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# Spin relaxation in a quantum well by phonon scatterings

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The spin relaxation due to the spin-orbit interaction (SOI) is studied theoretically in a quantum well with electrons occupying only the ground subband. First, it is shown that the coefficient of the Rashba SOI is proportional to $b_{\rm off} - 1$, in which the parameter $b_{\rm off}$, determined by the band offsets and the band gaps, passes through unity, for example, by changing $x$ in $\rm Ga_{0.47}In_{0.53}As$(well)/$\rm Al_xGa_{1-x}As_ySb_{1-y}$(barrier). Second, it is derived that the transition matrix element of each spin-flip phonon scattering has the same proportionality factor $b_{\rm off} - 1$, in addition to the impurity scattering previously studied by the same authors [Phys. Rev. B **89**, 075314 (2014)]. These findings suggest the possibility of strongly suppressing the spin-relaxation rate by choosing appropriate materials.

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## I. INTRODUCTION

Controlling the spin relaxation is one of the challenges in research toward the information processing with use of spin. By suppressing the spin relaxation, the spin polarization induced in nonmagnetic materials can be made larger and maintained longer. Moreover, if we succeed in changing the spin-relaxation rate in a wider range by the applied gate voltage, the on-off current ratio in the proposed spin-lifetime FET [1,2] is improved. In this paper we show theoretically that the spin relaxation in a quantum well is significantly suppressed even at room temperature by choosing appropriate semiconducting materials for well and barrier layers.

The spin relaxation in quantum wells has been extensively studied in a number of theoretical [3–14] and experimental [15–27] works. One of the major mechanisms of the spin relaxation in $n$-doped quantum wells [28,29] is the Dyakonov-Perel (DP) mechanism [3,30–32] which is due to the spin precession around the effective magnetic field induced by the Dresselhaus spin-orbit interaction (SOI) [33] and the Rashba SOI [34–37]. Fortunately, it has been theoretically shown [3] that the DP mechanism due to the Dresselhaus SOI can be turned off in a quantum well parallel to the (110) plane of the zinc-blende structure for the spin polarization perpendicular to the well layer, which has been demonstrated in several experiments [16–18,22]. If we make a symmetric quantum well, the DP mechanism due to the Rashba SOI is also turned off.

To further reduce the spin-relaxation rate, we move on to suppressing another important mechanism of the spin relaxation, called the Elliott-Yafet mechanism [38–40], which is a spin-flip scattering process due to the SOI and the scattering potential. Although the spin relaxation due to the Elliott-Yafet mechanism has been theoretically studied in a quantum well, most works considered only spin-flip scattering processes within the ground subband (see for example Refs. [5,11]). Intersubband spin-flip scattering processes have been taken into account in the spin relaxation only by a few recent papers. Döhrmann *et al.* [19] have proposed a spin relaxation due to an intersubband transition, which is the spin-flip scattering induced by the Dresselhaus SOI and the impurity potential, to explain their observed result of the spin relaxation time in a (110) symmetric quantum well at higher temperatures such that electrons occupy the first-excited subband. Zhou and Wu [12] have made a calculation of the spin relaxation time by considering a virtual-intersubband process through the first-excited subband, which consists of an intersubband spin-flip process due to the Dresselhaus SOI and an intersubband scattering process due to the impurity potential, in a (110) symmetric quantum well with electrons occupying only the ground subband. Although such theoretical efforts have been made, there have been no reports on the contribution from virtual-intersubband spin-flip scatterings by the confining-potential-induced SOI to the spin relaxation before our previous paper [14].

In the previous paper [14] we have studied the spin relaxation in a quantum well with electrons occupying only the ground subband for the spin polarization perpendicular to the well. We have taken into account virtual-intersubband spin-flip scattering processes through excited subbands as well as intrasubband processes within the ground subband. In the virtual-intersubband processes, we have considered contributions from both the confining-potential-induced SOI and the Dresselhaus SOI. By considering the case where the scattering potential is given by a random distribution of impurities, we have shown that the following two methods are effective in suppressing the spin-flip scattering rate.

(A) Placing impurities in the center plane of a symmetric quantum well with use of $\delta$ doping [41,42]. Because both the impurity potential and the confining potential in this case are symmetric with respect to the center plane, all transition matrix elements for spin-flip scatterings vanish.

(B) Tuning band offsets of the conduction and valence bands so that intrasubband and virtual-intersubband processes interfere destructively. This destructive interference has been derived in the case where the scattering potential is the sum of central-force potentials due to each impurity.

When we consider the spin relaxation at room temperature, we must take into account spin-flip scatterings by phonons. The suppression by the symmetry of the potential is not applicable to phonon scatterings. However, the suppression by the interference between intrasubband and virtual-intersubband processes is worth being investigated in phonon scatterings. The contribution from virtual-intersubband spin-flip scatterings by phonons and the confining-potential-induced SOI to the spin relaxation remains to be studied theoretically.

In this paper we theoretically study the spin relaxation in a quantum well due to the Elliott-Yafet mechanism as well as the DP mechanism by including the contribution from virtual-intersubband spin-flip scatterings by phonons and the confining-potential-induced SOI. We show that the spin relaxation by the Elliott-Yafet mechanism is suppressed for both impurity and phonon scatterings by tuning band offsets to the position of the destructive interference in the case where the spin polarization is perpendicular to the well and electrons occupy only the ground subband. In addition we show that the spin relaxation by the DP mechanism, which appears in an asymmetric confining potential, is also suppressed for the same values of band offsets. This suppression, which is made by choosing appropriate constituent semiconductors, does not require the symmetry of the confining potential. A reduction of the spin-relaxation rate by orders of magnitude is expected by the present method of controlling the spin-relaxation rate by the band offset (depending on constituent semiconductors), which is waiting for an experimental realization.

We present our Hamiltonian in Sec. II for an electron in the conduction band of a quantum well, which interacts with phonons as well as impurities. Here we assume that the impurity potential is a slowly varying electrostatic potential with arbitrary spatial dependence, which is more general than that in our previous theory [14]. An important feature of the SOI in a quantum well is that there is an additional factor $b_{\rm off}$ in the SOI induced by the band offset between well and barrier layers. The factor $b_{\rm off}$, which is determined by the band offset of each of the conduction band and split valence bands as well as the band gaps, varies widely with the constituent compounds in well and barrier layers.

We find in Sec. III that the effective magnetic field induced by the Rashba SOI in the presence of the gate voltage is proportional to $b_{\rm off}-1$. The resulting spin-relaxation rate is proportional to $(b_{\rm off}-1)^2$ and is therefore reduced in quantum wells with small $|b_{\rm off}-1|$.

We show in Sec. IV that the same reduction factor $(b_{\rm off}-1)^2$ appears in the spin-flip scattering rate for both impurity and phonon scatterings, which means that the spin relaxation in the Elliott-Yafet mechanism is also suppressed in quantum wells with small $|b_{\rm off}-1|$. The suppression here is due to the interference between intrasubband and virtual-intersubband processes. In this paper we do not consider spin-flip scatterings induced by the Dresselhaus SOI which gives only a virtual-intersubband process [12]. The contribution from the Dresselhaus SOI has been estimated to be only 1% of the total spin-flip scattering rate in the case of impurity scatterings in a GaAs/AlGaAs quantum well [14].

## II. HAMILTONIAN

We consider an electron in the conduction band of a quantum-well structure which is formed by two different semiconductors with the zinc-blende structure. The Hamiltonian $H_{\rm QW}$ describing an electron confined in a quantum well perpendicular to the $z$ axis is

$$H_{\rm QW}=\frac{\hat{\boldsymbol{p}}^2}{2m}+V_{\rm W}(z), \tag{1}$$

where $\hat{\boldsymbol{p}}=(\hat{p}_x,\hat{p}_y,\hat{p}_z)=-i\hbar\nabla=-i\hbar(\nabla_x,\nabla_y,\nabla_z)$ and $m$ is the effective mass of the conduction band. The confining potential $V_{\rm W}(z)$ is

$$V_{\rm W}(z)=V_{\rm bo}^{\rm c}(z)+V_{\rm es}(z). \tag{2}$$

Here $V_{\rm bo}^{\rm c}(z)$ is the potential due to the conduction-band offset at two interfaces between constituent semiconductors and is given for the well width $W$ and the potential height $V_0(>0)$ by

$$V_{\rm bo}^{\rm c}(z)=\begin{cases}0 & (|z|<W/2),\\ V_0 & (|z|>W/2).\end{cases} \tag{3}$$

The second term $V_{\rm es}(z)$ is the electrostatic potential, which is caused by the charge density within the quantum-well structure and by the applied gate voltage. The $z$ dependence of $V_{\rm es}(z)$ is not restricted to being symmetric with respect to $z=0$. We apply the periodic boundary conditions in the $x$ and $y$ directions. Then each eigenstate of $H_{\rm QW}$ is labeled by the subband index $n=0,1,2,\ldots$, the wave vector in the $xy$ plane $\boldsymbol{k}=(k_x,k_y)$, and the $z$ component of spin $\sigma=\uparrow,\downarrow$. The corresponding eigenvector $|n\boldsymbol{k}\sigma\rangle$ satisfies

$$H_{\rm QW}|n\boldsymbol{k}\sigma\rangle=\varepsilon_{n\boldsymbol{k}\sigma}|n\boldsymbol{k}\sigma\rangle, \tag{4}$$

where the eigenenergy is

$$\varepsilon_{n\boldsymbol{k}\sigma}=\varepsilon_n+\hbar^2k^2/2m, \tag{5}$$

with $k=|\boldsymbol{k}|$. Here $\varepsilon_n$ is the eigenvalue of the Hamiltonian associated with the motion along the $z$ direction,

$$\left[\frac{\hat{p}_z^2}{2m}+V_{\rm W}(z)\right]|n\rangle=\varepsilon_n|n\rangle, \tag{6}$$

where $|n\rangle$ is the corresponding eigenvector.

The perturbation we consider consists of the SOI caused by the quantum-well potential $V_{\rm W}^{\rm so}$, the Dresselhaus SOI $H_{\rm D}^{\rm so}$, the impurity potential $V_{\rm imp}$, its associated SOI $V_{\rm imp}^{\rm so}$, and the electron-phonon interaction with the induced SOI $H_{\rm ep}$. The SOI $V_{\rm W}^{\rm so}$ is induced by the band offsets and the electrostatic potential, and is derived in Appendix A to be

$$\begin{aligned}V_{\rm W}^{\rm so}&=-\hbar^{-1}\eta\,\boldsymbol{\sigma}\cdot\left[\nabla\left(b_{\rm off}V_{\rm bo}^{\rm c}+V_{\rm es}\right)\times\hat{\boldsymbol{p}}\right]\\&=\eta\left(b_{\rm off}\nabla_zV_{\rm bo}^{\rm c}+\nabla_zV_{\rm es}\right)(\sigma_xk_y-\sigma_yk_x),\end{aligned} \tag{7}$$

which is of the form of the Rashba SOI. Here $\boldsymbol{\sigma}=(\sigma_x,\sigma_y,\sigma_z)$ is the Pauli spin matrix, while $\eta$ is the effective coupling constant of the SOI for an electron in the conduction band of the semiconductor in the well layer ($|z|<W/2$), given by Eq. (A10). Since $V_{\rm W}^{\rm so}$ does not have a term with $\sigma_z$, non-spin-flip matrix elements are absent: $\langle n'\boldsymbol{k}\sigma|V_{\rm W}^{\rm so}|n\boldsymbol{k}\sigma\rangle=0$.

The factor $b_{\rm off}$, which appears in front of $V_{\rm bo}^{\rm c}$ in Eq. (7), is given with band offsets $\Delta E_{\rm c}$, $\Delta E_{\rm v}$, and $\Delta E_{\rm s}$, by

$$b_{\rm off}=\frac{\Delta E_{\rm v}/[E_{\rm g}(E_{\rm g}-\Delta E_{\rm v})]-\Delta E_{\rm s}/\left[E_{\rm g}^{\rm s}\left(E_{\rm g}^{\rm s}-\Delta E_{\rm s}\right)\right]}{\Delta E_{\rm c}\left\{1/(E_{\rm g})^2-1/\left(E_{\rm g}^{\rm s}\right)^2\right\}}, \tag{8}$$

where $E_{\rm g}=E_{\rm c}-E_{\rm v}$, $E_{\rm g}^{\rm s}=E_{\rm g}+\Delta_{\rm so}$ with $\Delta_{\rm so}=E_{\rm v}-E_{\rm s}$, $\Delta E_{\rm c}=E_{\rm c}^{\rm b}-E_{\rm c}=V_0$, $\Delta E_{\rm v}=E_{\rm v}^{\rm b}-E_{\rm v}$, and $\Delta E_{\rm s}=E_{\rm s}^{\rm b}-E_{\rm s}$ with $E_{\rm c}$ ($E_{\rm c}^{\rm b}$) the energy of the conduction-band bottom, $E_{\rm v}$ ($E_{\rm v}^{\rm b}$) that of the valance-band top, $E_{\rm s}$ ($E_{\rm s}^{\rm b}$) that of the split-off-band top in the well layer (the barrier layers). Figure 1 presents

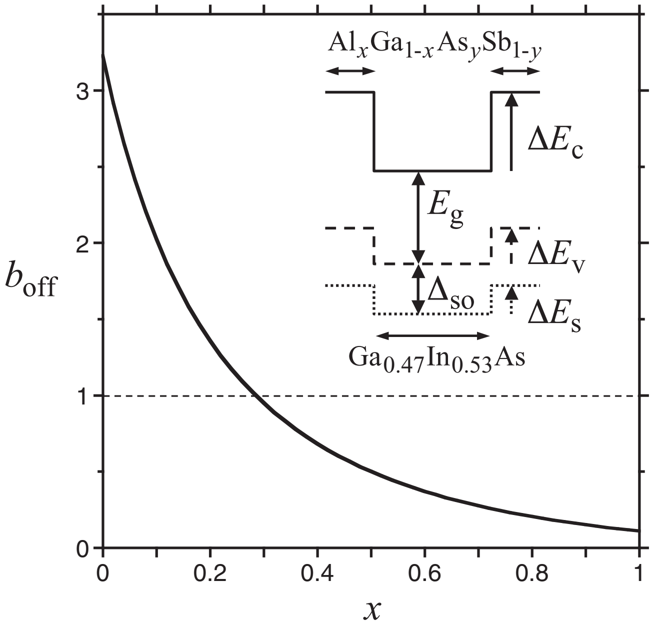

FIG. 1. The factor $b_\mathrm{off}$, defined by Eq. (8), as a function of the Al fraction $x$ in a type-II quantum well $\mathrm{Ga}_{0.47}\mathrm{In}_{0.53}\mathrm{As}$(well)/$\mathrm{Al}_x\mathrm{Ga}_{1-x}\mathrm{As}_y\mathrm{Sb}_{1-y}$(barrier) lattice matched to InP. We have used band parameters in Ref. [43] and employed the linear interpolation of band offsets between $x = 0$ and $x = 1$ to obtain $\Delta E_\mathrm{c}[\mathrm{eV}] = 0.436 + 1.43x$, $\Delta E_\mathrm{v}[\mathrm{eV}] = 0.444 - 0.354x$, and $\Delta E_\mathrm{s}[\mathrm{eV}] = 0.373 - 0.371x$. Corresponding to $\mathrm{Ga}_{0.47}\mathrm{In}_{0.53}\mathrm{As}$, $E_\mathrm{g}[\mathrm{eV}] = 0.816$ and $\Delta_\mathrm{so}[\mathrm{eV}] = 0.330$.

$b_\mathrm{off}$ as a function of the Al fraction $x$ in a type-II quantum well $\mathrm{Ga}_{0.47}\mathrm{In}_{0.53}\mathrm{As}$(well)/$\mathrm{Al}_x\mathrm{Ga}_{1-x}\mathrm{As}_y\mathrm{Sb}_{1-y}$(barrier), where $y$ is determined so that $\mathrm{Al}_x\mathrm{Ga}_{1-x}\mathrm{As}_y\mathrm{Sb}_{1-y}$ is lattice matched to InP. The curve $b_\mathrm{off}(x)$ and the line $b_\mathrm{off} = 1$ cross between $x = 0$ and $x = 1$. The exact value of $x$, at which $b_\mathrm{off}$ is unity, may deviate from that indicated in Fig. 1 because of the uncertainty in the values of band parameters. On the other hand, $b_\mathrm{off} = -0.68 < 0$ in a typical type-I quantum well $\mathrm{GaAs}/\mathrm{Al}_x\mathrm{Ga}_{1-x}\mathrm{As}$ at $x = 0.4$. The sign of $b_\mathrm{off}$ is determined by that of $\Delta E_\mathrm{v}/\Delta E_\mathrm{c}$ [negative (positive) for type-I (type-II) quantum wells] in the case where $\Delta E_\mathrm{v} \sim \Delta E_\mathrm{s}$ and $\Delta_\mathrm{so}$ is large enough that the first term in the numerator of Eq. (8) is dominant over the second one.

The Dresselhaus SOI $H_\mathrm{D}^\mathrm{so}$ is given by

$$H_\mathrm{D}^\mathrm{so} = -\frac{\gamma}{2\hbar^3}\boldsymbol{\sigma}\cdot\boldsymbol{h}, \tag{9}$$

where $\gamma$ is the coupling constant of the Dresselhaus SOI. In (110) quantum wells $\boldsymbol{h} = (h_x, h_y, h_z)$ is given by

$$\begin{aligned} h_x &= \left(-\hat{p}_x^2 - 2\hat{p}_y^2 + \hat{p}_z^2\right)\hat{p}_z,\\ h_y &= 4\hat{p}_x\hat{p}_y\hat{p}_z,\\ h_z &= \hat{p}_x\left(\hat{p}_x^2 - 2\hat{p}_y^2 - \hat{p}_z^2\right), \end{aligned} \tag{10}$$

where the Cartesian unit vectors are taken as

$$\begin{aligned} \boldsymbol{e}_x &= (-\boldsymbol{e}_{[100]} + \boldsymbol{e}_{[010]})/\sqrt{2},\\ \boldsymbol{e}_y &= \boldsymbol{e}_{[001]},\\ \boldsymbol{e}_z &= (\boldsymbol{e}_{[100]} + \boldsymbol{e}_{[010]})/\sqrt{2}, \end{aligned} \tag{11}$$

with $\boldsymbol{e}_{[100]}$, $\boldsymbol{e}_{[010]}$, and $\boldsymbol{e}_{[001]}$ the unit vectors along the crystal axes.

The impurity potential $V_\mathrm{imp}(\boldsymbol{r})$ with $\boldsymbol{r} = (x,y,z)$ is an electrostatic potential, which originates from charges due to ionized impurities and to spatial variations in the electron density by the screening. We assume that ionized impurities are randomly distributed over the cross section at each $z$, while their distribution along the $z$ direction may be nonuniform such as in the modulation doping. We define $V_\mathrm{imp}(\boldsymbol{r})$ so that $\int V_\mathrm{imp}(\boldsymbol{r})dxdy = 0$, that is $V_\mathrm{imp}(\boldsymbol{r})$ does not include the average over the cross section at each $z$ of the electrostatic potential due to ionized impurities and the screening. This average is the electrostatic potential independent of $x$ and $y$ so that we include it in $V_\mathrm{es}(z)$. Since $\int V_\mathrm{imp}(\boldsymbol{r})dxdy = 0$, the matrix elements diagonal in momentum are zero: $\langle n'\boldsymbol{k}\sigma|V_\mathrm{imp}|n\boldsymbol{k}\sigma\rangle = 0$. The SOI originating from $V_\mathrm{imp}(\boldsymbol{r})$, denoted by $V_\mathrm{imp}^\mathrm{so}$, is given by

$$V_\mathrm{imp}^\mathrm{so} = -\frac{\eta}{\hbar}\boldsymbol{\sigma}\cdot(\nabla V_\mathrm{imp}\times\hat{\boldsymbol{p}}), \tag{12}$$

which satisfies $\langle n'\boldsymbol{k}\bar{\sigma}|V_\mathrm{imp}^\mathrm{so}|n\boldsymbol{k}\sigma\rangle = 0$ with $\bar{\sigma}$ the spin opposite to $\sigma$. Furthermore, we assume that $V_\mathrm{imp}(\boldsymbol{r})$ is a slowly varying function of $\boldsymbol{r}$ so that the effective-mass approximation is applicable. We impose no additional restrictions on the $\boldsymbol{r}$ dependence of $V_\mathrm{imp}(\boldsymbol{r})$. When we calculate the spin-relaxation rate in the lowest order of the SOI, we neglect non-spin-flip matrix elements $\langle n'\boldsymbol{k}'\sigma|V_\mathrm{imp}^\mathrm{so}|n\boldsymbol{k}\sigma\rangle$, since they only give corrections of higher orders in the SOI.

The electron-phonon interaction $H_\mathrm{ep}$ is given by

$$H_\mathrm{ep} = \sum_{\boldsymbol{q}\lambda}\check{v}_{\boldsymbol{q}\lambda}\left(b_{\boldsymbol{q}\lambda} + b_{-\boldsymbol{q}\lambda}^\dagger\right), \tag{13}$$

where $\check{v}_{\boldsymbol{q}\lambda}$ is the potential induced by a bulk phonon mode with wave vector $\boldsymbol{q} = (q_x, q_y, q_z)$ and branch $\lambda$ plus its associated SOI:

$$\check{v}_{\boldsymbol{q}\lambda} = v_{\boldsymbol{q}\lambda} + v_{\boldsymbol{q}\lambda}^\mathrm{so}. \tag{14}$$

Here $b_{\boldsymbol{q}\lambda}$ and $b_{\boldsymbol{q}\lambda}^\dagger$ are the annihilation and creation operators, respectively, of a bulk phonon $\boldsymbol{q}\lambda$. Since only phonons with small $|\boldsymbol{q}|$ contribute to phonon scatterings, the complex coefficient $v_{\boldsymbol{q}\lambda}$ is a slowly varying function of $\boldsymbol{r}$ and can be treated within the effective-mass approximation [44]. Then the associated SOI is given in the same form as the impurity-potential-induced SOI:

$$v_{\boldsymbol{q}\lambda}^\mathrm{so} = -\frac{\eta}{\hbar}\boldsymbol{\sigma}\cdot(\nabla v_{\boldsymbol{q}\lambda}\times\hat{\boldsymbol{p}}). \tag{15}$$

For $H_\mathrm{ep}$ to be Hermitian, the following relations hold:

$$v_{\boldsymbol{q}\lambda}^* = v_{-\boldsymbol{q}\lambda}, \quad v_{\boldsymbol{q}\lambda}^{\mathrm{so}\,\dagger} = v_{-\boldsymbol{q}\lambda}^\mathrm{so}, \tag{16}$$

where the Hermitian conjugate of an operator $A$ is denoted by $A^\dagger$.

## III. SPIN RELAXATION IN THE DP MECHANISM

The effective magnetic field leading to the spin relaxation in the DP mechanism is proportional to $\langle 0|[\nabla_z(b_\mathrm{off}V_\mathrm{bo}^\mathrm{c} + V_\mathrm{es})]|0\rangle$ for the Rashba SOI in Eq. (7). The effective magnetic field of the Rashba SOI is absent in a symmetric quantum well, since $\varphi_0(z) = \langle z|0\rangle$, $V_{\rm bo}^{\rm c}(z)$, and $V_{\rm es}(z)$ are even functions of $z$, leading to

$$\langle 0|\big[\nabla_z\big(b_{\rm off}V_{\rm bo}^{\rm c} + V_{\rm es}\big)\big]|0\rangle = 0 \ \ \text{(symmetric wells)}. \tag{17}$$

Even in an asymmetric quantum well, the effective magnetic field is zero when $b_{\rm off}$ is equal to unity:

$$\langle 0|\big[\nabla_z\big(V_{\rm bo}^{\rm c} + V_{\rm es}\big)\big]|0\rangle = 0, \tag{18}$$

as shown by Ando [45,46]. This is because, for any *bound* eigenstate $|n\rangle$ of the Hamiltonian $\hat{p}_z^2/2m + V(z)$ with arbitrary potential $V(z)$, the expectation value of the force $-\nabla_z V(z)$ is zero, that is $\langle n|(-\nabla_z V)|n\rangle = 0$. When $b_{\rm off} \neq 1$, we obtain using Eq. (18)

$$\langle 0|\big[\nabla_z\big(b_{\rm off}V_{\rm bo}^{\rm c} + V_{\rm es}\big)\big]|0\rangle = (b_{\rm off} - 1)\langle 0|(-\nabla_z V_{\rm es})|0\rangle. \tag{19}$$

This indicates that the effective magnetic field due to the Rashba SOI is proportional to $b_{\rm off} - 1$. Therefore, we find that the spin-relaxation rate in the DP mechanism due to the Rashba SOI is proportional to $(b_{\rm off} - 1)^2$ and is significantly reduced in quantum wells with small $|b_{\rm off} - 1|$.

The other effective magnetic field, which is induced by the Dresselhaus SOI in Eq. (9), is proportional to $\langle 0|\boldsymbol{h}|0\rangle$. In (110) quantum wells with $\boldsymbol{h}$ given by Eq. (10), the effective magnetic field is along the growth ($z$) direction [3,10]:

$$\langle 0|h_x|0\rangle = \langle 0|h_y|0\rangle = 0, \tag{20}$$

even in asymmetric quantum wells, since $\langle 0|\hat{p}_z|0\rangle = \langle 0|\hat{p}_z^3|0\rangle = 0$ in any $V_{\rm W}(z)$ if $|0\rangle$ is a bound state. This leads to the absence of the spin precession for the spin polarization in the $z$ direction. Therefore, the DP mechanism due to the Dresselhaus SOI does not work in (110) quantum wells for this spin direction.

## IV. SPIN RELAXATION BY SPIN-FLIP SCATTERINGS

The spin-relaxation time $\tau_s$ and the spin-relaxation rate $\tau_s^{-1}$, for the spin polarization along the $z$ axis, are obtained from

$$\frac{dS_z}{dt} = -\frac{S_z}{\tau_s}, \tag{21}$$

where $S_z$ is the spin polarization of electrons, which is defined by the sum of the spin angular momentum of each electron in the $z$ direction ($\tau_s$ is the longitudinal spin-relaxation time $T_1$ since it describes the relaxation of $S_z$ after $S_z$ is created). We derive the equation of motion for $S_z$ from that for the density operator in the presence of spin-flip scatterings.

### A. Spin-flip scatterings by impurities

We first review the equation of motion for the density operator in a general system of noninteracting electrons, which we later apply to electron spin-flip scatterings at impurities. We divide the Hamiltonian $H$ into the unperturbed Hamiltonian $H_0$ and the perturbation $H_1$:

$$H = H_0 + H_1, \tag{22}$$

and denote the eigenvalue and the eigenvector of $H_0$ by $\varepsilon_\nu$ and $|\nu\rangle$, respectively:

$$H_0|\nu\rangle = \varepsilon_\nu|\nu\rangle. \tag{23}$$

The perturbation $H_1$ causes the transition between different eigenstates of $H_0$. With use of $H_{\nu\nu'} \equiv \langle\nu|H|\nu'\rangle$, the Hamiltonian in the second quantization $\tilde{H}$ is written as

$$\tilde{H} = \sum_{\nu,\nu'} H_{\nu\nu'} a_\nu^\dagger a_{\nu'}, \tag{24}$$

where $a_\nu$ and $a_\nu^\dagger$ are the annihilation and creation operators, respectively, of an electron in an eigenstate $\nu$. A wave function for such a system of electrons is written as

$$|\Psi_{\rm e}\rangle = \sum_{\boldsymbol{n}} c_{\boldsymbol{n}}(t)|\boldsymbol{n}\rangle, \tag{25}$$

where $\boldsymbol{n}$ is a collection of the occupation numbers $n_\nu(= 0,1)$ for all one-electron states $\nu$ and $|\boldsymbol{n}\rangle$ represents the corresponding many-electron state. With use of the coefficient $c_{\boldsymbol{n}}(t)$, the many-electron density matrix is defined by

$$\tilde{\rho}_{\boldsymbol{n}\boldsymbol{n}'} \equiv \langle c_{\boldsymbol{n}'}^*(t)c_{\boldsymbol{n}}(t)\rangle, \tag{26}$$

where brackets denote the statistical average. With use of $\tilde{\rho}_{\boldsymbol{n}\boldsymbol{n}'}$, the statistically averaged expectation value of an operator $\tilde{A}$ of the form $\tilde{A} = \sum_{\nu,\nu'} A_{\nu\nu'} a_\nu^\dagger a_{\nu'}$ is given by

$$\langle\langle\Psi_{\rm e}|\tilde{A}|\Psi_{\rm e}\rangle\rangle = {\rm Tr}(\tilde{\rho}\tilde{A}), \tag{27}$$

where ${\rm Tr}(\cdots)$ is the trace operation with respect to $|\boldsymbol{n}\rangle$. The temporal evolution of the corresponding density operator $\tilde{\rho}$ is described by

$$i\hbar\frac{d\tilde{\rho}}{dt} = [\tilde{H},\tilde{\rho}]. \tag{28}$$

According to Kohn and Luttinger [47], we introduce a density matrix $\rho_{\nu\nu'}$, which is defined, for one-electron states $\nu$ and $\nu'$, by

$$\rho_{\nu\nu'} \equiv {\rm Tr}(\tilde{\rho}a_{\nu'}^\dagger a_\nu). \tag{29}$$

Such a density matrix can be used to calculate [47] $\langle\langle\Psi_{\rm e}|\tilde{A}|\Psi_{\rm e}\rangle\rangle$ by virtue of

$${\rm Tr}(\tilde{\rho}\tilde{A}) = {\rm tr}(\rho A), \tag{30}$$

where the second trace operation ${\rm tr}(\cdots)$ is with respect to one-electron states $|\nu\rangle$, and $A$ is the one-electron operator corresponding to $\tilde{A}$. When $A = 1$, this formula reduces to the normalization condition ${\rm tr}(\rho) = N$, where $N$ is the number of electrons. The corresponding density operator satisfies the following equation of motion [47]:

$$i\hbar\frac{d\rho}{dt} = [H,\rho]. \tag{31}$$

Here we assume that the electron system is in an incoherent state such that

$$\tilde{\rho}_{\boldsymbol{n}\boldsymbol{n}'} = 0 \ \ (\boldsymbol{n} \neq \boldsymbol{n}') \ \ (t = 0), \tag{32}$$

when the perturbation $H_1$ is turned on. This leads to

$$\rho_{\nu\nu'} = 0 \ \ (\nu \neq \nu') \ \ (t = 0). \tag{33}$$

(When an electron is injected from an external lead to the quantum well and enters a superposition state $c_\nu|\nu\rangle + c_{\nu'}|\nu'\rangle$, we have $\rho_{\nu\nu'} \neq 0$. However, the presence of dephasing processes in the quantum well turns $c_\nu|\nu\rangle + c_{\nu'}|\nu'\rangle$ into $|\nu\rangle$ and $|\nu'\rangle$ and therefore gives $\rho_{\nu\nu'} = 0$.) On the other hand, each of diagonal elements $\rho_{\nu\nu}$ represents the occupation probability of eigenstate $\nu$ and its value at $t = 0$ is denoted by $f_\nu$:

$$\rho_{\nu\nu} = f_\nu \quad (t = 0). \tag{34}$$

We assume that the perturbation $H_1$ is independent of $t$ after it is turned on. Then we obtain the time evolution of the occupation probability from Eq. (31) in the lowest order of $H_1$ as

$$\frac{d\rho_{\nu\nu}}{dt} = \sum_{\nu'(\neq\nu)} (W_{\nu\nu'} f_{\nu'} - W_{\nu'\nu} f_\nu), \tag{35}$$

with the transition rate from $\nu$ to $\nu'$

$$W_{\nu'\nu} = \frac{2\pi}{\hbar} |(H_1)_{\nu'\nu}|^2 \delta(\varepsilon_{\nu'} - \varepsilon_\nu). \tag{36}$$

Equation (35) shows that the population change of noninteracting electrons does not include the factor $1 - f_\nu$ expressing the Pauli exclusion principle, which has already been proved by Kohn and Luttinger [47].

Now we calculate the time derivative of $S_z$ in the lowest order both in the SOI and in the impurity potential by choosing the following $H_0$ and $H_1$:

$$H_0 = H_{\mathrm{QW}} + V_{\mathrm{W}}^{\mathrm{so}}, \quad H_1 = V_{\mathrm{imp}} + V_{\mathrm{imp}}^{\mathrm{so}}. \tag{37}$$

The Rashba SOI $V_{\mathrm{W}}^{\mathrm{so}}$ appearing as the second term of $H_0$ conserves the in-plane momentum $\hbar\boldsymbol{k}$ but mixes states with different subband indices and different spins. Then each eigenstate of $H_0$ in a linear combination of eigenstates of $H_{\mathrm{QW}}$, $|n\boldsymbol{k}\sigma\rangle$, becomes up to the first order of $V_{\mathrm{W}}^{\mathrm{so}}$

$$|\nu\_n\boldsymbol{k}\sigma\rangle = |n\boldsymbol{k}\sigma\rangle + \sum_{n'(\neq n)} |n'\boldsymbol{k}\bar{\sigma}\rangle \frac{\langle n'\boldsymbol{k}\bar{\sigma}|V_{\mathrm{W}}^{\mathrm{so}}|n\boldsymbol{k}\sigma\rangle}{\varepsilon_{n\boldsymbol{k}\sigma} - \varepsilon_{n'\boldsymbol{k}\bar{\sigma}}}, \tag{38}$$

which is denoted by $|\nu\_n\boldsymbol{k}\sigma\rangle$ to indicate the corresponding unperturbed state. In the above expansion we excluded terms due to intrasubband matrix elements ($n' = n$) of $V_{\mathrm{W}}^{\mathrm{so}}$ since they were already taken into account in the DP spin relaxation in the previous section. We neglected second-order terms of $|\nu\_n\boldsymbol{k}\sigma\rangle$ with respect to $V_{\mathrm{W}}^{\mathrm{so}}$, which are products of two spin-flip matrix elements of $V_{\mathrm{W}}^{\mathrm{so}}$, since they only give corrections of higher orders in the SOI to the spin-relaxation rate just as non-spin-flip matrix elements, which are absent for $V_{\mathrm{W}}^{\mathrm{so}}$, would do.

The spin polarization $S_z$ is, using Eq. (30),

$$S_z = \frac{\hbar}{2}\mathrm{tr}(\rho\sigma_z) = \frac{\hbar}{2}\sum_\nu \rho_\nu \sigma_z^\nu, \tag{39}$$

with an abbreviation

$$\rho_\nu \equiv \rho_{\nu\nu}, \quad \sigma_z^\nu \equiv (\sigma_z)_{\nu\nu}. \tag{40}$$

Here we have used $\rho_{\nu\nu'} = 0$ $(\nu \neq \nu')$ by assuming that the system is in an incoherent state with $\tilde{\rho}_{\boldsymbol{n}\boldsymbol{n}'} = 0$ $(\boldsymbol{n} \neq \boldsymbol{n}')$ at the time when the spin polarization is measured, just as at $t = 0$ [Eq. (32)]. In this paper we consider the case where electrons occupy only the ground subband: $\rho_{\nu\_n\boldsymbol{k}\sigma} = 0$ for $n \geqslant 1$. Then we have for its time derivative

$$\frac{dS_z}{dt} = \frac{\hbar}{2}\sum_{\boldsymbol{k}\sigma} \frac{d\rho_{\nu\_0\boldsymbol{k}\sigma}}{dt}\sigma_z^{\nu\_0\boldsymbol{k}\sigma}. \tag{41}$$

Here

$$\sigma_z^{\nu\_0\boldsymbol{k}\sigma} = \sigma(1 - w_{0\boldsymbol{k}\sigma}), \tag{42}$$

with

$$w_{0\boldsymbol{k}\sigma} = \sum_{n'(\neq 0)} \left| \frac{\langle n'\boldsymbol{k}\bar{\sigma}|V_{\mathrm{W}}^{\mathrm{so}}|0\boldsymbol{k}\sigma\rangle}{\varepsilon_{0\boldsymbol{k}\sigma} - \varepsilon_{n'\boldsymbol{k}\bar{\sigma}}} \right|^2. \tag{43}$$

When we expand $dS_z/dt = (\hbar/2)\mathrm{tr}[(d\rho/dt)\sigma_z]$ in a perturbation series with respect to $V_{\mathrm{W}}^{\mathrm{so}}$ and $H_1$, each term includes an even number of spin flips. Since terms with no spin flips do not contribute to $dS_z/dt$, the lowest-order terms giving the spin relaxation are of the second order in the SOI. In this order of the SOI, the contribution from $-\sigma w_{0\boldsymbol{k}\sigma}$ in Eq. (42) to $dS_z/dt$ is absent, as is shown in Appendix B, and we have

$$\frac{dS_z}{dt} = \sum_{\boldsymbol{k}\sigma} \frac{d\rho_{\nu\_0\boldsymbol{k}\sigma}}{dt}\frac{\hbar}{2}\sigma. \tag{44}$$

With use of Eq. (35) we obtain

$$\frac{dS_z}{dt} = \sum_{\boldsymbol{k}\boldsymbol{k}'\sigma} (-\hbar\sigma) W_{\nu\_0\boldsymbol{k}'\bar{\sigma},\nu\_0\boldsymbol{k}\sigma} f_{\nu\_0\boldsymbol{k}\sigma}, \tag{45}$$

which shows that the change of $S_z$ is generated only by transitions with a spin flip, as expected. The transition rate, given by Eq. (36), becomes

$$W_{\nu\_0\boldsymbol{k}'\bar{\sigma},\nu\_0\boldsymbol{k}\sigma} = \frac{2\pi}{\hbar}|(H_1)_{\nu\_0\boldsymbol{k}'\bar{\sigma},\nu\_0\boldsymbol{k}\sigma}|^2 \delta(\varepsilon_{0\boldsymbol{k}'\bar{\sigma}} - \varepsilon_{0\boldsymbol{k}\sigma}), \tag{46}$$

where we substituted $\varepsilon_{\nu\_0\boldsymbol{k}\sigma} = \varepsilon_{0\boldsymbol{k}\sigma}$ since we give the transition rate in the second order of the SOI. In the first order of the impurity potential and of the SOI, the transition matrix element is obtained to be

$$\begin{aligned}(H_1)_{\nu\_0\boldsymbol{k}'\bar{\sigma},\nu\_0\boldsymbol{k}\sigma} = {}& \langle 0\boldsymbol{k}'\bar{\sigma}|V_{\mathrm{imp}}^{\mathrm{so}}|0\boldsymbol{k}\sigma\rangle \\ &+ \sum_{n'(\neq 0)} \frac{\langle 0\boldsymbol{k}'\bar{\sigma}|V_{\mathrm{imp}}|n'\boldsymbol{k}\bar{\sigma}\rangle\langle n'\boldsymbol{k}\bar{\sigma}|V_{\mathrm{W}}^{\mathrm{so}}|0\boldsymbol{k}\sigma\rangle}{\varepsilon_{0\boldsymbol{k}\sigma} - \varepsilon_{n'\boldsymbol{k}\bar{\sigma}}} \\ &+ \sum_{n'(\neq 0)} \frac{\langle 0\boldsymbol{k}'\bar{\sigma}|V_{\mathrm{W}}^{\mathrm{so}}|n'\boldsymbol{k}'\sigma\rangle\langle n'\boldsymbol{k}'\sigma|V_{\mathrm{imp}}|0\boldsymbol{k}\sigma\rangle}{\varepsilon_{0\boldsymbol{k}'\bar{\sigma}} - \varepsilon_{n'\boldsymbol{k}'\sigma}}.\end{aligned} \tag{47}$$

In the previous paper [14] we have made a perturbation calculation with unperturbed Hamiltonian $H_{\mathrm{QW}}$ and perturbation $V_{\mathrm{W}}^{\mathrm{so}} + V_{\mathrm{imp}} + V_{\mathrm{imp}}^{\mathrm{so}}$ and obtained the same transition matrix element as Eq. (47). The reason for this coincidence is that in both calculations the spin-flip scattering takes place in the same time domain of $t > 0$ with scattering potential turned on, and that both calculations take terms of the same order of the SOI and the scattering potential.

Substituting the time derivative of $S_z$, given by Eq. (45), and

$$S_z = \sum_{\boldsymbol{k}\sigma} \frac{\hbar}{2}\sigma f_{0\boldsymbol{k}\sigma} \tag{48}$$

into Eq. (21), we obtain the spin-relaxation rate in the lowest order of the SOI and the impurity potential. The spin-relaxation rate thus obtained, in general, depends on the electron distribution $f_{0\boldsymbol{k}\sigma}$. If we employ an equilibrium distribution with the temperature $T$ and the spin-dependent chemical potential $\mu_\sigma$

satisfying $k_BT \ll \varepsilon_F - \varepsilon_0$ and $|\mu_\uparrow - \mu_\downarrow| \ll \varepsilon_F - \varepsilon_0$ ($\varepsilon_F$: the Fermi energy), the spin-relaxation rate becomes independent of $T$ and $\mu_\uparrow - \mu_\downarrow$ [14].

### B. Spin-flip scatterings by phonons

To obtain the rate of spin relaxation due to phonon scatterings, we employ the equation of motion for the density operator given by Argyres [48], which is derived in the lowest order of the electron-phonon interaction and for the equilibrium phonon distribution at $t = 0$. We again assume that the initial state of electrons is incoherent [Eq. (32)].

We consider a system of electrons and phonons with a Hamiltonian in the second quantization written as

$$H_{\rm T} = \tilde{H}_0 + \tilde{H}_{\rm ep} + H_{\rm p}. \tag{49}$$

Here $\tilde{H}_0$ represents a Hamiltonian of noninteracting electrons, given by

$$\tilde{H}_0 = \sum_\nu \varepsilon_\nu a_\nu^\dagger a_\nu. \tag{50}$$

The second term in Eq. (49), $\tilde{H}_{\rm ep}$, is the second quantized form of $H_{\rm ep}$ in Eq. (13) written as

$$\tilde{H}_{\rm ep} = \sum_{\boldsymbol{q}\lambda}\left[\sum_{\nu,\nu'}(\check{v}_{\boldsymbol{q}\lambda})_{\nu\nu'}a_\nu^\dagger a_{\nu'}\right](b_{\boldsymbol{q}\lambda} + b_{-\boldsymbol{q}\lambda}^\dagger). \tag{51}$$

The explicit form of $\check{v}_{\boldsymbol{q}\lambda}$ is not used in the derivation for a while. The third term $H_{\rm p}$ is given by

$$H_{\rm p} = \sum_{\boldsymbol{q}\lambda}\hbar\omega_{\boldsymbol{q}\lambda}\left(b_{\boldsymbol{q}\lambda}^\dagger b_{\boldsymbol{q}\lambda} + \frac{1}{2}\right), \tag{52}$$

where $\omega_{\boldsymbol{q}\lambda}$ is the angular frequency which satisfies $\omega_{\boldsymbol{q}\lambda} = \omega_{-\boldsymbol{q}\lambda}$.

A wave function describing such a system of electrons and phonons is written as

$$|\Psi_{\rm ep}\rangle = \sum_{\boldsymbol{n},\boldsymbol{m}} a_{\boldsymbol{nm}}(t)|\boldsymbol{nm}\rangle, \tag{53}$$

where $\boldsymbol{n}$ ($\boldsymbol{m}$) is a collection of the occupation numbers $n_\nu$ ($m_\mu$) for all one-electron states $\nu$ [all phonon modes $\mu = (\boldsymbol{q},\lambda)$]. The corresponding density matrix is defined by

$$R_{\boldsymbol{nn}'}^{\boldsymbol{mm}'} \equiv \langle a_{\boldsymbol{n}'\boldsymbol{m}'}^*(t)a_{\boldsymbol{nm}}(t)\rangle. \tag{54}$$

We introduce the reduced density matrix for electrons $\tilde{\rho}_{\boldsymbol{nn}'}$, defined by

$$\tilde{\rho}_{\boldsymbol{nn}'} \equiv \sum_{\boldsymbol{m}} R_{\boldsymbol{nn}'}^{\boldsymbol{mm}}, \tag{55}$$

and employ the same definition of the density matrix $\rho_{\nu\nu'}$ given in Eq. (29). Then we can express the statistically averaged expectation value of an electron operator $\tilde{A}$ of the form $\tilde{A} = \sum_{\nu,\nu'} A_{\nu\nu'}a_\nu^\dagger a_{\nu'}$, with use of $\tilde{\rho}_{\boldsymbol{nn}'}$ and $\rho_{\nu\nu'}$,

$$\langle\langle\Psi_{\rm ep}|\tilde{A}|\Psi_{\rm ep}\rangle\rangle = {\rm Tr}(\tilde{\rho}\tilde{A}) = {\rm tr}(\rho A). \tag{56}$$

The equation of motion for the density operator $\rho(t)$ is derived, according to the Argyres theory [48], to be [49]

$$\frac{d\rho(t)}{dt} = \frac{1}{i\hbar}[H_0,\rho(t)] + C(t). \tag{57}$$

The collision operator $C(t)$ is given by [50]

$$C(t) = \frac{1}{\hbar^2}\int_0^t d\tau \sum_{\boldsymbol{q}\lambda}\sum_{\alpha=\pm1}\left(N_{\boldsymbol{q}\lambda} + \frac{1+\alpha}{2}\right)U_{\boldsymbol{q}\lambda}^\alpha[B_{\boldsymbol{q}\lambda},\check{v}_{\boldsymbol{q}\lambda}] + {\rm H.c.}, \tag{58}$$

where H.c. denotes the Hermitian conjugate $\alpha = +1$ $(-1)$ for the electron scattering with emission (absorption) of a phonon,

$$N_{\boldsymbol{q}\lambda} = [\exp(\hbar\omega_{\boldsymbol{q}\lambda}/k_BT) - 1]^{-1} \tag{59}$$

is the average number of phonons in equilibrium, and

$$U_{\boldsymbol{q}\lambda}^\alpha = \exp[-i\alpha\omega_{\boldsymbol{q}\lambda}(t-\tau)]. \tag{60}$$

The operator $B_{\boldsymbol{q}\lambda}$ is given by

$$B_{\boldsymbol{q}\lambda} = U_0[1-\rho(0)]\check{v}_{\boldsymbol{q}\lambda}^\dagger\rho(0)U_0^{-1}, \tag{61}$$

with

$$U_0 = \exp[-iH_0(t-\tau)/\hbar]. \tag{62}$$

The time derivative of the diagonal element $\rho_{\nu\nu}$ is derived from Eq. (57) in the lowest order of the electron-phonon interaction as

$$\frac{d\rho_{\nu\nu}}{dt} = \sum_{\nu'(\neq\nu)}\left[W_{\nu\nu'}^{\rm ep}(1-f_\nu)f_{\nu'} - W_{\nu'\nu}^{\rm ep}(1-f_{\nu'})f_\nu\right], \tag{63}$$

where the factor $1 - f_\nu$ expresses the Pauli exclusion principle and the transition rate is given by

$$W_{\nu'\nu}^{\rm ep} = \sum_{\boldsymbol{q}\lambda}\sum_{\alpha=\pm1}W_{\nu'\nu}^{\boldsymbol{q}\lambda\alpha}, \tag{64}$$

with

$$W_{\nu'\nu}^{\boldsymbol{q}\lambda\alpha} = \frac{2\pi}{\hbar}|(\check{v}_{\boldsymbol{q}\lambda}^\dagger)_{\nu'\nu}|^2\left(N_{\boldsymbol{q}\lambda} + \frac{1+\alpha}{2}\right)\delta(\varepsilon_{\nu'} + \alpha\hbar\omega_{\boldsymbol{q}\lambda} - \varepsilon_\nu). \tag{65}$$

Now we consider electrons in a quantum well with $H_0$ in Eq. (37), and choose $\check{v}_{\boldsymbol{q}\lambda}$ in Eq. (14) for the electron-phonon interaction. We again employ the formula for the time derivative of $S_z$ in Eq. (44) and substitute Eq. (63) into $d\rho_{\nu\_0\boldsymbol{k}\sigma}/dt$ in Eq. (44). Then we obtain

$$\frac{dS_z}{dt} = \sum_{\boldsymbol{kk}'\sigma}(-\hbar\sigma)W_{\nu\_0\boldsymbol{k}'\bar{\sigma},\nu\_0\boldsymbol{k}\sigma}^{\rm ep}(1-f_{\nu\_0\boldsymbol{k}'\bar{\sigma}})f_{\nu\_0\boldsymbol{k}\sigma}. \tag{66}$$

In the first order of the electron-phonon interaction and of the SOI, the matrix element of $\check{v}_{\boldsymbol{q}\lambda}^\dagger (=\check{v}_{-\boldsymbol{q}\lambda})$ in Eq. (65) becomes

$$\begin{aligned}(\check{v}_{\boldsymbol{q}\lambda}^\dagger)_{\nu\_0\boldsymbol{k}'\bar{\sigma},\nu\_0\boldsymbol{k}\sigma} = {}&\langle 0\boldsymbol{k}'\bar{\sigma}|v_{\boldsymbol{q}\lambda}^{\rm SO\,\dagger}|0\boldsymbol{k}\sigma\rangle\\ &+\sum_{n'(\neq0)}\frac{\langle 0\boldsymbol{k}'\bar{\sigma}|v_{\boldsymbol{q}\lambda}^*|n'\boldsymbol{k}\bar{\sigma}\rangle\langle n'\boldsymbol{k}\bar{\sigma}|V_{\rm W}^{\rm so}|0\boldsymbol{k}\sigma\rangle}{\varepsilon_{0\boldsymbol{k}\sigma} - \varepsilon_{n'\boldsymbol{k}\bar{\sigma}}}\\ &+\sum_{n'(\neq0)}\frac{\langle 0\boldsymbol{k}'\bar{\sigma}|V_{\rm W}^{\rm so}|n'\boldsymbol{k}'\sigma\rangle\langle n'\boldsymbol{k}'\sigma|v_{\boldsymbol{q}\lambda}^*|0\boldsymbol{k}\sigma\rangle}{\varepsilon_{0\boldsymbol{k}'\bar{\sigma}} - \varepsilon_{n'\boldsymbol{k}'\sigma}}.\end{aligned} \tag{67}$$

Here we have considered transitions with initial and final states in the ground subband and neglected those from a state in the ground subband to a state in excited subbands because the energy separation between the ground and the first excited subbands is 300 meV for a quantum well with width of 7.5 nm (as the quantum well used in the spin-relaxation experiment [16]), while the maximum phonon energy is only about 30 meV.

### C. $b_{\text{off}}$ dependence of spin-flip scattering rate

Here we show that

$$\begin{aligned}T_{\boldsymbol{k}'\bar{\sigma}\boldsymbol{k}\sigma} \equiv{}& \langle 0\boldsymbol{k}'\bar{\sigma}|v_{\text{so}}|0\boldsymbol{k}\sigma\rangle \\ &+\sum_{n(\neq 0)}\frac{\langle 0\boldsymbol{k}'\bar{\sigma}|v|n\boldsymbol{k}\bar{\sigma}\rangle\langle n\boldsymbol{k}\bar{\sigma}|V_{\text{W}}^{\text{so}}|0\boldsymbol{k}\sigma\rangle}{\varepsilon_0-\varepsilon_n} \\ &+\sum_{n(\neq 0)}\frac{\langle 0\boldsymbol{k}'\bar{\sigma}|V_{\text{W}}^{\text{so}}|n\boldsymbol{k}'\sigma\rangle\langle n\boldsymbol{k}'\sigma|v|0\boldsymbol{k}\sigma\rangle}{\varepsilon_0-\varepsilon_n} \\ \propto{}& b_{\text{off}}-1, \end{aligned} \tag{68}$$

where $v$ is an arbitrary function of $\boldsymbol{r}$, which is in general complex, and

$$v_{\text{so}} = -\frac{\eta}{\hbar}\boldsymbol{\sigma}\cdot(\nabla v\times\hat{\boldsymbol{p}}). \tag{69}$$

$T_{\boldsymbol{k}'\bar{\sigma}\boldsymbol{k}\sigma}$ with a substitution $v(\boldsymbol{r}) = V_{\text{imp}}(\boldsymbol{r})$ becomes $(H_1)_{\nu\_0\boldsymbol{k}'\bar{\sigma},\nu\_0\boldsymbol{k}\sigma}$ in Eq. (47), while $T_{\boldsymbol{k}'\bar{\sigma}\boldsymbol{k}\sigma}$ with $v(\boldsymbol{r}) = v_{\boldsymbol{q}\lambda}^{*}(\boldsymbol{r})$ is $(\check{v}_{\boldsymbol{q}\lambda}^{\dagger})_{\nu\_0\boldsymbol{k}'\bar{\sigma},\nu\_0\boldsymbol{k}\sigma}$ in Eq. (67). Since these matrix elements are found to be proportional to $b_{\text{off}}-1$, *the spin-flip scattering rate for both impurities and phonons vanishes when* $b_{\text{off}}=1$ if we neglect terms of the order higher than $\eta v(\boldsymbol{r})$ and those due to the Dresselhaus SOI. The same proportionality relation Eq. (68) was derived in our previous paper [14] for impurity scatterings in the case where the impurity potential is the sum of central-force potentials due to each impurity and the electrostatic potential is absent, $V_{\text{es}}(z)=0$. According to the proof presented below, Eq. (68) is true for any spatial dependence of the scattering potential $v(\boldsymbol{r})$ in the presence of $V_{\text{es}}(z)$.

To prove Eq. (68), we divide the matrix element of $v(\boldsymbol{r})$ between $|\boldsymbol{k}'\rangle$ and $|\boldsymbol{k}\rangle$, denoted by $v_{\boldsymbol{q}}(z)$, into the real part $v_{\boldsymbol{q}}^{R}(z)$ and the imaginary part $v_{\boldsymbol{q}}^{I}(z)$ as

$$\langle\boldsymbol{k}'|v|\boldsymbol{k}\rangle \equiv v_{\boldsymbol{q}}(z) = v_{\boldsymbol{q}}^{R}(z)+i\,v_{\boldsymbol{q}}^{I}(z), \tag{70}$$

with $\boldsymbol{q}=\boldsymbol{k}'-\boldsymbol{k}$. Since we have $\langle 0\boldsymbol{k}'\bar{\sigma}|v_{\text{so}}|0\boldsymbol{k}\sigma\rangle = (\eta/2)K_{1\sigma}\langle 0|(\nabla_z v_{\boldsymbol{q}})|0\rangle$, $\langle 0\boldsymbol{k}'\bar{\sigma}|v|n\boldsymbol{k}\bar{\sigma}\rangle = \langle 0|v_{\boldsymbol{q}}|n\rangle$, $\langle n\boldsymbol{k}\bar{\sigma}|V_{\text{W}}^{\text{so}}|0\boldsymbol{k}\sigma\rangle = \eta(k_y-is_\sigma k_x)\langle n|[\nabla_z(b_{\text{off}}V_{\text{bo}}^{\text{c}}+V_{\text{es}})]|0\rangle$, where $K_{1\sigma}=(k_y+k_y')-is_\sigma(k_x+k_x')$ with $s_\sigma=1\,(\sigma=\uparrow)$ and $s_\sigma=-1\,(\sigma=\downarrow)$, $T_{\boldsymbol{k}'\bar{\sigma}\boldsymbol{k}\sigma}$ in Eq. (68) becomes

$$T_{\boldsymbol{k}'\bar{\sigma}\boldsymbol{k}\sigma} = \frac{\eta}{2}K_{1\sigma}\left\{F\left[v_{\boldsymbol{q}}^{R}(z)\right]+i\,F\left[v_{\boldsymbol{q}}^{I}(z)\right]\right\}, \tag{71}$$

where the functional $F[\tilde{v}(z)]$ of a real function $\tilde{v}(z)$ is defined by

$$\begin{aligned}F[\tilde{v}(z)] ={}& \langle 0|(\nabla_z\tilde{v})|0\rangle \\ &+2\sum_{n\geqslant 1}\frac{\langle 0|\left[\nabla_z\left(b_{\text{off}}V_{\text{bo}}^{\text{c}}+V_{\text{es}}\right)\right]|n\rangle\langle n|\tilde{v}|0\rangle}{\varepsilon_0-\varepsilon_n}.\end{aligned} \tag{72}$$

Here we introduce a wave function $|\psi_0^{v}\rangle$ for the ground state of the following Schrödinger equation with an additional potential $\tilde{v}(z)$,

$$\left[\frac{\hat{p}_z^2}{2m}+V_{\text{bo}}^{\text{c}}(z)+V_{\text{es}}(z)+\tilde{v}(z)\right]\left|\psi_0^{v}\right\rangle = \varepsilon_0^{v}\left|\psi_0^{v}\right\rangle, \tag{73}$$

where $\varepsilon_0^{v}$ is the corresponding eigenvalue. The potential $\tilde{v}(z)$ $[= v_{\boldsymbol{q}}^{R}(z), v_{\boldsymbol{q}}^{I}(z)]$ can be thought of as an effective potential which the electron, in its motion along $z$, feels at a scattering. Since $|\psi_0^{v}\rangle$ is given, up to the first order of $\tilde{v}(z)$, by

$$\left|\psi_0^{v}\right\rangle = |0\rangle+\sum_{n\geqslant 1}|n\rangle\frac{\langle n|\tilde{v}|0\rangle}{\varepsilon_0-\varepsilon_n}, \tag{74}$$

we have

$$\begin{aligned}F[\tilde{v}(z)] &= \left\langle\psi_0^{v}\right|\left[\nabla_z\left(b_{\text{off}}V_{\text{bo}}^{\text{c}}+V_{\text{es}}+\tilde{v}\right)\right]\left|\psi_0^{v}\right\rangle \\ &\quad\text{in the first order of } \tilde{v}(z).\end{aligned} \tag{75}$$

The same argument, which leads to Eq. (18), shows that $\langle\psi_0^{v}|[\nabla_z(V_{\text{bo}}^{\text{c}}+V_{\text{es}}+\tilde{v})]|\psi_0^{v}\rangle$ vanishes in any order of $\tilde{v}(z)$. This means that all the forces acting on the electron are balanced because $|\psi_0^{v}\rangle$ adjusts itself to the additional potential $\tilde{v}(z)$. Such a force balance gives

$$\begin{aligned}F[\tilde{v}(z)] &= (b_{\text{off}}-1)\left\langle\psi_0^{v}\right|\left(\nabla_z V_{\text{bo}}^{\text{c}}\right)\left|\psi_0^{v}\right\rangle \\ &\quad\text{in the first order of } \tilde{v}(z).\end{aligned} \tag{76}$$

Since $F[\tilde{v}(z)]\propto b_{\text{off}}-1$, we finally obtain Eq. (68) which states that the transition matrix element $T_{\boldsymbol{k}'\bar{\sigma}\boldsymbol{k}\sigma}$ is proportional to $b_{\text{off}}-1$. The disappearance of the transition matrix element for all spin-flip scatterings at $b_{\text{off}}=1$ means that the spin-flip scattering process within the ground subband [the first term of Eq. (68)] and those through one of excited subbands (the second and third terms) *interfere destructively* for every spin-flip scattering at the same value of $b_{\text{off}}=1$. Interestingly, the zeroth-order term of the right-hand side of Eq. (76), which becomes $(b_{\text{off}}-1)\langle 0|(\nabla_z V_{\text{bo}}^{\text{c}})|0\rangle$, coincides with Eq. (19), which determines the effective magnetic field due to the Rashba SOI.

## V. CONCLUSIONS

We have theoretically investigated the spin relaxation in (110) quantum wells for the spin polarization perpendicular to the well with electrons occupying only the ground subband. We have taken into account two major mechanisms of the spin relaxation, the Dyakonov-Perel mechanism as well as the spin-flip scattering (the Elliott-Yafet mechanism) due to impurities and phonons. We have shown that the spin-relaxation rate is proportional to $(b_{\text{off}}-1)^2$ in both mechanisms, with use of the additional factor $b_{\text{off}}$ of the SOI caused by the band offset, if the spin relaxation due to the Dresselhaus-SOI-induced spin-flip scattering is neglected. The factor $b_{\text{off}}$ depends on the band offset of each of the conduction band and split valence bands as well as the band gaps. Since the dependence on the constituent semiconductors of the spin-relaxation rate is largely determined by its proportionality factor $(b_{\text{off}}-1)^2$, the spin-relaxation rate can be orders of magnitude reduced by choosing appropriate constituent semiconductors.

## APPENDIX A

Here we derive $V_{\rm W}^{\rm so}$, given by Eq. (7), with the factor $b_{\rm off}$, defined by Eq. (8), on the basis of the $\boldsymbol{k}\cdot\boldsymbol{p}$ theory developed for heterostructures [51–53]. Our system is a quantum well with width $W$, which consists of two different semiconductors with the zinc-blende structure, $S_{\rm W}$ and $S_{\rm B}$: the semiconductor $S_{\rm W}$ is in the well layer ($|z| < W/2$), while the semiconductor $S_{\rm B}$ is in the barrier layers ($|z| > W/2$).

The SOI induced by the band offsets and the electrostatic potential $V_{\rm es}(z)$ is given, for an electron with momentum $(k_x, k_y)$ in the conduction band, by [51–53]

$$V_{\rm W}^{\rm so} = \frac{P^2}{3} G(z)(\sigma_x k_y - \sigma_y k_x), \tag{A1}$$

with use of the Kane matrix element [54] $P$ given by

$$P = -i\frac{\hbar}{m_0}\langle S|\hat{p}_x|X\rangle, \tag{A2}$$

where $m_0$ is the electron rest mass, while $|S\rangle$ and $|X\rangle$ are the $s$-type wave function at the conduction-band bottom and the $p$-type wave function at the valence-band top, respectively. In the above equation for $V_{\rm W}^{\rm so}$,

$$G(z) = \nabla_z\left(\frac{1}{E-\tilde{E}_{\rm v}} - \frac{1}{E-\tilde{E}_{\rm s}}\right), \tag{A3}$$

where $E$ is the energy of the electron, while $\tilde{E}_{\rm v}$ ($\tilde{E}_{\rm s}$) is the diagonal element of the $8\times 8$ Kane Hamiltonian, corresponding to the heavy-hole plus light-hole bands (the split-off band). They are given by

$$\begin{aligned} \tilde{E}_{\rm v} &= E_{\rm K0} - E_{\rm g} + V_{\rm bo}^{\rm v}(z) + V_{\rm es}(z), \\ \tilde{E}_{\rm s} &= E_{\rm K0} - E_{\rm g}^{\rm s} + V_{\rm bo}^{\rm s}(z) + V_{\rm es}(z), \end{aligned} \tag{A4}$$

where

$$E_{\rm K0} = \frac{\hbar^2}{2m_0}\left(k_x^2 + k_y^2 + \hat{k}_z^2\right), \tag{A5}$$

with $\hat{k}_z = -i\nabla_z$, $E_{\rm g}$ is the band gap of $S_{\rm W}$, $E_{\rm g}^{\rm s} = E_{\rm g} + \Delta_{\rm so}$ with $\Delta_{\rm so}$ the spin-orbit splitting of $S_{\rm W}$, and $V_{\rm bo}^{\rm v}(z)$ ($V_{\rm bo}^{\rm s}(z)$) is the potential due to the band offset for an electron in the heavy-hole plus light-hole bands (the split-off band). These potentials together with that for the conduction band, $V_{\rm bo}^{\rm c}(z)$ in Eq. (3), are expressed by

$$V_{\rm bo}^{\rm c}(z) = \Delta E_{\rm c} h(z), \quad V_{\rm bo}^{\rm v}(z) = \Delta E_{\rm v} h(z), \quad V_{\rm bo}^{\rm s}(z) = \Delta E_{\rm s} h(z), \tag{A6}$$

where

$$h(z) = \begin{cases} 0 & (|z| < W/2), \\ 1 & (|z| > W/2), \end{cases} \tag{A7}$$

while $\Delta E_{\rm c}(= V_0)$, $\Delta E_{\rm v}$, and $\Delta E_{\rm s}$ are the band offsets introduced in Eq. (8).

Here we neglect $E_{\rm K0}$, $E$, and $V_{\rm es}(z)$ in $G(z)$, compared to $E_{\rm g}$ and $E_{\rm g}^{\rm s}$, while we take into account the contribution of $\nabla_z V_{\rm es}(z)$ to $G(z)$. Then we obtain an approximate expression for $G(z)$,

$$G(z) = G_{\rm v} + G_{\rm s} + G_{\rm es}, \tag{A8}$$

with

$$\begin{aligned} G_{\rm v} &= \nabla_z\left(E_{\rm g} - V_{\rm bo}^{\rm v}\right)^{-1}, \\ G_{\rm s} &= -\nabla_z\left(E_{\rm g}^{\rm s} - V_{\rm bo}^{\rm s}\right)^{-1}, \\ G_{\rm es} &= \left[\left(E_{\rm g} - V_{\rm bo}^{\rm v}\right)^{-2} - \left(E_{\rm g}^{\rm s} - V_{\rm bo}^{\rm s}\right)^{-2}\right]\nabla_z V_{\rm es}. \end{aligned} \tag{A9}$$

Furthermore, we neglect $V_{\rm bo}^{\rm v}$ and $V_{\rm bo}^{\rm s}$ in $G_{\rm es}$ since they are nonzero only in the barrier layers where the absolute value of the wave function is small. Using $(E_{\rm g} - V_{\rm bo}^{\rm v})^{-1} = [(E_{\rm g} - \Delta E_{\rm v})^{-1} - (E_{\rm g})^{-1}]h(z) + (E_{\rm g})^{-1}$ and a similar equation for $(E_{\rm g}^{\rm s} - V_{\rm bo}^{\rm s})^{-1}$, we finally obtain the expression for $V_{\rm W}^{\rm so}$ in Eq. (7) with

$$\eta = \frac{P^2}{3}\left[\frac{1}{(E_{\rm g})^2} - \frac{1}{\left(E_{\rm g}^{\rm s}\right)^2}\right], \tag{A10}$$

and with the formula for $b_{\rm off}$ in Eq. (8).

In our previous paper [14] the formula for $b_{\rm off}$ was derived with an additional approximation: $E_{\rm g} - \Delta E_{\rm v} \approx E_{\rm g}$ and $E_{\rm g}^{\rm s} - \Delta E_{\rm s} \approx E_{\rm g}^{\rm s}$. This approximation is found to be not accurate, in particular, in quantum wells with large positive values of $\Delta E_{\rm v}$ and $\Delta E_{\rm s}$. In fact, for a type-II quantum well $\mathrm{Ga_{0.47}In_{0.53}As}$(well)/$\mathrm{GaAs_{0.5}Sb_{0.5}}$(barrier), the additional approximation gives a value of $b_{\rm off} = 1.2$, while the present formula Eq. (8) gives a larger value of $b_{\rm off} = 3.2$.

## APPENDIX B

We here show that the contribution from $-\sigma w_{0\boldsymbol{k}\sigma}$ in Eq. (42) to $dS_z/dt$ is absent in the second order of the SOI. First we note that $w_{0\boldsymbol{k}\sigma} = Ak^2$, where $A$ is a constant independent of $\boldsymbol{k}$ and $\sigma$, since

$$\langle n'\boldsymbol{k}\bar{\sigma}|V_{\rm W}^{\rm so}|0\boldsymbol{k}\sigma\rangle = \eta(k_y - is_\sigma k_x)\langle n'|\left[\nabla_z\left(b_{\rm off}V_{\rm bo}^{\rm c} + V_{\rm es}\right)\right]|0\rangle, \tag{B1}$$

with $s_\sigma = 1$ ($\sigma = \uparrow$) and $s_\sigma = -1$ ($\sigma = \downarrow$). Because $w_{0\boldsymbol{k}\sigma}$ is of the second order of the SOI, we evaluate $d\rho_{\nu\_0\boldsymbol{k}\sigma}/dt$ in front of $w_{0\boldsymbol{k}\sigma}$ in the zeroth order. Then we obtain

$$\sum_{\boldsymbol{k}} \frac{d\rho_{\nu\_0\boldsymbol{k}\sigma}}{dt}(-\sigma w_{0\boldsymbol{k}\sigma}) = (-\sigma)A\frac{2m}{\hbar^2}\frac{dE_\sigma}{dt}, \tag{B2}$$

where $E_\sigma = \sum_{\boldsymbol{k}} \rho_{0\boldsymbol{k}\sigma}\hbar^2k^2/2m$ is the sum (with respect to spin-$\sigma$ electrons which occupy the state in the probability $\rho_{0\boldsymbol{k}\sigma}$) of the energy relative to $\varepsilon_0$. $E_\sigma$ does not change at any elastic impurity scattering. In the presence of phonon scatterings we also have $dE_\sigma/dt = 0$ when each of electron and phonon systems is in an equilibrium state with the common temperature.

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