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Breakdown of the universality of the Kadowaki-Woods Ratio in multi-band metals

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We calculate the Kadowaki-Woods ratio (KWR) in Fermi liquids with arbitrary band structures. We find that, contrary to the single-band case, the ratio is not generally independent of the effects of electronic correlations (universal). This is very surprising given the experimental findings of a near universal KWR in many multiband strongly correlated metals. We identify a limit where the universality of the ratio is recovered. We discuss the KWR in Dirac semimetals and find that the KWR is independent of correlations, but strongly dependent on the doping of the system: for massless fermions, the KWR is proportional to the inverse square of the carrier density.

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I. INTRODUCTION

Fermi-liquid theory describes the low-temperature behavior of the vast majority of metals extremely well [1–6]. One of the beauties of Fermi-liquid theory is that it reduces the description of the interacting electron fluid to a small number of (Landau) parameters. Therefore, ratios in which these parameters cancel, such as the Wilson-Sommerfeld ratio and Wiedemann-Franz law [1,7], provide important tests of Fermi-liquid theory.

In a Fermi liquid, the electronic contributions to the resistivity, $[\rho_{el}(T) = AT^2]$, and heat capacity, $[C_{el}(T) = \gamma T]$, are both governed by the effective mass m^*—roughly speaking, $A \propto m^{*2}$ and $\gamma \propto m^*$. So the Kadowaki-Woods ratio, A/γ^2, should be constant in a Fermi liquid [8–12]. More precisely, one might expect correlations to leave the Kadowaki-Woods ratio (KWR) unrenormalized because a Kramers-Kronig transformation relates the real and imaginary parts of the self-energy [12–14], which determine the electronic contributions to the heat capacity and resistivity, respectively. This means that the KWR is somewhat similar to a fluctuation-dissipation theorem.

First Rice [15] and later Kadowaki and Woods [16] found that A/γ^2 is approximately constant within classes of materials (transition metals and heavy-fermion compounds, respectively). However, the ratio differs by two orders of magnitude between these two classes. It was subsequently discovered that the Kadowaki-Woods ratio in transition metals and organic charge-transfer salts can be even larger than in the heavy fermions (see Refs. [14,17] and references therein).

It was long believed [12,18] that the size of the KWR gave an indication of the strength of the electron-electron scattering [19]. However, more recently, it has been argued that this is incorrect [14,17]. Rather, the large variations in the KWR between different classes of materials can be explained almost entirely by taking into account noninteracting properties of the materials (e.g., electron density and dimensionality) [14,17]. Furthermore, it has been shown [14] that the modified KWR takes the same predicted value, $A/\gamma^2 f = 81/4\pi\hbar k_B^2 e^2$ (where f is a material-specific function of the noninteracting band structure, defined below), in a large range of transition metals, charge-transfer salts, heavy-fermion compounds, and elemental metals; a result which has since been verified in many other materials [20–26].

Deviations from this predicted universal value of the KWR could provide an indication of non-Fermi-liquid behavior. However, most previous calculations of the (modified) KWR [8,10,14] have focused on simple, single-band models with toy dispersion relations, e.g., spherical Fermi surfaces.

In this paper, we derive a modified KWR for Fermi liquids with arbitrary dispersion relations, including band structures with multiple bands crossing the Fermi surface (henceforth when we refer to the number of bands in a material, this counts only those bands that cross the Fermi level, since bands away from the Fermi surface will only enter weakly via high-order scattering terms). We find that the universality of the KWR evident in the single-band expression is not a general feature of the multiple-band case. In the most general case, the strength of electronic correlations *does* affect the value of the KWR. This is extremely puzzling as the KWR is found to be close to its universal value in many multiband systems (see, in particular, Fig. 2 of [14]). However, if the renormalization is the same in all bands (in a sense made precise in Sec. V), then correlations do cancel in the KWR.

Previous studies of the KWR in systems with orbital degeneracy [27,28] or multiple bands [17] have found that either of these can cause some variation in the KWR. However, this body of work has assumed that the electronic correlations are identical in all bands/orbitals. An important example, where this may have led to an incorrect interpretation of the measured KWR, is $\mathrm{Na}_x\mathrm{CoO}_2$. The large KWR was initially interpreted as evidence of "giant electron-electron scattering" [18] without any possible effects of multiple bands in this system [29,30] or other material-specific factors such as the unit cell volume or the dimensionality of the system [14,17].

We also demonstrate that in Dirac semimetals, the KWR depends strongly on the carrier density n, $R_{KW} \propto 1/n^2$. In contrast, for a massive semimetal with an isotropic quadratic dispersion, $R_{KW} \propto 1/n^{7/3}$ [14]. In semimetals, the carrier density is both small and tunable, allowing for a large variation in n, and a direct test of this prediction.

The remainder of this paper is laid out as follows. In Sec. II, we introduce the phenomenological local Fermi-liquid theory on which our calculations are based. In Sec. III, we calculate the resistivity of an arbitrary multiband Fermi liquid by calculating the conductivity from linear response theory. In Sec. IV, we calculate the effect of multiple bands on the heat

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capacity. In Sec. V, we combine these results to determine the form of the Kadowaki-Woods ratio in arbitrary band structures. In Sec. VI, we apply this form of the KWR to simple models of Dirac semimetals.

II. PHENOMENOLOGICAL LOCAL FERMI-LIQUID THEORY

There are two very different ways one might proceed to evaluate the KWR. One could start from a microscopic model for a specific material or class of materials and proceed to evaluate the self-energy within some approximation and thence the resistivity and heat capacity. Indeed, many authors have taken this approach [6,8–11,27]. However, this approach necessarily entails limitations; no single microscopic Hamiltonian applies to all of the wide range of materials for which the KWR has been measured. An alternative approach, which we follow below, is motivated by the experimental observation that the universality of the KWR seems to be a feature of Fermi liquids in general, and so we wish to describe the most general Fermi liquid possible. This motivates the study of phenomenological forms of the self-energy for Fermi liquids. Microscopic and phenomenological approaches to the KWR are extremely complementary and, probably, both are required for a complete understanding of what this ratio tells us.

Miyake *et al.* [12] introduced a phenomenological form for the self-energy of a single-band local Fermi liquid. The natural generalization (see Supplemental Material [31]) of this model to include multiple Fermi surfaces is to assume that in each band $\tilde{a} = (a,\sigma)$, where a denotes the band and σ the spin (giving $2N_b$ spin bands), the imaginary part of the self-energy takes the phenomenological form

$$\Sigma''_{\tilde{a}\tilde{a}}(\omega) = -\frac{\hbar}{2\tau_{0;\tilde{a}}} - s_{\tilde{a}}\frac{\omega^2 + (\pi k_B T)^2}{(\omega^*_{\tilde{a}})^2} \quad \text{for } |\omega^2 + (\pi k_B T)^2| < (\omega^*_{\tilde{a}})^2, \tag{1a}$$

$$= -\left(\frac{\hbar}{2\tau_{0;\tilde{a}}} + s_{\tilde{a}}\right) F\left\{\left[\frac{\omega^2 + (\pi k_B T)^2}{(\omega^*_{\tilde{a}})^2}\right]^{\frac{1}{2}}\right\} \quad \text{for } |\omega^2 + (\pi k_B T)^2| > (\omega^*_{\tilde{a}})^2, \tag{1b}$$

where the energy ω is measured from the Fermi level, $\tau^{-1}_{0;\tilde{a}}$ is the impurity scattering rate, $s_{\tilde{a}}$ is the maximum electron-electron scattering rate compatible with a Fermi liquid (see below), $F(x)$ is a monotonically decreasing function with $F(1) = 1$ and $F(\infty) = 0$, and $\omega^*_{\tilde{a}}$ is an energy scale characterizing the strength of the many-body correlations (we will show below that the quasiparticle weight $Z_{\tilde{a}} \approx \omega^*_{\tilde{a}}/4s_{\tilde{a}}$).

Equation (1) makes a number of important assumptions. Therefore, it is important to understand why this is a reasonable *ansatz*.

Local (momentum-independent) self-energy. Given a diagonal self-energy, the effective mass of the quasiparticles in the $\tilde{a}$th spin band of a Fermi liquid is given by

$$\frac{m^*_{\tilde{a}}}{m_{\tilde{a}}} = \left|\frac{1 - \partial\Sigma'_{\tilde{a}\tilde{a}}(\boldsymbol{k},\omega)/\partial\omega}{1 + \partial\Sigma'_{\tilde{a}\tilde{a}}(\boldsymbol{k},\omega)/\partial\varepsilon_{\boldsymbol{k}\tilde{a}}}\right|_{k=k_F;\,\omega=0}, \tag{2}$$

where $\Sigma'_{\tilde{a}\tilde{a}}(\boldsymbol{k},\omega)$ is the real part of the self-energy for the $\tilde{a}$th spin band, $m_{\tilde{a}}$ is the bare (band) mass, $\varepsilon_{\boldsymbol{k}\tilde{a}}$ is the bare dispersion, and k_F is the Fermi energy [32]. The KWR is of interest in strongly correlated materials where $m^*_{\tilde{a}} \gg m_{\tilde{a}}$, which implies $|\partial\Sigma'_{\tilde{a}\tilde{a}}/\partial\omega| \gg 1$ and hence that the frequency dependence of the self-energy is of primary importance. Furthermore, in this limit, it can be shown [33,34] that $m^*_{\tilde{a}}/m_{\tilde{a}} \simeq 1/Z_{\tilde{a}}$, where the quasiparticle weight is $Z_{\tilde{a}} = [|1 - \partial\Sigma'_{\tilde{a}\tilde{a}}(\boldsymbol{k},\omega)/\partial\omega|_{k=k_F;\,\omega=0}]^{-1}$.

Further motivation for considering a local self-energy comes from the successes of dynamical mean-field theory (DMFT) [35,36]. First, it can be shown that in infinite dimensions, the self-energy is exactly momentum independent [35]. Furthermore, in finite dimensions, one can construct a self-consistent theory on the assumption of a local self-energy, DMFT. This has been shown to give a good description of many transport and thermodynamic properties of the metallic states of materials discussed here—such as heavy fermions, transition-metal oxides, and organic molecular crystals [35–39]. Thus, the phenomenological theory constructed from Eq. (1) can be motivated as a "poor man's DMFT." A momentum-independent self-energy is also found in the Fermi-liquid regime of Kotliar-Ruckenstein slave boson theories [40,41].

Low energy/temperature behavior. The assumption that the metal is a Fermi liquid at low energies and temperatures places significant constraints on the form of the low-energy part of the self-energy. From this assumption, we have $\Sigma'' \sim \overline{\omega}^2$, where $\overline{\omega} \equiv \sqrt{\omega^2 + (\pi k_B T)^2}$ [2,6]. The energy factor ω measures the area of phase space available for an electron-electron scattering process at zero temperature. The thermal factor includes the extra phase space allowed by the thermal distribution of electrons at finite temperature [42,43].

Maximum resistivity in the Fermi-liquid regime. The low-energy quadratic growth of Σ'' cannot continue indefinitely; the fact that the self-energy is a causal response function means that $\Sigma'' \to 0$ as $\overline{\omega} \to \infty$. Furthermore, there is a maximum resistivity compatible with a Fermi-liquid regime [37,44–47]. There are several ways one might estimate the value of this maximum. Interestingly, the estimates below give the same answer to within factors of order unity, suggesting a deep underlying connection. (i) A Fermi liquid is characterized by the coherent transport of quasiparticles. The Mott-Ioffe-Regel limit states that for coherent transport, the mean free path $l \gtrsim c \sim 1/k_F$, where c is the lattice constant [37,44–47]. For systems with relatively weak electron-phonon interactions, the quadratic scattering rate observed persists up to the Mott-Ioffe-Regel limit [23,39,48,49]. (ii) As the self-energy arises from the elastic scattering of (coherent) quasiparticles, conservation of particle number implies that the scattering cross section cannot exceed the unitarity limit [7,50]. For s-wave scattering at unitarity, only one length scale remains in the problem and one finds that $l \gtrsim c \sim 1/k_F$. It has been observed that the scattering rate is of the order of the unitarity limit for s-wave scattering in many strongly correlated systems, most notably when Kondo physics is at play [7,12,42,51].

It is straightforward to show that (up to factors of order unity) these constraints imply that the maximum imaginary part of the self-energy consistent with a Fermi liquid is $\Sigma''_{\tilde{a}\tilde{a}} = s_{\tilde{a}} = W^0_{\tilde{a}}$, where $W^0_{\tilde{a}}$ is the energy from the Fermi energy to the nearest band edge, i.e., it is approximately (but typically less than) the bare half-bandwidth of the relevant band, i.e., the scattering energy $\hbar/\tau$ cannot exceed the energy of a quasiparticle in a Fermi liquid. At times it will be helpful to specialize to the quadratic dispersion relation appropriate for free fermions, whence one finds, independent of the dimension of the system, that $s_{\tilde{a}} = E^0_{F\tilde{a}} \simeq n_{\tilde{a}}/D_{0;\tilde{a}}$, where $E^0_{F\tilde{a}}$ is the bare Fermi energy, $n_{\tilde{a}}$ is the density of charge carriers in band $\tilde{a}$, and $D_{0;\tilde{a}}$ is the bare density of states in band $\tilde{a}$ at the Fermi energy. For free fermions, we will choose the factor of order unity so as to reproduce the one-band expression introduced by Miyake *et al.* [12] and set $s_{\tilde{a}} = 2n_{\tilde{a}}/3\pi D_{0\tilde{a}}$.

We will not be concerned here with the details of what happens once the scattering rate has reached the Mott-Ioffe-Regel limit. However, for completeness, we note two important possibilities: (i) resistivity saturation [52] and (ii) entering the "bad metal" regime [36,37]. The latter is associated with the absence of well-defined quasiparticles at high temperatures [47] and is a clear indication that Fermi-liquid theory is no longer applicable.

Interactions between electrons in different bands. Equation (1) is the natural form for a local Fermi liquid provided interband interactions are weak compared to the intraband interactions [53]. If the two are comparable (as in some multiorbital models), then the self-energy is multiplied by an additional factor of $N_b - 1$ [27].

Many-body correlations. The remaining issue to be determined is the size of the coefficient of the quadratic term in $\Sigma''_{\tilde{a}\tilde{a}}$. This is the key issue that determines the strength of the many-body correlations. It is mathematically convenient to parametrize this as $s_{\tilde{a}}/(\omega^*_{\tilde{a}})^2$ as this means that the maximum resistivity compatible with a Fermi liquid is realized when $\overline{\omega} = \omega^*$. Thus, ω^* parametrizes all of the many-body effects present in the material.

High energies/temperatures. Fermi-liquid theory is an effective low-energy theory and therefore has little to say about the form of self-energy at high energies and/or temperatures. Nevertheless, the Kramers-Kronig transformation, which we will apply below, is a Hilbert transformation that extends over all $\overline{\omega}$. Therefore, we do require some constraints on the high-energy behavior. Here, we assume a monotonic decrease in Σ'' following Miyake *et al.* [12]. We will show in Sec. IV [see particularly Eq. (11)] that reasonable changes in the high-energy form of Σ'' only change our results by factors of order unity. This is because we only concern ourselves with the low-energy limit of Σ' relevant to Fermi-liquid quasiparticles, so that when we perform the Kramers-Kronig transform, the poles are restricted to the low-energy (quadratic) part of Σ''.

Umklapp scattering. Below we focus on the pure limit, $\tau_{0;\tilde{a}} \to \infty$. As normal electron-electron scattering conserves momentum, an electrical resistance can only arise if there are Umklapp processes [32]. However, it can be shown that all scattering processes contribute to the resistance when Umklapp processes are present [54,55].

III. CONDUCTIVITY FROM THE KUBO FORMULA

In general, the contribution to the intraband self-energy from interband terms scales quadratically with the intraband self-energy, such that, as long as the self-energy is small, the intraband contribution will dominate (see Supplemental Material [31]). When intraband scattering is the dominant contribution to the scattering rate, the diagonal component of the conductivity tensor, σ_{xx}, for a material with N_b bands crossing the Fermi surface in the low-temperature limit is [56]

$$\sigma_{xx} = ue^2\hbar \int_{-\infty}^{\infty} \frac{d^3\mathbf{k}}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{4\pi} \sum_{\tilde{a}}^{2N_b} \frac{2\pi Z_{\tilde{a}} \delta(\omega - Z_{\tilde{a}}\omega_{\mathbf{k}\tilde{a}})}{\Sigma''_{\tilde{a}\tilde{a}}(\omega)} v^2_{\mathbf{k}x\tilde{a}} \times \left[\frac{dn_f(\omega)}{d\omega}\right], \tag{3}$$

where $v_{\mathbf{k}x\tilde{a}}$ is the x component of the group velocity of an electron in spin band $\tilde{a}$, and u measures the extent to which umklapp scattering processes allow the imaginary part of the self-energy to contribute to the conductivity. As one expects that u is $O(1)$ [34,54,55], we henceforth set $u = 1$.

In Eq. (3), we have neglected vertex corrections. It is well known that for electron impurity, scattering differences between the single-particle lifetime and the transport lifetime are accounted for by the effects of vertex corrections [32]. In the limit of vanishing vertex corrections, the two lifetimes are equivalent, corresponding to the relaxation time approximation [32]. It has been argued that the same is true for electron-electron scattering [57]. Furthermore, in infinite dimensions, where the self-energy is strictly local, it can also be shown that there are no vertex corrections to the conductivity [35,58], which suggests that the two approximations are not independent. It is also interesting to note that in many of the materials for which the KWR has been studied, the transport and quasiparticle lifetimes do appear to be very similar [59].

The sharply peaked derivative of the Fermi-Dirac distribution at low temperatures implies that

$$\sigma_{xx} \approx \sum_{\tilde{a}=1}^{2N_b} \frac{e^2\hbar\langle v^2_{\mathbf{k}x\tilde{a}}\rangle}{2} \int_{-\infty}^{\infty} \frac{d^3\mathbf{k}}{(2\pi)^3} Z_{\tilde{a}}\delta(Z_{\tilde{a}}\mu - Z_{\tilde{a}}\omega_{\mathbf{k}\tilde{a}}) \times \int_{-\infty}^{\infty} d\omega \left[\frac{-1}{\Sigma''_{\tilde{a}\tilde{a}}(\omega)}\right]\left[-\frac{dn_f(\omega)}{d\omega}\right], \tag{4}$$

where $\langle\cdots\rangle$ indicates an average over the Fermi surface. The conductivity is clearly then the sum, in series, of the conductivities of the individual bands:

$$\sigma_{xx} = \sum_{\tilde{a}=1}^{2N_b} \sigma_{xx\tilde{a}} = \sum_{\tilde{a}=1}^{2N_b} \frac{e^2\hbar\langle v^2_{\mathbf{k}x\tilde{a}}\rangle}{2} D_{0;\tilde{a}} \times \int_{-\infty}^{\infty} d\omega \left[\frac{1}{\Sigma''_{\tilde{a}\tilde{a}}(\omega)}\right]\left[\frac{dn_f(\omega)}{d\omega}\right], \tag{5}$$

where the bare density of states of band $\tilde{a}$ at the Fermi level is $D_{0;\tilde{a}} = (2\pi)^{-3}\int_{-\infty}^{\infty} d^3\mathbf{k}[\delta(\omega_{\mathbf{k}\tilde{a}} - \mu)]$ and $\sigma_{xx\tilde{a}}$ is the conductivity of spin band $\tilde{a}$. It immediately follows that if interband scattering is neglected, the resistivities of the bands must add in parallel, consistent with Matthiessen's rule [1].

In the limit of vanishing impurity scattering, $\tau^{-1}_{0;\tilde{a}} \to 0$, it follows straightforwardly from Eqs. (1) and (5) that the conductivity is given by

$$\sigma_{xx} = -\sum_{\tilde{a}=1}^{2N_b} \frac{e^2\hbar}{2} \frac{D_{0;\tilde{a}}\langle v_{0x\tilde{a}}^2\rangle(\omega_{\tilde{a}}^*)^2}{s_{\tilde{a}}} \times \int_{-\infty}^{\infty} d\omega \frac{1}{\omega^2+(\pi k_B T)^2}\frac{dn_f(\omega)}{d\omega}. \tag{6}$$

After computing the energy integral, the A coefficient for an N_b-band system is

$$A = \frac{24k_B^2}{e^2\hbar}\left[\sum_{\tilde{a}=1}^{2N_b}\langle v_{0x\tilde{a}}^2\rangle \frac{D_{0;\tilde{a}}(\omega_{\tilde{a}}^*)^2}{s_{\tilde{a}}}\right]^{-1} = \left(\sum_{\tilde{a}=1}^{2N_b} A_{\tilde{a}}^{-1}\right)^{-1}, \tag{7}$$

and, thus, we see that the coefficients of the individual bands, $A_{\tilde{a}}$, add in parallel.

IV. THE HEAT CAPACITY VIA KRAMERS-KRONIG TRANSFORM

It follows from the extensivity of the heat capacity that in a multiband system, the total heat capacity is given by the sum in series of the heat capacity due to each individual band [1],

$$\gamma = \frac{C_{el;N_b}(T)}{T} = \frac{\pi^2k_B^2}{3}\sum_{\tilde{a}=1}^{2N_b}\frac{D_{0;\tilde{a}}}{Z_{\tilde{a}}} = \frac{\pi^2k_B^2}{3}\sum_{\tilde{a}=1}^{2N_b} D_{0;\tilde{a}}\left[1-\frac{\partial\Sigma_{\tilde{a}\tilde{a}}'(\omega)}{\partial\omega}\right]. \tag{8}$$

To determine how the heat capacity is influenced by interactions and calculate the relevant γ coefficients, we need to first find the real part of the self-energy in each band. In a Fermi liquid, the self-energy is causal and $\lim_{\omega\to\infty}\Sigma''(\omega)\to 0$; thus, the self-energy satisfies the conditions for the Kramers-Kronig relations in the frequency domain [13]. Knowledge of the form of the imaginary part of the self-energy is therefore sufficient to determine the real part, which appears in the definition of the quasiparticle weight and therefore in the expression for the heat capacity. The real part of the self-energy within each band is then

$$\begin{aligned}\Sigma_{\tilde{a}\tilde{a}}'(\omega) &= \frac{1}{\pi}P\int_{-\infty}^{\infty}d\omega'\frac{\Sigma_{\tilde{a}\tilde{a}}''(\omega')}{\omega'-\omega}\\ &= \frac{-s_{\tilde{a}}}{\pi}\left\{P\int_{-\infty}^{-\omega_{\tilde{a}}^*}\frac{d\omega'}{\omega'-\omega}F\left[\frac{|\omega'|}{\omega_{\tilde{a}}^*}\right] + P\int_{-\omega_{\tilde{a}}^*}^{\omega_{\tilde{a}}^*}\frac{d\omega'}{\omega'-\omega}\left[\frac{\omega'^2}{(\omega_{\tilde{a}}^*)^2}\right] + P\int_{\omega_{\tilde{a}}^*}^{\infty}\frac{d\omega'}{\omega'-\omega}F\left[\frac{|\omega'|}{\omega_{\tilde{a}}^*}\right]\right\},\end{aligned} \tag{9}$$

where we have again taken the limit of vanishing impurity scattering, and have restricted the pole of the integral, $\omega'=\omega$, to occur below the cutoff energy scale, $|\omega'|\leqslant|\omega_{\tilde{a}}^*|$ (i.e., we consider only low-energy excitations). This second assumption is reasonable because we are interested only in the low-energy contributions to the specific heat. Nevertheless, it is important to stress that the theory below, which is developed on the basis of this assumption, will only be valid for $\overline{\omega}<\omega^*$.

The second term in Eq. (9) contributes a logarithmic term to the result, which we approximate by the lowest-order terms in a Taylor-series expansion, while the first and third terms contribute linearly to the self-energy. Neglecting terms of order $O(\omega/\omega_{\tilde{a}}^*)^3$ and higher, we find that the real part of the self-energy for a low-energy quasiparticle in band $\tilde{a}$ is

$$\Sigma_{\tilde{a}\tilde{a}}'(\omega) = -\frac{4s_{\tilde{a}}\omega}{\pi\omega_{\tilde{a}}^*}\xi, \tag{10}$$

where

$$\xi = \frac{1}{2}\left[1+\int_1^{\infty}dy\frac{F(y)}{y^2}\right]. \tag{11}$$

It follows straightforwardly from the definition of $F(y)$ that $1/2\leqslant\xi\leqslant1$; henceforth we take $\xi=1$ for simplicity. Inserting this expression for the real parts of the self-energy into the heat-capacity expression (8), and taking the strong scattering ($4s_{\tilde{a}}/\pi\omega_{\tilde{a}}^*\gg1$) limit, which corresponds physically to $m^*\gg m_0$, we obtain

$$\gamma = \frac{4\pi k_B^2}{3}\sum_{\tilde{a}=1}^{2N_b}\frac{s_{\tilde{a}}D_{0;\tilde{a}}}{\omega_{\tilde{a}}^*}. \tag{12}$$

Note that Eqs. (8) and (12) imply that $\omega_{\tilde{a}}^*\approx4s_{\tilde{a}}Z_{\tilde{a}}$ ($=4Z_{\tilde{a}}n_{\tilde{a}}/3\pi D_{0;\tilde{a}}$ for free electrons), which gives a straightforward interpretation of this energy scale.

V. THE KADOWAKI-WOODS RATIO

Taking the above expressions for A and γ in a multiple-band system, given by Eqs. (7) and (12), we find that the KWR is given by

$$R_{KW;N_b} = \frac{A}{\gamma^2} = \frac{81}{6\pi^2e^2\hbar k_B^2}\left[\sum_{\tilde{b}=1}^{2N_b}\langle v_{0x\tilde{b}}^2\rangle\frac{D_{0;\tilde{b}}(\omega_{\tilde{b}}^*)^2}{s_{\tilde{b}}}\right]^{-1} \times\left[\sum_{\tilde{a}=1}^{2N_b}\frac{s_{\tilde{a}}D_{0;\tilde{a}}}{\omega_{\tilde{a}}^*}\right]^{-2}. \tag{13}$$

Alternatively, one may define a new ratio,

$$\frac{A}{\gamma^2}f_{x;N_b} = \frac{81}{4\pi e^2\hbar k_B^2}, \tag{14}$$

where we have defined an, in general, nonuniversal function for an N_b-band Fermi liquid with the resistivity measured in the x direction,

$$f_{x;N_b} = \frac{3\pi}{2}\left[\sum_{\tilde{a}=1}^{2N_b}\frac{s_{\tilde{a}}D_{0;\tilde{a}}}{\omega_{\tilde{a}}^*}\right]^2\left[\sum_{\tilde{b}=1}^{2N_b}\langle v_{0x\tilde{b}}^2\rangle\frac{D_{0;\tilde{b}}(\omega_{\tilde{b}}^*)^2}{s_{\tilde{b}}}\right]. \tag{15}$$

For a single band with $D_{0,\uparrow}=D_{0,\downarrow}=D_0/2$, $s_\uparrow=s_\downarrow=s$, and $\omega_\uparrow^*=\omega_\downarrow^*=\omega^*$, the f is universal (independent of the strength of the electronic correlations), but remains strongly dependent on the noninteracting band structure of the material,

$$f_{x;1} = \frac{3\pi}{2}s\langle v_{0x}^2\rangle D_0^3. \tag{16}$$

Upon taking the free-electron expression for s, with $n_\uparrow=n_\downarrow=n/2$, this expression simplifies to that calculated in

Ref. [14]:

$$f_{x;1} = n\langle v_{0x}^2\rangle D_0^2. \tag{17}$$

It is clear from comparing Eqs. (15)–(17) that the factor of v_{0x}^2 arises from the resistivity, one factor of D_0 comes from the resistivity and two come from the square of the specific heat, and the factor of s^2 in the heat capacity squared is partially canceled by the factor of $1/s$ in the resistivity. The remaining factor of $s \sim n/D_0$ is for free electrons.

The inclusion of multiple Fermi-surface sheets significantly complicates the form of $f_{x;N_b}$. Most importantly, $\omega_{\tilde{a}}^*$, which describes the electronic correlations, does not cancel out of the multiband expression as it does for the single-band KWR [14]. Therefore, our calculation predicts that the Kadowaki-Woods ratio is not, in general, independent of electronic correlations. This is rather surprising as observed values of the KWR (including the values for many multiband systems) are in almost universal agreement with the prediction from the single-band calculation that electronic correlations do not influence the KWR [14].

It is therefore important to ask how renormalization effects might cancel in the multiband case and hence universality might be recovered.

The simplest limiting case for which the effects of many-body correlations cancel out of the KWR is when $\omega_{\tilde{a}}^*$ is independent of the band index $\tilde{a}$. This is a straightforward extension of the earlier assumption of the locality of the self-energy, by assuming that it is independent of band index as well as momentum. This assumption yields

$$f_{x;N_b} = \frac{3\pi}{2}\left[\sum_{\tilde{a}=1}^{2N_b} s_{\tilde{a}} D_{0;\tilde{a}}\right]^2 \left[\sum_{\tilde{b}=1}^{2N_b} \langle v_{0x\tilde{b}}^2\rangle \frac{D_{0;\tilde{b}}}{s_{\tilde{b}}}\right]. \tag{18}$$

Although this calculation has been performed with exactly uniform correlation strengths for simplicity, clearly the result will hold approximately while the correlation strengths are close to uniform.

Other limits also produce a universal KWR. For example, for free fermions, $D_{0;\tilde{b}} \propto n_{\tilde{b}}$ in any dimension, so if the carrier density in one band is much larger than all others, the correlations cancel from the KWR. But it seems unlikely that this is relevant to the behavior of a broad range of materials. Note, in particular, that if we have a single heavy band, it will dominate the heat capacity, but be shorted out of the resistivity. Thus, the limit of a single heavy band is far from universal [for example, in the two-band case, the KWR is scaled by a factor $(\omega_{\text{light}}^*/\omega_{\text{heavy}}^*)^2$ in the limit $\omega_{\text{light}}^* \gg \omega_{\text{heavy}}^*$]. Therefore, the above calculation seems to suggest that the correlation strength (as measured by $\omega_{\tilde{a}}^* = 4Z_{\tilde{a}}s_{\tilde{a}} \sim Z_{\tilde{a}}W_{\tilde{a}}$) does not vary strongly between different bands in strongly correlated systems.

If, further, all of the bands are identical, i.e., if the carrier density n, the Fermi velocity $\langle v_{0x}^2\rangle$, and density of states D_0 are equal for all bands, as well as uniform ω^*, we have

$$f_{x;N_b} = N_b^2\langle v_{0x}^2\rangle s D_0^3 = N_b^2 f_{x;1}, \tag{19}$$

and

$$R_{KW;N_b} = \frac{81}{4N_b^2\pi e^2\hbar k_B^2\langle v_{0x}^2\rangle s D_0^3} = \frac{R_{KW;1}}{N_b^2}. \tag{20}$$

Thus we see that the expression for the KWR in the single-band case is modified by a simple factor of $1/N_b^2$.

At first glance, this expression appears rather similar to the finding of Kontani *et al.* [27,28] that in the multiorbital periodic Anderson model with N_o impurity states, the KWR is reduced by a factor of $1/N_o(N_o-1)$. However, closer examination reveals that the results are actually very different. In particular Kontani's factor arises because of interorbital terms in the self-energy, whereas our $1/N_b^2$ factor arises purely from the electronic structure. We do not obtain Kontani's factor because of our assumption that interband interactions are irrelevant at low energies. In contrast, the model Hamiltonian studied by Kontani [27] explicitly sets the intra- and interorbital interactions to the same strength. Which approach is appropriate will depend on the material. This therefore adds another layer of nonuniversality to the KWR.

VI. DIRAC SEMIMETALS

The general expression, given by Eq. (14), can be applied to systems of arbitrary band structure to calculate the generalized Kadowaki-Woods ratio, taking into account the effects of multiple bands. Even for materials where the quasiparticle weight is similar for all bands, such efforts will, in general, involve first-principles band structure calculations. In this section, we calculate the KWR for a simple, linear dispersion ($\varepsilon_{\mathbf{k}\tilde{a}} = \hbar v_F|\mathbf{k}|$) appropriate for Dirac semimetals. These models present analytically tractable and instructive examples of complicated band structures for which the presence of multiple bands is important (here, the number of bands may be taken as equivalent to the number of Dirac cones, as each cone will form a sheet in the Fermi surface). Furthermore, the multiple Fermi surfaces are expected to be identical, allowing us to use the simplified form of the KWR, and the carrier density will be low, suggesting that correlations will play an important role.

We apply the KWR expression to a simple model of a two-band Dirac semimetal, e.g., graphene or Cd_3As_2 [60–62]. The spatial symmetry between the two bands and the spin symmetry simplifies the calculation greatly; applying the expression for N_b identical bands, given by Eq. (19), and taking the free-electron expression for s, we find

$$f_{x;2,D} = \frac{2n^2}{\pi\hbar^2}, \tag{21}$$

so

$$\frac{A}{\gamma^2} = \frac{1}{f_{x;2,D}}\frac{81}{4\pi e^2\hbar k_B^2} = \frac{81\hbar}{8e^2k_B^2n^2}. \tag{22}$$

Note that $f_{x;2,D}$ is larger by a factor of $N_b^2 = 4$ than the corresponding f in a single-band calculation with the same dispersion. It is interesting to note that the Kadowaki-Woods ratio for these materials depends straightforwardly on the electronic density, which is tunable over a wide range via chemical doping [61,62], providing a potential experimental test of this expression.

VII. CONCLUSIONS

We have shown that, in general, the Kadowaki-Woods ratio of a multiband local Fermi liquid is changed by electronic correlations. This is in marked contrast to the single-band case, where the KWR is independent of the strength of the electronic correlations. It is therefore puzzling that the experimental data suggest that within classes of materials, the KWR is remarkably consistent, and that the modified KWR is remarkably consistent across many chemically diverse, strongly correlated metals. The simplest explanation is that the correlations are indeed very similar across all bands in these materials. We have also shown that a nonparabolic dispersion does not significantly alter the form of the KWR, provided the fermions remain massive. In the case of uniform renormalization across bands, we have further demonstrated that $R_{KW} \propto 1/N_b^2$ in a system of N_b bands. In semimetals, the low carrier density opens the possibility of large variations in the carrier density, n. In Dirac semimetals, we have shown that the massless fermion dispersion relations lead to $R_{KW} \propto 1/n^2 N_b^2$.

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