$$

The new operators satisfy the boson commutation relations,

$$[b_{\mu k}, b_{\nu k'}] = 0, \quad [b_{\mu k}, b_{\nu k'}^{+}] = \delta_{\mu\nu}\delta_{kk'}, \quad (28)$$

and are interpreted as creation and annihilation operators of lattice vibrations, whose quanta are called phonons. In terms of these operators, the displacement and the momentum density operators are

$$u_i = \left(\frac{\hbar}{2\rho V}\right)^{1/2} \sum_{k,\mu} \varepsilon_{i\mu}(\vec{k})\,\omega_{p\mu}^{-1/2}\left(b_{\mu k}^{+} e^{-i\,\vec{k}\cdot\vec{r}} + b_{\mu k} e^{i\,\vec{k}\cdot\vec{r}}\right), \quad (29)$$

$$\rho\,\dot{u}_i = \left(\frac{\rho\,\hbar}{2\,V}\right)^{1/2} \sum_{k,\mu} i\varepsilon_{i\mu}(\vec{k})\,\omega_{p\mu}^{1/2}(b_{\mu k}^{+} e^{i\,\vec{k}\cdot\vec{r}} - b_{\mu k} e^{-i\,\vec{k}\cdot\vec{r}}). \quad (30)$$

Note that $\rho\,\dot{u}_i$ is the canonical momentum density associated with the elastic displacement. One can also introduce a linear momentum density carried by the elastic waves, which has been shown to be [54]

$$g_p^i = \frac{\rho}{2}\left(\frac{\partial^2\vec{u}}{\partial x_i \partial t} \cdot \vec{u} - \frac{\partial\vec{u}}{\partial t} \cdot \frac{\partial\vec{u}}{\partial x_i} \right). \quad (31)$$

Using the transformations to phonon operators given by Eqs. (29) and (30), integration of Eq. (31) in the volume gives the total phonon linear momentum

$$\vec{p}_p = \sum_{k,\mu} \hbar\vec{k}\, b_{\mu k}^{+} b_{\mu k}, \quad (32)$$

where $\hbar\vec{k}$ is the momentum of one phonon. Phonons may also carry angular momentum, as recently discussed in Ref. [61]. The angular momentum of an elastic solid is the sum of two components, an orbital angular momentum corresponding to the macroscopic rotation and a spin angular momentum corresponding to small-radius circular shear displacements. For a rigid solid, only the elastic spin angular momentum exists, which is given by [61]

$$\vec{S}_p = \int d^3r\,\rho\,\vec{u} \times \frac{\partial\vec{u}}{\partial t}. \quad (33)$$

Using the transformations (29) and (30) in Eq. (33), one can write the elastic spin angular momentum in terms of the two transverse phonon operators [61]

$$\vec{S}_p = i\hbar\sum_k \frac{\vec{k}}{k}\,(b_{2k}^{+} b_{1k} - b_{1k}^{+} b_{2k}). \quad (34)$$

In order to write Eq. (34) in a diagonal form, we introduce creation and annihilation operators for transverse circularly polarized phonons denoted by (+) and (−):

$$b_{k(+)} = 2^{-1/2}(b_{1k} + i b_{2k}), \quad (35)$$

$$b_{k(-)} = 2^{-1/2}(b_{1k} - i b_{2k}). \quad (36)$$

With the circular polarization operators (35) and (36), one can show that the elastic Hamiltonian and the commutation relations have the same form as Eqs. (27) and (28), respectively, while Eq. (34) becomes [61]

$$\vec{S}_p = \hbar\sum_k \frac{\vec{k}}{k}\,(-\,b_{k(+)}^{+} b_{k(+)} + b_{k(-)}^{+} b_{k(-)}). \quad (37)$$

This result shows that a circularly polarized (+) or (−) phonon carries an angular momentum parallel or antiparallel to its wave vector that can be interpreted as the spin of the phonon [61]. As expected, a linearly polarized phonon carries no angular momentum since it is a superposition of (+) and (−) phonons. Multiplying Eq. (37) by the phonon velocity and dividing by the volume, we obtain the phonon spin current density

$$\vec{J}_p = \frac{\hbar}{V}\sum_k \vec{v}_{pk}\,(-b^+_{k(+)}b_{k(+)} + b^+_{k(-)}b_{k(-)}), \qquad (38)$$

which has the same form as the magnon spin current density in Eq. (13). As we show in Sec. IV, the linear and the angular momenta of magnons and phonons are important time-invariant quantities for spatially uniform media.

III. INTERACTING MAGNONS AND PHONONS

Due to the spin-orbit interaction, the elastic displacement in a magnetic medium is coupled to the spin excitation. This is what ultimately relaxes the magnetization dynamics in any material and also gives rise to the magnetostrictive properties of ferromagnets. Thus, we expect that if a spin wave has frequency and wave vector close to those of an elastic wave, they become strongly coupled, giving rise to hybrid excitations, called magnetoelastic waves, or magnon-phonon excitations.

A. The magnetoelastic interaction

The magnetoelastic interaction can be expressed by a phenomenological Hamiltonian, which is a function of $\vec{M}$ and $\vec{u}$. For a cubic crystal, with the static field applied along one of the [100] directions, the lowest-order term of the interaction Hamiltonian is given by [4,40]

$$\mathrm{H}_{me} = \int d^3r\,\frac{b_2}{2M^2}\,M_iM_j\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right), \qquad (39)$$

where the repeated indices indicate summation with $i \neq j$, and b_2 is one of the magnetoelastic constants. Using the transformations (8), (29), and (30), this Hamiltonian can be written in terms of the boson operators. We will assume that the wave vectors of interest lie on the x-z plane of the Cartesian system, and take $\hat{\varepsilon}(\vec{k},1)=\hat{x}$, $\hat{\varepsilon}(\vec{k},2)=\hat{y}$. The component of the Eq. (39) quadratic in the boson operators is given by

$$\begin{aligned}\mathrm{H}_{me} = i\left(\frac{b_2^2\gamma\hbar^2}{4\rho M}\right)^{1/2} &\times \sum_k \big[k\omega_{pt}^{-1/2}\cos 2\theta\,(a_k + a^+_{-k})(b^+_{1k}+b_{1-k})\\ &- ik\omega_{pt}^{-1/2}\cos\theta\,(a_k - a^+_{-k})(b^+_{2k}+b_{2-k})\\ &- k\omega_{pl}^{-1/2}\sin 2\theta\,(a_k + a^+_{-k})(b^+_{3k}+b_{3-k})\big],\end{aligned} \qquad (40)$$

where ω_{pt} and ω_{pl} are the shear and longitudinal phonon frequencies. We shall now confine our attention only to waves propagating along the magnetic field ($\theta = 0$), because in this case the equations for the field variables are simple to solve. The physical aspects of the general case are essentially the same inferred in this particular situation. Taking $\theta = 0$ in Eq. (40) we obtain

$$\begin{aligned}\mathrm{H}_{me} = i\left(\frac{b_2^2\gamma\hbar^2}{4\rho M}\right)^{1/2}\sum_k \big[k\omega_{pt}^{-1/2}a_k(b^+_{1k} - ib^+_{2k}\\ + b_{1-k} - b_{2-k}) - \mathrm{H.c.}\big].\end{aligned} \qquad (41)$$

Note that longitudinal phonons do not couple with magnons propagating along the magnetic field. Using the transformations to circularly polarized phonons given by Eqs. (35) and (36), the total Hamiltonian for the magnon-phonon system becomes

$$\begin{aligned}\mathrm{H}_t &= \mathrm{H}_m + \mathrm{H}_e + \mathrm{H}_{me}\\ &= \sum_k \hbar\omega_m(k)\,a^+_k a_k + \sum_{k,\mu}\hbar\omega_{p\mu}(k)\,b^+_{k\mu}b_{\mu k}\\ &\quad + \sum_k i\,\hbar(\sigma_k/2)[a^+_k(b_{k(+)} + b^+_{-k(-)})\\ &\quad - a_k(b^+_{k(+)} + b_{-k(-)})],\end{aligned} \qquad (42)$$

where

$$\sigma_k = b_2\left(\frac{2\gamma k}{\rho\,v_{pt}M}\right)^{1/2} \qquad (43)$$

is a parameter that expresses the coupling between magnons and phonons, and v_{pt} is the velocity of the transverse phonon, given by $v_{pt} = (c_{44}/\rho)^{1/2}$ for a wave propagating along a [100] axis in a cubic crystal, or $v_{pt} = (\mu_t/\rho)^{1/2}$ in a more general case, where μ_t is the shear modulus.

B. Eigenstates of the magnon-phonon system

In this section we study some properties of the normal-mode collective excitations of a magnetoelastic crystal under a static uniform magnetic field. We consider magnetoelastic waves propagating along the z direction, so that spin waves are coupled only to shear elastic waves. In order to simplify a little further the total Hamiltonian of the system, let us consider the equations of motion of the magnon and phonon operators in the Heisenberg representation. Using the Heisenberg equation $dA/dt = \partial A/\partial t + (1/i\hbar)[A,\mathrm{H}_t]$, we obtain the following equations of motion for the magnon and phonon operators:

$$\frac{da^+_k}{dt} = i\omega_m a^+_k + \frac{\sigma_k}{2}b^+_{k(+)} + \frac{\sigma_k}{2}b_{-k(-)}, \qquad (44)$$

$$\frac{db^+_{k(+)}}{dt} = i\omega_p b^+_{k(+)} - \frac{\sigma_k}{2}a^+_k, \qquad (45)$$

$$\frac{db^+_{k(-)}}{dt} = i\omega_p b^+_{k(-)} - \frac{\sigma_k}{2}a_{-k}. \qquad (46)$$

In the stationary state, all operators have a $\exp(i\omega t)$ variation, so that the magnetoelastic dispersion relation resulting from Eqs. (44)–(46) is

$$\left(\omega^2 - \omega_p^2\right)(\omega - \omega_m) - \tfrac{1}{2}\omega_p\sigma_k^2 = 0, \qquad (47)$$

which is a well-known result [4,42,44]. If there is no magnetoelastic coupling, $\sigma_k = 0$ and the three roots of Eq. (47) are ω_m and $\pm\omega_p$, the two signs corresponding to (+) and (−) circularly polarized phonons. Figure 1 shows the dispersion relation, frequency $f = \omega/2\pi$ versus

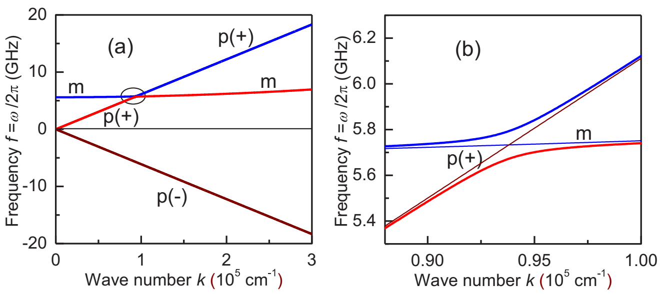

FIG. 1. (Color online) Magnetoelastic dispersion curves for z-directed waves in YIG for an applied magnetic field $H = 2.0$ kOe. (a) Full dispersion with three branches. The negative branch corresponds to (−) circularly polarized phonons, p(−), that have negligible coupling with magnons. In the positive branches m and p(+) denote the regions where the excitation is essentially a pure magnon or (+) phonon. (b) Blowup of the crossover region showing the splitting of the magnetoelastic dispersions (thick lines) and the pure magnon and phonon dispersions (thin lines).

wave number, calculated from the roots of Eq. (47), with the magnon and phonon frequencies of Eqs. (6) and (26), using the following parameters for YIG: $H = 2.0$ kOe, $4\pi M = 1.76$ kG, $\gamma = 2.8 \times 2\pi \times 10^6\,\mathrm{s}^{-1}/\mathrm{Oe}$, $D = 5.4 \times 10^{-9}$ Oe cm^2, $b_2 = 7.0 \times 10^6$ erg/cm^3, $\rho = 5.2$ g/cm^3, and $v_p = 3.84 \times 10^5$ cm/s. The dispersion has three branches because the magnetoelastic excitation involves one magnon and two transverse phonon modes. The negative linear dispersion corresponds to the (−) circularly polarized phonon that has negligible coupling with magnons. The two positive branches correspond to the hybridized magnon-(+) circularly polarized phonon. As expected, the magnetoelastic coupling is strongest in the region where the magnon and phonon curves cross, called crossover region. The zoom of the crossover region in Fig. 1(b) shows that for $H = 2$ kOe the magnon and phonon dispersions cross at a frequency 5.73 GHz and wave number $k_{\text{cross}} = 0.938 \times 10^5\,\mathrm{cm}^{-1}$. It also shows that the frequency splitting is 0.12 GHz, which is quite small compared to the magnon and phonon frequencies.

The analysis of the equations of motion (44)–(46) shows that an excitation with frequency and wave number far from the crossover has an almost pure magnon or phonon character. However, in the crossover region the normal modes are mixtures of magnetic and elastic excitations. The phenomenon that we investigate in this paper is the change of character of an excitation, from magnetic to elastic or vice versa, caused by the time variation of the applied field. From Eqs. (44)–(46) we find that, in the stationary state, the expectation values of the positive and negative circularly polarized phonon operators are related by

$$\langle b^+_{k(-)}\rangle = \left(\frac{\omega - \omega_p}{\omega + \omega_p}\right)\langle b^+_{k(+)}\rangle, \tag{48}$$

which shows that in a large portion of the two upper branches of the dispersion diagram, the influence of the negative circularly polarized phonons is negligible. Therefore, we can neglect the negative phonon operators in Eq. (42). Dropping the (+) index in the phonon operators left, we can write the Hamiltonian as

$$\mathrm{H}_t = \sum_k \Big[\hbar\omega_m(k)a_k^+a_k + \hbar\omega_p(k)\,b_k^+b_k + i\frac{1}{2}\hbar\sigma_k(a_k^+b_k - b_k^+a_k)\Big]. \tag{49}$$

This Hamiltonian can be diagonalized by a canonical transformation to new operators obtained by the linear combinations of the magnon and the phonon operators,

$$d_k = u_k b_k - i v_k a_k, \tag{50a}$$
$$c_k = u_k a_k - i v_k b_k, \tag{50b}$$

where

$$u_k = \left(\frac{\omega_s + \omega_\delta}{2\omega_s}\right)^{1/2}, \quad v_k = \left(\frac{\omega_s - \omega_\delta}{2\omega_s}\right)^{1/2}, \tag{51a}$$
$$u_k^2 + v_k^2 = 1, \tag{51b}$$

and

$$\omega_\delta = (\omega_p - \omega_m)/2,\ \omega_s = \left(\omega_\delta^2 + \sigma_k^2/4\right)^{1/2}. \tag{52}$$

The transformation (50) is such that the new operators satisfy the boson commutation relations

$$[c_k, c_{k'}^+] = [d_k, d_{k'}^+] = \delta_{kk'}, \tag{53}$$
$$[c_k, d_{k'}] = [c_k, d_{k'}^+] = 0, \tag{54}$$
$$[c_k, c_{k'}] = [d_k, d_{k'}] = 0, \tag{55}$$

and the Hamiltonian has the diagonal form

$$\mathrm{H} = \sum_k [\hbar\omega_c(k)c_k^+c_k + \hbar\omega_d(k)\,d_k^+d_k], \tag{56}$$

where

$$\omega_c(k) = (\omega_p + \omega_m)/2 + \omega_s, \tag{57}$$
$$\omega_d(k) = (\omega_p + \omega_m)/2 - \omega_s, \tag{58}$$

which are the normal-mode frequencies corresponding to the two upper branches of Fig. 1.Notice that these frequencies can also be obtained from Eq. (47) by elimination of the negative root $-\omega_p$ and solving the resulting second-degree equation. Equations (53)–(56) lead to the interpretation that c_k^+, c_k, d_k^+, and d_k are the creation and annihilation operators of quanta of collective magnetoelastic excitations, or magnon-phonon hybrid excitations, with energies $\hbar\omega_c(k)$ and $\hbar\omega_d(k)$. Note that far from the crossover region, where the difference between the magnon and phonon frequencies is much larger than the splitting of the two branches $|\omega_p - \omega_m| \gg \sigma_k$, we have the following limits:

$$\begin{array}{llll} \omega_p > \omega_m & \omega_c \to \omega_p & \text{and} & \omega_d \to \omega_m \\ (v_k \to 0) & c_k \to b_k & d_k \to a_k, & \end{array} \tag{59}$$

$$\begin{array}{llll} \omega_m > \omega_p & \omega_c \to \omega_m & \text{and} & \omega_d \to \omega_p \\ (u_k \to 0) & c_k \to -ia_k & d_k \to -ib_k. & \end{array} \tag{60}$$

The stationary states of the Hamiltonian (56) may be obtained by applying integral powers of the creation operators to the vacuum state. The single-mode states can be written in normalized form as

$$|n_{ck}\rangle = [(c_k^+)^{n_k}/(n_{ck}\,!)^{1/2}]\,|0\rangle, \tag{61}$$

$$|n_{dk}\rangle = [(d_k^+)^{n_k}/(n_{dk}\,!)^{1/2}]\,|0\rangle. \tag{62}$$

The mean occupation numbers of magnons and phonons in these states are given by

$$\langle n_{ck}|a_k^+ a_k|n_{ck}\rangle = \langle n_{dk}|b_k^+ b_k|n_{dk}\rangle = v_k^2 n_k, \tag{63}$$

$$\langle n_{ck}|b_k^+ b_k|n_{ck}\rangle = \langle n_{dk}|a_k^+ a_k|n_{dk}\rangle = u_k^2 n_k, \tag{64}$$

which are in agreement with the limits (59) and (60). Note also that, since $u_k^2 + v_k^2 = 1$, the mean number of magnons plus the mean number of phonons in any state is the total number of the magnetoelastic quanta in that state.

The stationary states (61) and (62) can also be expanded in terms of pure magnon and pure phonon eigenstates. As discussed in Sec. II A, these states have a well-defined number of quanta and uncertain phase. Coherent magnetoelastic waves should have a well-defined phase and involve a large and uncertain number of magnons and phonons. In order to establish a correspondence between classical and quantum magnetoelastic waves, one must use the magnetoelastic coherent states, defined as the eigenstates of the annihilation operators

$$c_k|\alpha_{ck}\rangle = \alpha_{ck}|\alpha_{ck}\rangle, \quad d_k|\alpha_{dk}\rangle = \alpha_{dk}|\alpha_{dk}\rangle. \tag{65}$$

These can be expanded in terms of the eigenstates of the Hamiltonian

$$|\alpha_{ck}\rangle = e^{-|\alpha_{ck}|^2/2}\sum_{n_{ck}}(\alpha_{ck})^{n_{ck}}/(n_{ck}\,!)^{1/2}\,|n_{ck}\rangle, \tag{66}$$

$$|\alpha_{dk}\rangle = e^{-|\alpha_{dk}|^2/2}\sum_{n_{dk}}(\alpha_{dk})^{n_{dk}}/(n_{dk}\,!)^{1/2}\,|n_{dk}\rangle, \tag{67}$$

and they have magnetization and elastic displacement components with a well-defined phase, as expected for a classical wave.

IV. THE MAGNON-PHONON CONVERSION IN A TIME-DEPENDENT MAGNETIC FIELD

In this section, we consider that the ferromagnetic medium is subject to a dynamic variation in parameters, such as a change in the magnetization produced by a pulsed laser [49,51], or an applied magnetic field that varies in time. This results in a change in the magnon frequency that may change the character of the magnetoelastic excitation. For simplicity we will consider that the only time-varying quantity is the magnetic field. Let us assume, for example, that prior to an instant of time t_1 the applied field H_1 is constant, between t_1 and t_2 it increases monotonically in time, and after t_2 it remains constant at a larger value $H_2 > H_1$. We assume also that, prior to t_1, a magnetoelastic-wave pulse with essentially pure elastic character generated by a piezoelectric transducer is propagating in the material. If the transverse elastic wave is linearly polarized, it can be seen as the superposition of (+) and (−) circularly polarized waves. Figure 2(a) illustrates the

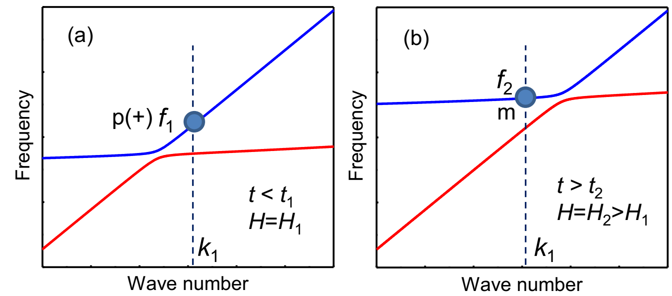

FIG. 2. (Color online) Illustration of the conversion of an elastic excitation (phonon) into a spin wave (magnon) in a time-varying magnetic field. (a) At $t < t_1$, a (+) elastic wave propagates in the medium with frequency f_1 and wave number k_1. (b) At $t > t_2$ the field is at a higher value and the excitation is converted to an essentially pure spin wave with frequency f_2 and the same wave number k_1.

magnetoelastic dispersion at $t < t_1$, showing that the phonon has frequency f_1 and wave number $k_1 > k_{\text{cross}}$ in the upper branch. If the variation of the magnetic field is slow enough so that the excitation stays in the same eigenmode, at $t > t_2$ it should remain in the upper branch. As will be shown later, the wave number of the excitation remains constant during the process, because the field is spatially uniform. Thus, as the field changes the frequency of the excitation changes and goes through the crossover region, so that the final state is essentially a spin wave with wave number k_1 and frequency f_2, as shown in Fig. 2(b). Conversely, if the initial state is a magnon excitation as in Fig. 2(b) and the field decreases in time, it is converted into a phonon excitation. Similarly, if the initial phonon state has wave number $k_1 < k_{\text{cross}}$ in the lower branch, it can be converted into a magnon by a step-down time variation in the field. The conversion from coherent elastic waves produced by a piezoelectric transducer in a YIG cylinder into a magnetic excitation, detected by the current induced in a nearby fine wire, was observed sometime ago [62]. As we show in the following sections, the magnon-phonon conversion efficiency, which will be defined later, depends on the time rate of change of the field.

A. Equations of motion in a time-dependent magnetic field

If the magnetic field varies in time, the magnon and phonon operators do not have the $\exp(i\omega t)$ time dependence. The quantity ω does not have the meaning of angular frequency of a periodic function, but its relation with the energy, $\omega = E/\hbar$, is still valid. Furthermore, since the transformations used to define a_k and b_k do not involve time-dependent quantities, these operators are not explicit functions of time. Thus, Eq. (42) is valid and so is the Hamiltonian in Eq. (49), provided we neglect the coupling between (−) phonons and magnons. For a magnetic field $H(t)$ with any time dependence, we can obtain the time evolution of the magnon and phonon operators using the Heisenberg equations, considering for the Hamiltonian

$$\mathrm{H}_t = \sum_k\left[\hbar\omega_m(t)\,a_k^+ a_k + \hbar\omega_p\,b_k^+ b_k + i\frac{1}{2}\hbar\sigma_k(a_k^+ b_k - b_k^+ a_k)\right]$$

$$= \sum_k[\hbar\omega_c(t)\,c_k^+ c_k + \hbar\omega_d(t)\,d_k^+ d_k], \tag{68}$$

where $\omega_m(t)$ is a function of time. In the case of z-directed waves, $\omega_m(t) = \gamma H(t) + \gamma D k^2$, so that $\omega_m(t)$ is proportional to $H(t)$. Therefore we have for the magnon and phonon operators,

$$\frac{da_k^+}{dt} = i\omega_m(t)\, a_k^+ + \frac{1}{2}\sigma_k b_k^+, \tag{69}$$

$$\frac{db_k^+}{dt} = i\omega_p\, b_k^+ - \frac{1}{2}\sigma_k a_k^+. \tag{70}$$

The equations of motion for the normal-mode magnetoelastic operators can be obtained from Eqs. (50), (69), and (70), or directly from the diagonal Hamiltonian (68). In this case, one has to note that the partial derivatives of the operators with respect to time are not zero. We can show that

$$\frac{dc_k^+}{dt} = i\omega_{ck}(t)\, c_k^+ + i\beta_k(t) d_k^+, \tag{71}$$

$$\frac{dd_k^+}{dt} = i\omega_{dk}(t)\, d_k^+ + i\, \beta_k(t) c_k^+, \tag{72}$$

where $\beta_k(t)$ is a parameter approximately proportional to the time rate of change of the magnetic field $\dot{H} \equiv dH/dt$,

$$\beta(t) = \frac{\gamma\sigma_k}{8\,\omega_s^2}\dot{H}(t). \tag{73}$$

Equations (69) and (70) have the same form as the semiclassical equations [42] for the transformed magnetization and elastic displacement variables, whereas Eqs. (71) and (72) are the same as for the normal-mode magnetoelastic variables. Notice that if $\dot{H} = 0$, and $\beta = 0$ the equations for c_k and d_k are not coupled to each other. In this case, the states corresponding to the two positive branches of the dispersion diagram of Fig. 1 are orthogonal to each other at any instant of time. However, if $\beta \neq 0$, one can couple the excitations of the two branches and the situation illustrated in Fig. 2 is plausible. The foregoing equations have been formulated in the Heisenberg picture, which is characterized by time-dependent operators and a time-independent wave vector. Therefore, if the system is initially in a state for which the expectation values of the magnon and the phonon operators are not zero, the time evolution of the expectation values is governed by Eqs. (69)–(73).

B. Explicit time-dependent invariants

The invariance properties of a system play an important role in quantum as well as in classical physics. In the problem we are considering, an invariant with respect to time is expected to have two roles. First, for the situation illustrated in Fig. 2, we have to define a magnon-phonon conversion efficiency in terms of a quantity which is conserved in the process. As the system is not conservative, the efficiency cannot be defined as the ratio between the energies of the two states. Second, it is possible to study the evolution of the state of a system with a time-dependent Hamiltonian by means of a simple theory [63] based on the expansion of the state in terms of the eigenstates of invariant operators. In the semiclassical theory of a magnetoelastic medium, in a time-varying magnetic field it has been shown [44] that, due to the spatial uniformity of the field, the total linear momentum density is conserved. Here we consider the linear momentum density as the sum of the magnon and phonon momenta given by Eqs. (11) and (32),

$$\vec{P}_t = \vec{P}_m + \vec{P}_p = \sum_k (\hbar\vec{k}\, a_k^+ a_k + \hbar\vec{k}\, b_k^+ b_k), \tag{74}$$

where b_k^+ and b_k denote the operators for the (+) circularly polarized phonons, since it is the only polarization that matters here. With Eqs. (50) and (51) one can see that $c_k^+ c_k + d_k^+ d_k = a_k^+ a_k + b_k^+ b_k$, so that the total momentum of magnons and phonons is equal to the total momentum of the hybrid magnon-phonon excitations,

$$\vec{P}_t = \sum_k (\hbar\vec{k}\, c_k^+ c_k + \hbar\vec{k}\, d_k^+ d_k). \tag{75}$$

The equation of motion for $\vec{P}_t(t)$ is

$$\frac{d\vec{P}_t}{dt} = \frac{\partial\vec{P}_t}{\partial t} + \frac{1}{i\hbar}[\vec{P}_t,\mathrm{H}]. \tag{76}$$

The commutator in Eq. (76) is zero, a conclusion easily drawn from the expressions of H and $\vec{P}_t$ in terms of the normal-mode magnetoelastic operators, Eqs. (68) and (76). The partial derivative of $\vec{P}_t$ with respect to time is also zero, which can be seen from Eq. (74) because a_k and b_k are not explicit functions of time. Therefore, $d\vec{P}_t/dt = 0$ and $\vec{P}_t$ is an explicit time-dependent invariant. Since the total number of quanta is conserved, the wave vector $\vec{k}$ is also invariant. This is consistent with Eqs. (71) and (72), which show that modes with different $\vec{k}$ are not coupled by the time variation of the magnetic field. Clearly, this property is true only because the "reflected particles" represented by the operator a_{-k} in (44) and (46) were neglected. However, the conclusion for $\vec{P}_t$ holds true in general.

With Eq. (37) and the similar result for magnons, considering only (+) circularly polarized phonons, we can write the total spin angular momentum of the magnon-phonon system as

$$\vec{S}_t = \vec{S}_m + \vec{S}_p = -\hbar\sum_k (\hat{k}\, a_k^+ a_k + \hat{k}\, b_k^+ b_k), \tag{77}$$

where $\hat{k} = \vec{k}/k$. Based on the same arguments used to discuss the properties of the linear momentum, the total spin angular momentum of the hybrid magnon-phonon excitations can be written as

$$\vec{S}_t = -\hbar\sum_k (\hat{k}\, c_k^+ c_k + \hat{k}\, d_k^+ d_k), \tag{78}$$

which is also an invariant quantity for a spatially uniform time-varying magnetic field.

C. Solutions of the Heisenberg equations: The magnon-phonon conversion efficiency

In this section we present solutions of the Heisenberg equations of motion for the operators in the coupled magnon-phonon system in a time-varying field, introduced in Sec. IV A. Although Eqs. (69) and (70) are operator equations, their linear character means that they can be solved in terms of c-number linear equations. We write their solutions in the form

$$a_k^+(t) = q(t)\, a_k^+(t_0) + p(t)\, b_k^+(t_0), \tag{79}$$

$$b_k^+(t) = s(t)\, a_k^+(t_0) + r(t)\, b_k^+(t_0), \tag{80}$$

where we have omitted the index k in the c-number functions to simplify the notation. The momentum invariance implies that

$$|q|^2 + |s|^2 = 1, \quad |p|^2 + |r|^2 = 1, \quad q\,p^* + s\,r^* = 0. \tag{81}$$

The initial conditions at $t = t_0$ are

$$q(t_0) = r(t_0) = 1, \quad p(t_0) = s(t_0) = 0. \tag{82}$$

From Eqs. (69), (79), and (80) we obtain for two of the functions

$$\frac{dq(t)}{dt} = i\omega_m(t)\,q(t) + \frac{1}{2}\sigma_k\,s(t), \tag{83}$$

$$\frac{ds(t)}{dt} = i\omega_p\,s(t) - \frac{1}{2}\sigma_k\,q(t). \tag{84}$$

Similarly, with Eq. (70) we obtain

$$c_k^+(t) = x(t)\,c_k^+(t_0) + w(t)\,d_k^+(t_0), \tag{85}$$

$$d_k^+(t) = y(t)\,c_k^+(t_0) + z(t)\,d_k^+(t_0), \tag{86}$$

where

$$|x|^2 + |y|^2 = 1, \quad |w|^2 + |z|^2 = 1, \quad x\,w^* + y\,z^* = 0, \tag{87}$$

and

$$x(t_0) = z(t_0) = 1, \quad w(t_0) = y(t_0) = 0. \tag{88}$$

Analogously, we obtain

$$\frac{dx(t)}{dt} = i\omega_{ck}(t)\,x(t) + i\,\beta(t)y(t), \tag{89}$$

$$\frac{dy(t)}{dt} = i\omega_{dk}(t)\,y(t) + i\,\beta(t)\,x(t). \tag{90}$$

With Eqs. (79)–(90) one can calculate the evolution of any quantity of interest in the coupled magnon-phonon system for a time-varying field with given initial conditions. For instance, let us assume that at instant t_0 we have in the system a pure magnon excitation characterized by a state $|\psi_0\rangle$. This is only an approximation because it is not possible to have a magnon excitation without some phonon admixture. However, if $\vec{k}$ is very far from the crossover region, this approximation may be very good. If after t_0 the applied field varies in time, there will be a transfer of linear and angular momenta to phonon excitations, as revealed by Eqs. (79) and (80). Since the sum of the magnon and the phonon mean momenta is conserved, it is convenient to define a conversion efficiency from the magnon to the phonon state as the ratio between the mean momenta, either linear or angular, in the two states. Using (76), (79), and (80), and considering that $\langle\psi_0|b_k|\psi_0\rangle = \langle\psi_0|b_k^+ b_k|\psi_0\rangle = 0$, we find for the magnon-phonon conversion efficiency

$$\eta_{mp}(t) = \frac{\langle S_p(t)\rangle}{\langle S_m(t_0)\rangle} = |s(t)|^2. \tag{91}$$

Notice that this is valid for any magnon state $|\psi_0\rangle$. Analogously, we see that if the system is initially in a phonon state, the phonon-magnon conversion efficiency is given by $|p(t)|^2$. Similarly, we can define a conversion factor for the two magnetoelastic normal-mode excitations, which represents the transfer of momenta between the two branches of Fig. 2. It can be shown that

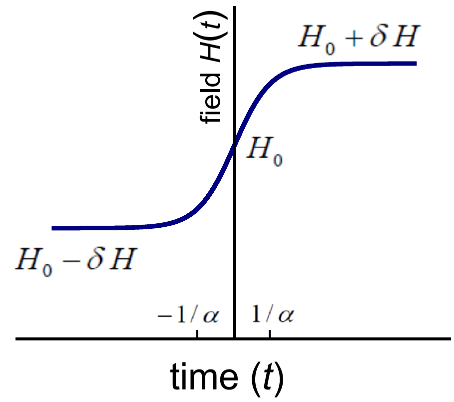

FIG. 3. (Color online) Variation of the magnetic field with a $\tanh(\alpha\,t/2)$ time dependence.

$$\eta_{cd}(t) = \langle S_d(t)\rangle/\langle S_c(t_0)\rangle = |y(t)|^2, \tag{92}$$

$$\eta_{dc}(t) = \langle S_c(t)\rangle/\langle S_d(t_0)\rangle = |w(t)|^2, \tag{93}$$

which are valid for conditions analogous to those used to derive (91). To obtain (92) we assume that the system is initially in a pure c_k state, and for (93) it is initially in a pure d_k state.

The systems of linear equations (83)–(84) and (89)–(90) cannot be solved for a general time dependence of the applied field. However, it is possible to find their solution for particular cases of interest. In the case of a slowly varying field (the slowness condition will be specified later), i.e., in an adiabatic approximation, it is convenient to work with Eqs. (71)–(72) and (89)–(90), because in this case the coupling between the c_k and d_k modes is small. Consider, for instance, that in this approximation we have the situation depicted in Fig. 3. The system is initially in a phonon state in branch c_k, and the field increases so that the frequency goes through the crossover region. The phonon-magnon conversion efficiency is therefore given by $1 - |y(t)|^2$. The solution of (89) and (90), in the limit where $\gamma\dot{H} \ll \sigma_k^2$, is identical to the solution of the semiclassical equations for a similar problem [44],

$$\eta_{pm} = 1 - |y(\infty)|^2 \approx 1 - (\pi^2/9)\,\exp(-\dot{H}_{\text{crit}}/\dot{H}_{\text{cross}}), \tag{94}$$

where $\dot{H}_{\text{cross}}$ is the absolute value of the field gradient at the instant when the magnon and phonon frequencies cross each other, and

$$\dot{H}_{\text{crit}} = \frac{\pi\,\sigma_k^2}{2\gamma} \tag{95}$$

is a critical field-gradient evaluated at the wave number of the excitation. Another situation of interest is that of the sudden change of the field, characterized by the condition $\gamma\dot{H} \gg \sigma_k^2$. In this case the coupling between modes c_k and d_k is strong so that their character of quasinormal modes loses meaning. In this case, however, the coupling between the magnon and phonon operators is small, and Eqs. (69)–(70) and (83)–(84) can be solved approximately. Again, considering the situation of Fig. 2, we see that the phonon-magnon conversion efficiency

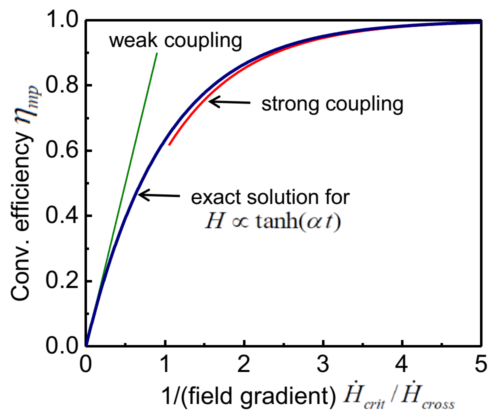

FIG. 4. (Color online) Calculated magnon-phonon conversion efficiency as a function of the field time-gradient at the crossover.

is given by $|p(t)|^2$, where $p(t)$ is a solution of the equations for p and r which are identical to (83) and (84). The analogy with the semiclassical case gives immediately [44]

$$\eta_{pm} = |p(\infty)|^2 \approx \dot{H}_{\text{crit}}/\dot{H}_{\text{cross}}. \tag{96}$$

Finally, we note that Eqs. (83) and (84) can be integrated exactly for a magnetic field with a time variation $H(t) = H_0 + \delta H \tanh(\alpha t)$, where $t = 0$ is the instant when the magnon and phonon frequencies cross each other, so that $\dot{H}_{\text{cross}} = \alpha\,\delta H$. Figure 3 shows a plot of this function, which is a rounded steplike variation of the type necessary to produce the conversion illustrated in Fig. 2. It has been shown [64] that equations similar to (83) and (84) can be transformed into a hypergeometric equation exactly soluble for the time dependence $\tanh(\alpha t)$. For the same situation previously considered, it can be shown that the phonon-magnon conversion efficiency, defined as the ratio between the magnon momentum at $t \to \infty$ and the phonon momentum at $t \to -\infty$, is given exactly by

$$\eta_{pm} = 1 - \exp(-\dot{H}_{\text{crit}}/\dot{H}_{\text{cross}}). \tag{97}$$

From the symmetry of Eqs. (89) and (90) one can see that if the system is initially in a magnon state and the field varies so that it is converted into a phonon state, the magnon-phonon conversion efficiency η_{mp} is given by the same equations as η_{pm}. Figure 4 shows plots of the magnon-phonon conversion efficiencies expressed by Eqs. (94), (96), and (97) valid, respectively, for $\dot{H}_{\text{cross}} \ll \dot{H}_{\text{crit}}$, $\dot{H}_{\text{cross}} \gg \dot{H}_{\text{crit}}$, and $H \propto \tanh(\alpha t)$. We call the condition $\dot{H}_{\text{cross}} \ll \dot{H}_{\text{crit}}$ the strong magnon-phonon coupling regime, because in this case there is little transfer of momenta between the two magnetoelastic normal modes, and therefore as the frequency passes the crossover there is a large conversion from elastic into magnetic excitation. The opposite situation, where $\dot{H}_{\text{cross}} \gg \dot{H}_{\text{crit}}$, represents a weak magnon-phonon coupling, because in this case there is a large transfer of momenta between the two normal modes. Figure 4 shows that a conversion efficiency of at least 80% is achieved with a field gradient $\dot{H}_{\text{cross}} < \dot{H}_{\text{crit}}/2$. Using for YIG the same parameters employed to calculate the curves of Fig. 1, we obtain with Eq. (95) $\dot{H}_{\text{crit}} = 5.2 \times 10^{10}$ Oe/s. Thus, a large conversion efficiency can be attained with steplike field variations with gradients smaller than 260 Oe/10 ns, a condition easily realized in experiments.

V. ADDITIONAL REMARKS ABOUT MAGNON-PHONON CONVERSION AND CONCLUSION

To conclude the study of the quantum aspects of the magnon-phonon conversion in a time-dependent magnetic field, let us consider the situation where the system is initially in a "pure" phonon state $|\psi_0\rangle$. Due to a time variation of the field, the system may end up in a magnon state with a momentum conversion efficiency given by $|p(t)|^2$. This result is valid for any initial state which has a nonzero mean momentum. It can be, for instance, a stationary state of the Hamiltonian, or a coherent state. Assume that the system is initially in a coherent phonon state, which is the type expected to be generated by a piezoelectric transducer in typical ultrasonics experiments. Thus, $|\psi_0\rangle = |\alpha_k, k\rangle$, where

$$b_k|\alpha_k,k\rangle = \alpha_k|\alpha_k,k\rangle. \tag{98}$$

Before the field starts changing in time, the variance of the operator $b_k^+ + b_k$, which is related to the displacement operator, is given by

$$\Delta^2(b_k^+ + b_k) = \langle\psi_0|(b_k^+ + b_k)^2|\psi_0\rangle - \langle\psi_0|b_k^+ + b_k|\psi_0\rangle^2 = 1. \tag{99}$$

From this result, it is possible to show that the product of the variances of $b_k^+ + b_k$ and of its canonical conjugate is given by $\hbar^2/4$, which is the minimum value allowed by the uncertainty principle. This is a well-known property of coherent states [59]. An indication of the time evolution of the coherence of the system, and consequently of the precision of measurements, is given by the time dependence of the variances of observables. Using the Heisenberg equations of motion and the commuting properties of the magnon and phonon operators, we find that for an initial coherent phonon state we have

$$\Delta^2(b_k^+ + b_k) = 1, \quad \Delta^2(a_k^+ + a_k) = 1. \tag{100}$$

Therefore, a system described by the Hamiltonian in Eq. (68) that is initially in a coherent state maintains its coherence properties regardless of the time dependence of the field.

In conclusion, we have presented a quantum theory of the interaction between magnons and phonons in a ferromagnetic material with magnetic parameters that can be varied dynamically. We have shown that if a parameter variation produces changes in the magnon frequency, one can convert magnons into phonons, or vice versa. In the case of a spatially uniform time-varying magnetic field, if the field time-gradient at the magnetoelastic crossover region is much smaller than a critical value, an initial magnon state can be completely converted into a circularly polarized phonon with conservation of linear momentum and spin angular momentum, or vice versa. This means that phonons created by the conversion of magnons have spin angular momentum and carry spin current. We have also shown that if the system is initially in a quantum coherent state, its coherence properties are maintained regardless of the time dependence of the field.

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