

# Tensor network trial states for chiral topological phases in two dimensions and a no-go theorem in any dimension

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(Received 5 August 2013; revised manuscript received 28 September 2015; published 19 November 2015)

Trial wave functions that can be represented by summing over locally coupled degrees of freedom are called tensor network states (TNSs); they have seemed difficult to construct for two-dimensional topological phases that possess protected gapless edge excitations. We show it can be done for chiral states of free fermions, using a Gaussian Grassmann integral, yielding  $p_x \pm ip_y$  and Chern insulator states, in the sense that the fermionic excitations live in a topologically nontrivial bundle of the required type. We prove that any strictly short-range quadratic parent Hamiltonian for these states is gapless; the proof holds for a class of systems in any dimension of space. The proof also shows, quite generally, that sets of compactly supported Wannier-type functions do not exist for band structures in this class. We construct further examples of TNSs that are analogs of fractional (including non-Abelian) quantum Hall phases; it is not known whether parent Hamiltonians for these are also gapless.

DOI: [10.1103/PhysRevB.92.205307](https://doi.org/10.1103/PhysRevB.92.205307)

PACS number(s): 73.43.Cd, 02.40.-k, 71.10.Pm

## I. INTRODUCTION

Our theoretical understanding of quantum phases of matter frequently relies on the use of trial ground-state wave functions whose properties serve as the paradigm for an entire phase, such as the Bardeen-Cooper-Schrieffer (BCS) wave function in the theory of superconductivity [1], and the Laughlin wave function in the fractional quantum Hall effect [2]; both examples have generalizations that describe distinct phases. In recent work, there has been progress in understanding the structure of ground states of generic short-range lattice Hamiltonians, especially for “topological” phases. A Hamiltonian can be defined as being in a topological phase if it possesses a gap in the bulk energy spectrum above the ground-state energy. For many ground states, a representation as a *tensor network state* (TNS), in which the amplitude of a basis configuration of the local degrees of freedom is expressed as a product of tensors that involve auxiliary degrees of freedom which are then summed over, can be found. In our definition of a TNS, we further impose that the auxiliary degrees of freedom live in finite-dimensional local Hilbert spaces, with only short-range couplings; when they live on the links of a lattice in  $d$  space dimensions and couple only to the physical degrees of freedom on the end of that link, the state is known as a *projected entangled pair state* (PEPS) [3], or in the one-dimensional case as a *matrix product state* (MPS) [4]. TNSs are also sometimes referred to as “tensor product states,” but we avoid this term because it both risks confusion with product states that have no entanglement, and conflicts with mathematical terminology for tensor product spaces.

In one dimension it is known that MPSs provide enough variational freedom to approximate the ground state in any topological phase [5]. In more than one dimension, it is known that many trial states and exact ground states of special lattice Hamiltonians that lie in a topological phase that does not exhibit topologically protected (e.g., chiral) gapless edge excitations can be viewed as TNSs, and it seems natural to expect the approximation results from one dimension to generalize. But, for chiral topological phases in two dimensions, such as quantum Hall states and  $p \pm ip$  BCS paired states, it has seemed difficult to obtain a TNS

in the phase, even as a trial state. Most attempts did not impose locality [6], while Béri and Cooper [7], by truncating a flat-band Hamiltonian, obtained some local tensor networks that approximate expectation values in certain trial states.

In this paper, we exhibit some fairly simple trial TNSs that belong to chiral topological phases in two dimensions, in the sense that the states exhibit the desired topological structure. We begin by constructing explicit examples of translationally invariant Gaussian (i.e., free-) fermionic TNSs, in each of which the TNS is defined locally by a few tensors. The basic examples are TNS versions of a topologically nontrivial  $p \pm ip$  BCS paired state of fermions [8], and of a filled band with Chern number equal to one [9]. In each of these, there is a Hamiltonian with the TNS as ground state and a gapped but dispersionless (“flat-band”) fermion excitation-energy spectrum; these Hamiltonians have power-law-decaying matrix elements in position space [10]. Moreover, a consequence of the ground state being a free-fermion TNS is that there are single-fermion destruction operators that annihilate the TNS, and are strictly short range, in the sense that their support is *compact* (i.e., a bounded region). Using these, there are many ways to construct a “parent” Hamiltonian that has uniformly bounded support for its terms, each of which annihilates the TNS. The constructions used in these examples generalize to other free-fermion phases, including other symmetry classes, and to higher dimensions. (For a one-dimensional lattice, they yield the matrix product ground state of the Kitaev chain [11].) We also calculate the entanglement spectrum of some of our two-dimensional examples, which exhibits some unusual features.

However, for the states in free-fermion topological phases with chiral edge excitations in two dimensions that we construct, we find that the parent Hamiltonian is always *gapless* in the bulk. (In addition, these TNSs always have power law, rather than the desired exponential, correlations for some local operators, as noted in some examples in Ref. [10].) We prove a general no-go theorem, which states that for any free-fermion TNS in a class of topological phases (those in which the vector bundle formed by the filled bands is topologically nontrivial as a complex vector bundle, ignoring additional symmetries; these include the chiral examples above) in *any* space dimension  $d$ , any short-range single-particle parent

Hamiltonian always has a gapless bulk energy spectrum. (Such a TNS also has power-law correlations in the bulk.) A gap can be produced for the given TNS only by using a Hamiltonian with long-range matrix elements. A slight variation of the same argument also proves, for the same class of free-fermion band structures, that a set of compactly supported single-particle states that, when Fourier transformed, span the states in the filled band at every point  $\mathbf{k}$  in the Brillouin zone (“Wannier-type” states), cannot exist unless the filled-band bundle is topologically trivial when viewed as a complex vector bundle, ignoring symmetries.

Back in two dimensions, we go further by utilizing the free-fermion TNSs and (similarly to earlier authors [6,7]) imposing local constraints on a system of several copies of a TNS, producing further TNSs. We argue that such constructions produce the chiral topological phases associated with a variety of Chern-Simons theories [12] or fractional quantum Hall states, including non-Abelian topological phases [13]. We do not at present have parent Hamiltonians for these TNSs, though these should exist based on general principles of TNSs; we expect that they would be gapless.

The paper is organized as follows: In Sec. II, we describe our construction of free-fermion TNSs and their properties. In Sec. III, we state and prove the no-go theorem. In Sec. IV, we briefly describe the constructions for nonfree chiral topological phases in two dimensions. Some appendixes explain some additional points or give examples.

## II. GAUSSIAN TENSOR NETWORK STATES FOR TOPOLOGICAL PHASES

In this section, we describe the basic constructions in the two-dimensional case. We begin with generalities on Gaussian fermionic TNSs, and continue with examples of topologically nontrivial phases. Then, we discuss the annihilation operators that (in real space) have compact support, and the parent Hamiltonians (which are quadratic in fermion operators) for which the TNS is the exact ground state. There is one set of Hamiltonians that each has “flat-band” energy spectrum, but is not short range, while there is another set each of which is short range but gapless. We also show that the entanglement spectrum of these TNSs has a peculiar form.

### A. Gaussian fermionic TNS

The square lattice  $\mathbb{Z}^2 \subset \mathbb{R}^2$  is generated by the two vectors  $\mathbf{i} = (1,0)$  and  $\mathbf{j} = (0,1)$ . Our physical degrees of freedom are fermions, with  $n$  orbitals per site; the creation/annihilation modes obey the canonical anticommutation relations  $\{c_{\mathbf{x},\alpha}^\dagger, c_{\mathbf{x}',\alpha'}\} = \delta_{\alpha,\alpha'} \delta_{\mathbf{x},\mathbf{x}'}$ . The fermion vacuum  $|0\rangle$  is annihilated by all the  $c_{\mathbf{x},\alpha}$ . A class of translation-invariant Gaussian TNSs is constructed as follows: to every site  $\mathbf{x}$ , we associate a set of (real) Grassmann variables  $\xi_{\mathbf{x}}^p$ ,  $p \in \{1, \dots, P\}$ . To every edge  $(\mathbf{x}'\mathbf{x})$  ( $\mathbf{x}' \in \{\mathbf{x} + \mathbf{i}, \mathbf{x} + \mathbf{j}\}$ ), we associate the expression

$$e^{\sum_{p,q} \xi_{\mathbf{x}+\mathbf{i}}^p A_{pq}^h \xi_{\mathbf{x}}^q} \quad \text{or} \quad e^{\sum_{p,q} \xi_{\mathbf{x}+\mathbf{j}}^p A_{pq}^v \xi_{\mathbf{x}}^q} \quad (1)$$

with  $A_{pq}^h, A_{pq}^v \in \mathbb{C}$ . The superscripts  $h, v$  stand for “horizontal” and “vertical”; in what follows, we will leave these superscripts implicit. To every site  $\mathbf{x}$ , we associate a weight

and a generating function of physical particles onsite:

$$e^{\sum_{p,q} \xi_{\mathbf{x}}^p B_{pq} \xi_{\mathbf{x}}^q} e^{\sum_{q,\alpha} \xi_{\mathbf{x}}^q \kappa_q^\alpha c_{\mathbf{x},\alpha}^\dagger}, \quad (2)$$

with  $B_{pq} = -B_{qp} \in \mathbb{C}$  and  $\kappa_q^\alpha \in \mathbb{C}$ . Grassmann variables anticommute with the physical creation/annihilation operators; thus, the two exponentials in (1) and those in (2) all commute, both onsite and at different sites. After taking the product, we integrate out the Grassmann variables to obtain a translation-invariant Gaussian TNS:

$$|\psi\rangle \propto \int [d\xi] \prod_{\text{edges } (\mathbf{x}\mathbf{y})} e^{\xi_{\mathbf{x}}^t \cdot A \cdot \xi_{\mathbf{y}}} \prod_{\text{sites } \mathbf{z}} e^{\xi_{\mathbf{z}}^t \cdot B \cdot \xi_{\mathbf{z}}} e^{\xi_{\mathbf{z}}^t \cdot \kappa \cdot c_{\mathbf{z}}^\dagger} |0\rangle. \quad (3)$$

Here, we have used the compact notation  $\xi_{\mathbf{x}} = (\xi_{\mathbf{x}}^1, \dots, \xi_{\mathbf{x}}^P)$ ,  $c_{\mathbf{x}}^\dagger = (c_{\mathbf{x},1}^\dagger, \dots, c_{\mathbf{x},n}^\dagger)$ , and matrices  $A, B, \kappa$ . In (3), some ordering of the Grassmann variables must be chosen to define the “measure”  $\int [d\xi]$ . By construction, the state  $|\psi\rangle$  is always a free-fermion BCS paired state. It is possible to write the state (3) in a form that is closer to the usual form of TNS [3], with the Grassmann variables living on the edges rather than on the sites; see Appendix A.

### B. Example in $p_x - ip_y$ phase

Our first example has  $n = 1$  orbital per site; we use  $P = 2$  Grassmann variables on each site. The  $A, B, \kappa$  matrices are

$$\xi_{\mathbf{x}+\mathbf{i}}^t \cdot A^h \cdot \xi_{\mathbf{x}} = \begin{pmatrix} \xi_{\mathbf{x}+\mathbf{i}}^1 & \xi_{\mathbf{x}+\mathbf{i}}^2 \end{pmatrix} \begin{pmatrix} -i & \lambda \\ -\lambda & -i \end{pmatrix} \begin{pmatrix} \xi_{\mathbf{x}}^1 \\ \xi_{\mathbf{x}}^2 \end{pmatrix}, \quad (4a)$$

$$\xi_{\mathbf{x}+\mathbf{j}}^t \cdot A^v \cdot \xi_{\mathbf{x}} = \begin{pmatrix} \xi_{\mathbf{x}+\mathbf{j}}^1 & \xi_{\mathbf{x}+\mathbf{j}}^2 \end{pmatrix} \begin{pmatrix} 1 & \lambda \\ -\lambda & -1 \end{pmatrix} \begin{pmatrix} \xi_{\mathbf{x}}^1 \\ \xi_{\mathbf{x}}^2 \end{pmatrix}, \quad (4b)$$

$$\xi_{\mathbf{x}}^t \cdot B \cdot \xi_{\mathbf{x}} = \begin{pmatrix} \xi_{\mathbf{x}}^1 & \xi_{\mathbf{x}}^2 \end{pmatrix} \begin{pmatrix} 0 & -2\lambda \\ 2\lambda & 0 \end{pmatrix} \begin{pmatrix} \xi_{\mathbf{x}}^1 \\ \xi_{\mathbf{x}}^2 \end{pmatrix}, \quad (4c)$$

$$\xi_{\mathbf{x}}^t \cdot \kappa \cdot c_{\mathbf{x}}^\dagger = \begin{pmatrix} \xi_{\mathbf{x}}^1 & \xi_{\mathbf{x}}^2 \end{pmatrix} \begin{pmatrix} \kappa_1 \\ 0 \end{pmatrix} (c_{\mathbf{x}}^\dagger). \quad (4d)$$

$\lambda \in \mathbb{R}$  and  $\kappa_1 \in \mathbb{C}$  are two variational parameters.

With these matrices, (3) gives a state  $|\psi_D\rangle$ ; its behavior is easily analyzed in momentum space. The Fourier modes of the particle creation operator are defined by  $c_{\mathbf{x}}^\dagger = \int \frac{d^2\mathbf{k}}{(2\pi)^2} e^{-i\mathbf{k}\cdot\mathbf{x}} c_{\mathbf{k}}^\dagger$ , where the integral is over the first Brillouin zone  $[-\pi, \pi]^2$ . Similarly, for the Grassmann variables  $\xi_{\mathbf{x}} = \int \frac{d^2\mathbf{k}}{(2\pi)^2} e^{-i\mathbf{k}\cdot\mathbf{x}} \xi_{\mathbf{k}}$ . The terms  $\prod_{\mathbf{z}} e^{\xi_{\mathbf{z}}^t \cdot \kappa \cdot c_{\mathbf{z}}^\dagger}$  and  $e^S \equiv \prod_{(\mathbf{x}\mathbf{y})} e^{\xi_{\mathbf{x}}^t \cdot A \cdot \xi_{\mathbf{y}}} \prod_{\mathbf{z}} e^{\xi_{\mathbf{z}}^t \cdot B \cdot \xi_{\mathbf{z}}}$  in (3) become, respectively,

$$\exp\left(\int \frac{d^2\mathbf{k}}{(2\pi)^2} \xi_{-\mathbf{k}}^t \cdot \kappa \cdot c_{\mathbf{k}}^\dagger\right), \quad (5a)$$

$$\exp\left[\int \frac{d^2\mathbf{k}}{(2\pi)^2} \begin{pmatrix} \xi_{-\mathbf{k}}^1 & \xi_{-\mathbf{k}}^2 \end{pmatrix} S_{\mathbf{k}} \begin{pmatrix} \xi_{\mathbf{k}}^1 \\ \xi_{\mathbf{k}}^2 \end{pmatrix}\right], \quad (5b)$$

where  $S_{\mathbf{k}}$  is the  $2 \times 2$  matrix

$$\begin{pmatrix} \sin k_x + i \sin k_y & -\lambda(2 - \cos k_x - \cos k_y) \\ \lambda(2 - \cos k_x - \cos k_y) & \sin k_x - i \sin k_y \end{pmatrix}. \quad (6)$$

The integral over all the Fourier modes of the Grassmann variables is easily performed. It yields the familiar BCS form

(see, e.g., Ref. [8])

$$|\psi_D\rangle \propto \exp\left(\frac{1}{2} \int \frac{d^2\mathbf{k}}{(2\pi)^2} g_{\mathbf{k}} c_{\mathbf{k}}^\dagger c_{-\mathbf{k}}^\dagger\right) |0\rangle, \quad (7)$$

with a *pairing function*  $g_{\mathbf{k}}$  which is the (1,1) matrix element of the inverse matrix of  $S_{\mathbf{k}}$ , and so is a component of the propagator of the Grassmann variables:

$$\frac{g_{\mathbf{k}}}{\kappa_1^2} = \frac{\int [d\xi] e^{S\xi} \xi_{\mathbf{k}}^1 \xi_{-\mathbf{k}}^1}{\int [d\xi] e^S} = [S_{\mathbf{k}}^{-1}]_{11}. \quad (8)$$

Explicitly,

$$\frac{g_{\mathbf{k}}}{\kappa_1^2} = \frac{\sin k_x - i \sin k_y}{(\sin k_x)^2 + (\sin k_y)^2 + \lambda^2 [2 - \cos k_x - \cos k_y]^2}, \quad (9)$$

and  $g_{-\mathbf{k}} = -g_{\mathbf{k}}$  for all  $\mathbf{k}$ . As  $\mathbf{k} \rightarrow \mathbf{0}$ ,  $g_{\mathbf{k}}$  diverges as  $g_{\mathbf{k}} \sim \kappa_1^2/(k_x + ik_y)$ , and is nondiverging at other  $\mathbf{k}$ . Hence, in position space,  $g(\mathbf{x}) \sim \kappa_1^2/(x + iy)$  as  $|\mathbf{x}| \rightarrow \infty$ , where  $\mathbf{x} \equiv \mathbf{x}_i - \mathbf{x}_j$  represents the separation of the members  $i, j$  of a pair. These properties are sufficient to show the state is in the nontrivial  $p - ip$  phase [8] in symmetry class D; they can also be related to a Chern number in  $\mathbf{k}$  space (see following). One can show that the average density of particles scales as  $\sim |\kappa_1|^4 \ln(1/|\kappa_1|)$  as  $\kappa_1 \rightarrow 0$ .

### C. Annihilation operators and parent Hamiltonians

A state of the Gaussian (or BCS) form in Eq. (7) is annihilated by (unnormalized) “destruction” mode operators  $c_{\mathbf{k}} - g_{\mathbf{k}} c_{-\mathbf{k}}^\dagger$  for all  $\mathbf{k}$ . In our example,  $g_{\mathbf{k}} = v_{\mathbf{k}}/u_{\mathbf{k}}$  is clearly a ratio of two “trigonometric polynomials” (TPs; i.e., polynomials in  $\sin k_x, \cos k_x, \sin k_y, \cos k_y$ )  $u_{\mathbf{k}}$  and  $v_{\mathbf{k}}$ , which we assume have no common TP factor other than a constant times an integer power of  $e^{ik_x}$ , times another of  $e^{iky}$ . Then, we define

$$d_{\mathbf{k}} = u_{\mathbf{k}} c_{\mathbf{k}} - v_{\mathbf{k}} c_{-\mathbf{k}}^\dagger. \quad (10)$$

If we normalize the destruction operators (10) as

$$\widehat{d}_{\mathbf{k}} = \widehat{u}_{\mathbf{k}} c_{\mathbf{k}} - \widehat{v}_{\mathbf{k}} c_{-\mathbf{k}}^\dagger, \quad (11)$$

where  $\widehat{u}_{\mathbf{k}} = u_{\mathbf{k}}/\sqrt{|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2}$ ,  $\widehat{v}_{\mathbf{k}} = v_{\mathbf{k}}/\sqrt{|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2}$ , then  $\widehat{d}_{\mathbf{k}}, \widehat{d}_{\mathbf{k}}^\dagger$  obey canonical anticommutation relations  $\{\widehat{d}_{\mathbf{k}}, \widehat{d}_{\mathbf{k}'}^\dagger\} = (2\pi)^2 \delta(\mathbf{k} - \mathbf{k}')$ . For the “flat-band” Hamiltonian

$$\widehat{H}_D = \int \frac{d^2\mathbf{k}}{(2\pi)^2} \widehat{d}_{\mathbf{k}}^\dagger \widehat{d}_{\mathbf{k}}, \quad (12)$$

the fermion excitations  $\widehat{d}_{\mathbf{k}}^\dagger |\psi_D\rangle$  have nonzero and  $\mathbf{k}$ -independent energy for all  $\mathbf{k}$ . Expanded in  $c_{\mathbf{k}}, c_{\mathbf{k}}^\dagger$ , the coefficients in  $\widehat{H}_D$  are ratios of TPs, and are not real analytic in  $k_x, k_y$  at  $\mathbf{k} = \mathbf{0}$ , but are elsewhere in the Brillouin zone. Hence, in position space,  $\widehat{H}_D$  contains terms that decay as powers of distance [10].

On the other hand, the operators  $d_{\mathbf{k}}$  contain coefficients that are TPs, and so the inverse Fourier transform gives operators  $d_{\mathbf{x}}$  that annihilate the TNS and are truly local: they have compact support that surrounds  $\mathbf{x}$ . The existence of such operators is not an accident. Any TNS has by construction the property that if the sites of the system are bipartitioned into two sets  $A$  and  $B$ , and  $A$  is finite, then the rank of the Schmidt decomposition of the TNS can be bounded by some constant to the power

of the surface area or perimeter of the region  $A$ . For free fermions, the reduced density matrix again has the form of the exponential of a free-fermion Hamiltonian, and so for a free-fermion TNS the number of fermion modes that can appear in the entanglement Hamiltonian is some constant times the surface area, and so much smaller than the volume of region  $A$  in general. It follows that for such regions  $A$ , there must be linear combinations of fermion operators, supported in region  $A$ , that annihilate the reduced density matrix, when acting on it from the left. The operators  $d_{\mathbf{x}}$  with support in region  $A$  are a basis set for these operators in our case. We note that these operators anticommute with one another, but do not in general anticommute with the operators  $d_{\mathbf{x}'}$  at  $\mathbf{x}' \neq \mathbf{x}$ , except when their supports are disjoint. By contrast, the  $\widehat{d}_{\mathbf{x}}$  operators obey canonical anticommutation relations and are true Wannier functions, but are not supported locally near  $\mathbf{x}$ , instead they have long power-law tails due to nonanalytic behavior in  $\mathbf{k}$  space at  $\mathbf{k} = \mathbf{0}$  [14].

Using the operators  $d_{\mathbf{x}}$ , we can form other Hamiltonians that annihilate the TNS, for example:

$$H_D = \int \frac{d^2\mathbf{k}}{(2\pi)^2} d_{\mathbf{k}}^\dagger d_{\mathbf{k}} = \sum_{\mathbf{x}} d_{\mathbf{x}}^\dagger d_{\mathbf{x}}. \quad (13)$$

This Hamiltonian is a sum of terms that have compact support, and each annihilates  $|\psi_D\rangle$ , so it is a parent Hamiltonian, however, from its  $\mathbf{k}$ -space form we can see that it is gapless at  $\mathbf{k} = \mathbf{0}$ : by expressing it in terms of  $\widehat{d}_{\mathbf{k}}, \widehat{d}_{\mathbf{k}}^\dagger$ , we find that the energy of a fermion excitation is  $|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2$ , which is  $\propto \mathbf{k}^2$  near  $\mathbf{k} = \mathbf{0}$ .

### D. Chern-band example

The preceding construction can be generalized to include more orbitals, more singularities in  $g_{\mathbf{k}}$ , more dimensions, or more symmetry (e.g., time reversal); we describe one more example. We take two copies of the previous example, by using  $n = 2$  orbitals per site; we view particles occupying either type of orbital as two distinct types of particles, that have opposite charges under a U(1) symmetry. In the TNS construction, we use  $P = 4$  Grassmann variables, or  $P = 2$  complex Grassmann variables, with the matrices constructed to respect the U(1) charge mentioned. Then, we can obtain (see Appendix B for more details) a TNS that is a BCS state with pairing only between opposite particle types,

$$|\psi_A\rangle \propto \exp\left(\int \frac{d^2\mathbf{k}}{(2\pi)^2} g_{\mathbf{k}}^{21} c_{\mathbf{k},2}^\dagger c_{-\mathbf{k},1}^\dagger\right) |0\rangle, \quad (14)$$

which conserves the U(1) symmetry; here,  $g_{\mathbf{k}}^{21}$  is the same as  $g_{\mathbf{k}}$  above. If we now perform a particle-hole transformation on the type-1 fermions, and so that  $c_{\mathbf{k},1}^\dagger \rightarrow c_{-\mathbf{k},1}$ , and write  $|0\rangle$  as  $|1,0\rangle$ , which is annihilated by all  $c_{\mathbf{x},1}^\dagger$  and  $c_{\mathbf{x},2}$ , we arrive at a particle-number-conserving state  $|\psi_A\rangle$  that represents a filled band (symmetry class A). There are now two types of operator  $d_{\mathbf{k},\bar{1}}, d_{\mathbf{k},\bar{2}}$  that annihilate  $|\psi_A\rangle$ ; one of these is

$$d_{-\mathbf{k},\bar{1}} = u_{\mathbf{k}} c_{\mathbf{k},1}^\dagger + v_{\mathbf{k}} c_{\mathbf{k},2}, \quad (15)$$

which is a creation operator associated with states in the filled band. The state  $|1,0\rangle$  (i.e., for  $g_{\mathbf{k}} = 0$  for all  $\mathbf{k}$ ) describes topologically trivial bands. For general  $g_{\mathbf{k}}$  (which here does

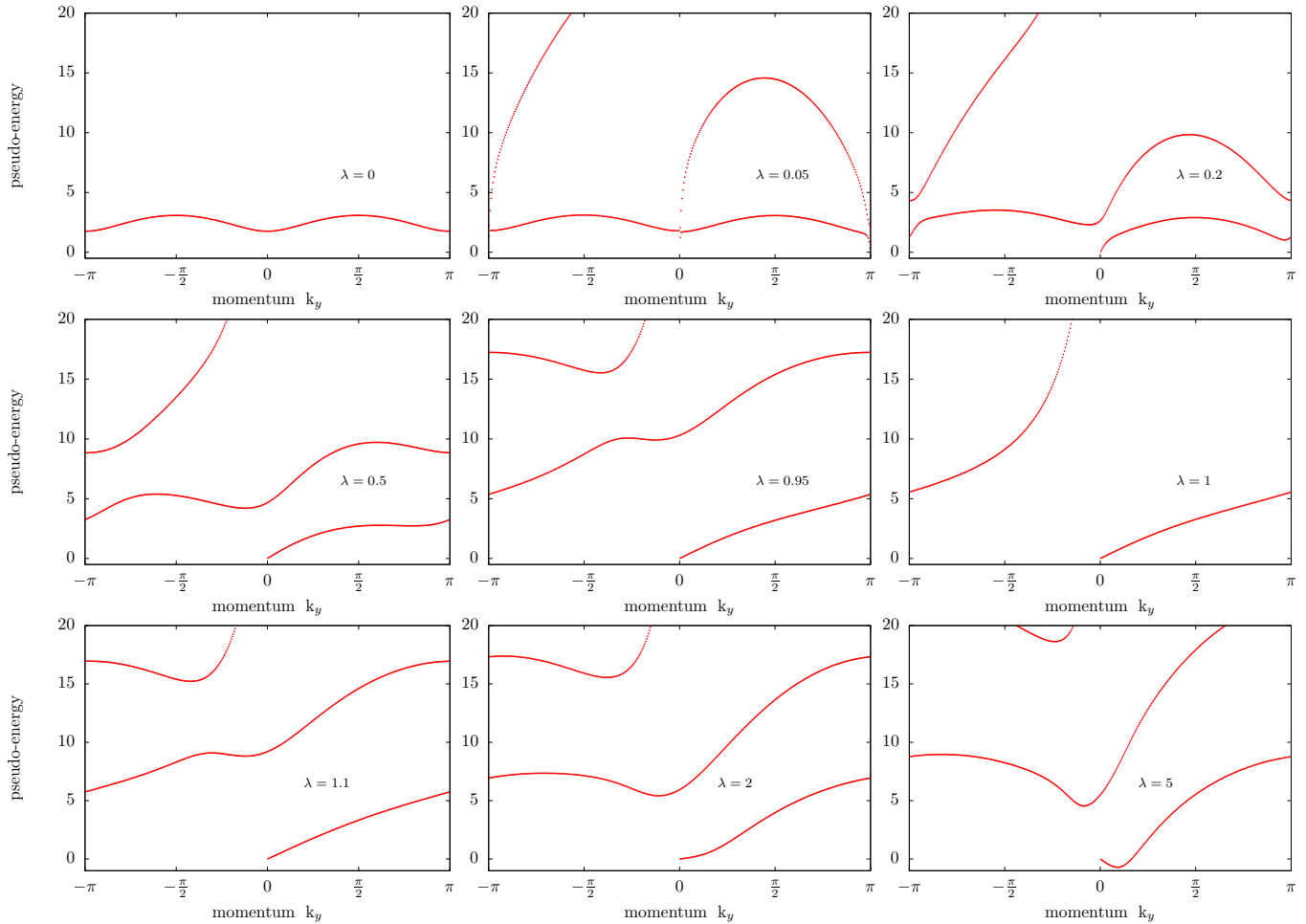


FIG. 1. (Color online) Single-particle entanglement spectrum of the  $p - ip$  state  $|\psi_D\rangle$  on an infinite cylinder of circumference  $L = 200$ . We take antiperiodic boundary conditions for the fermions around the cylinder, such that  $k_y \in \frac{2\pi}{L}(\mathbb{Z} + \frac{1}{2})$ . Here,  $|\kappa_1| = 1$ , and we vary the parameter  $\lambda \geq 0$ .

not have to be an odd function of  $\mathbf{k}$ , the filled band is nontrivial when its Chern number is nonzero; the other band has opposite Chern number [9]. The Chern number can be obtained (up to a choice of sign convention) from the (generically isolated) points at which  $g_{\mathbf{k}}$  diverges, as the sum of the winding numbers of each; the winding number can be defined as the winding of  $g_{\mathbf{k}}/|g_{\mathbf{k}}|$  as  $\mathbf{k}$  traverses a small circle about the point of divergence in the counterclockwise direction. Thus, in our case, the Chern number is 1. There are Hamiltonians  $H_A$  ( $\hat{H}_A$ ) with similar properties as  $H_D$  ( $\hat{H}_D$ ) above:  $H_A$  is short range but gapless, while  $\hat{H}_A$  has flat-band form and is gapped, but has power-law decaying matrix elements.

### E. Entanglement spectrum

For chiral phases of matter, one expects the entanglement spectrum to exhibit gapless chiral modes [15]. We have checked this for some of the Gaussian TNSs constructed earlier. Here, we illustrate this with our  $p - ip$  example, which is a translation-invariant BCS state  $|\psi_D\rangle$  defined by its pairing function

$$g_{\mathbf{k}} = \frac{(\kappa_1)^2(\sin k_x - i \sin k_y)}{(\sin k_x)^2 + (\sin k_y)^2 + \lambda^2[2 - \cos k_x - \cos k_y]}. \quad (16)$$

We put this Gaussian TNS on an infinite cylinder  $(x, y) \in \mathbb{Z} \times [0, L]$ , and chose the bipartition  $A \cup B$ ,  $B = \{(x, y) | x < 0\}$ , and  $A = \{(x, y) | x \geq 0\}$ . Notice that the bipartition is translation invariant in the  $y$  direction. Since we are dealing with a Gaussian state  $|\psi_D\rangle$ , its reduced density matrix  $\rho_A = \text{Tr}_B |\psi_D\rangle\langle\psi_D|$  is the exponential of a sum of fermion bilinears. In other words,  $-\ln \rho_A$  is a quadratic operator. The entanglement spectrum (i.e., the spectrum of  $-\ln \rho_A$ ) is a free-fermion spectrum, generated by a set of single-particle pseudoenergies  $\varepsilon_j$ . Finally,  $\rho_A$  (and thus  $-\ln \rho_A$ ) commutes with  $T_y$ , the generator of translations  $(x, y) \mapsto (x, y + 1)$ . Thus, every single-particle pseudoenergy is associated to a fixed momentum sector  $k_y$ . We plot the single-particle entanglement spectrum  $\varepsilon_p(k_y)$ , which in general has more than one branch (hence the subscript  $p$ ).

By construction, the number of branches in the single-particle entanglement spectrum must be bounded by the number of Grassmann variables per site that we use to define our TNS. Here, there are two Grassmann variables per site, so there are at most two branches in the single-particle spectrum (see Figs. 1 and 2). Since  $|\psi_D\rangle$  only depends on  $\lambda^2$ , we can focus on  $\lambda \geq 0$ . For generic values of  $\lambda$ , there are two branches in the single-particle spectrum, but when  $\lambda = 0$  or 1, one of the

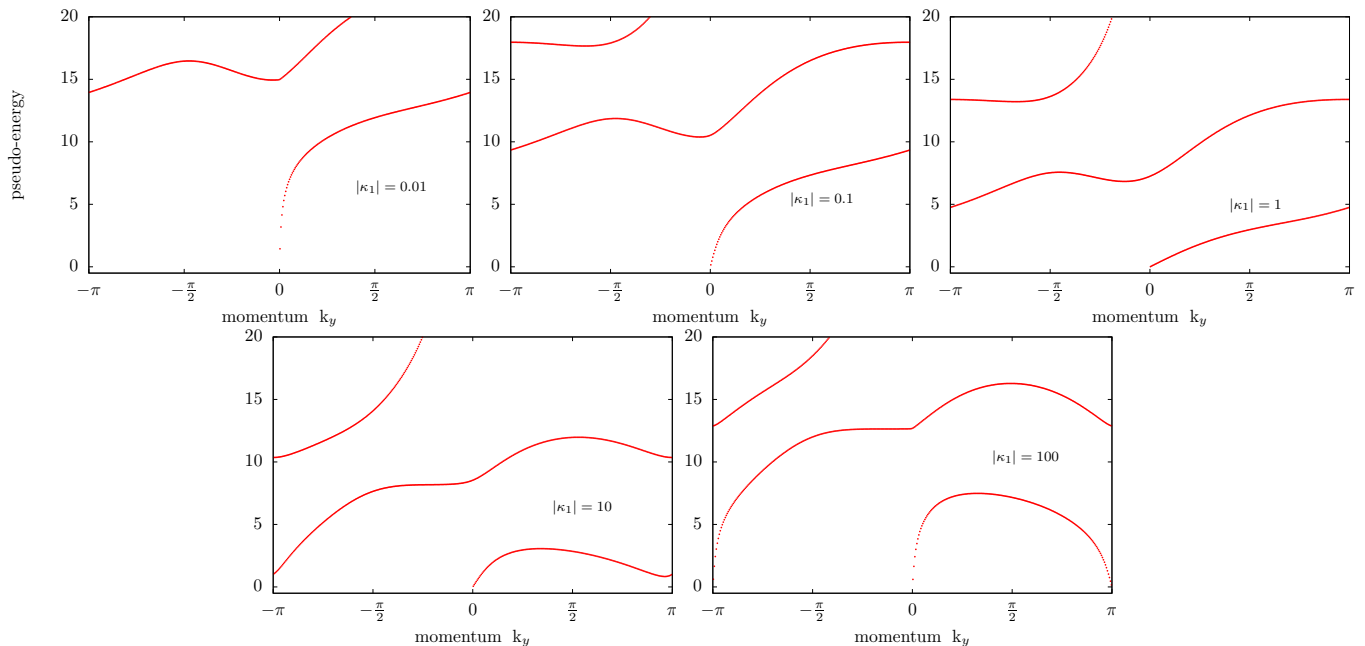


FIG. 2. (Color online) Same as in Fig. 1, with  $\lambda$  fixed ( $\lambda = 0.8$ ). We vary  $|\kappa_1|$ .

two branches disappears (it goes to infinity, meaning that the rank of  $\rho_A$  is smaller for these values of  $\lambda$ ), and one is left with a single branch. The reduction of the rank can also be traced back to the properties of the pairing function  $g_{\mathbf{k}}$  [16]. Indeed,  $g_{\mathbf{k}}$  may be viewed as a rational function of the variable  $e^{ik_x}$ , and has generically four simple poles, but has only two when  $\lambda = 0$  or 1. This modifies the form of the Fourier transform  $g_{k_y}(x) = \int \frac{dk_x}{2\pi} e^{ik_x x} g_{\mathbf{k}}$ , which determines the single-particle entanglement spectrum [16].

We clearly observe that, as long as  $\lambda \neq 0$ , there is one chiral gapless edge mode starting at  $k_y = 0$ , as expected. At  $\lambda = 0$ , this branch disappears; this is also expected since at  $\lambda = 0$  the degree of the mapping  $\mathbf{k} \mapsto g_{\mathbf{k}}$  suddenly changes, and the state  $|\psi_D\rangle$  does not belong to a chiral topological phase anymore.

### III. NO-GO THEOREM

In this section, we explain and prove the general no-go theorem for translation-invariant free-fermion TNSs. First, we explain various generalities about bands, bundles, and their parametrization using a Grassmannian manifold, then introduce relevant notions of analytic and polynomial bundles. Then, we state and prove the no-go theorem, starting with special cases before giving the proof in the general case. We also state a converse, that a gapped parent Hamiltonian exists in topologically trivial cases, and end with some remarks. Finally, we explain a variation of the proof that shows that complete sets of compactly supported Wannier functions cannot exist unless the filled-band bundle is topologically trivial.

#### A. Generalities on bands and bundles

We begin by generalizing the construction in the Chern-band example. In general, we consider a TNS for a system with  $n$  orbitals per site on a  $d$ -dimensional lattice, with  $m \leq n$  filled bands (as in class A; those for class D map onto these on

taking two copies). In fact, we will begin with more general statements (occasionally indicating how to specialize them for TNSs), before turning to the no-go theorem. The general free-fermion ground state in this symmetry class has the form

$$\exp\left(\int \frac{d^d \mathbf{k}}{(2\pi)^d} \sum_{\alpha, \bar{\alpha}} g_{\mathbf{k}, \alpha \bar{\alpha}} c_{\mathbf{k}, \bar{\alpha}}^\dagger c_{\mathbf{k}, \alpha}\right) |11 \dots, 00 \dots 0\rangle. \quad (17)$$

Here,  $\alpha = 1, \dots, m$ ,  $\bar{\alpha} = m + 1, \dots, n$ , and the reference state  $|11 \dots, 00 \dots 0\rangle$  is annihilated by  $c_{\mathbf{k}, \alpha}^\dagger$  and  $c_{\mathbf{k}, \bar{\alpha}}$  for all  $\mathbf{k}$ , or equivalently by  $c_{\mathbf{x}, \alpha}^\dagger$  and  $c_{\mathbf{x}, \bar{\alpha}}$  for all  $\mathbf{x}$ . We write  $g_{\mathbf{k}, \alpha \bar{\alpha}}$  as  $g_{\mathbf{k}}$ , which is an  $m \times (n - m)$  matrix of functions of  $\mathbf{k}$  in the Brillouin zone, say  $[-\pi, \pi]^d$  for the hypercubic lattice; thus,  $\mathbf{k}$  is always real, except where explicitly stated otherwise. (In the case of a TNS, the entries of  $g_{\mathbf{k}}$  are ratios of TPs; conversely, in Appendix C we show that such a form of  $g_{\mathbf{k}}$  can always be obtained from a TNS, possibly one of more general form.)

It will be useful later if we point out here some arbitrariness in the specification of the state, in that the choice of which orbitals are filled and which are empty on every site in the reference state was arbitrary. We could have chosen a different reference state, in particular by replacing  $|11 \dots, 00 \dots 0\rangle$  with a state annihilated by a different set of  $m$  creation operators and by the annihilation operators for the remaining  $n - m$  orbitals orthogonal to these, for every site (the same set for each site  $\mathbf{x}$ ); these sets of creation operators could be any linear combination of those used above. To preserve the canonical anticommutation relations, the transformation on the set of all  $n$  orbitals at each  $\mathbf{x}$  should be unitary; further, as it acts locally, it is independent of  $\mathbf{k}$  in  $\mathbf{k}$  space. In order that the TNS be unchanged, the effect of such a transformation on  $g_{\mathbf{k}}$  has to be a fractional linear transformation of the form

$$g_{\mathbf{k}} \rightarrow g'_{\mathbf{k}} = (A + g_{\mathbf{k}} B)^{-1} (C + g_{\mathbf{k}} D) \quad (18)$$

for matrices  $A, B, C, D$  of appropriate sizes, such that

$$\mathcal{U} = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \quad (19)$$

is the  $\mathbf{k}$ -independent  $n \times n$  unitary matrix. [The reason why the transformation of  $g_{\mathbf{k}}$  takes this form may become clearer when we discuss the Grassmannian manifold  $G(m, n)$  in the following.] We know that at some points in  $\mathbf{k}$  space, say at  $\mathbf{k} = \mathbf{k}_0$ ,  $g_{\mathbf{k}}$  may diverge; we can make  $g_{\mathbf{k}}$  finite at any given  $\mathbf{k}_0$ , using such a transformation if necessary. (Whether the resulting  $g_{\mathbf{k}}$  actually exists as a continuous function at  $\mathbf{k} = \mathbf{k}_0$  depends on the properties of the function  $g_{\mathbf{k}}$  with which we started; we discuss this issue following.)

Generally (not only for TNSs), any one-fermion operator that annihilates the general free-fermion ground state must take the form

$$d_{-\mathbf{k}} = \sum_{\alpha} u_{\mathbf{k},\alpha} c_{\mathbf{k},\alpha}^{\dagger} + \sum_{\bar{\alpha}} v_{\mathbf{k},\bar{\alpha}} c_{\mathbf{k},\bar{\alpha}}^{\dagger}, \quad (20)$$

for the filled bands, or a similar form for the empty bands. At present, these operators can be considered for any given  $\mathbf{k}$ , without regard to how they depend on  $\mathbf{k}$ . Here,  $u_{\mathbf{k}}$  and  $v_{\mathbf{k}}$  are a solution to the equation

$$v_{\mathbf{k}} = u_{\mathbf{k}} g_{\mathbf{k}}, \quad (21)$$

where  $u_{\mathbf{k}}$  is a  $1 \times m$  matrix (or  $m$ -component row vector) and  $v_{\mathbf{k}}$  is a  $1 \times (n - m)$  matrix (or  $n - m$  component row vector), for the given  $\mathbf{k}$ , wherever it makes sense, that is wherever  $g_{\mathbf{k}}$  is finite. Where  $g_{\mathbf{k}}$  is not finite, we can make a transformation as above to render it finite; this acts as a unitary change of basis on the  $n$ -dimensional vectors  $(u, v)$ . The set of all solutions (viewed basis independently) to this equation for all  $\mathbf{k}$  forms a space consisting of a vector space of dimension at least  $m$  for each  $\mathbf{k}$ . (These vectors correspond to the complex conjugates of the single-particle states in the filled band at that  $\mathbf{k}$ .)

When  $g_{\mathbf{k}}$  is not too singular (we will explain the precise statement of this), this space has the structure of a vector bundle [17]; the base space of the bundle is the Brillouin zone, and the fiber at each  $\mathbf{k}$  is the space of solutions to the equation at that  $\mathbf{k}$ . To qualify as a bundle, it must be possible to find, for any  $\mathbf{k}_0$ , a neighborhood of  $\mathbf{k}_0$  in which there is a set of  $m$  solutions which are linearly independent for each  $\mathbf{k}$  in the neighborhood, and each of which vary continuously with  $\mathbf{k}$ . (The dimension of the fiber as a complex vector space is also called the *rank* of the bundle.) We refer to this bundle as the filled-band bundle. Our key assumption (other than specializing to TNSs) is that  $g_{\mathbf{k}}$  is such that the corresponding single-particle states in the filled bands define an  $m$ -dimensional vector bundle over the Brillouin zone [17]; otherwise, the ground state does not lie in a topological phase. This means precisely that in the neighborhood of any  $\mathbf{k}_0$ , it is possible to transform so that  $g_{\mathbf{k}}$  is finite and continuous (as a function of  $\mathbf{k}$ ) in that neighborhood. (As an example of the opposite situation, in the situation when the ground state is associated with an energy spectrum that has a Dirac point, the space of solutions does not form a bundle;  $g_{\mathbf{k}}$  itself, while finite, is not continuous at the Dirac point, and so cannot even be uniquely defined there by continuity; see for example Ref. [8]. This behavior is associated with the filled band “touching,” or becoming degenerate with, an “unfilled” band at the Dirac point.)

Under these conditions, the states in the unfilled bands form an empty-band bundle, which is complementary to the filled-band bundle, in the sense that the orthogonal direct sum of the two, which consists of the full  $n$ -dimensional space of states at each  $\mathbf{k}$ , is a bundle of rank  $n$ . This rank- $n$  bundle is topologically trivial because it arises from the tight-binding model. (We can also describe this by saying that both the filled- and empty-band bundles are embedded as subbundles in the trivial rank- $n$  bundle.) The (single-particle) Hilbert space of the tight-binding model is a product of local single-particle Hilbert spaces in real space, and so in  $\mathbf{k}$  space the states can be spanned by a set of  $\mathbf{k}$ -independent vectors. We recall that a bundle of rank  $m$  is topologically trivial (as a complex vector bundle, with no reference to additional symmetries) if and only if there is a set of  $m$  sections that are linearly independent at each and every  $\mathbf{k}$  value (it follows that these sections are nonvanishing everywhere); here and elsewhere a *section* is a choice of a vector in the bundle at each  $\mathbf{k}$ , and so is defined for every  $\mathbf{k}$  in the base space (here, the Brillouin zone), and is further required to vary continuously with  $\mathbf{k}$  for all  $\mathbf{k}$ . For the case of  $d = 2$  dimensions, a nonzero Chern class (or number) implies that the bundle is nontrivial [17]. When the filled-band bundle is nontrivial, so is that of the empty bands, so that their direct sum is trivial [17].

More explicitly, the condition that we have a well-defined filled-band bundle can be expressed by saying that, in some neighborhood of any  $\mathbf{k}_0$ , and possibly after a transformation,  $g_{\mathbf{k}}$  is finite, and can be written in terms of a set of  $m$  solutions  $(u_{\mathbf{k}}, v_{\mathbf{k}})$  which we assemble into matrices, so that

$$g_{\mathbf{k}} = U_{\mathbf{k}}^{-1} V_{\mathbf{k}} \quad (22)$$

in the neighborhood, where  $U_{\mathbf{k}}$  is  $m \times m$  and  $V_{\mathbf{k}}$  is  $m \times (n - m)$  (the rows are the vectors  $u_{\mathbf{k}}, v_{\mathbf{k}}$ ). This is a consequence of the existence of linearly independent sections over the neighborhood of any point in the base space; by the choice of transformation,  $U_{\mathbf{k}}$  is invertible throughout the neighborhood. It should go without saying that there may be no one choice of transformation that makes these statements hold for all  $\mathbf{k}$ .

The function  $g_{\mathbf{k}}$ , and the use of different coordinates related by a fractional linear transformation, can be understood in terms of a Grassmannian manifold. The choice of an  $m$ -dimensional subspace in the (fixed)  $n$ -dimensional space at each  $\mathbf{k}$  can be parametrized as corresponding to a point in the complex Grassmannian

$$G(m, n) = \frac{U(n)}{U(m) \times U(n - m)}. \quad (23)$$

For  $m = 1$ ,  $G(m, n)$  reduces to complex projective space  $\mathbf{CP}^{n-1}$ , and if also  $n = 2$ , this becomes  $\mathbf{CP}^1 \cong S^2$ , the 2-sphere. For the latter it is well known that if the vector is  $(u, v)$  ( $u$  and  $v$  complex numbers), then  $g = v/u$  represents any point on the sphere as a point in the plane by stereographic projection, except for one pole which is mapped to infinity. By unitary rotation in  $U(2)$  of the sphere, the pole omitted can be mapped to any finite point (the transformation is fractional linear as above, with  $A, \dots, D$  replaced by complex numbers). The use of the matrices  $U, V$ , with  $g = U^{-1}V$ , and the fractional linear transformations of  $g$ , generalize this to any  $G(m, n)$ . Thus, a bundle of rank  $m$  that is continuously embedded as a subbundle of a trivial rank- $n$  bundle can be

viewed as a continuous function from the base space (the Brillouin zone) into the Grassmannian  $G(m, n)$ . Note that as we have described it, our bundle is more than just a bundle as defined topologically [17]: it has a given embedding into the trivial  $n$ -dimensional bundle, and these are what are classified by the maps to the Grassmannian.

**B. Analytic bundles**

We introduce a further notion that does not hold for general bundles (or even for general subbundles), and may be unconventional, but will be crucial for our discussion. A bundle of rank  $m$  which is embedded into a trivial bundle will be said to be *analytic* at a point  $\mathbf{k}_0$  in the base space if, in a neighborhood of  $\mathbf{k}_0$  in the base space, there is a set of  $m$  sections (defined over the neighborhood) that are linearly independent and all of their components (relative to the given trivialization of the rank- $n$  bundle) are analytic functions of  $\mathbf{k}$ , both properties holding in the neighborhood. (Here, analytic means as functions of the  $d$  real variables  $k_\mu$ .) We define in a similar way the notion of analytic in an open set in the base space. The term analytic, when used without qualification as to the neighborhood or open set in which it holds, means the bundle is analytic at all points in the base space (for us, the Brillouin zone). We note that analyticity extends to *complex*  $\mathbf{k}$  in a complex neighborhood of  $\mathbf{k}_0$  whenever it holds at a real  $\mathbf{k}_0$ ; this will be important later (for the theory of analytic functions in several variables, see e.g. Ref. [18]). We also note that the filled-band bundle is analytic if and only if the complementary empty-band bundle is.

The significance of analyticity is that if the embedded bundle (for example, the filled-band bundle) is analytic, then a projection operator onto the vectors in the fiber varies analytically with  $\mathbf{k}$  also. Thus, for an analytic bundle, taking the inverse Fourier transform of the projection operators onto the two bundles, the resulting flat-band single-particle Hamiltonian will also be short range, at least in the exponentially decaying sense. Conversely, when the bands and associated bundles are obtained from a short-range single-particle Hamiltonian that has an energy gap between filled and empty bands at all  $\mathbf{k}$ , the projection operators onto the filled and empty bands are analytic (from basic results in matrix analysis on perturbations of eigenvalue problems); this follows by taking the Fourier transform of a Hamiltonian whose ‘‘hopping’’ matrix elements decay exponentially or faster with distance, so that in  $\mathbf{k}$  space the Hamiltonian is analytic in  $\mathbf{k}$ . Given an  $n \times n$  projection operator  $P_{\mathbf{k}}$  onto the filled band, which is of rank  $m$  and analytic at all  $\mathbf{k}$ , it is always possible to construct (e.g., by diagonalizing  $P_{\mathbf{k}}$ ) an  $m \times n$  matrix function  $\phi_{\mathbf{k}}$  in a neighborhood of any  $\mathbf{k}_0$ , such that  $\phi_{\mathbf{k}}$  is analytic and  $P_{\mathbf{k}} = \phi_{\mathbf{k}}^\dagger \phi_{\mathbf{k}}$  in the neighborhood. The rows of  $\phi_{\mathbf{k}}$  provide the local analytic sections, so the filled-band bundle is analytic.

Like the definition of a bundle, the definition of an analytic bundle can be expressed in terms of  $g_{\mathbf{k}}$  [after a suitable transformation of the form in (18)]. By definition of an embedded bundle, in some neighborhood of any  $\mathbf{k}$ , say of  $\mathbf{k}_0$ , the matrix  $W_{\mathbf{k}} = (U_{\mathbf{k}}, V_{\mathbf{k}})$  formed from the  $m$  solutions in the neighborhood has nonvanishing minors of rank  $m$ . Without loss of generality (that is, by permuting the columns, which is a special case of a fractional linear transformation), we can

assume that  $\det U \neq 0$  at  $\mathbf{k}_0$ . If the bundle is analytic in a neighborhood of  $\mathbf{k} = \mathbf{k}_0$ , then we can assume  $W_{\mathbf{k}}$  is in fact analytic in the neighborhood, so  $\det U \neq 0$  also in a (possibly smaller) neighborhood of  $\mathbf{k}_0$ . Then, it follows that  $g_{\mathbf{k}}$  is analytic in that neighborhood also. Conversely, if (with a choice of a transformation)  $g_{\mathbf{k}}$  is analytic in a neighborhood of  $\mathbf{k}_0$  (and so does not diverge in the neighborhood), then there are solutions  $W_{\mathbf{k}}$  that are analytic in that neighborhood (for example, take  $U_{\mathbf{k}} = I$ ,  $V_{\mathbf{k}} = g_{\mathbf{k}}$ ). Then if for all  $\mathbf{k}$  there is a transformation such that  $g_{\mathbf{k}}$  is analytic at that  $\mathbf{k}$ , then the bundle is analytic.

**C. TNSs give rise to polynomial bundles**

We now turn to the specific aspects that arise when we consider a TNS; in this section, we return to bundles that are not necessarily analytic. For a TNS, each entry of  $g_{\mathbf{k}}$  is a ratio of TPs (i.e., trigonometric polynomials; see Sec. II). Then, it is natural to seek solutions of Eq. (21) such that  $u_{\mathbf{k}}$ ,  $v_{\mathbf{k}}$  also have TP entries. Each such solution is a section of the filled-band bundle, and we call such a section a *TP section*. The TP sections then give rise to the local (compact-support) operators in position space that annihilate the ground state, via inverse Fourier transform of Eq. (20). Of course, these do not give all the sections in  $\mathbf{k}$  space of the filled-band bundle, either globally or over smaller regions. More general sections can be obtained by taking linear combinations of the TP sections with arbitrary continuous complex functions of  $\mathbf{k}$  as coefficients. Nonetheless, we will find that consideration of TP sections, and of the simpler polynomial sections to be described next, will be very useful.

When considering solutions, or TP sections, say for the hypercubic lattice, it is possible to work in terms of polynomials in  $X_\mu = e^{ik_\mu}$  ( $\mu = 1, \dots, d$ ) only, and not their reciprocals (which would be allowed when working in TPs), because given a solution  $u_{\mathbf{k}}$ ,  $v_{\mathbf{k}}$  in TPs, we can multiply all components by positive powers of each  $e^{ik_\mu}$  so that negative powers are eliminated. (This trivial change in the solutions has the effect of translation in real space, which is something we make use of anyway in obtaining operators with compact support.) Likewise,  $g_{\mathbf{k}}$  itself can be expressed in terms of ratios of such polynomials. Thus, henceforth we will be interested in solutions of Eq. (21) with entries in the polynomial ring

$$R = \mathbb{C}[X_1, \dots, X_d] \tag{24}$$

of such polynomials (and not in the TPs); we refer to these solutions simply as *polynomial sections*. Solutions (with entries in  $R$ ) to the Eq. (21) are plentiful. We emphasize that polynomial sections map to polynomial sections under basis changes that use constant unitary matrices  $U$  as in a fractional linear transformation of  $g_{\mathbf{k}}$ , Eq. (18) (a corresponding statement holds for TP sections).  $g_{\mathbf{k}}$  itself can be written in terms of some sets of  $m$  solutions as in Eq. (22), where now the entries of  $U_{\mathbf{k}}$  and  $V_{\mathbf{k}}$  are in the ring  $R$  and, excluding a trivial case in which  $g_{\mathbf{k}}$  is constant,  $U_{\mathbf{k}}$  is invertible except possibly on a set of measure zero in the Brillouin zone. Thus, for example, if the lowest common denominator (in  $R$ ) of the entries of  $g_{\mathbf{k}}$  is  $D_{\mathbf{k}}$ , we can write  $U_{\mathbf{k}} = D_{\mathbf{k}} I_m$ ,  $V_{\mathbf{k}} = U_{\mathbf{k}} g_{\mathbf{k}}$ . (Note that  $R$  is a unique factorization domain [19,20], so that factorization can be used here. The lowest common denominator is unique up to multiplication by an invertible element of  $R$ ; the invertible

elements are just the nonzero constants in  $\mathbf{C}$ .) For lattices other than hypercubic, we can similarly work in the ring of polynomials in some combinations of exponentials of components of  $\mathbf{k}$ . For example, for the triangular lattice in  $d = 2$  dimensions with lattice spacing 1, we can work with polynomials in  $X_1 = e^{ik_x}$  and  $X_2 = e^{ik_x/2 + \sqrt{3}ik_y/2}$ . This works for any lattice because the monomials in exponentials of wave vectors that appear correspond to translations on the lattice, which form an Abelian group, with  $d$  generators. The indeterminates in the polynomials constituting the ring correspond to these generators. Thus, as a ring,  $R$  always has the same structure for each  $d$ .

Similarly to Eq. (21), the operators that annihilate the TNS by destroying states in the empty band contain coefficients that are solutions of

$$\tilde{v}_{\mathbf{k}} = g_{\mathbf{k}} \tilde{u}_{\mathbf{k}}, \quad (25)$$

where  $\tilde{u}_{\mathbf{k}}$  and  $\tilde{v}_{\mathbf{k}}$  are column vectors of size  $n - m$  and  $m$ , respectively, and are now to be viewed as solutions in  $R$ . A set of  $n - m$  of these that are linearly independent at almost all values of  $\mathbf{k}$  can be assembled into matrices  $\tilde{U}_{\mathbf{k}}$ ,  $\tilde{V}_{\mathbf{k}}$  of sizes  $(n - m) \times (n - m)$  and  $m \times (n - m)$ , respectively, so  $g_{\mathbf{k}} = \tilde{V}_{\mathbf{k}} \tilde{U}_{\mathbf{k}}^{-1}$ . Finally, using the  $m \times n$  matrix  $W = (U, V)$ , and forming the  $n \times (n - m)$  matrix  $Z$  with the blocks

$$\begin{pmatrix} \tilde{V} \\ -\tilde{U} \end{pmatrix} \quad (26)$$

(we will sometimes leave the  $\mathbf{k}$  dependence implicit, as here), the equations take the form

$$WZ = 0 \quad (27)$$

(linear equations with entries in  $R$ ) which may either be solved to find solutions for  $w_{\mathbf{k}} = (u_{\mathbf{k}}, v_{\mathbf{k}})$  as rows of  $W$ , given any  $Z$  of the form here for the given  $g_{\mathbf{k}}$ , or likewise for columns

$$\begin{pmatrix} \tilde{v}_{\mathbf{k}} \\ -\tilde{u}_{\mathbf{k}} \end{pmatrix}, \quad (28)$$

when  $W$  is given. [These equations essentially assert that the states ( $n$ -component vectors) in the filled band are orthogonal to those in the empty band at the same  $\mathbf{k}$ .] In this form, the equations are manifestly invariant under a unitary transformation given by  $W \rightarrow W\mathcal{U}$ ,  $Z \rightarrow \mathcal{U}^{-1}Z$  for a constant unitary  $n \times n$  matrix  $\mathcal{U}$ , the same one as in the transformation (18) and following.

Finally, bundles that can be defined as a subbundle of a trivial bundle determined by solutions to Eqs. (21) or (27), i.e., linear equations with polynomial coefficients, will be called *polynomial bundles*. Note that according to our definitions, if there are points  $\mathbf{k}_0$  in the base space at which there are more than  $m$  linearly independent solutions for  $w_{\mathbf{k}}$  (over  $\mathbf{C}$ ), then the bundle is defined by continuity at those points; it was assumed that that is possible. It is also possible that solutions in  $R$  span a space of dimension smaller than  $m$  at some  $\mathbf{k}_0$ ; again, since we obtain general sections by taking linear combinations of those solutions using functions of  $\mathbf{k}$ , we may be able to obtain continuous sections that are linearly independent throughout a neighborhood of  $\mathbf{k}_0$ , and it is our assumption that we can, so the bundle is well defined. (We will soon see, however, that in the context we wish to discuss, these issues do not actually arise.)

#### D. Statement and proof of no-go theorem

In this section, we state and prove the no-go theorem for translation-invariant TNSs for free-fermion systems. We define the general TNS in  $d$  dimensions to be of the general free-fermion form as in the present Sec. III, with the additional condition that each entry of  $g_{\mathbf{k}}$  as a function of  $\mathbf{k}$  in the Brillouin zone is a ratio of polynomials taken from  $R$ . We are interested in parent Hamiltonians as defined earlier; these are translation-invariant one-body Hamiltonians that annihilate the TNS, in which the single-particle matrix elements are strictly short range, that is, they vanish unless the two sites involved are separated by less than some constant. (We note that in class A, in which there is a conserved particle number, a chemical potential times particle number term can be added to the Hamiltonian. In our definition, the chemical potential has already been fixed to lie between the filled and empty states. If there is an energy gap between them, then the chemical potential can be changed without affecting the value of the gap, or the ground state, as long as the chemical potential remains in the gap. Moving the chemical potential out of this range would change the ground state and violate our assumptions.)

The no-go theorem arises from the confrontation between the properties of being analytic and being polynomial. We have seen that if the Hamiltonian has a gap throughout the Brillouin zone, then the filled-band bundle is analytic. At the same time, we have seen that for a TNS, the filled-band bundle is a polynomial bundle (and that is the only consequence of having a TNS that we will use). We will ask if the bundle is topologically nontrivial as a complex vector bundle. All of these properties have been discussed in the immediately preceding sections. In view of the earlier discussion, the relevant theorem can now be stated as follows:

*No-go theorem.* If a complex vector bundle is both polynomial and analytic, then it is topologically trivial.

By contraposition, the no-go theorem implies that for a TNS in which the filled-band bundle is nontrivial as a complex bundle, there can be no parent Hamiltonian with a gap in its spectrum at the Fermi energy. Indeed, there can be no gapped Hamiltonian even if its matrix elements are allowed to decay exponentially. We will also show that, conversely, when the filled-band bundle is analytic, there is a gapped parent Hamiltonian for any TNS. We remark here that the mathematical statement and proof of the no-go theorem hold not only for bundles over the Brillouin zone, which can be defined as the set  $\{|X_{\mu}| = 1 : \mu = 1, \dots, d\}$  in  $\mathbf{C}^d$ , but also hold without modification for bundles with any domain in  $\mathbf{C}^d$  as the base space, in particular, with analyticity required throughout the domain. However, in the arguments, we treat the case of the Brillouin zone, parametrized by  $\mathbf{k}$ , which is real except when otherwise stated.

We will begin with a result that will be used repeatedly in the later proofs, and then give proofs of the theorem in a couple of special cases, as these are more easily understood, and they build towards the general case. After stating some additional results that will be used, we will proceed to the proof in the general case.

*Proposition.* If  $g_{\mathbf{k}}$  has entries that are ratios of polynomials in  $R$  and there is a  $\mathbf{k}_0$  such that  $g_{\mathbf{k}}$  is analytic in a neighborhood of  $\mathbf{k}_0$ , then the lowest common denominator  $D_{\mathbf{k}}$  of  $g_{\mathbf{k}}$  is



nonvanishing at  $\mathbf{k}_0$ , and there are a set of  $m$  solutions  $w_{\mathbf{k}}$  (with entries in  $R$ ) to Eq. (21) that are linearly independent in the neighborhood of  $\mathbf{k}_0$ .

We remark that the conclusion is similar to that in the discussion of analytic bundles above; the point here is that  $m$  polynomial sections  $w_{\mathbf{k}}$  can be used to span the bundle locally, in the neighborhood of  $\mathbf{k}_0$ .

*Proof.* Analyticity at  $\mathbf{k}_0$  implies analyticity in some complex neighborhood of  $\mathbf{k}_0$ , that is, at nearby complex values of  $k_{\mu}$ . If the denominator,  $D'_{\mathbf{k}}$  say, in an entry in  $g_{\mathbf{k}}$  vanishes at  $\mathbf{k}_0$ , then it does so on a variety, the solution set of the polynomial equation  $D'_{\mathbf{k}} = 0$ . To ensure analyticity, the zero on the variety must be canceled not only at  $\mathbf{k}_0$  but everywhere in the complex neighborhood; otherwise, this entry of  $g_{\mathbf{k}}$  diverges at some complex  $\mathbf{k}$  points arbitrarily close to  $\mathbf{k}_0$  and so is not analytic at  $\mathbf{k}_0$  [18]. For ratios of polynomials, this means that the numerator must contain a polynomial factor that cancels those factors in the denominator that vanish in a neighborhood of  $\mathbf{k}_0$ . Hence, the lowest common denominator  $D_{\mathbf{k}}$  cannot vanish at  $\mathbf{k}_0$ . We can obtain a matrix of the form  $W_{\mathbf{k}}$  (with entries in  $R$ ) using  $U_{\mathbf{k}} = D_{\mathbf{k}} I_m$ , and the rows of  $U_{\mathbf{k}}$  are linearly independent in a neighborhood of  $\mathbf{k}_0$  because  $\det U_{\mathbf{k}} = D_{\mathbf{k}}^m \neq 0$  there. ■

We remark that with this result, issues (mentioned above) of whether solutions to Eq. (21) really define a bundle do not arise once we assume that in some neighborhood of any  $\mathbf{k}_0$ ,  $g_{\mathbf{k}}$  is analytic after it has been transformed to make it finite at  $\mathbf{k}_0$ .

*Corollary.* If a complex vector bundle of rank  $m$  is both analytic and polynomial, then for any  $\mathbf{k}$ , there is a set of  $m$  polynomial sections that are linearly independent in some neighborhood of  $\mathbf{k}$ . (This follows immediately from the proposition.)

*Proof of the no-go theorem in the case  $m = 1$  (so the filled-band bundle has rank one).* It is easy to see that all solutions with entries in  $R$  to Eq. (21) (i.e., polynomial sections) are multiples of that obtained by taking  $u_{\mathbf{k}} = D_{\mathbf{k}}$  to be the lowest common denominator of the entries of  $g_{\mathbf{k}}$  (in  $R$ ). Call this basic solution  $w_{\mathbf{k}}$ . By the corollary, as the bundle is analytic, for any  $\mathbf{k} = \mathbf{k}_0$  there is a polynomial section that is nonvanishing in a neighborhood of  $\mathbf{k}_0$ . Since all polynomial sections are multiples of  $w_{\mathbf{k}}$ ,  $w_{\mathbf{k}}$  cannot vanish anywhere, which shows that the bundle is trivial. ■

As a second proof, note that the definition of  $w_{\mathbf{k}}$  implies that there can be no polynomial factor common to all its entries, while analyticity implies that if  $w_{\mathbf{k}}$  vanished (as a vector) at some  $\mathbf{k}_0$ , then there would have to be such a common factor, yielding a contradiction.

*Proof of the no-go theorem for  $m \geq 1$  under an additional hypothesis:* Namely, we assume that all solutions to Eq. (21) can be expressed as linear combinations (with coefficients in  $R$ ) of a set of  $m$  basic solutions (or “generators”) with entries in  $R$ ; we can write these polynomial sections as the matrix  $W$ . Then, the argument is similar to the  $m = 1$  case. By the corollary, as the bundle is analytic, for any  $\mathbf{k}_0$ , there is a set of  $m$  polynomial sections that are linearly independent in a neighborhood of  $\mathbf{k}_0$ . As the generators (rows of  $W$ ) are assumed to span all solutions, they too must be linearly independent at  $\mathbf{k}_0$ . As this holds for all  $\mathbf{k}$ , the set of generators gives the trivialization, and the bundle is trivial. ■

Before turning to the proof in the general case, we first discuss the issues involved and the background that it uses.

The solutions to matrix equations like  $wZ = 0$  (where  $w$  is an unknown row vector and  $Z$  is a known  $n \times p$  matrix, for some  $p$ ) in the polynomial ring  $R$  may not all be expressible as linear combinations (over  $R$ ) of a single set of  $m$  solutions that are linearly independent over  $R$ . (This basic fact about equations over a polynomial ring  $R$  can be found in any text on commutative algebra, such as Refs. [19,20].) Thus, the additional hypothesis used in the preceding special case may not hold. Then, even though a particular set of  $m$  solutions may be linearly independent at some  $\mathbf{k}_0$  (as implied by the proposition), that particular set might be linearly dependent elsewhere, and there may not be one set of  $m$  solutions that generates all such locally linearly independent solution sets as linear combinations over  $R$ , so the argument breaks down. Likewise, it is not clear either that there is a set of  $n - m$  column vectors  $z$  that suffice to fully determine all the solutions  $w$ ; it is possible that more have to be used. The existence of such a set of size  $n - m$  would again lead to a simple proof as above (applied to the empty-band bundle). [The possible nonexistence of such a set is the reason we do not assume that the form  $WZ = 0$  can be used, with  $Z$  a known  $n \times (n - m)$  matrix, but instead we continue to work with  $g$ , which determines all the solutions. Note that when we stated above that there are matrices with entries in  $R$  such that  $g = U^{-1}V$  or  $g = \tilde{V}\tilde{U}^{-1}$ , we never said in the general case that those matrices are unique, not even up to multiplication by invertible matrices.] This is the most challenging aspect of the problem. To make further progress, we are forced to use more machinery.

The space of solutions to Eq. (21) with entries in  $R$ , that is, the space of polynomial sections, forms what is called a module over the ring  $R$  (analogous to a vector space over a field): that is, the space forms an Abelian group under addition, with the zero solution as the identity element, and any element of the space times an element of  $R$  yields another element of the space. Our module  $M$  of solutions to Eq. (21) is a submodule of a “free” module of rank  $n$  over  $R$  (the space of all  $n$ -component row vectors with entries in  $R$ ), and so because  $R$  is Noetherian (the Hilbert basis theorem),  $M$  has finite sets of generators, linear combinations (over  $R$ ) of which span the space [19,20]. Given such a finite set of generators, we can form a matrix  $\phi_1$ , the rows of which are the generators. However, unlike the case in which the ring is a field (and the module is then a vector space),  $M$  may not be free, that is (as already stated above) it may not be possible to find a set of generators that are linearly independent (over  $R$ ) [19,20]. This implies that in this case  $\phi_1$  must have more than  $m$  rows. Given a choice of a set of generators, the linear relations (or syzygies) among them also form a module over the same ring  $R$  because the space of linear combinations of generators that equal zero obeys all the conditions in the definition of a module. The module of relations in turn must be finitely generated but may not be free, so a set of generators for it also obeys some relations, and so on. The Hilbert syzygy theorem [20,21], to be explained next, says that for the polynomial ring  $R$  this so-called chain of syzygies can be constructed so that it terminates in a bounded number of steps.

The syzygy theorem can be described in the following way [21] (the application to our problem is immediate): First, we note that a free module of any rank  $n$  say, is isomorphic (over the polynomial ring  $R$ ) to the module consisting of all

$n$ -component row vectors with entries in  $R$ . Then, giving a set of generators for a module  $M$  is equivalent to specifying a map (a homomorphism of modules over  $R$ ) of a free module onto  $M$ . As our modules consist of spaces of row vectors with entries in  $R$ , this map  $\phi_1$  can be described as in the previous paragraph by a matrix that we also denote by  $\phi_1$ , whose rows are the generators. Due to the generators not being linearly independent, the map of modules has a kernel (null space), a subspace of the free module that is annihilated by  $\phi_1$ , and the kernel is a module which can itself be described in a similar way, and so on. The Hilbert syzygy theorem says that this construction can be carried out so that there exists a so-called free resolution of the module (an exact sequence of modules) of the form

$$0 \longrightarrow F_d \xrightarrow{\phi_d} F_{d-1} \xrightarrow{\phi_{d-1}} \dots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \quad (29)$$

in which  $F_\mu$  ( $\mu = 1, \dots, d$ ) is free and has rank  $n_\mu$  over  $R$ , the rank of the free module  $F_0$  is  $n_0 = n$ , and  $0$  at the left end is the trivial module. Each map  $\phi_\mu$  can be viewed as an  $n_\mu \times n_{\mu-1}$  matrix with entries in  $R$ , and  $\phi_{\mu+1}\phi_\mu = 0$  for  $\mu = 1, \dots, d$ , with  $\phi_{d+1} = 0$ . (The case in which the sequence is shorter can be thought of as  $F_\mu = 0$  for  $\mu$  larger than some  $\mu_0 < d$ .) The exactness of the sequence means that the kernel of one map is precisely the image of its predecessor (as modules over  $R$ ). The module  $M$  is the image of  $\phi_1$ , and also is by definition the kernel of an additional map, corresponding to the equation  $wZ = 0$  for  $w \in M$ , so that the exact sequence can in fact be extended one more step at the right. (It is in that form that the syzygy theorem is usually stated.) The structure of such a free resolution of  $M$  is described further by the Buchsbaum-Eisenbud theorem [21], which says that the rank (over  $R$ ) of each map  $\phi_\mu$ , which we write as  $\text{rank}_R \phi_\mu$ , obeys the usual rank-nullity theorem of linear algebra, which here reads as

$$\text{rank}_R \phi_{\mu+1} + \text{rank}_R \phi_\mu = n_\mu, \quad (30)$$

and also describes how the ranks of maps *evaluated at any particular complex  $\mathbf{k}$*  behave: the rank over  $\mathbf{C}$  of the matrix of  $\phi_\mu$  at some  $\mathbf{k}$ , which we write as  $\text{rank}_{\mathbf{C}} \phi_\mu(\mathbf{k})$ , can be less than  $\text{rank}_R \phi_\mu$ :  $\text{rank}_{\mathbf{C}} \phi_\mu(\mathbf{k}) \leq \text{rank}_R \phi_\mu$  in general.

Even though in general the module of polynomial sections of the filled-band bundle is not free (as assumed in the proof under that additional hypothesis), the syzygy theorem shows that the general case is not much worse because the sequence in Eq. (29) terminates at the left. With that additional information, we can now present a proof in the general case.

*Proof of the no-go theorem in the general case:* We first describe the filled-band vector bundle  $E$ . At each  $\mathbf{k}$ , vectors in  $E$  are solutions  $w_{\mathbf{k}}$  to Eq. (21) (transformed to make  $g_{\mathbf{k}}$  finite, if necessary). As the bundle is analytic, it follows from the corollary that for any  $\mathbf{k}_0$  there is a set of solutions (with entries in  $R$ ) that are linearly independent in a neighborhood of  $\mathbf{k}_0$ . Hence,  $\phi_1$  has “full rank” (namely  $m$ ) over  $\mathbf{C}$ , that is  $\text{rank}_{\mathbf{C}} \phi_1(\mathbf{k}) = \text{rank}_R \phi_1 = m$  at all  $\mathbf{k}$ , and the bundle  $E$  can be identified with the image of  $\phi_1$ , viewed as evaluated at each  $\mathbf{k}$  in the Brillouin zone.

Then, we consider the sequence of vector bundles over the Brillouin zone

$$0 \longrightarrow E_d \xrightarrow{\phi_d} E_{d-1} \xrightarrow{\phi_{d-1}} \dots \xrightarrow{\phi_2} E_1 \xrightarrow{\phi_1} E_0. \quad (31)$$

The bundles  $E_\mu$  and  $E_0$  are trivial bundles of the same ranks as the corresponding  $F_\mu$  and  $F_0$ . Maps of vector bundles are maps of the vector space (fiber) over each  $\mathbf{k}$  in one bundle into that at the same  $\mathbf{k}$  in the other bundle, and the maps must vary continuously with  $\mathbf{k}$  [17]. In our case, the maps  $\phi_\mu$  are those that come from the syzygy theorem, viewed as evaluated at the value  $\mathbf{k}$  in question, so we use the same notation. (What we have done here can be described in the following way: A vector bundle can be understood as a “locally free” module over the ring of continuous complex functions on the base space, which here is the Brillouin zone. Given the exact sequence of modules over  $R$  given by the syzygy theorem, we have made a change of rings to obtain the sequence of vector bundles with the same maps  $\phi_\mu$ .) However, while the composite maps of  $\phi_{\mu+1}$  followed by  $\phi_\mu$  do vanish (that is,  $\phi_{\mu+1}\phi_\mu = 0$  for all  $\mu$ ), we have not yet shown that the sequence is exact. For a sequence of maps of vector bundles to be exact, it must be exact when evaluated at any given  $\mathbf{k}$ . It follows that for a sequence of the form shown, for each map the rank [ $\text{rank}_{\mathbf{C}} \phi_\mu(\mathbf{k})$ ] must be the same everywhere in the base space. That is, all the maps should have full rank at all  $\mathbf{k}$ , while we have seen that for the maps coming from the syzygy theorem this might fail.

To prove that the sequence of vector bundles is exact, we use the analyticity of the bundle  $E$ . As  $\phi_1$  has full rank  $m$  at any  $\mathbf{k}_0$ , we can assume (by permuting rows) that the first  $m$  rows of  $\phi_1$  are linearly independent (over  $\mathbf{C}$ ) at  $\mathbf{k}_0$ . Further, we can assume (by permuting columns) that the first  $m$  columns of this set of rows are linearly independent. We view this top-left  $m \times m$  block of  $\phi_1$  as the  $m \times m$  matrix  $U$  (which is invertible in the neighborhood of  $\mathbf{k}_0$ ). Rows of  $\phi_1$  (vectors  $w$ ) are determined by their first  $m$  entries (the vectors  $u$ ), so we can reduce  $\phi_1$  to an  $n_1 \times m$  matrix  $[\phi_1]$  consisting of the first  $m$  columns of  $\phi_1$ . Then, we can define an  $(n_1 - m) \times m$  matrix  $G_{\mathbf{k}}$  by the block matrix equation

$$\begin{pmatrix} I_m \\ G \end{pmatrix} = [\phi_1]U^{-1}. \quad (32)$$

Hence,  $G$  is analytic at  $\mathbf{k}_0$ , and we have

$$\phi_2 \begin{pmatrix} I_m \\ G \end{pmatrix} = \phi_2[\phi_1]U^{-1} = 0. \quad (33)$$

The idea is that  $G_{\mathbf{k}}$  is analogous to  $g_{\mathbf{k}}$ ; it characterizes the image of  $\phi_2$  as a bundle [for each  $\mathbf{k}$ ,  $G_{\mathbf{k}}$  is a point in  $G(n_1 - m, n_1)$ ]. At this point, we essentially repeat for  $G_{\mathbf{k}}$  the arguments used earlier for  $g_{\mathbf{k}}$  to conclude that  $\phi_2$  has full rank at  $\mathbf{k}_0$ , and hence at all real  $\mathbf{k}$ . Working back up the chain with this argument, we establish that all the evaluated maps  $\phi_\mu(\mathbf{k})$  have full rank at all real  $\mathbf{k}$ .

Given that the sequence of vector bundles is exact, the triviality of the image  $E$  (the bundle of interest) of  $\phi_1$  in  $E_0$  follows by a standard argument: Because of exactness, the image of  $E_d$  is faithful, and hence is a trivial bundle of rank  $n_d$  embedded in  $E_{d-1}$ , and  $E_{d-1}$  is also trivial. It follows that the quotient bundle  $E_{d-1}/E_d$  (i.e., the quotient in the fiber at each  $\mathbf{k}$ ) is also trivial. By exactness, this quotient bundle is isomorphic to the image of  $\phi_{d-1}$ , and the argument can be repeated. Hence, all the image bundles are trivial, ending with  $E$ , the image of  $\phi_1$ . That is the conclusion of the theorem. ■

The standard fact used in this step is that any short exact sequence of vector bundles splits: each bundle is equivalent to a direct sum of the image and the quotient [17]. We combined this with triviality to obtain the result.

This argument can be seen as the natural extension of those for the special cases with additional hypotheses above, for which the proofs were simply specializations of the general one. Note that we do not in the end claim to show in the more general cases that there is a set of  $m$  generators for the module  $M$  (that is,  $m$  polynomial sections) that are linearly independent at all real  $\mathbf{k}$ ; the trivialization whose existence is proved by the theorem may contain vectors with nonpolynomial entries. The Hilbert syzygy theorem was essential to the argument, as it established that the exact sequence of vector bundles terminates at the left. If we had been working only with analytic functions, as would be the case for states that are not TNSs, then no such finite exact sequence need exist globally over the Brillouin zone, although they would exist over smaller open neighborhoods [18]. It is only in the case of polynomial sections and an analytic bundle that we obtain such a general result. The syzygy theorem was the key statement that enabled us to argue from the hypothesis involving a local property (analyticity) to a result involving a global property (topological triviality).

**E. Parent Hamiltonian for analytic bundles**

In order to show that the obstruction to finding a gapped parent Hamiltonian is due to the nontriviality of the filled- and empty-band bundles, and does not extend to other examples with trivial bundles, we should show that when the bundle associated with a TNS is analytic, a gapped parent Hamiltonian does exist. When both bundles are analytic (trivial or not), a projection operator Hamiltonian (projector onto states in the empty band, minus the projector onto the states in the filled band) has a spectral gap of 2 between the filled and empty states, and its matrix elements (when rows and columns are labeled by positions  $x$  as well as by orbital labels) decay exponentially with the distance. For a TNS, we want to obtain a strictly short-range parent Hamiltonian, so in general a projection operator will not work. We can state our general result as a corollary to the proof of the theorem:

*Corollary.* Under the conditions of the no-go theorem, that is, when the filled-band bundle is both polynomial and analytic, a gapped single-particle parent Hamiltonian exists.

*Proof.* We use the matrix  $\phi_1$  from the proof of the no-go theorem, whose rows are a set of generators for all solutions to Eq. (21) over the ring  $R$ . When the filled-band bundle is analytic, the proposition implies that at any  $\mathbf{k}$  these generators span the  $m$ -dimensional fiber of the bundle. Then, at all  $\mathbf{k}$ ,  $\phi_1^\dagger \phi_1$  is positive semidefinite, its null space consists of the empty-band states, and it has rank  $m$  (compare with the examples of parent Hamiltonians in the previous section). It is noteworthy that, because the number of generators  $n_1$  is finite, the entries of this matrix are finite. Taking a matrix-valued function of  $\mathbf{k}$  constructed similarly for the empty-band bundle (say  $\tilde{\phi}_1 \tilde{\phi}_1^\dagger$ ), and subtracting  $\phi_1^\dagger \phi_1$ , we obtain a single-particle Hamiltonian as a matrix with entries that are TPs. The many-body version of this Hamiltonian has the TNS as ground state, and has an energy gap between the filled and empty bands at every  $\mathbf{k}$  in

the Brillouin zone. As the matrix elements are TPs, it is strictly short range in position space, as required. ■

**F. Remarks**

We emphasize that there are topological phases of free fermions in which the bundles are topologically nontrivial because of symmetry requirements, but are trivial when viewed as complex vector bundles ignoring symmetries. Our no-go theorem does not exclude the possibility of a TNS for one of these to have a gapped parent Hamiltonian; an example is the one-dimensional Kitaev chain [11] (there are no nontrivial complex vector bundles in one dimension). However, we do not know if this is possible in general. In cases in which the bundle is nontrivial as a complex vector bundle (that is, ignoring symmetry), our result prevents a gapped short-range parent Hamiltonian from existing. This is the situation in the  $p \pm ip$  examples in Sec. II.

**G. Compactly supported Wannier-type functions**

In this section, we wish to explain how the no-go theorem can be recast to give a result about what may be called compactly supported Wannier-type functions. Usually, Wannier functions are single-particle states constructed from states in a single band (by inverse Fourier transform from  $\mathbf{k}$  space) that are in some sense well localized in position space. Often, what is used is a section of a band bundle (say, a line bundle for a single band, such as one of the filled bands) that is normalized at each  $\mathbf{k}$ . This, together with its translates in real space, spans all single-particle states in the band. More generally, one can consider a set of  $m$  such sections (say, orthonormal ones) for  $m$  bands, or for a bundle of rank  $m$ . It has been recognized [14] that for a topologically nontrivial band, such a normalized section cannot be constructed such that it is continuous in  $\mathbf{k}$  (sometimes this is described as the absence of a suitable choice of gauge over the Brillouin zone because at each  $\mathbf{k}$  the choice reduces to the choice of a phase for the state). The reason is that such a choice amounts to a nonvanishing section of the bundle, which by definition does not exist for a nontrivial line bundle. (Similar remarks apply for  $m$  bands forming a higher-rank nontrivial bundle.) For a “section” that is not a true section because it fails to be continuous at some points in the Brillouin zone (say, on a set of measure zero), the states at those points are not in the Fourier decomposition of the Wannier functions, and the functions will have power-law tails in real space, and not be well localized.

In some applications, it is of interest to consider *compactly supported* Wannier-type functions, that is, functions constructed from a single band or set of bands, that have compact support in real space, and span all single-particle functions in the band (see, for example, Ref. [22]). In  $\mathbf{k}$  space, such compactly supported functions correspond to polynomial sections of the bundle associated with a single band or sets of bands (such as our filled-band bundle; we will continue to use that terminology here). (Once again, because of the use of translations in real space, it is sufficient to consider only polynomials in the ring  $R$ .) Notice that here we are no longer demanding (ortho)normalization at each  $\mathbf{k}$ , but we retain a weaker version of that property: in order not to overlook any

single-particle states, in either finite or infinite systems, we should require that in  $\mathbf{k}$  space the polynomial sections span the  $m$ -dimensional space of single-particle states at every  $\mathbf{k}$  in the Brillouin zone. To increase the likelihood that that is possible, we allow the set to contain more than  $m$  polynomial sections. Thus, we use the following:

*Definition.* A set of compactly supported Wannier-type functions for the filled bands is a set of polynomial sections that span the single-particle states in the filled-band bundle at every  $\mathbf{k}$  in the Brillouin zone (notice the definition includes the “completeness” property in  $\mathbf{k}$  space.) One might expect that, while the notion of compact support (or polynomials sections) is more restrictive than usual, allowing for more than  $m$  sections allows each of them to vanish at some  $\mathbf{k}$ , and then the set might get around the discontinuities associated with nonvanishing (or with orthonormal sets of) sections of nontrivial bundles. Nonetheless, we have the following version of the no-go theorem:

*Theorem.* If a set of compactly supported Wannier-type functions exists for a filled-band bundle, then the bundle is trivial when viewed as a complex vector bundle.

*Proof.* In this case, we take the given set of sections as generators for a module  $M$  over  $R$ , without using a system of linear equations to define them. By the syzygy theorem, there is a free resolution over  $R$  ending in  $M$ . The generators over  $R$  form the rows of a matrix of polynomials  $\phi_1$ , as before. (In the present case, the resolution may have one more term at the left than previously [21], but this makes no difference.) By hypothesis, the bundle over the Brillouin zone has full rank at each  $\mathbf{k}$ . Then, the same argument as in the proof of the no-go theorem shows that the generators of the kernel (over  $R$ ) of  $\phi_1$  form a matrix  $\phi_2$  which has full rank over  $\mathbb{C}$  at each  $\mathbf{k}$ . Repeating the argument as before shows that each map in the resolution has full rank over  $\mathbb{C}$  at all  $\mathbf{k}$  in the Brillouin zone, and hence the filled-band bundle spanned by the generators (at all  $\mathbf{k}$ ) is trivial as a complex vector bundle. ■

The corollary in Sec. III D above, together with the use of the Hilbert basis theorem as above before the discussion of the syzygy theorem, shows that for any analytic polynomial bundle, sets of compactly supported Wannier-type functions exist. Conversely, we also point out that if a set of compactly supported Wannier-type functions exists, then the bundle is analytic. To show this, one only has to take, for any  $\mathbf{k}_0$ , a subset of  $m$  of the sections that are linearly independent at  $\mathbf{k}_0$ ; this can always be done. Those  $m$  polynomial sections are analytic at  $\mathbf{k}_0$ , and so (e.g., by considering determinants) are then linearly independent throughout a neighborhood of  $\mathbf{k}_0$ ; this means the bundle is analytic.

#### IV. NON-FREE-FERMION PHASES

In the wave function in the free Chern-band example, without the particle-hole transformation, the component that has  $N_+$  particles of U(1) charge +1 (say, those in the 1 orbital), and  $N_-$  of charge -1 (say, those in the 2 orbital) has  $N_+ = N_-$ , and is a determinant that when all particles are well separated has the form (ignoring boundary conditions, for simplicity)

$$\det \left( \frac{1}{z_i - w_j} \right) = \frac{\prod_{i < j} (z_i - z_j) \prod_{k < l} (w_k - w_l)}{\prod_{m,n} (z_m - w_n)}, \quad (34)$$

where we use complex coordinates [e.g.  $z = x + iy$  in place of  $\mathbf{x} = (x, y)$ ]  $z_i, i = 1, \dots, N_+$  for charge +1,  $w_k, k = 1, \dots, N_-$  for charge -1 particles, and the equality of the two sides is just the Cauchy determinant identity. The norm square of the state  $|\psi_A\rangle$  can then be identified as the partition function of a Coulomb plasma (see, e.g., Ref. [23]), with fugacity  $|\kappa_1|^2$  for either type of particle. We will assume  $|\kappa_1|$  is small so that particles are well separated. Then, it is known that this plasma is in a screening phase, which confirms the topological identification of the state.

Next, we will take  $Q$  copies of this TNS, and impose the constraint that the number of particles in each orbital must be either 0 or  $Q$  (one from each copy) [6,7]. Then, the composite of  $Q$  fermions [with U(1) charge either +1 or -1] will be regarded as the physical particle of the state, and is a boson for  $Q$  even, fermion for  $Q$  odd. The resulting wave function for these particles is then the  $Q$ th power of that for the one-copy case. The important point is that such a “product” construction yields a state that is again a TNS (because the constraints are local), but not Gaussian/free fermion for  $Q > 1$ .

The norm square of the product wave function is (at long distances) again a plasma, but with exponent  $Q$  times larger. A renormalization-group (RG) analysis applies to these plasmas [23], and shows that for asymptotically small positive fugacity, screening occurs for  $Q \leq 2$ . For  $Q = 2$ , the screening length is exponentially large in  $1/|\kappa_1|$  as  $|\kappa_1| \rightarrow 0$ . In this case, the TNS is in the same topological phase as the Laughlin  $\nu = \frac{1}{2}$  state for bosons, or SU(2) level 1, phase. The state possesses fractional-statistics vortex excitations (semions) that carry fractional U(1) charge, and there are two (or more, possibly depending on the parent Hamiltonian) ground states for the system on a finite torus with periodic boundary conditions. For  $Q > 2$ , the plasma does not screen at small  $|\kappa_1|$ , but does at sufficiently large  $|\kappa_1|$ , and then corresponds to the  $1/Q$  Laughlin state. This construction can be extended to obtain, for example, the non-Abelian SU(2) level  $k > 1$  Chern-Simons theory [12], or the Read-Rezayi states for bosons [24,25]. For these TNSs, we do not at present have parent Hamiltonians.

#### V. CONCLUSION

It is possible to construct wave functions that are TNSs for phases of matter that possess chiral gapless edge modes. This may have important implications for numerical simulation or theoretical analysis of such phases. However, the parent Hamiltonians for the free-fermion TNSs are gapless, as we proved in our no-go theorem; it is not clear if the same will be true for the non-free-fermion chiral TNSs.

#### ACKNOWLEDGMENTS

We thank B. Bauer, F. Verstraete, and M. Zaletel for asking (independently) the question that motivated this work, and B. Bradlyn, D. Freed, G. Moore, R. Howe, and Z. Wang for discussions. This work was supported by a Yale Postdoctoral Prize Fellowship (J.D.) and by NSF Grants No. DMR-1005895 and No. DMR-1408916 (N.R.). We thank the Simons Center for Geometry and Physics at SUNY Stony Brook, where the work was initiated, for its hospitality for both authors and its support for N.R., and the Aspen Center for Physics for its

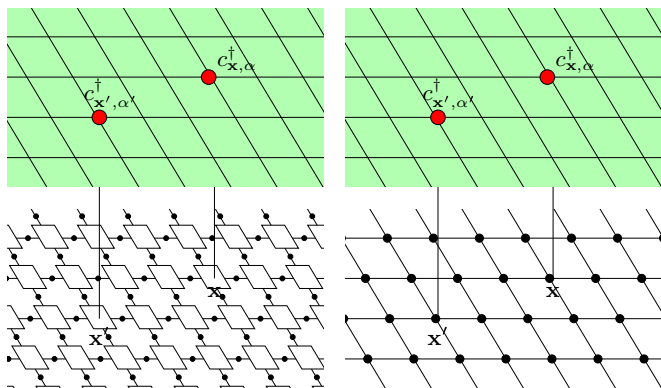


FIG. 3. (Color online) Particles on the square lattice  $\mathbb{Z}^2$  (in green) are created by operators  $c_{x, \alpha}^\dagger$  acting on the vacuum  $|0\rangle$ , generating a Hilbert (Fock) space of physical states for each site  $\mathcal{H}_x$ . One associates one tensor  $T_x \in \mathcal{H}_x \otimes V^h \otimes (V^h)^* \otimes V^v \otimes (V^v)^*$  per site  $\mathbf{x}$ . The spaces  $V^{h,v}$  are finite-dimensional vector spaces, there is one copy of the space  $V^{h,v}$  and its dual  $(V^{h,v})^*$  per (horizontal or vertical) edge of the lattice, represented by solid dots (left figure). Tracing over all the spaces  $V^{h,v}$ , namely, using the canonical evaluation map  $V \otimes V^* \mapsto \mathbb{C}$  on each edge, we get a state in the many-body Hilbert space  $\bigotimes_{\mathbf{x} \in \mathbb{Z}^2} \mathcal{H}_x$ . In the main text, we used a different version of a TNS, with auxiliary variables living on the sites, as in the right figure; it can, however, be reformulated as a usual TNS.

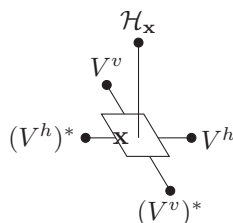
hospitality and partial support for N.R. under NSF Grant No. PHY-1066293 in 2013 while the first version of this work was completed.

#### APPENDIX A: TRANSFORMING A TNS WITH AUXILIARY VARIABLES ON SITES TO A TNS WITH AUXILIARY VARIABLES ON EDGES

The Gaussian TNS we construct in Sec. II, with auxiliary variables attached to the physical sites (illustrated in Fig. 3, right) can be recast in the form of usual (and in the present case, still Gaussian) TNS (Fig. 3, left). First, let us define what a ‘‘usual TNS’’ is. In the case of bosonic degrees of freedom, a TNS is obtained from a function  $\mathbf{x} \mapsto T_x$  which, to every site of the square lattice  $\mathbf{x} \in \Lambda$ , associates a tensor

$$T_x \in \mathcal{H}_x \otimes V^h \otimes (V^h)^* \otimes V^v \otimes (V^v)^*, \quad (\text{A1})$$

where  $\mathcal{H}_x$  is the physical space on the site, and  $V^h$ ,  $(V^h)^*$  [ $V^v$ ,  $(V^v)^*$ ] is the horizontal (vertical) auxiliary space and its dual. Such a tensor is conveniently represented as follows:



For fermionic degrees of freedom, the construction parallels the one for bosons, but  $\mathcal{H}_x$ ,  $V^h$ , and  $V^v$  must be  $\mathbb{Z}_2$ -graded vector spaces, while  $(V^h)^*$  and  $(V^v)^*$  are their  $\mathbb{Z}_2$ -graded duals, and the tensor product  $\otimes$  is  $\mathbb{Z}_2$  graded (see Appendix A in [26]

for definitions). It is customary to represent  $\mathbb{Z}_2$ -graded vector spaces as spaces of polynomials of Grassmann variables. Indeed, one can view  $V^h$  as (isomorphic to) the  $\mathbb{Z}_2$ -graded vector space generated by a set of  $P$  Grassmann variables living on the edge at position  $\mathbf{x} + \mathbf{i}/2$ :

$$\text{span}\{(\eta_{\mathbf{x}+\mathbf{i}/2}^1)^{n_1} \dots (\eta_{\mathbf{x}+\mathbf{i}/2}^P)^{n_P} \mid n_1, \dots, n_P = 0, 1\}. \quad (\text{A2})$$

The  $\mathbb{Z}_2$  grading is then the number (mod 2) of Grassmann variables in each Grassmann monomial. Similarly, one can think of  $V^v$  as being (isomorphic to) the  $\mathbb{Z}_2$ -graded vector space generated by  $P$  Grassmann variables on the edge at position  $\mathbf{x} + \mathbf{j}/2$ . The (left) dual  $V^*$  is then generated by another set of  $P$  Grassmann variables:

$$\text{span}\{(\eta_{\mathbf{x}+\mathbf{j}/2}^{1*})^{n_1} \dots (\eta_{\mathbf{x}+\mathbf{j}/2}^{P*})^{n_P} \mid n_1, \dots, n_P = 0, 1\}. \quad (\text{A3})$$

With these notations, the canonical evaluation map is nothing but the Berezin integral

$$v^* \otimes u \in V^* \otimes V \mapsto \int [d\eta^1 d\eta^{1*} \dots d\eta^P d\eta^{P*}] e^{\sum_p \eta^{p*} \eta^p} v^* u. \quad (\text{A4})$$

The tensor  $T_x$  must have degree 0, namely, it must be a sum of terms with an even total number of Grassmann variables and physical fermions. Finally, the physical state, which is a state in  $\bigotimes_{\mathbf{x} \in \mathbb{Z}^2} \mathcal{H}_x$ , is defined as  $\text{Tr}[\bigotimes_{\mathbf{x}} T_x]$ , where one traces over all the auxiliary spaces  $V^{h,v}$  using the canonical evaluation map (see also Fig. 3, left).

The Gaussian TNS that we exhibit in Sec. II have not been expressed in the form of local tensors attached to the sites  $\mathbf{x} \in \mathbb{Z}^2$ . Instead, they are translation-invariant Gaussian states of the form

$$\int [d\xi] \prod_{\text{edges } (xy)} e^{\xi_x^t \cdot A \cdot \xi_y} \prod_{\text{sites } z} e^{\xi_z^t \cdot B \cdot \xi_z} e^{\xi_z^t \cdot \kappa \cdot c_x^\dagger} |0\rangle, \quad (\text{A5})$$

where  $\xi_x = (\xi_x^1, \dots, \xi_x^P)$ ,  $A^{h,v}$  and  $B$  are  $P \times P$  matrices, while  $\kappa$  is  $P \times n$ . To recast this expression in the form of a  $\mathbb{Z}_2$ -graded TNS, we introduce new Grassmann variables  $\eta_{\mathbf{x}+\mathbf{i}/2} = (\eta_{\mathbf{x}+\mathbf{i}/2}^1, \dots, \eta_{\mathbf{x}+\mathbf{i}/2}^P)$ ,  $\eta_{\mathbf{x}+\mathbf{j}/2} = (\eta_{\mathbf{x}+\mathbf{j}/2}^1, \dots, \eta_{\mathbf{x}+\mathbf{j}/2}^P)$ ,  $\eta_{\mathbf{x}+\mathbf{i}/2}^* = (\eta_{\mathbf{x}+\mathbf{i}/2}^{1*}, \dots, \eta_{\mathbf{x}+\mathbf{i}/2}^{P*})$ ,  $\eta_{\mathbf{x}+\mathbf{j}/2}^* = (\eta_{\mathbf{x}+\mathbf{j}/2}^{1*}, \dots, \eta_{\mathbf{x}+\mathbf{j}/2}^{P*})$ , and

$$W_x = \xi_x^t \cdot A^h \cdot \eta_{\mathbf{x}+\mathbf{i}/2} + \xi_x^t \cdot A^v \cdot \eta_{\mathbf{x}+\mathbf{j}/2} + \xi_x^t \cdot B \cdot \xi_x \\ - \xi_x^t \cdot \eta_{\mathbf{x}-\mathbf{i}/2}^* - \xi_x^t \cdot \eta_{\mathbf{x}-\mathbf{j}/2}^*, \quad (\text{A6})$$

such that the integration over the  $\eta$  variables on each edge gives back the exponential weights in (A5). But, instead of tracing out the  $\eta$  variables, we now integrate out the onsite variables  $\xi_x$ . This gives a  $\mathbb{Z}_2$ -graded tensor for each site, as we want:

$$T_x = \int [d\xi_x] e^{W_x} e^{\xi_x^t \cdot \kappa \cdot c_x^\dagger} \in \mathcal{H}_x \otimes V^h \otimes (V^h)^* \otimes V^v \otimes (V^v)^*. \quad (\text{A7})$$

#### APPENDIX B: CHERN-BAND EXAMPLE: EXPLICIT FORM

In Sec. II, we sketch the construction of a Gaussian TNS which corresponds to a filled band with Chern number 1. Here, we give a few more details about this state. We

start with a BCS state of the form (A5), with  $P = 4$  Grassmann variables (equivalently, one could write this state

with  $P = 2$  complex Grassmann variables). The matrices  $A$ ,  $B$ , and  $\kappa$  are chosen such that

$$\xi_{\mathbf{x}+i}^t \cdot A^h \cdot \xi_{\mathbf{x}} = \begin{pmatrix} \xi_{\mathbf{x}+i}^1 & \xi_{\mathbf{x}+i}^2 & \xi_{\mathbf{x}+i}^3 & \xi_{\mathbf{x}+i}^4 \end{pmatrix} \begin{pmatrix} -i & \lambda & 0 & 0 \\ -\lambda & -i & 0 & 0 \\ 0 & 0 & -i & \lambda \\ 0 & 0 & -\lambda & -i \end{pmatrix} \begin{pmatrix} \xi_{\mathbf{x}}^1 \\ \xi_{\mathbf{x}}^2 \\ \xi_{\mathbf{x}}^3 \\ \xi_{\mathbf{x}}^4 \end{pmatrix}, \quad (\text{B1a})$$

$$\xi_{\mathbf{x}+j}^t \cdot A^v \cdot \xi_{\mathbf{x}} = \begin{pmatrix} \xi_{\mathbf{x}+j}^1 & \xi_{\mathbf{x}+j}^2 & \xi_{\mathbf{x}+j}^3 & \xi_{\mathbf{x}+j}^4 \end{pmatrix} \begin{pmatrix} 1 & \lambda & 0 & 0 \\ -\lambda & -1 & 0 & 0 \\ 0 & 0 & 1 & \lambda \\ 0 & 0 & -\lambda & -1 \end{pmatrix} \begin{pmatrix} \xi_{\mathbf{x}}^1 \\ \xi_{\mathbf{x}}^2 \\ \xi_{\mathbf{x}}^3 \\ \xi_{\mathbf{x}}^4 \end{pmatrix}, \quad (\text{B1b})$$

$$\xi_{\mathbf{x}}^t \cdot B \cdot \xi_{\mathbf{x}} = \begin{pmatrix} \xi_{\mathbf{x}}^1 & \xi_{\mathbf{x}}^2 & \xi_{\mathbf{x}}^3 & \xi_{\mathbf{x}}^4 \end{pmatrix} \begin{pmatrix} 0 & -2\lambda & 0 & 0 \\ 2\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\lambda \\ 0 & 0 & 2\lambda & 0 \end{pmatrix} \begin{pmatrix} \xi_{\mathbf{x}}^1 \\ \xi_{\mathbf{x}}^2 \\ \xi_{\mathbf{x}}^3 \\ \xi_{\mathbf{x}}^4 \end{pmatrix}, \quad (\text{B1c})$$

$$\xi_{\mathbf{x}}^t \cdot \kappa \cdot c_{\mathbf{x}}^\dagger = \kappa_1 \frac{\xi_{\mathbf{x}}^1 - i\xi_{\mathbf{x}}^3}{\sqrt{2}} c_{\mathbf{x},1}^\dagger + \kappa_2 \frac{\xi_{\mathbf{x}}^1 + i\xi_{\mathbf{x}}^3}{\sqrt{2}} c_{\mathbf{x},2}^\dagger. \quad (\text{B1d})$$

Here,  $\lambda \in \mathbb{R}$ , and  $\kappa_1, \kappa_2 \in \mathbb{C}$  are free parameters. In momentum space, this state takes the form

$$\exp \left( \int \frac{d^2\mathbf{k}}{(2\pi)^2} g_{\mathbf{k}} c_{\mathbf{k},2}^\dagger c_{-\mathbf{k},1}^\dagger \right) |1\rangle, \quad (\text{B2})$$

where the function  $g_{\mathbf{k}}$  is the following propagator in the auxiliary theory of the Grassmann variables:

$$\begin{aligned} \frac{g_{\mathbf{k}}}{\kappa_1 \kappa_2} &= \frac{\int [\prod d\xi^p] e^S \frac{\xi_{\mathbf{k}}^1 - i\xi_{\mathbf{k}}^3}{\sqrt{2}} \frac{\xi_{-\mathbf{k}}^1 + i\xi_{-\mathbf{k}}^3}{\sqrt{2}}}{\int [\prod d\xi^p] e^S} \\ &= \frac{\int [\prod d\xi^p] e^S \xi_{\mathbf{k}}^1 \xi_{-\mathbf{k}}^1}{\int [\prod d\xi^p] e^S} \\ &= \frac{\sin k_x - i \sin k_y}{(\sin k_x)^2 + (\sin k_y)^2 + \lambda^2 [2 - \cos k_x - \cos k_y]^2}. \end{aligned} \quad (\text{B3})$$

We see that our BCS state depends only on the product  $\kappa_1 \kappa_2$ , so we can take  $\kappa_2 = \kappa_1$  without loss of generality. Finally, performing the particle-hole transformation mentioned in the main text, namely  $|0\rangle \rightarrow |1,0\rangle$  and  $c_{\mathbf{x},1}^\dagger \rightarrow c_{\mathbf{x},1}$ , we obtain the state

$$\exp \left( \int \frac{d^2\mathbf{k}}{(2\pi)^2} g_{\mathbf{k}} c_{\mathbf{k},2}^\dagger c_{\mathbf{k},1} \right) |1,0\rangle, \quad (\text{B4})$$

which corresponds to a filled band; one can check that it has Chern number 1 as soon as  $\lambda \neq 0$ .

### APPENDIX C: ALTERNATIVE WAY TO EXPRESS A PAIRED STATE WITH ANY $g_{\mathbf{k}} = v_{\mathbf{k}}/u_{\mathbf{k}}$ AS A GAUSSIAN TNS

For any  $g_{\mathbf{k}} = v_{\mathbf{k}}/u_{\mathbf{k}}$ , where  $u_{\mathbf{k}}$  and  $v_{\mathbf{k}}$  are trigonometric polynomials, we can obtain the state from a Gaussian TNS that is slightly more general in form. For simplicity, we consider only the  $n = 1$  examples, as in Sec. II B. First write  $g_{\mathbf{k}}$  as  $g_{\mathbf{k}} = v_{\mathbf{k}} v_{-\mathbf{k}} / (v_{-\mathbf{k}} u_{\mathbf{k}})$ . We will use a single Grassmann variable  $\xi_{\mathbf{x}}$  on each site. For the  $\kappa$  terms, we write in  $\mathbf{k}$  space  $v_{\mathbf{k}} \xi_{-\mathbf{k}} c_{\mathbf{k}}^\dagger$  for each  $\mathbf{k}$ , and for the  $A$  and  $B$  terms in  $\mathbf{k}$  space  $\xi_{-\mathbf{k}} v_{-\mathbf{k}} u_{\mathbf{k}} \xi_{\mathbf{k}}$ . Note that in real space, both kinds of terms are strictly short range. Then, clearly integrating out the  $\xi$  variables produces the desired form. Notice that if  $g_{\mathbf{k}}$  is odd under  $\mathbf{k} \rightarrow -\mathbf{k}$ , then so is  $v_{-\mathbf{k}} u_{\mathbf{k}}$ , as required in the Grassmann bilinear form.

This form differs from our original TNS expression in that the coupling of  $c_{\mathbf{x}}$  to  $\xi_{\mathbf{x}}$  is no longer just onsite. Moreover, there is now only one variable  $\xi$  per site ( $P = 1$ ). The first change compensates the second, so that when both forms exist, the results, including the entanglement spectrum, must be the same. Then, the rank of the entanglement spectrum cannot be read off simply from the number of  $\xi$  variables in the present form. However, due to the TPs (alternatively, polynomials in  $R$ ) in the coefficients  $A$ ,  $B$ ,  $\kappa$ , the range of these terms in real space is bounded, and this should limit the rank of the entanglement spectrum in a similar way. Notice also that many results in the main text use only the fact that  $g_{\mathbf{k}}$  is a ratio of TPs, and so hold for the class discussed here.

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