Spin dynamics with inertia in metallic ferromagnets

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The nonadiabatic contribution of environmental degrees of freedom yields an effective inertia of spin in the effective spin dynamics. In this paper, we study several aspects of the inertia of spin in metallic ferromagnets: (i) a concrete expression of the spin inertia m_s: $m_s = \hbar S_c/(2g_{\rm sd})$, where S_c is the spin polarization of conduction electrons and $g_{\rm sd}$ is the *sd* coupling constant; (ii) a dynamical behavior of spin with inertia, discussed from the viewpoints of a spinning top and of a particle on a sphere; (iii) the behavior of spin waves and domain walls in the presence of inertia and the behavior of spin with inertia under a time-dependent magnetic field.

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I. INTRODUCTION

Different from other fundamental quantities such as mass and charge, spin is a dynamical quantity, and its dynamics have been widely studied and applied in science and technology. In particular, the recent rapid growth of spintronics provides a stage where a deeper understanding of the spin dynamics directly leads to practical applications.

The dynamics of spin is governed by the spin Berry phase [1], and its equation of motion includes only the first-order time derivative of spin. This is natural because spin is an angular momentum and its equation of motion takes the familiar form: the time derivative of the angular momentum (i.e., spin) is given by the torque acting on it. Without any torque, the solution of the equation of motion of spin is only a static one. This is in contrast with, for example, the case of a massive point particle, which has inertia and can move at nonzero speed as its free motion. In this sense, spin does not have inertia.

However, for systems where spin interacts with other environmental degrees of freedom, the spin dynamics is affected by those environmental degrees of freedom, and the dynamical law of spin is changed to be an effective one. For example, in metallic ferromagnets, the dynamics of localized spins (i.e., spins of atoms on lattice cites) is affected by conduction electrons. A typical effect is spin damping (e.g., Ref. [2]), where the energy and the angular momentum of spins are transferred to the environmental degrees of freedom (i.e., the conduction electrons) and, as a result, spins relax to their ground state within a certain time scale. Without this effect, spins undergo the Larmor precession around an applied magnetic field forever. In this respect, the existence of environmental degrees of freedom other than spins changes the dynamical behavior of spins significantly. In the equation of motion of spins, this damping is represented by the Gilbert damping term [3]. This damping term includes only the first-order time derivative of spins. Therefore spins still do not have inertia even when we take into account the Gilbert damping effect.

The effects of the environment other than the Gilbert damping can be studied systematically by the derivative expansion, where the effects are expanded in powers of the time (and spatial) derivative of spin. From that point of view, the Gilbert damping term gives the leading order term in that expansion. In the higher orders, terms appear that include the second-order time derivative, the third-order time derivative, and so on, in the equation of motion of spin. These terms with higher-order time derivatives are interesting in that, like the Gilbert damping term, they change the dynamical law of spin itself, more than give additional torque on spin. In particular, from a comparison with the form of the Newton's equation of motion of a massive point particle, the term with the second-order time derivative of spin plays the role of the inertia of spin. Furthermore, as we will see in this paper, such inertia gives nonadiabatic contribution of the environment to the effective dynamics of the localized spin. Nonadiabaticity here means that the spin of the environmental degrees of freedom does not instantaneously align with the localized spin. This nonadiabaticity induced by the spin inertia is independent of applied fields and, in this sense, is intrinsic to the system. When such nonadiabaticity is included, the gyromagnetic relation, i.e., the relation between angular momentum and localized spin, is generalized: the angular momentum does not point in the same direction as the localized spin. Thus the inertia of spin gives an intrinsic deviation of the direction of the angular momentum from that of the localized spin, which will bring rich variety to spin dynamics.

Such spin inertia has been discussed in the literature (e.g., Ref. [4]). In particular, recent progress in ultrafast magnetization [5,6] motivated several works. In Refs. [7,8], the inertia of spin was introduced phenomenologically and was shown to give an additional nutation to the motion of spin. The time scale where the effect of the inertia is significant was discussed, based on the work of Brown [9], to be of the subpicosecond order. In Ref. [10], the equivalence between the dynamics of spin with inertia and of a spinning top was discussed. A microscopic derivation of the spin inertia was performed in Refs. [11] and [12]. In Ref. [11], an extended breathing Fermi surface model was used, and a relation between the Gilbert damping coefficient and the spin inertia was given in terms of physical quantities (such as Fermi-Dirac occupation numbers) of conduction electrons. The time scale for the nutational motion to be damped by the Gilbert damping was estimated to be of the subpicosecond order. In Ref. [12], a general expression of the contribution of the conduction electrons to spin dynamics was discussed. The spin effective dynamics was shown to be nonlocal in general, which can be approximated as local dynamics by the derivative expansion

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of spin. The inertial term of spin arises in that derivative expansion and its general expression was given in terms of the Green's function of the conduction electrons.

Since the inertia of spin is conceptually interesting in its own and gives the first step toward an understanding of the nonadiabatic contribution of environmental degrees of freedom to spin effective dynamics, further investigations are worthwhile. Although the expressions of the inertia of spin were given as integral forms [11,12], an explicit expression of the inertia of spin in terms of the parameters of a model has not been obtained so far. Furthermore, the effect of the inertia of spin on the dynamics of spin has been discussed only for spatially homogeneous spin configurations under a time-independent magnetic field. In this paper, we present a detailed theoretical study of the effects induced by the spin inertia based on an *sd* model. In Sec. II, we derive a concrete expression of spin inertia in terms of the parameters in the *sd* model. In Sec. III, the basic behavior of spin with finite inertia is studied with the help of its two equivalents: a symmetric spinning top and a massive charged particle on a sphere subject to a monopole field.

In Sec. IV, we study spatially inhomogeneous configurations of spins, and discuss that spin waves and magnetic domain walls acquire an additional oscillation mode due to spin inertia. We also study the behavior of spin under a large and time-dependent magnetic field, and find an unusual behavior of spin where the velocity of spin is parallel to the direction of the time-derivative of the magnetic field.

II. SPIN EFFECTIVE ACTION AND INERTIA

In this section, we discuss that the inertia of spin arises naturally in its effective dynamics, which takes into account the effects of the environmental degrees of freedom. We follow mainly the line of Ref. [12]. Consider a system where a classical field of localized spins, $\boldsymbol{S}(\boldsymbol{x},t) = S_l \boldsymbol{n}(\boldsymbol{x},t)$ (with $S_l = |\boldsymbol{S}|$ fixed), and a field $c(\boldsymbol{x},t)$ representing the other environmental degrees of freedom are interacting with each other. For concreteness, we consider the case of metallic ferromagnets in this paper, where the conduction electrons play the role of the environment for the localized spins. The conduction electrons are represented by annihilation and creation operators, c and $\bar{c}$. The total action is given by $\mathcal{S}_s[\boldsymbol{n}] + \mathcal{S}_e[c,\bar{c},\boldsymbol{n}]$, where $\mathcal{S}_s[\boldsymbol{n}]$ is the action of spin $\boldsymbol{n}$ and $\mathcal{S}_e[c,\bar{c},\boldsymbol{n}]$ is that of the conduction electrons with their interaction with spin $\boldsymbol{n}$. When we are interested only in the dynamics of spin $\boldsymbol{n}$, it is convenient to integrate out c and $\bar{c}$, and derive the effective action of spin $\boldsymbol{n}$. The contribution $\Delta\mathcal{S}_{\text{eff}}$ from electrons is given by path integration as

$$\exp\left(\frac{i}{\hbar}\Delta\mathcal{S}_{\text{eff}}[\boldsymbol{n}]\right) \equiv \int \mathcal{D}\bar{c}\mathcal{D}c \exp\left(\frac{i}{\hbar}\mathcal{S}_e[c,\bar{c},\boldsymbol{n}]\right). \quad (1)$$

The sum of this $\Delta\mathcal{S}_{\text{eff}}[\boldsymbol{n}]$ and the original spin action $\mathcal{S}_s[\boldsymbol{n}]$ gives the total spin effective action.

It is difficult to calculate $\Delta\mathcal{S}_{\text{eff}}[\boldsymbol{n}]$ exactly, so we should rely on a perturbative analysis. We here perform a derivative expansion, where $\Delta\mathcal{S}_{\text{eff}}[\boldsymbol{n}]$ is expanded in powers of $\partial_\mu \boldsymbol{n}$ ($\mu = t,x,y,z$). When the system is isotropic, the general form is

$$\Delta\mathcal{S}_{\text{eff}}[\boldsymbol{n}] = \int \frac{d^3x}{a^3}dt\left[S_c\dot{\phi}(\cos\theta - 1) - \frac{J_c S_c^2}{2}(\partial_i \boldsymbol{n})^2 + \frac{m_s}{2}\dot{\boldsymbol{n}}^2\right] + \mathcal{O}((\partial_\mu \boldsymbol{n})^3), \quad (2)$$

where $\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and $i = x,y,z$. We have divided the Lagrangian density entirely by the lattice volume a^3 so that each coefficient represents a quantity per each lattice cite. The first term is the spin Berry phase with S_c the spin polarization of the conduction electrons and the second is the spin-spin exchange interaction, with J_c the coupling constant, induced by electrons. In the final term, there arises the inertial term of spin with inertia m_s. This m_s has the dimension of [kg m^2], the same as that of moments of inertia.

In Ref. [12], a general expression of spin inertia m_s was derived in term of the Green's function of the conduction electrons. Let us here calculate a concrete expression of spin inertia. As in Ref. [12], a typical example of $\mathcal{S}_e[c,\bar{c},\boldsymbol{n}]$ is an *sd* model [13], where conduction electrons interact with localized spins $\boldsymbol{n}$ as

$$\mathcal{S}_e[c,\bar{c},\boldsymbol{n}] = \int d^3x dt \bar{c}\left(i\hbar\partial_t + \frac{\hbar^2\partial_i^2}{2m} + \epsilon_F + g_{\text{sd}}\boldsymbol{n}\cdot\boldsymbol{\sigma}\right)c. \quad (3)$$

Here, m is the mass of the conduction electrons, ϵ_F is the Fermi energy, g_{sd} is the *sd* coupling constant, and $\boldsymbol{\sigma}$ is the Pauli matrix vector. To obtain the derivative expansion of $\Delta\mathcal{S}_{\text{eff}}[\boldsymbol{n}]$, we perform an $SU(2)$ gauge transformation $c \to U(\boldsymbol{x},t)c$ with an $SU(2)$ matrix U acting on the spinor indices, so that the *sd* interaction becomes diagonal, $\bar{c}(\boldsymbol{n}\cdot\boldsymbol{\sigma})c \to \bar{c}\sigma_3 c$. Due to this unitary transformation, there appears a so-called spin gauge field $A_\mu \equiv -iU^\dagger\partial_\mu U$ in Eq. (3) through $\partial_\mu c \to U(\partial_\mu + iA_\mu)c$. This A_μ contains the first-order derivative of spin, $\partial_\mu \boldsymbol{n}$. Therefore expanding $\exp(i\mathcal{S}_e[c,\bar{c},\boldsymbol{n}]/\hbar)$ in powers of A_μ, we can calculate $\Delta\mathcal{S}_{\text{eff}}[\boldsymbol{n}]$ perturbatively in powers of $\partial_\mu \boldsymbol{n}$. The spin polarization S_c and the inertia m_s can be calculated as (see Appendix A for details)

$$S_c = a^3\frac{\hbar}{2}\frac{k_{F+}^3 - k_{F-}^3}{6\pi^2}, \quad m_s = \frac{\hbar S_c}{2g_{\text{sd}}} \quad (4)$$

with $\hbar^2 k_{F\pm}^2/(2m) \equiv \epsilon_F \pm g_{\text{sd}}$. The inertia can be rewritten as

$$m_s = (k_F a)^3\frac{\hbar^2}{8\pi^2}\frac{1}{\epsilon_F}f(g_{\text{sd}}/\epsilon_F)$$
$$\text{with } f(x) \equiv \frac{1}{3x}\left[(1+x)^{\frac{3}{2}} - (1-x)^{\frac{3}{2}}\right]. \quad (5)$$

For $0 < x < 1$, the function $f(x)$ is only a slightly decreasing function from $f(0) = 1$ to $f(1) = 0.94$. Therefore spin inertia does not depend much on the *sd* coupling constant g_{sd}.

Adding this $\Delta\mathcal{S}_{\text{eff}}[\boldsymbol{n}]$ to the original spin action of the form

$$\mathcal{S}_s[\boldsymbol{n}] = \int \frac{d^3x}{a^3}dt\left[S_l\dot{\phi}(\cos\theta - 1) - \frac{J_l S_l^2}{2}(\partial_i \boldsymbol{n})^2\right], \quad (6)$$

with J_l the exchange coupling between spins, we obtain the total spin effective action as $\mathcal{S}_{\text{eff}}[\boldsymbol{n}] = \mathcal{S}_s[\boldsymbol{n}] + \Delta\mathcal{S}_{\text{eff}}[\boldsymbol{n}]$.

Including the Zeeman coupling with external magnetic field,[1] we obtain

$$\mathcal{S}_{\text{eff}}[\boldsymbol{n}] = \int \frac{d^3x}{a^3} dt \left[S\boldsymbol{B}\cdot\boldsymbol{n} + S\dot{\phi}(\cos\theta - 1) - \frac{JS^2}{2}(\partial_i \boldsymbol{n})^2 + \frac{m_s}{2}\dot{\boldsymbol{n}}^2 \right] + \mathcal{O}((\partial_\mu \boldsymbol{n})^3), \tag{7}$$

with $S \equiv S_c + S_l$ the total spin amplitude per lattice cite and $J \equiv (J_c S_c^2 + J_l S_l^2)/S^2$. We have set the gyromagnetic ratio in the Zeeman coupling as unity.

As we will see below, the spin with finite inertia has a typical precession mode with frequency $\omega_0 \sim S/m_s$. Using the expressions of m_s (4) or (5), and assuming $S \sim \hbar, k_F a \sim \pi$ and $\epsilon_F \sim 1$ eV, the energy scale of this frequency becomes $\hbar\omega_0 \sim \epsilon_F \sim 1$ eV, so that its period is $2\pi/\omega_0 \sim 0.1$ ps. Therefore, as far as this simple estimation suggests, the existence of the inertia is significant for the dynamics of subpicosecond scale in the case of metallic ferromagnets.

The equation of motion derived from this effective action (7) is

$$S\dot{\boldsymbol{n}} = -S\boldsymbol{B}\times\boldsymbol{n} - JS^2\partial_i^2\boldsymbol{n}\times\boldsymbol{n} + m_s\ddot{\boldsymbol{n}}\times\boldsymbol{n}. \tag{8}$$

Thus the inertia produces an acceleration-dependent term. We can rewrite this equation of motion, by taking the vector product with $\boldsymbol{n}$, as

$$m_s\ddot{\boldsymbol{n}} = S\boldsymbol{n}\times\dot{\boldsymbol{n}} + S\boldsymbol{B} + JS^2\partial_i^2\boldsymbol{n} - (S\boldsymbol{B}\cdot\boldsymbol{n} + m_s\dot{\boldsymbol{n}}^2 - JS^2(\partial_i\boldsymbol{n})^2)\boldsymbol{n}. \tag{9}$$

The Gilbert damping effect adds a term $-\alpha S\dot{\boldsymbol{n}}$, with α a dimensionless constant, to the right-hand side of the equation of motion (9). Therefore the Gilbert damping plays the same role as the familiar linear damping force for a point particle, and the time scale for this damping term to be significant is $t_{\text{damp}} \sim m_s/(\alpha S)$. On the other hand, the time scale for the inertial term to be effective is, as we will see below, $t_{\text{inertia}} \sim m_s/S$. Therefore, for $\alpha \ll 1$, we can neglect the Gilbert damping term as long as we are interested in the dynamics within the time scale t_{inertia}.

The equation of motion (8) can be rewritten (when $\boldsymbol{B} = 0$) in the conservation form of the angular momentum current $(\boldsymbol{j}^0, \boldsymbol{j}^i)$,

$$\partial_0\boldsymbol{j}^0 + \partial_i\boldsymbol{j}^i = 0$$
$$\text{with } \boldsymbol{j}^0 = S\boldsymbol{n} + m_s\boldsymbol{n}\times\dot{\boldsymbol{n}}, \quad \boldsymbol{j}^i = JS^2\partial_i\boldsymbol{n}\times\boldsymbol{n} \tag{10}$$

($\partial_0 \equiv \partial/\partial t$). Note that the angular momentum $\boldsymbol{j}^0$ (per lattice cite), which is the Noether charge corresponding to the invariance of the action (7) under SO(3) rotations in the internal spin space (see Appendix B for details), is no longer proportional to $\boldsymbol{n}$ but includes the nonadiabatic contribution of the conduction electrons, $m_s\boldsymbol{n}\times\dot{\boldsymbol{n}}$. Originally, the total angular momentum consists of that of the localized spin and that of the conduction electrons: $\boldsymbol{j}^0 = S_l\boldsymbol{n} + (\hbar a^3/2)\langle\bar{c}\boldsymbol{\sigma}c\rangle$ with S_l the amplitude of the localized spin. In the lowest order of the derivative expansion, i.e., in the adiabatic limit, the spin of the conduction electron aligns with the localized spin, so that $(\hbar a^3/2)\langle\bar{c}\boldsymbol{\sigma}c\rangle = S_c\boldsymbol{n}$ with S_c the spin polarization of the conduction electrons. Beyond the adiabatic limit, the direction of the spin of the conduction electron is generally different from that of the localized spin. The derivative expansion incorporates this difference systematically, and the next order term in $(\hbar a^3/2)\langle\bar{c}\boldsymbol{\sigma}c\rangle$ is given by $m_s\boldsymbol{n}\times\dot{\boldsymbol{n}}$. Thus the inertia m_s is related to the nonadiabatic contribution of the environmental degrees of freedom to the spin effective dynamics. In this respect, it is reasonable that the typical energy scale where the inertia m_s is relevant to the spin dynamics is that of the Fermi energy, $\hbar\omega_0 \sim \hbar S/m_s \sim \epsilon_F$ [from Eq. (5) with $k_F a \sim \pi$ fixed]: when the energy scale of the localized spin becomes comparable to that of the conduction electrons, the spins of electrons can no longer follow perfectly the direction of the localized spins. In other words, to find physical situations where the inertia has a measurable effect, materials with low ϵ_F are more promising, such as organic conductors [15] and heavy-fermion systems [16].

Remarkably, the relation between the inertia m_s and the spin polarization S_c in Eq. (4), $m_s = \hbar S_c/(2g_{\text{sd}})$, is easily obtained without any microscopic calculation, as follows. As we have discussed in the last paragraph, the angular momentum $\Delta\boldsymbol{j}^0$ derived from $\Delta\mathcal{S}_{\text{eff}}[\boldsymbol{n}]$ represents the spin polarization of the conduction electrons,

$$\Delta\boldsymbol{j}^0 \equiv S_c\boldsymbol{n} + m_s\boldsymbol{n}\times\dot{\boldsymbol{n}} = \frac{\hbar a^3}{2}\langle\bar{c}\boldsymbol{\sigma}c\rangle. \tag{11}$$

Since $(\hbar a^3/2)\langle\bar{c}\boldsymbol{\sigma}c\rangle$ obeys the following equation of motion,

$$\partial_0\left(\frac{\hbar a^3}{2}\langle\bar{c}\boldsymbol{\sigma}c\rangle\right) = -\frac{2g_{\text{sd}}}{\hbar}\boldsymbol{n}\times\left(\frac{\hbar a^3}{2}\langle\bar{c}\boldsymbol{\sigma}c\rangle\right) \tag{12}$$

(we consider here only the spatially homogeneous case and set $\partial_i\boldsymbol{n} = 0$, for simplicity), $\Delta\boldsymbol{j}^0$ also satisfies

$$\partial_0\Delta\boldsymbol{j}^0 = -\frac{2g_{\text{sd}}}{\hbar}\boldsymbol{n}\times\Delta\boldsymbol{j}^0. \tag{13}$$

Substitution of Eq. (11) into Eq. (13) leads to

$$\left(S_c - \frac{2g_{\text{sd}}}{\hbar}m_s\right)\dot{\boldsymbol{n}} + \mathcal{O}(\partial_0^2) = 0. \tag{14}$$

Since this equation is true for an arbitrary $\boldsymbol{n}$, we arrive at the relation $m_s = \hbar S_c/(2g_{\text{sd}})$. Thus we can obtain m_s from S_c without any detailed calculation for this simple *sd* model. The point is that Eq. (13) is not the equation of motion of $\boldsymbol{n}$, although it involves the time derivative of $\boldsymbol{n}$. [The equation of motion of $\boldsymbol{n}$ is $\partial_0\boldsymbol{j}^0 = 0$, or, $\partial_0(S_l\boldsymbol{n} + \Delta\boldsymbol{j}^0) = 0$.] Before integrating out the electrons, it was the equation of motion of $\langle\bar{c}\boldsymbol{\sigma}c\rangle$ (12) and, after integrating out the electrons, Eq. (13) determines the structure of $\Delta\mathcal{S}_{\text{eff}}[\boldsymbol{n}]$. Conversely, the relation $m_s = \hbar S_c/(2g_{\text{sd}})$ must hold in order for the effective action $\Delta\mathcal{S}_{\text{eff}}$ to reproduce the equation of motion of $\langle\bar{c}\boldsymbol{\sigma}c\rangle$. We can repeat this procedure to arbitrary orders in the derivative expansion: first, write down all possible terms in the effective action $\Delta\mathcal{S}_{\text{eff}}$, with their coefficients left undetermined; second, derive the angular momentum $\Delta\boldsymbol{j}^0$ from that $\Delta\mathcal{S}_{\text{eff}}$ and substitute it into Eq. (13); then we can obtain the recursion

[1] To be precise, there emerge other coupling terms between spin and electromagnetic fields in addition to the Zeeman coupling, such as in Ref. [14], by integrating out conduction electrons. In this paper, we simply assume that they are negligible.

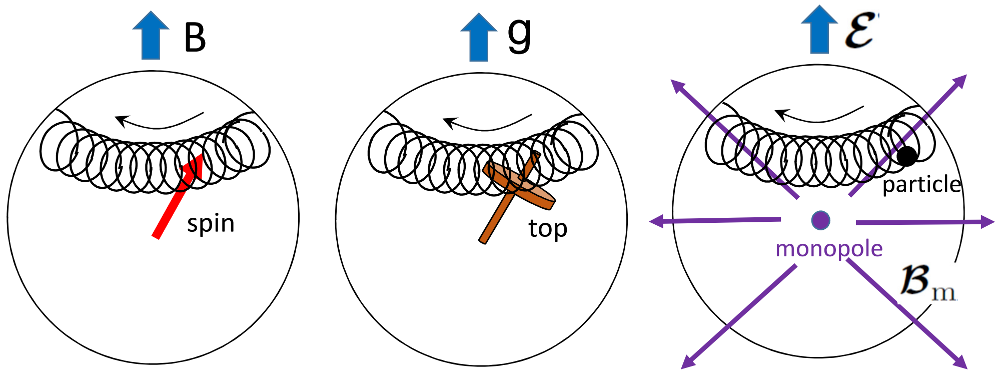

FIG. 1. (Color online) The dynamics of spin with inertia, a symmetric spinning top, and a massive charged particle on a sphere subject to a monopole magnetic field $\mathcal{B}_{\rm m}$ are classically equivalent. They undergo a precession motion accompanied by nutation under applied fields, which are a magnetic field B for spin, a gravitational field g for the top, and an electric field $\mathcal{E}$ for the particle, respectively. See the main text for details.

relations between the coefficients.[2] For example, $\Delta\mathcal{S}_{\rm eff}$ to the fourth-order can be obtained as follows (see Appendix C for details):

$$\Delta\mathcal{S}_{\rm eff}[\boldsymbol{n}] = S_c\int\frac{d^3x}{a^3}dt\bigg[\dot\phi(\cos\theta-1)+\frac{1}{4}\frac{\hbar}{g_{\rm sd}}\dot{\boldsymbol{n}}^2 - \frac{1}{8}\left(\frac{\hbar}{g_{\rm sd}}\right)^2\boldsymbol{n}\cdot(\dot{\boldsymbol{n}}\times\ddot{\boldsymbol{n}})+\frac{1}{16}\left(\frac{\hbar}{g_{\rm sd}}\right)^3\ddot{\boldsymbol{n}}^2 - \frac{5}{64}\left(\frac{\hbar}{g_{\rm sd}}\right)^3(\dot{\boldsymbol{n}}^2)^2\bigg]+\mathcal{O}(\partial_0^5). \tag{15}$$

The same procedure can be applied also to the terms in the effective action $\Delta\mathcal{S}_{\rm eff}$, which involve the spatial derivative.

III. DYNAMICAL BEHAVIOR OF SPIN WITH INERTIA

In this section, we describe the classical dynamics of spin with inertia. For that purpose, it is helpful to use two equivalent pictures, which are summarized in Fig. 1. One is a symmetric spinning top, and the other is a massive charged particle on a sphere subject to a monopole magnetic field. We consider here only a spatially homogeneous spin, $\partial_i\boldsymbol{n}=0$, under a time-independent magnetic field, for simplicity.

A. Equivalence to a symmetric spinning top

The equivalence between the classical dynamics of spin and a spinning top has been recognized in the literature, e.g., Refs. [17] and [10]. The content of this section is essentially a recapitulation of these facts, which we describe here in order for this paper to be self-contained.

First, let us write down the Lagrangian of spin with finite inertia and that of a spinning top. The Lagrangian of spin (7) is

$$L_{\rm spin} = \frac{m_s}{2}(\dot\theta^2+\dot\phi^2\sin^2\theta)+S\dot\phi(\cos\theta-1)+BS\cos\theta, \tag{16}$$

while the Lagrangian of a spinning top in terms of the Euler angles (θ,ϕ,ψ) is

$$L_{\rm top} = \frac{I_1}{2}(\dot\theta^2+\dot\phi^2\sin^2\theta)+\frac{I_3}{2}(\dot\psi+\dot\phi\cos\theta)^2+\mu gl\cos\theta, \tag{17}$$

where I_1 and I_3 are the principle moments of inertia, μ is the mass of the top, g is the gravitational acceleration constant, and l is the distance between the center of mass and the fixed extremity of the top. Here, we have taken a symmetric spinning top and set two moments of inertia equal, $I_1=I_2$. We take the positive directions of the external magnetic field $\boldsymbol{B}$ and the gravity both in the positive z direction. Note that $L_{\rm spin}$ is a function of (θ,ϕ), while $L_{\rm top}$ is that of (θ,ϕ,ψ). Let us see below that they have equivalent dynamics concerning (θ,ϕ).

The equivalence can be directly seen at the level of their equations of motion. The equation of motion of spin is

$$\begin{aligned} m_s(\ddot\theta-\dot\phi^2\sin\theta\cos\theta)+S\dot\phi\sin\theta+BS\sin\theta&=0,\\ \frac{d}{dt}[m_s\dot\phi\sin^2\theta+S(\cos\theta-1)]&=0, \end{aligned} \tag{18}$$

while the equation of motion of the spinning top is

$$\begin{aligned} I_1(\ddot\theta-\dot\phi^2\sin\theta\cos\theta)+I_3(\dot\psi+\dot\phi\cos\theta)\dot\phi\sin\theta+\mu gl\sin\theta&=0,\\ \frac{d}{dt}[I_1\dot\phi\sin^2\theta+I_3(\dot\psi+\dot\phi\cos\theta)\cos\theta]&=0,\\ \frac{d}{dt}[I_3(\dot\psi+\dot\phi\cos\theta)]&=0. \end{aligned} \tag{19}$$

From the last equation in Eq. (19), the canonical momentum M_3 conjugate to ψ,

$$M_3 \equiv I_3(\dot\psi+\dot\phi\cos\theta), \tag{20}$$

is conserved. Substituting this M_3 for $\dot\psi$ in the other two equations in (19), we obtain the same equations as (18) with

[2]Such recursion relations can be obtained also via the equation of motion of spins, as follows. The original equation of motion of a localized spin is $S_l\dot{\boldsymbol{n}} = -S_l\boldsymbol{B}\times\boldsymbol{n} - J_lS_l^2\partial_i^2\boldsymbol{n}\times\boldsymbol{n} - g_{\rm sd}a^3\langle\bar c\boldsymbol{\sigma}c\rangle\times\boldsymbol{n}$. Substitution of the expression of $\langle\bar c\boldsymbol{\sigma}c\rangle$ [Eq. (11)] into this original equation of motion gives a term proportional to $\dot{\boldsymbol{n}}$, i.e., $-g_{\rm sd}a^3\langle\bar c\boldsymbol{\sigma}c\rangle\times\boldsymbol{n} = -(2g_{\rm sd}m_s/\hbar)\dot{\boldsymbol{n}}+\mathcal{O}(\partial_0^2)$. The coefficient of this term is identical with the spin polarization S_c of the conduction electron, which gives the renormalization of the spin amplitude, $S_l\to S_l+S_c$. Therefore we obtain the relation $m_s=\hbar S_c/(2g_{\rm sd})$.

replacements

$$m_s \leftrightarrow I_1, \quad S \leftrightarrow M_3, \quad BS \leftrightarrow \mu g l. \tag{21}$$

Thus the classical behaviors of θ and ϕ are the same for spin and a spinning top.

We can see the correspondence more explicitly through their Hamiltonians:

$$H_{\text{spin}} = \frac{p_\theta^2}{2m_s} + \frac{1}{2m_s}\frac{(M_\phi - S\cos\theta)^2}{\sin^2\theta} - BS\cos\theta \tag{22}$$

and

$$H_{\text{top}} = \frac{p_\theta^2}{2I_1} + \frac{1}{2I_1}\frac{(M_\phi - M_3\cos\theta)^2}{\sin^2\theta} - \mu g l\cos\theta + \frac{M_3^2}{2I_3}, \tag{23}$$

where $p_\theta \equiv \partial L/\partial\dot\theta$ and $M_\phi \equiv \partial L/\partial\dot\phi$ are the canonical momenta of θ and ϕ, respectively. These two Hamiltonians are completely the same under the replacements (21). [The last term in Eq. (23) is just a constant and does not contribute to the dynamics of θ and ϕ.]

The Lagrangians (16) and (17) are related by the Legendre transformation about ψ:

$$L_{\text{spin}}(\theta,\dot\theta,\dot\phi;S) = L_{\text{top}}(\theta,\dot\theta,\dot\phi,\dot\psi) - S\dot\psi|_{\dot\psi=\dot\psi(\theta,\dot\phi,S)} \tag{24}$$

with replacements (21). In the right-hand side, $\dot\psi$ is substituted by S (or M_3) via Eq. (20). The situation is quite similar to that of the familiar centrifugal force problem of a point particle. There, the original Lagrangian is given as $L(r,\dot r,\dot\phi) = (m/2)(\dot r^2 + r^2\dot\phi^2) - U(r)$, and we can obtain the ϕ-reduced Lagrangian by the Legendre transformation about ϕ: $L_{\text{red}}(r,\dot r;M) \equiv L - M\dot\phi = (m/2)\dot r^2 - M^2/(2mr^2) - U(r)$ with M the canonical momentum conjugate to ϕ. In exchange for reducing ϕ, there appears a fictitious potential $M^2/(2mr^2)$. Likewise, the spin dynamics is the ψ-reduced dynamics of a spinning top, and the spin Berry phase [the second term in Eq. (16)] appears as the fictitious potential arising from the ψ-reduction. When we perform the Legendre transformation also about θ and ϕ on both sides in Eq. (24), we are led to the same Hamiltonians (22) and (23).

Since the classical dynamics of spin with inertia and of a spinning top are equivalent, a spin with inertia behaves in the same manner as a spinning top does (Fig. 2). When a magnetic field is not applied, spin undergoes free precession: the spin precesses around the total angular momentum $\boldsymbol{j}^0 = S\boldsymbol{n} + m_s\boldsymbol{n}\times\dot{\boldsymbol{n}}$ [Eq. (10)], which is a constant of motion. When a magnetic field is applied, spin undergoes a Larmor precession accompanied by nutation: the total angular momentum $\boldsymbol{j}^0$ precesses around the magnetic field (the Larmor precession), and the spin precesses around $\boldsymbol{j}^0$ at each time (the nutation) [7]. The free precession solution of the equation of motion (18) (with $\boldsymbol{B} = 0$) is

$$\theta = \theta_0(\text{const.}), \quad \dot\phi = \frac{S}{m_s\cos\theta_0}. \tag{25}$$

Therefore the free precession frequency, or the nutation frequency ω_0, is $\omega_0 \sim S/m_s$, assuming that $\cos\theta_0$ is the order of unity (that is, the radius of the spin free precession or nutation is not very large). This gives the typical time scale for the inertial term, $t_{\text{inertia}} \sim m_s/S$.

Finally, we mention that the usual spin with zero inertia, $m_s = 0$, can be regarded as follows. It corresponds to the case of $I_1(= I_2) = 0$ under replacements (21). This means that the other principle moment of inertia, I_3, also vanishes since, for an actual rigid body, any one of the principle moments of inertia is equal to or less than the sum of the other two, e.g., $I_3 \leqslant I_1 + I_2$ [18]. Thus setting $I_1 = I_2 = 0$ leads to $I_3 = 0$. Therefore there does not exist a spinning top corresponding to spin with zero inertia, in a nonrelativistic framework.

B. Equivalence to a massive charged particle

Since the spin Berry phase corresponds to a monopole gauge field [1], the classical dynamics of spin is also equivalent to that of a charged particle on a sphere subject to a monopole background. Let us describe this equivalence and use it to understand the behavior of spin with inertia.

A magnetic monopole yields a magnetic field in the radial direction with a strength inversely proportional to the square of the distance from the monopole. Being put at the origin $\boldsymbol{x} = 0$, the gauge field $\boldsymbol{\mathcal{A}}_{\text{m}}$ and the magnetic field $\boldsymbol{\mathcal{B}}_{\text{m}}$ of the monopole are

$$\boldsymbol{\mathcal{A}}_{\text{m}} = q\frac{1-\cos\theta}{r\sin\theta}\boldsymbol{e}_\phi, \quad \boldsymbol{\mathcal{B}}_{\text{m}} = \nabla\times\boldsymbol{\mathcal{A}}_{\text{m}} = \frac{q}{r^3}\boldsymbol{x}, \tag{26}$$

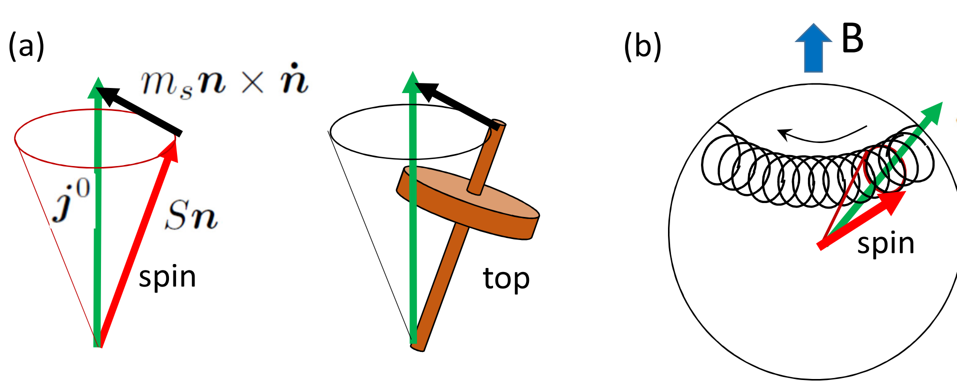

FIG. 2. (Color online) (a) Without any magnetic field, the general motion of spin with finite inertia is a free precession motion, just like a spinning top. The spin $S\boldsymbol{n}$ precesses around the total angular momentum $\boldsymbol{j}^0$. (b) When a constant magnetic field $\boldsymbol{B}$ is applied, the total angular momentum $\boldsymbol{j}^0$ precesses around $\boldsymbol{B}$. Therefore the "free precession cone" in (a) precesses around $\boldsymbol{B}$ as a whole, which corresponds to the Larmor precession in the absence of the inertia. What was called the free precession in (a) is now called the nutation.

where $\boldsymbol{x} = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, $\boldsymbol{e}_\phi$ is the unit vector in ϕ direction, and q is the magnetic charge of the monopole. The Lagrangian of a massive charged particle coupled to this monopole magnetic field and a constant electric field $\boldsymbol{\mathcal{E}}$ is

$$L = \frac{m}{2}\dot{\boldsymbol{x}}^2 - \boldsymbol{\mathcal{A}}_\mathrm{m}\cdot\dot{\boldsymbol{x}} - \Phi = \frac{m}{2}\dot{\boldsymbol{x}}^2 + q\dot{\phi}(\cos\theta - 1) + \boldsymbol{\mathcal{E}}\cdot\boldsymbol{x}, \tag{27}$$

where we take the electromagnetic scalar potential $\Phi = -\boldsymbol{\mathcal{E}}\cdot\boldsymbol{x}$ so that $-\nabla\Phi = \boldsymbol{\mathcal{E}}$. This Lagrangian is identical with that of a spin with inertia (16) when we assume that the particle is constrained on a unit sphere and identify the direction of a particle $\boldsymbol{x}$ with spin direction $\boldsymbol{n}$, a mass m with spin inertia m_s, a magnetic charge q with spin amplitude S, and an electric field $\boldsymbol{\mathcal{E}}$ acting on the particle with a magnetic field $S\boldsymbol{B}$ acting on the spin (we use calligraphic letters for electromagnetic fields on the particle).

Being restricted on the unit sphere, the dynamical degrees of freedom of the particle are the spherical angles θ and ϕ, and the equation of motion is, in a vectorial form,

$$\boldsymbol{e}_\theta\left(\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta}\right) + \frac{\boldsymbol{e}_\phi}{\sin\theta}\left(\frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}} - \frac{\partial L}{\partial\phi}\right) = 0 \quad \text{or} \quad m\ddot{\boldsymbol{x}} = q\boldsymbol{x}\times\dot{\boldsymbol{x}} + \boldsymbol{\mathcal{E}} - (\boldsymbol{\mathcal{E}}\cdot\boldsymbol{x} + m\dot{\boldsymbol{x}}^2)\boldsymbol{x}, \tag{28}$$

which is identical with Eq. (9) (with $\partial_i\boldsymbol{n} = 0$). The last term proportional to $\boldsymbol{x}$ in the right-hand side of the second line is the constraint force, which keeps $|\boldsymbol{x}| = 1$.

The particle is on a unit sphere and is subject to the monopole magnetic field $\boldsymbol{\mathcal{B}}_\mathrm{m} = q\boldsymbol{x}$ (on the sphere $|\boldsymbol{x}| = 1$) and the electric field $\boldsymbol{\mathcal{E}}$. The general motion of the particle is, in the absence of the electric field $\boldsymbol{\mathcal{E}}$, the cyclotron motion, and, in the presence of the electric field $\boldsymbol{\mathcal{E}}$, the $\boldsymbol{\mathcal{E}}\times\boldsymbol{\mathcal{B}}_\mathrm{m}$ drift motion.[3]

In view of the equivalence between the dynamics of a spin with inertia and of a charged particle on a sphere, the free precession of the spin described in the last section corresponds to the cyclotron motion of a particle with a frequency $\omega_0 \sim q/m = S/m_s$, and the Larmor precession of spin accompanied by nutation corresponds to a $\boldsymbol{\mathcal{E}}\times\boldsymbol{\mathcal{B}}_\mathrm{m}$ drift motion.

We summarize in Table I the three pictures for the classical dynamics described by the Lagrangian (16). We have seen in this section that a spin with finite inertia has an intrinsic precession mode (i.e., free precession or nutation). The magnetic field causing this precession is the monopole magnetic field intrinsic to spin, i.e., the spin Berry curvature. The frequency of the intrinsic precession mode, $\omega_0 \sim S/m_s$, is infinite when the inertia of spin m_s is zero, but goes down to a finite value when m_s becomes nonzero.

As related works, we mention Refs. [19–24] where the nutational motion of the spin in Josephson junctions or tunnel junctions between ferromagnets was discussed by examining the Landau-Lifshitz-Gilbert equation with a time-dependent Gilbert damping coefficient.

IV. BEHAVIOR OF SPIN WITH SPATIAL INHOMOGENEITY AND UNDER TIME-DEPENDENT MAGNETIC FIELD

We have so far assumed spatially homogeneous configurations of spin under time-independent magnetic fields. Let us include the spatial variation of the spin and the time-dependence of the applied magnetic fields. The most typical behavior of a spin under a time-dependent magnetic field is the resonance phenomenon. In Ref. [8], it is shown that a spin with finite inertia m_s has a resonance peak at the intrinsic frequency $\omega_0 \sim S/m_s$. We discuss other behaviors of spin, such as spin wave and domain wall motion. We also discuss an unusual behavior of the spin under a large and time-dependent magnetic field.

A. Spin waves

Let us consider spin wave solutions of spin with inertia, and see that there exists a new gapful spin wave mode with unusual handedness. We assume that the spins are almost in the positive z direction, $\boldsymbol{n} = (n_x, n_y, \sqrt{1 - n_x^2 - n_y^2}) \cong (n_x, n_y, 1)$ with $n_x, n_y \ll 1$, and keep n_x and n_y only up to their first order. Then, the equation of motion is,[4] from action (7),

$$m_s\ddot{\boldsymbol{n}} = S\boldsymbol{n}\times\dot{\boldsymbol{n}} + JS^2\partial_i^2\boldsymbol{n}, \tag{29}$$

where we have set $\boldsymbol{B} = 0$. Since the equation of motion is of second order in the time derivative, we have now two spin wave modes. When we substitute a plane wave $n_x + in_y = Ae^{-i(\omega t - kx)}$ to the equation of motion, where A is an arbitrarily small constant, we obtain the dispersion relation (Fig. 3),

$$\omega = \frac{1}{2}\frac{S}{m_s}\left(-1 \pm \sqrt{1 + 4Jm_s k^2}\right) \sim \begin{cases} JSk^2 + \mathcal{O}(k^4), \\ -\left(\frac{S}{m_s} + JSk^2\right) + \mathcal{O}(k^4). \end{cases} \tag{30}$$

[3] Our convention for the sign of the magnetic field is such that the equation of motion of a particle is $m\ddot{\boldsymbol{x}} = \boldsymbol{\mathcal{B}}\times\dot{\boldsymbol{x}} + \boldsymbol{\mathcal{E}}$, which yields the drift motion of the particle in the $\boldsymbol{\mathcal{B}}\times\boldsymbol{\mathcal{E}}$ direction, rather than the $\boldsymbol{\mathcal{E}}\times\boldsymbol{\mathcal{B}}$ direction.

[4] The assumption $n_x, n_y \ll 1$ means that we consider the problem in the tangent plane on $n_z = 1$, so the constraint force in (9) vanishes.

TABLE I. Three pictures for the classical dynamics described by the Lagrangian (16).

Picture	Inertia	Strength of the intrinsic magnetic field
Spin	The inertia m_s	The amplitude S of the Berry curvature
Charged particle	The mass m	The magnetic charge q
Symmetric spinning top	The moment of inertia I_1	The spinning angular momentum M_3

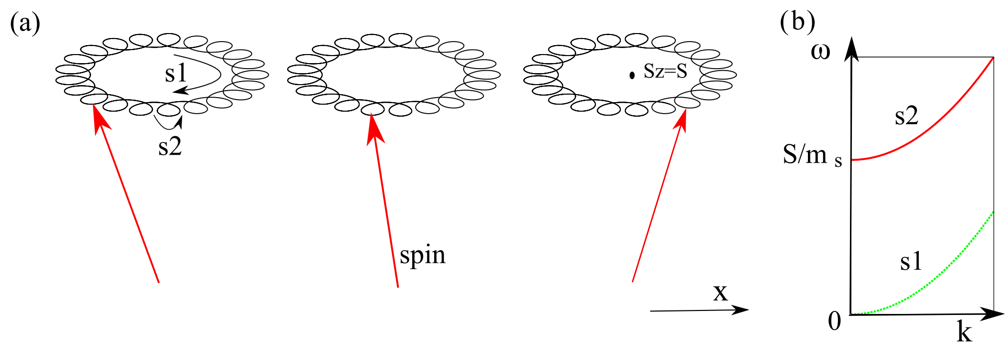

FIG. 3. (Color online) (a) The trajectory of a spin, which fluctuates around $S_z = S$ point (this fluctuation propagates in space as a spin wave). In the figure, two fluctuation modes are superimposed. One is the usual gapless mode rotating clockwise seen from the positive z direction (denoted by s1), while the other is a gapful mode rotating counterclockwise (denoted by s2). (b) The dispersion relations for the two modes of a spin wave. Besides the usual gapless spin wave mode (in green color and denoted by s1), a spin with inertia has a gapful spin wave mode (in red color and denoted by s2), which is the nutation of a spin propagating in space by the spin-spin exchange. See Eq. (30).

One is the usual spin wave mode and the other is the new spin wave mode, the latter of which is essentially the free precession of a spin mediated in space by the interaction J with neighboring spins. We have expanded the dispersion relation (30) in powers of k and cut higher-order terms because we are in the framework of the derivative expansion in deriving action (7).

The usual spin wave mode is clockwise seen from the positive z direction, while the new spin wave mode is counterclockwise. In fact, to the leading order of the dispersion relation (30), the former obeys the equation of motion $S\dot{\boldsymbol{n}} = -J\partial_i^2\boldsymbol{n}\times\boldsymbol{e}_z = +Jk^2\boldsymbol{n}\times\boldsymbol{e}_z$, while the latter $m_s\ddot{\boldsymbol{n}} = -S\dot{\boldsymbol{n}}\times\boldsymbol{e}_z$, where $\boldsymbol{e}_z$ is the unit vector in the z direction. Therefore their handednesses are opposite. The total angular momentum $\boldsymbol{j}^0$ in (10) also rotates counterclockwise for the new spin wave mode.

When the spin and the angular momentum align in the same direction, $\boldsymbol{j}^0 = S\boldsymbol{n}$, the fluctuation of a spin around a stable configuration is always clockwise. This can be understood as follows (see, e.g., Ref. [25]). Let a stable configuration be $n_z = 1$. The Poisson commutation relation of the angular momenta, $[j_x^0, j_y^0] = j_z^0 \cong S$, leads to that of spin, $[n_x, n_y] = 1/S$. Therefore the x and y components of spin are canonically conjugate to each other. Moreover, the Hamiltonian of spin expanded around the stable point is $H \cong D[(n_x)^2 + (n_y)^2]$ with $D > 0$. These two facts, i.e., the canonical commutation relation of n_x and n_y, and the form of the Hamiltonian, indicate that the fluctuation of spin around $n_z = 1$ has the same dynamical structure as that of a one-dimensional harmonic oscillator with coordinate x and its canonical momentum p governed by the commutation relation $[x,p] = 1$ and the Hamiltonian $H = D(x^2 + p^2)$. The trajectory of the harmonic oscillator is always clockwise in the phase space. Therefore the fluctuation of spin is also always clockwise. However, when the inertia of spin is finite, the spin and the angular momentum generally do not align due to the additional contribution $m_s\boldsymbol{n}\times\dot{\boldsymbol{n}}$ in $\boldsymbol{j}^0$ (10), and moreover the Hamiltonian of a spin such as Eq. (22) does not consist only of the potential term but also of the kinetic term. Therefore the fluctuation of a spin with inertia is not restricted to the clockwise motion.

B. Domain wall

Let us see that a domain wall, consisting of spins with inertia, also has an intrinsic precession mode. When easy-axis anisotropy $(KS^2/2)\cos^2\theta$ is included to the action (7), the system has a static domain wall solution, $\cos\theta = \tanh[(z - X)/\lambda], \phi = \phi_0$ (we take z axis as the wall normal), where X and ϕ_0 are integration constants and $\lambda = \sqrt{J/K}$. Promoting X and ϕ_0 to dynamical variables, substituting the domain wall solution into the action (7) and integrating it over space, we obtain the Lagrangian of X and ϕ_0:

$$L[X,\phi_0] = \frac{M_\mathrm{w}}{2}\left(\dot{X}^2 + \lambda^2\dot{\phi}_0^2\right) + \frac{SN_\mathrm{w}}{2\lambda}(\dot{X}\phi_0 - X\dot{\phi}_0) \quad (31)$$

with $M_\mathrm{w} = m_s N_\mathrm{w}/\lambda^2$ and $N_\mathrm{w} = 2\lambda A/a^3$, where A is the cross sectional area of the domain wall (we discarded irrelevant constant terms in the Lagrangian). We are going to show that this Lagrangian is of the same form as that of a charged particle on a cylinder with a constant magnetic field $\mathcal{B} = SN_\mathrm{w}/\lambda^2$ perpendicularly penetrating the surface of the cylinder. Take an orthogonal coordinate frame $\boldsymbol{x} = (x,y)$ on a cylinder, with $y \sim y + 2\pi\lambda$ the periodic direction. (The directions x and y and the direction outward normal to the surface of the cylinder make an orthogonal right-handed frame.) Then, the gauge potential $\boldsymbol{\mathcal{A}}$ for the constant magnetic field $\mathcal{B}$ in the outward normal direction of the surface can be taken as $\mathcal{A}_x = \mathcal{B}y/2, \mathcal{A}_y = -\mathcal{B}x/2$, and the Lagrangian of the particle is

$$L = \frac{m}{2}\dot{\boldsymbol{x}}^2 - \boldsymbol{\mathcal{A}}\cdot\dot{\boldsymbol{x}} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{\mathcal{B}}{2}(\dot{x}y - x\dot{y}). \quad (32)$$

Therefore, when we identify $(X,\lambda\phi_0)$ with (x,y), the Lagrangian (31) is that of a charged particle under a perpendicular magnetic field of flux $\mathcal{B} = SN_\mathrm{w}/\lambda^2$ (Fig. 4). The magnetic field $\mathcal{B}$ for the particle originates in the spin Berry phase, and the mass M_w of the particle comes from the inertia of spin in the action (7).

As a result, the particle undergoes cyclotron motion (in other words, the domain wall undergoes an intrinsic free precession mode), where X and $\lambda\phi_0$ oscillate, $X = r\cos(\omega_0 t), \lambda\phi_0 = r\sin(\omega_0 t)$ with r a constant, at frequency $\omega_0 = \mathcal{B}/M_\mathrm{w} = SN_\mathrm{w}/(\lambda^2 M_\mathrm{w}) = S/m_s$. This frequency

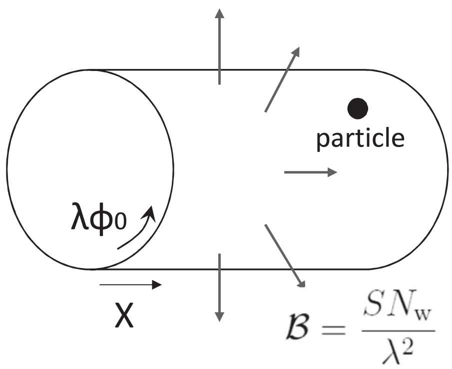

FIG. 4. The collective coordinates $(X,\lambda\phi_0)$ of a domain wall can be regarded as coordinates of a particle on a cylinder. This particle is subject to a magnetic field flux $\mathcal{B} = SN_w/\lambda^2$ penetrating the cylinder perpendicularly. The inertia of spin introduces the mass $M_w = m_s N_w/\lambda^2$ into this particle.

corresponds to that of the spin free precession discussed in Sec. III.

When an external field is applied on this domain wall, the particle on a cylinder $(X,\lambda\phi_0)$ feels the corresponding force on it, and it moves in the direction perpendicular to the force, accompanied by the cyclotron motion (nutation). That is, the particle undergoes $\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{B}}$ drift motion, where $\boldsymbol{\mathcal{E}}$ is the force and $\boldsymbol{\mathcal{B}}$ is the magnetic field penetrating the cylinder $(X,\lambda\phi_0)$ perpendicularly.

The discussion above is well valid when the inertia m_s is so large that the intrinsic frequency $\omega_0 \sim S/m_s$ is less than the frequency of spin waves: $S/m_s < KS^2/\hbar$. When S/m_s exceeds this frequency scale, the dynamics of the domain wall can be no longer described only by the collective coordinates X and ϕ_0, and spin wave excitations should also be included into the dynamics.

C. Parallel shift of spin under time-dependent magnetic field

In the spin nutation such as in Fig. 1, spin moves up and down parallel to the given magnetic field, but this motion is oscillatory and the guiding center (i.e., the center of the nutation) moves only perpendicularly to the magnetic field, just as the usual Larmor precession. Is it possible that the guiding center motion becomes different from the usual Larmor precession? In this section, we see that when a magnetic field is time-dependent, the guiding center motion has a finite component of velocity parallel to the time-derivative of the magnetic field.

Bearing the particle analogy discussed in Sec. III in mind, this parallel motion is most directly seen by looking at the behavior of a charged particle, located on a plane (x,y) under a constant and perpendicular magnetic field $\mathcal{B}$ and a time-dependent in-plane electric field $\boldsymbol{\mathcal{E}}(t)$ (recall that an electric field $\boldsymbol{\mathcal{E}}$ on the particle corresponds to a magnetic field $\boldsymbol{B}$ on the spin). The general solutions of the equation of motion are given by $(x,y) = (\xi,\eta) + (X,Y)$, such that

$$m\begin{pmatrix}\ddot{\xi}\\ \ddot{\eta}\end{pmatrix} = \mathcal{B}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\dot{\xi}\\ \dot{\eta}\end{pmatrix} \tag{33}$$

and

$$\begin{pmatrix}\dot{X}\\ \dot{Y}\end{pmatrix} = \frac{1}{\mathcal{B}}\frac{1}{1+\omega_c^{-2}(d^2/dt^2)}\begin{pmatrix}\omega_c^{-1}d/dt & -1\\ 1 & \omega_c^{-1}d/dt\end{pmatrix}\begin{pmatrix}\mathcal{E}_x\\ \mathcal{E}_y\end{pmatrix} \tag{34}$$

with $\omega_c = \mathcal{B}/m$. That is, we can divide the motion into the cyclotron motion (ξ,η) and the guiding center motion (X,Y) even when the electric field is time dependent.[5] We can see that the guiding center motion (X,Y) has a finite component of velocity parallel to the time derivative of the electric field, when the mass is finite.

As we saw in Sec. III, the classical dynamics of spin with inertia is a spherical version of this planar problem, with the role of an electric field for the particle played by a magnetic field for spin. Therefore the essential features of the dynamics will be the same (especially when the nutation radius is small and the curvature of the sphere can be neglected): (i) the spin nutation and the guiding center motion are decoupled; (ii) the effect of the inertia on the guiding center motion is significant when the time rate of change of the applied field is comparable to the intrinsic frequency S/m_s.

Therefore the effect of the inertia is not only the superimposition of the nutational motion on the Larmor precession. The inertia of spin gives an independent effect on the trajectory of the Larmor precession itself, cooperatively with the time-dependence of the applied field. Even when the nutation radius is small and there are no signs of nutation in the experimental data about the time-profile of the magnetization, it never means that the inertia has no effect on the dynamics, especially when the applied magnetic field has a very rapid time dependence, such as for ultrafast magnetization.

We illustrate this fact by numerical calculation (Fig. 5). We fix a magnetic field in the z direction. First, we apply a magnetic field in the negative z direction, then change it into the positive z direction. We can see that the spin moves parallel (i.e., in the positive z direction) to the time derivative of the magnetic field.[6] This phenomenon is similar to the usual spin flip by Gilbert damping, but is different in that we do not have to vary the direction of the magnetic field and moreover the time change rate of the magnetic field can be arbitrary (as long as our derivative expansion is valid), irrelevant to the damping coefficient.

This unusual behavior of spin can be simply understood by looking at the angular momentum rather than spin itself. As in (10), the angular momentum of spin with finite inertia is no longer $S\boldsymbol{n}$ but has an additional nonadiabatic contribution $m_s\boldsymbol{n} \times \dot{\boldsymbol{n}}$. When the Larmor frequency B is much smaller than the intrinsic frequency (nutation frequency), $B \ll S/m_s$, then the vector $m_s\boldsymbol{n} \times \dot{\boldsymbol{n}}$ points inward to the center of the nutation circle, and the trajectory of the angular momentum of spin is overlapped with that of spin [Fig. 6(a)]. On the other hand, when $B \sim S/m_s$, nutation does not form circles but rather ripples on the trajectory of spin. In this case, the vector

[5]As long as $\omega_c^{-1}d/dt < 1$ and the operator $(1+\omega_c^{-2}d^2/dt^2)^{-1}$ is well defined.

[6]The switching of the sign of the magnetic field from negative to positive is not essential for the shift of the spin in the z direction. Only the time dependence of the magnetic field is essential.

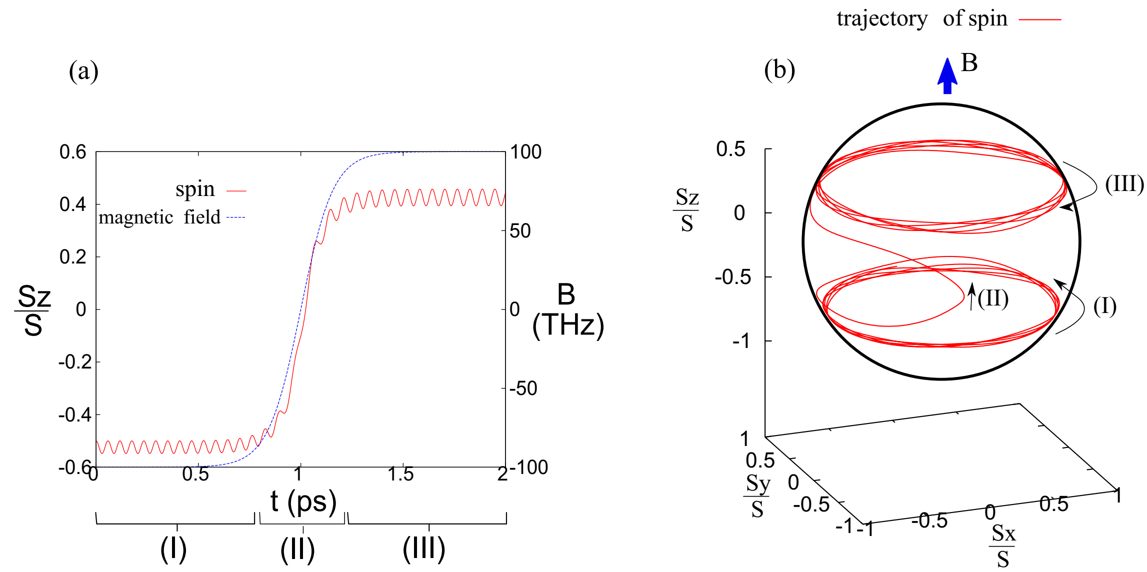

FIG. 5. (Color online) Response of a spin with finite inertia to a time-dependent magnetic field $B(t) = B_0 \tanh(f(t - t_0))$ in z direction with $B_0 = 100$ THz and $f = 2\pi$ THz $= 6.3$ THz. We take the spin amplitude as $S = \hbar$ and the inertia as $m_s/\hbar = 0.1$ ps$/(2\pi) = 0.016$ ps. (a) The time profiles of $S_z/S(= n_z)$ of the spin and of the applied magnetic field B. (b) The trajectory of the spin. The time is divided into three domains I–III. In I, the magnetic field points in the negative z direction and the spin precesses counterclockwise seen from the positive z direction. In II, the magnetic field switches to positive z direction and the z component of spin, S_z/S, also changes accordingly. In III, the magnetic field points in the positive z direction and the spin precesses clockwise.

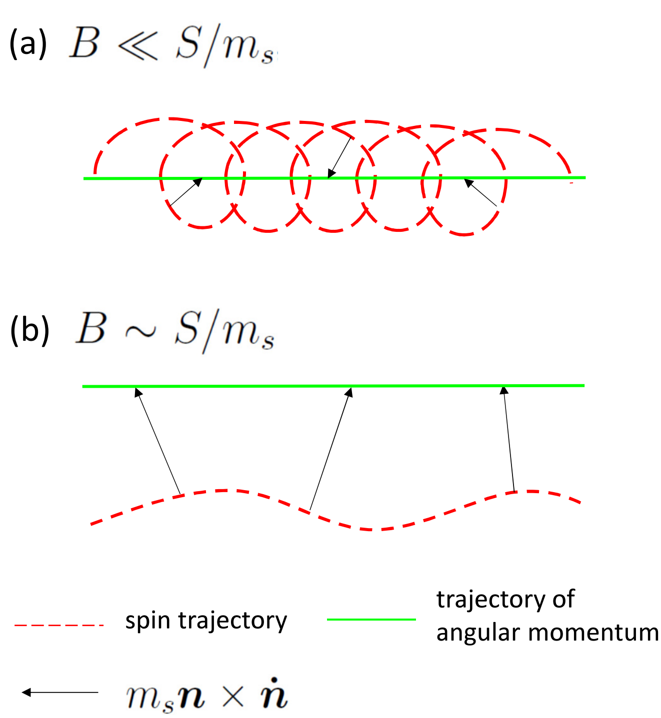

FIG. 6. (Color online) Trajectories of spin $S\boldsymbol{n}$ and angular momentum $\boldsymbol{j}^0 = S\boldsymbol{n} + m_s\boldsymbol{n} \times \dot{\boldsymbol{n}}$. (a) When $B \ll S/m_s$, spin nutation forms a circle shape. The additional component $m_s\boldsymbol{n} \times \dot{\boldsymbol{n}}$ points inward to the center of the nutation, and the trajectory of the angular momentum sits within that of spin. (b) When $B \sim S/m_s$, spin nutation forms a ripple shape. The additional component $m_s\boldsymbol{n} \times \dot{\boldsymbol{n}}$ points to one side of the spin trajectory. As a result, the trajectories of spin and the angular momentum deviate.

$m_s\boldsymbol{n} \times \dot{\boldsymbol{n}}$ points in almost the same direction at each time. As a result, the trajectory of the angular momentum deviates from that of the spin [Fig. 6(b)]. Figure 7 is a replot of the trajectory of spin in Fig. 5 together with that of the angular momentum. The vector $m_s\boldsymbol{n} \times \dot{\boldsymbol{n}}$ points in the positive z direction in the time domain I (described in Fig. 5), and in the negative z direction in the time domain III, resulting in the conservation of the z component of the angular momentum.

We can understand that this parallel shift of a spin in the direction of the time derivative of a magnetic field can be

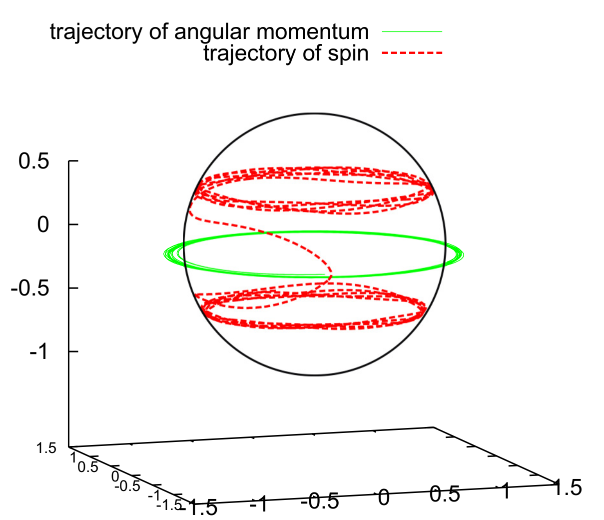

FIG. 7. (Color online) Replot of the trajectory of the spin in Fig. 5 (from slightly a different view angle), together with that of the angular momentum. The magnetic field is $B \sim 100$ THz and $S/m_s \sim 20\pi$ THz $= 62.8$ THz, corresponding to the case (b) in Fig. 6. Although the z-component of spin changes in time, that of the angular momentum is constant in motion. The axes are in unit of S.

understood also from the viewpoint of a spinning top, where the magnetic field on the spin corresponds to the gravitational field on the top. The time variation of the gravitational field can be realized fictitiously by putting the top in an elevator accelerated in the negative z direction. There, the top feels, besides the real gravitational field in the negative z direction, a fictitious inertial force in the positive z direction due to the acceleration. This inertial force in the positive z direction makes the top stand up in the positive z direction, which explains the shift of spin in the z direction. Thus the inertia of spin yields an inertial force, literally. In this respect, the connection between the inertia of the spin and the inertial effect on the spin [26,27] will also be interesting.

V. SUMMARY AND DISCUSSION

We examined the inertia of the spin in metallic ferromagnets, which arises in the derivative expansion of the spin effective action and represents a nonadiabatic contribution from the environmental degrees of freedom. We derived an explicit expression of spin inertia in an sd model. The equivalence between the dynamics of a spin with inertia, a spinning top, and a charged particle on a sphere was explained. The behavior of a spin with a spatially inhomogeneous configuration or under a time-dependent magnetic field is studied. The finite inertia has mainly two effects: the superimposition of the nutation on the usual Larmor precession and the parallel shift of the Larmor precession.

As explained several times in this paper, most of the unusual behavior of a spin with finite inertia originates in the nonadiabatic component $m_s \boldsymbol{n} \times \dot{\boldsymbol{n}}$ in the relation between the spin and the angular momentum, $\boldsymbol{j}^0 = S\boldsymbol{n} + m_s \boldsymbol{n} \times \dot{\boldsymbol{n}}$. Due to this difference between them, the spin acquires a kind of a separate freedom from the angular momentum, whose behavior is relatively restricted due to the conservation law. As a result, the spin undergoes the free precession or nutation discussed in Sec. III, and furthermore, a parallel shift along the direction of the time derivative of applied magnetic fields, discussed in Sec. IV. Since one of the main purposes of spintronics is understanding and application of diverse cooperative phenomena between localized spins and their environmental degrees of freedom, research beyond the adiabatic limit is well-motivated; in the adiabatic limit, the environmental degrees of freedom are rather obedient to the localized spins. In this paper, we estimated that in metallic ferromagnets such nonadiabaticity is effective for subpicosecond dynamics, i.e., for ultrafast magnetization. For broader applications, physical situations where nonadiabaticity is effective for subnanosecond dynamics are also desirable.

Apart from the context of effective dynamics (i.e., the dynamics including the effects of environmental degrees of freedom), it was shown that a staggered spin field in an antiferromagnet has an inertial term in its dynamics (e.g., Refs. [28,29]). In particular, for systems whose ground states can be intermediate states between a ferromagnet and an antiferromagnet (e.g., Ref. [30]), both the spin Berry phase term and the inertial term can be present in the dynamics of the order parameters. The discussion of this paper will be also applicable to such systems.

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APPENDIX A: SPIN EFFECTIVE ACTION

In this appendix, we calculate the spin effective action by integrating out conduction electrons coupled to the spin through the sd interaction, and derive the expression of the spin inertia (4). We always make a summation over repeated indices unless otherwise stated.

The action of a conduction electron, represented by field operators c and $\bar{c}$, coupled to a localized spin $\boldsymbol{n}$ is

$$\mathcal{S}_{\mathrm{e}}[c,\bar{c},\boldsymbol{n}] = \int d^3x dt \bar{c}\left(i\hbar\partial_t + \frac{\hbar^2\partial_i^2}{2m} + \epsilon_F + g_{\mathrm{sd}}\boldsymbol{n}\cdot\boldsymbol{\sigma}\right)c, \tag{A1}$$

with g_{sd} the sd coupling constant. What we calculate in this appendix is the contribution to the spin effective action from the conduction electrons, $\Delta\mathcal{S}_{\mathrm{eff}}[\boldsymbol{n}]$, defined as

$$\exp\left(\frac{i}{\hbar}\Delta\mathcal{S}_{\mathrm{eff}}[\boldsymbol{n}]\right) \equiv \int \mathcal{D}\bar{c}\mathcal{D}c \exp\left(\frac{i}{\hbar}\mathcal{S}_{\mathrm{e}}[c,\bar{c},\boldsymbol{n}]\right). \tag{A2}$$

Adding this $\Delta\mathcal{S}_{\mathrm{eff}}[\boldsymbol{n}]$ to the original spin action

$$\mathcal{S}_{\mathrm{s}}[\boldsymbol{n}] = \int \frac{d^3x}{a^3}\left[S_l\dot{\phi}(\cos\theta - 1) - J_l S_l^2(\partial_i\boldsymbol{n})^2\right], \tag{A3}$$

where θ and ϕ are the polar angles of the unit vector $\boldsymbol{n}$, we obtain the total spin effective action $\mathcal{S}_{\mathrm{eff}}[\boldsymbol{n}] = \mathcal{S}_{\mathrm{s}}[\boldsymbol{n}] + \Delta\mathcal{S}_{\mathrm{eff}}[\boldsymbol{n}]$.

First, we diagonalize the sd coupling in (A1). For this purpose, it is convenient and essential to employ the unitary transformed electron field a such that $c(\boldsymbol{x},t) = U(\boldsymbol{x},t)a(\boldsymbol{x},t)$. Here, the SU(2) matrix U is defined so that

$$U^\dagger(\boldsymbol{n}\cdot\boldsymbol{\sigma})U = \sigma_3. \tag{A4}$$

As such, U has an indefiniteness about the right U(1) gauge transformation, $U \to U\exp(i\chi(\boldsymbol{x},t)\sigma_3)$ with $\chi(\boldsymbol{x},t)$ an arbitrary function. [Due to the defining relation $c = Ua$ of a, this unitary transformation corresponds to that of a, i.e., $a \to \exp(-i\chi(\boldsymbol{x},t)\sigma_3)a$. The original electron field c remains invariant.] One choice of U is $U = \boldsymbol{m}\cdot\boldsymbol{\sigma}$ with $\boldsymbol{m} = (\sin\frac{\theta}{2}\cos\phi, \sin\frac{\theta}{2}\sin\phi, \cos\frac{\theta}{2})$. Geometrically, U rotates the electron spin in the positive z direction, $|\uparrow\rangle$, to the direction of $\boldsymbol{n}$. For the choice $U = \boldsymbol{m}\cdot\boldsymbol{\sigma}$ above, $U|\uparrow\rangle = |\boldsymbol{n}\rangle$ with $|\boldsymbol{n}\rangle \equiv \cos(\theta/2)|\uparrow\rangle + e^{i\phi}\sin(\theta/2)|\downarrow\rangle$. The indefiniteness of U, $U \to U\exp(i\chi\sigma_3)$, corresponds to the phase rotation of $|\boldsymbol{n}\rangle$, i.e., $|\boldsymbol{n}\rangle \to \exp(i\chi)|\boldsymbol{n}\rangle$. The sd coupling becomes diagonal in the a frame, $\bar{c}(\boldsymbol{n}\cdot\boldsymbol{\sigma})c = \bar{a}\sigma_3 a$, but, in compensation for the SU(2) gauge transformation by U, there appears a spin gauge field $A_\mu \equiv -iU^\dagger\partial_\mu U = (\boldsymbol{m}\times\partial_\mu\boldsymbol{m})\cdot\boldsymbol{\sigma}$ in the derivative terms, $\partial_\mu c = U(\partial_\mu + iA_\mu)a$.

Then, the action (A1) becomes [31]

$$\mathcal{S}_{\rm e} = \int d^3x dt \bar{a}\left(i\hbar(\partial_t + iA_t) + \frac{\hbar^2(\partial_i + iA_i)^2}{2m} + \epsilon_F + g_{\rm sd}\sigma_3\right)a$$
$$= \mathcal{S}_0[a,\bar{a}] + \int d^3x dt\left[A_t^a(-\hbar\bar{a}\sigma^a a) + A_i^a\left(\frac{i\hbar^2}{2m}\bar{a}\sigma_a \overleftrightarrow{\partial}_i a\right) + A_i^a A_i^a\left(-\frac{\hbar^2}{2m}\bar{a}a\right)\right]. \quad \text{(A5)}$$

Here, $\mathcal{S}_0[a,\bar{a}]$ is

$$\mathcal{S}_0[a,\bar{a}] \equiv \int d^3x dt \bar{a}\left(i\hbar\partial_t + \frac{\hbar^2\partial_i^2}{2m} + \epsilon_F + g_{\rm sd}\sigma_3\right)a, \quad \text{(A6)}$$

and $A_\mu = A_\mu^a\sigma^a (a = 1,2,3)$ is the spin gauge field (make a distinction between the electron field a and the $SU(2)$ index^a), which is, in terms of the polar angles of $\boldsymbol{n}$,

$$\boldsymbol{A}_\mu = \frac{1}{2}\begin{pmatrix} -\partial_\mu\theta\sin\phi - \sin\theta\cos\phi\partial_\mu\phi \\ \partial_\mu\theta\cos\phi - \sin\theta\sin\phi\partial_\mu\phi \\ (1-\cos\theta)\partial_\mu\phi \end{pmatrix}, \quad \text{(A7)}$$

where $(\boldsymbol{A}_\mu)^a = A_\mu^a$. Because A_μ is proportional to the first-order derivative of $\boldsymbol{n}$, the derivative expansion of $\Delta\mathcal{S}_{\rm eff}$ in powers of $\partial_\mu\boldsymbol{n}$ corresponds to that of A_μ:

$$\frac{i}{\hbar}\Delta\mathcal{S}_{\rm eff}[\boldsymbol{n}] = \frac{i}{\hbar}\int d^4x\left[A_t^a(-\hbar\langle\bar{a}\sigma^a a\rangle) + A_i^a\frac{i\hbar^2}{2m}\langle\bar{a}\sigma_a\overleftrightarrow{\partial}_i a\rangle + A_i^a A_i^a\left(-\frac{\hbar^2}{2m}\langle\bar{a}a\rangle\right)\right] + \frac{1}{2}\left(\frac{i}{\hbar}\right)^2\int d^4x d^4x'$$
$$\times\left[A_t^a(x)A_t^b(x')\hbar^2\langle\bar{a}\sigma^a a(x)\bar{a}\sigma^b a(x')\rangle + A_i^a(x)A_j^b(x')\left(\frac{i\hbar^2}{2m}\right)^2\langle\bar{a}\sigma^a\overleftrightarrow{\partial_i}a(x)\bar{a}\sigma^b\overleftrightarrow{\partial_j}a(x')\rangle\right.$$
$$\left. + A_t^a(x)A_i^b(x')\left(-\frac{i\hbar^3}{m}\right)\langle\bar{a}\sigma^a a(x)\bar{a}\sigma^b\overleftrightarrow{\partial_i}a(x')\rangle\right] + \mathcal{O}\left(A_\mu^3\right), \quad \text{(A8)}$$

where the time-ordered quantum expectation values are taken about $\mathcal{S}_0$,

$$\langle\cdots\rangle = \int\mathcal{D}\bar{a}\mathcal{D}a(\cdots)e^{\frac{i}{\hbar}\mathcal{S}_0[a,\bar{a}]}, \quad \text{(A9)}$$

and we take only the connected diagrams to evaluate the quantum expectation values. Because the nonperturbed action $\mathcal{S}_0$ (A6) is isotropic in space, $\langle\bar{a}\sigma^a\overleftrightarrow{\partial}_i a\rangle$ and $\langle\bar{a}\sigma^a a(x)\bar{a}\sigma^b\overleftrightarrow{\partial_i}a(x')\rangle$ vanish. We define the notation for the correlation functions as

$$\rho^{ab}(x,x') \equiv i\langle\bar{a}\sigma^a a(x)\bar{a}\sigma^b a(x')\rangle,$$
$$\chi_{ij}^{ab}(x,x') \equiv i\langle\bar{a}\sigma^a\overleftrightarrow{\partial_i}a(x)\bar{a}\sigma^b\overleftrightarrow{\partial_j}a(x')\rangle. \quad \text{(A10)}$$

Using the Fourier components, we expand these nonlocal quantities. For example,

$$\rho^{ab}(x,x') = \int\frac{d^4q}{(2\pi)^4}e^{iq\cdot(x-x')}\rho^{ab}(q)$$
$$= \delta^{(4)}(x-x')\rho^{ab}|_{q=0} + \frac{1}{i}\partial_\mu\delta^{(4)}(x-x')\left.\frac{\partial\rho^{ab}}{\partial q^\mu}\right|_{q=0} + \cdots. \quad \text{(A11)}$$

When we substitute this into the expansion (A8), terms other than the first term in (A11) increase the order of the derivative on $\boldsymbol{n}$ after partial integration, e.g., $A_t^a(x)A_t^b(x')\partial_\mu\delta^{(4)}(x-x') \to -\partial_\mu[A_t^a(x)A_t^b(x')]\delta^{(4)}(x-x')$. Therefore we take only the first term in (A11) as long as we are interested in at most $\mathcal{O}(\partial_\mu\boldsymbol{n})^2$ terms. After all, $\Delta\mathcal{S}_{\rm eff}[\boldsymbol{n}]$ is, to $(\partial_\mu\boldsymbol{n})^2$ order,

$$\Delta\mathcal{S}_{\rm eff}[\boldsymbol{n}] = \int d^4x\left[(-\hbar\langle\bar{a}\sigma^a a\rangle)A_t^a + \left(\frac{\hbar}{2}{\rm Re}\rho^{ab}\bigg|_{q=0}\right)A_t^a A_t^b + \left(-\frac{\hbar^2}{2m}\langle\bar{a}a\rangle\delta_{ab}\delta_{ij} - \frac{\hbar^3}{8m^2}{\rm Re}\chi_{ij}^{ab}\bigg|_{q=0}\right)A_i^a A_j^b\right] \quad \text{(A12)}$$

(the imaginary parts of χ_{ij}^{ab} and ρ^{ab} contribute to damping and are neglected here). We calculate each term in the following.

Firstly, we calculate $\langle\bar{a}\sigma^a a\rangle$. The time-ordered Green's function for $\mathcal{S}_0$ (A6) is

$$-i\langle a_\alpha(x)\bar{a}_\beta(x')\rangle = \delta_{\alpha\beta}\int\frac{d^3kd\omega}{(2\pi)^4}e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')-i\omega(t-t')}g_{\boldsymbol{k},\omega,\alpha}, \quad g_{\boldsymbol{k},\omega,\alpha} \equiv \frac{1}{\omega - \omega_{\boldsymbol{k}\alpha} + i\delta_\alpha}, \quad \text{(A13)}$$

[the indices $\alpha,\beta,\cdots = \pm$ correspond to spin up or down, and no summation over α in the first line of Eq. (A13)] with

$$\hbar\omega_{\boldsymbol{k}\pm} \equiv \frac{\hbar^2\boldsymbol{k}^2}{2m} - \epsilon_F \mp g_{\rm sd}, \quad \delta_\alpha = {\rm sgn}(\omega_{\boldsymbol{k}\alpha}) \times 0, \tag{A14}$$

where 0 denotes a positive infinitesimal. We define a 2×2 matrix $g_{\boldsymbol{k},\omega}$ by $(g_{\boldsymbol{k},\omega})_{\alpha\beta} \equiv \delta_{\alpha\beta} g_{\boldsymbol{k},\omega,\alpha}$ (no summation over α), or,

$$g_{\boldsymbol{k},\omega} \equiv \tfrac{1}{2}(g_{\boldsymbol{k},\omega,+} + g_{\boldsymbol{k},\omega,-} + \sigma^3(g_{\boldsymbol{k},\omega,+} - g_{\boldsymbol{k},\omega,-})). \tag{A15}$$

Then,

$$\langle \bar{a}\sigma^a a\rangle = -i\int \frac{d^3k d\omega}{(2\pi)^4} e^{i\omega 0}{\rm tr}[\sigma^a g_{\boldsymbol{k},\omega}] = -i\delta_3^a \sum_\pm (\pm)\int \frac{d^3kd\omega}{(2\pi)^4}\frac{e^{i\omega 0}}{\omega - \omega_{\boldsymbol{k}\pm} + i\delta_\pm} = \frac{k_{F+}^3 - k_{F-}^3}{6\pi^2}\delta_3^a, \quad \frac{\hbar^2 k_{F\pm}^2}{2m} \equiv \epsilon_F \pm g_{\rm sd}. \tag{A16}$$

Secondly, the number density of the conduction electrons $n \equiv \langle \bar{a}a\rangle$ is given by $n = (k_{F+}^3 + k_{F-}^3)/(6\pi^2)$.

Thirdly, we calculate ${\rm Re}\chi_{ij}^{ab}|_{q=0}$. Using

$${\rm tr}[\sigma^a g_{\boldsymbol{k},\omega}\sigma^b g_{\boldsymbol{k}+\boldsymbol{q},\omega+\Omega}] = \sum_\pm [(\delta_{ab} - \delta_{a3}\delta_{b3} \mp i\varepsilon_{ab3}) g_{\boldsymbol{k},\omega,\pm} g_{\boldsymbol{k}+\boldsymbol{q},\omega+\Omega,\mp} + \delta_{a3}\delta_{b3} g_{\boldsymbol{k},\omega,\pm} g_{\boldsymbol{k}+\boldsymbol{q},\omega+\Omega,\pm}] \tag{A17}$$

and

$$-i\int \frac{d\omega}{2\pi} g_{\boldsymbol{k},\omega,\alpha} g_{\boldsymbol{k}+\boldsymbol{q},\omega+\Omega,\beta} = \frac{\theta(\omega_{\boldsymbol{k}\alpha})\theta(-\omega_{\boldsymbol{k}+\boldsymbol{q}\beta})}{\omega_{\boldsymbol{k}+\boldsymbol{q}\beta} - \omega_{\boldsymbol{k}\alpha} - \Omega + i0} - \frac{\theta(-\omega_{\boldsymbol{k}\alpha})\theta(\omega_{\boldsymbol{k}+\boldsymbol{q}\beta})}{\omega_{\boldsymbol{k}+\boldsymbol{q}\beta} - \omega_{\boldsymbol{k}\alpha} - \Omega - i0}, \tag{A18}$$

we obtain

$$\begin{aligned}\chi_{ij}^{ab}(\boldsymbol{q},\Omega) =& -i\int \frac{d^3kd\omega}{(2\pi)^4}(2k_i + q_i)(2k_j + q_j){\rm tr}[\sigma^a g_{\boldsymbol{k},\omega}\sigma^b g_{\boldsymbol{k}+\boldsymbol{q},\omega+\Omega}] \\ =& \sum_\pm \int \frac{d^3k}{(2\pi)^3}(2k_i+q_i)(2k_j+q_j)\bigg\{(\delta_{ab} - \delta_{a3}\delta_{b3} \mp i\varepsilon_{ab3})\bigg[\frac{\theta(\omega_{\boldsymbol{k}\pm})\theta(-\omega_{\boldsymbol{k}+\boldsymbol{q}\mp})}{\omega_{\boldsymbol{k}+\boldsymbol{q}\mp} - \omega_{\boldsymbol{k}\pm} - \Omega + i0} - \frac{\theta(-\omega_{\boldsymbol{k}\pm})\theta(\omega_{\boldsymbol{k}+\boldsymbol{q}\mp})}{\omega_{\boldsymbol{k}+\boldsymbol{q}\mp} - \omega_{\boldsymbol{k}\pm} - \Omega - i0}\bigg] \\ &+ \delta_{a3}\delta_{b3}\bigg[\frac{\theta(\omega_{\boldsymbol{k}\pm})\theta(-\omega_{\boldsymbol{k}+\boldsymbol{q}\pm})}{\omega_{\boldsymbol{k}+\boldsymbol{q}\pm} - \omega_{\boldsymbol{k}\pm} - \Omega + i0} - \frac{\theta(-\omega_{\boldsymbol{k}\pm})\theta(\omega_{\boldsymbol{k}+\boldsymbol{q}\pm})}{\omega_{\boldsymbol{k}+\boldsymbol{q}\pm} - \omega_{\boldsymbol{k}\pm} - \Omega - i0}\bigg]\bigg\},\end{aligned} \tag{A19}$$

where $\theta(x) = 1$ for $x \geqslant 0$ and zero otherwise. We are interested here only in the real part, ${\rm Re}\chi_{ij}^{ab}$. Using $1/(x \pm i0) = {\rm P}(1/x) \mp i\pi\delta(x)$ and $\theta(x)\theta(-y) - \theta(-x)\theta(y) = \theta(x) - \theta(y) = \theta(-y) - \theta(-x)$, then

$$\begin{aligned}{\rm Re}\chi_{ij}^{ab}(\boldsymbol{q},\Omega) =& \sum_\pm \int \frac{d^3k}{(2\pi)^3}(2k_i+q_i)(2k_j+q_j)\bigg\{(\delta_{ab} - \delta_{a3}\delta_{b3}){\rm P}\frac{\theta(-\omega_{\boldsymbol{k}+\boldsymbol{q}\mp}) - \theta(-\omega_{\boldsymbol{k}\pm})}{\omega_{\boldsymbol{k}+\boldsymbol{q}\mp} - \omega_{\boldsymbol{k}\pm} - \Omega} \\ &+ \delta_{a3}\delta_{b3}{\rm P}\frac{\theta(-\omega_{\boldsymbol{k}+\boldsymbol{q}\pm}) - \theta(-\omega_{\boldsymbol{k}\pm})}{\omega_{\boldsymbol{k}+\boldsymbol{q}\pm} - \omega_{\boldsymbol{k}\pm} - \Omega} \pm \varepsilon_{ab3}\pi[\theta(-\omega_{\boldsymbol{k}+\boldsymbol{q}\mp}) - \theta(-\omega_{\boldsymbol{k}\pm})]\delta(\omega_{\boldsymbol{k}+\boldsymbol{q}\mp} - \omega_{\boldsymbol{k}\pm} - \Omega)\bigg\}.\end{aligned} \tag{A20}$$

Let us calculate the value of each component of ${\rm Re}\chi_{ij}^{ab}|_{q=0}$. First,

$${\rm Re}\chi_{ij}^{11}\big|_{q=0} = {\rm Re}\chi_{ij}^{22}\big|_{q=0} = \sum_\pm \int \frac{d^3k}{(2\pi)^3}4k_ik_j(\pm)\frac{\hbar}{2g_{\rm sd}}[\theta(-\omega_{\boldsymbol{k}\mp}) - \theta(-\omega_{\boldsymbol{k}\pm})] = -\frac{\hbar}{g_{\rm sd}}\frac{2(k_{F+}^5 - k_{F-}^5)}{15\pi^2}\delta_{ij}. \tag{A21}$$

These components do not depend on the order of taking the $\boldsymbol{q} \to 0$ and $\Omega \to 0$ limit. On the other hand, the value of the component ${\rm Re}\chi_{ij}^{33}$ depends on the order: from Eq. (A20), it is clear that when we take the $\boldsymbol{q} \to 0$ limit first, ${\rm Re}\chi_{ij}^{33}$ vanishes, while when we take the $\Omega \to 0$ limit first and then take the $\boldsymbol{q} \to 0$ limit, it gives 0/0 and, using the L'Hopital's rule,

$$\lim_{\boldsymbol{q}\to 0}\lim_{\Omega\to 0}{\rm Re}\chi_{ij}^{33} = \sum_\pm \int \frac{d^3k}{(2\pi)^3}4k_ik_j(-1)\delta(-\omega_{\boldsymbol{k}\pm}) = -\frac{4mn}{\hbar}\delta_{ij}. \tag{A22}$$

The order of the $\boldsymbol{q} \to 0, \Omega \to 0$ limits can be simply determined by the isotropy of the internal space (i.e., the space where the vector $\boldsymbol{n}$ is defined). For example, the combination $(A_i^1)^2 + (A_i^2)^2 = (1/4)(\partial_i \boldsymbol{n})^2$ [see Eq. (A7)] is isotropic [i.e., does not depend on the rotation $\boldsymbol{n} \to R\boldsymbol{n}$ with a constant SO(3) matrix R], but $(A_i^3)^2 = (1/4)(1-\cos\theta)^2(\partial_i\phi)^2$ is not an isotropic quantity. When we employ $\lim_{\Omega\to 0}\lim_{\boldsymbol{q}\to 0}{\rm Re}\chi_{ij}^{33} = 0$ as the value of ${\rm Re}\chi_{ij}^{33}|_{q=0}$, then the $(A_i^3)^2$ term appears in $\Delta\mathcal{S}_{\rm eff}$ (A12) due to the nonzero

$\langle \bar{a}a \rangle$ term. On the other hand, when we employ Eq. (A22) as the value of $\mathrm{Re}\chi_{ij}^{33}|_{q=0}$, then the $(A_i^3)^2$ term vanishes in $\Delta\mathcal{S}_{\rm eff}$ due to the cancellation with the $\langle \bar{a}a \rangle$ term. Therefore we should employ Eq. (A22) as the value of $\mathrm{Re}\chi_{ij}^{33}|_{q=0}$.[7]

For nondiagonal elements, it is clear that $\mathrm{Re}\chi_{ij}^{12}|_{q=0} = 0$ regardless of the order of the $\boldsymbol{q} \to 0$ and $\Omega \to 0$ limits.

Finally, let us calculate $\mathrm{Re}\rho^{ab}|_{q=0}$. The calculation is almost the same as that of $\mathrm{Re}\chi_{ij}^{ab}|_{q=0}$: just drop the factor $-(2k_i + q_i)(2k_j + q_j)$ from Eq. (A19). Then,

$$\mathrm{Re}\rho^{11}|_{q=0} = \mathrm{Re}\rho^{22}|_{q=0} = -\sum_{\pm}\int \frac{d^3k}{(2\pi)^3}(\pm)\frac{\hbar}{2g_{\rm sd}}[\theta(-\omega_{\boldsymbol{k}\mp}) - \theta(-\omega_{\boldsymbol{k}\pm})] = \frac{\hbar}{g_{\rm sd}}\frac{k_{F+}^3 - k_{F-}^3}{6\pi^2}. \tag{A23}$$

The order of taking the $\boldsymbol{q},\Omega \to 0$ limits matters again for $\mathrm{Re}\rho^{33}$. This time, $\lim_{\boldsymbol{q}\to 0}\lim_{\Omega\to 0}\mathrm{Re}\rho^{33} \neq 0$ while $\lim_{\Omega\to 0}\lim_{\boldsymbol{q}\to 0}\mathrm{Re}\rho^{33} = 0$. Again, we determine the order of the limits by the isotropy about $\boldsymbol{n}$. When $\mathrm{Re}\rho^{33}|_{q=0}$ does not vanish, $A_t^3 A_t^3$ appears in $\Delta\mathcal{S}_{\rm eff}$, which breaks the isotropy. Therefore, contrary to the case of $\mathrm{Re}\chi_{ij}^{33}|_{q=0}$, we employ $\lim_{\Omega\to 0}\lim_{\boldsymbol{q}\to 0}\mathrm{Re}\rho^{33} = 0$ as $\mathrm{Re}\rho^{33}|_{q=0}$. Finally, $\mathrm{Re}\rho^{12}|_{q=0}$ obviously vanishes.

After all, the expression of the contribution from the conduction electrons to the spin effective action is

$$\begin{aligned}\Delta\mathcal{S}_{\rm eff} &= \int d^4x\left\{-\hbar\frac{k_{F+}^3 - k_{F-}^3}{6\pi^2}A_t^3 + \frac{\hbar^2}{2m}\left(\frac{\hbar^2}{2m}\frac{1}{g_{\rm sd}}\frac{k_{F+}^5 - k_{F-}^5}{15\pi^2} - n\right)\left[\left(A_i^1\right)^2 + \left(A_i^2\right)^2\right] + \frac{\hbar^2}{2g_{\rm sd}}\frac{k_{F+}^3 - k_{F-}^3}{6\pi^2}\left[\left(A_t^1\right)^2 + \left(A_t^2\right)^2\right]\right\}\\ &= \int \frac{d^3x}{a^3}dt\left[S_c\dot{\phi}(\cos\theta - 1) - \frac{J_cS_c^2}{2}(\partial_i\boldsymbol{n})^2 + \frac{m_s}{2}\dot{\boldsymbol{n}}^2\right],\end{aligned} \tag{A24}$$

where a is the lattice constant and we have defined the spin polarization S_c of the conduction electron per lattice site, the spin-spin exchange coupling J_c and the inertia of spin m_s as

$$S_c \equiv a^3\frac{\hbar}{2}\frac{k_{F+}^3 - k_{F-}^3}{6\pi^2}, \quad J_c \equiv \frac{a^3}{S_c^2}\frac{\hbar^2}{4m}\left(n - \frac{\hbar^2}{m}\frac{1}{g_{\rm sd}}\frac{k_{F+}^5 - k_{F-}^5}{30\pi^2}\right), \quad m_s \equiv \frac{\hbar S_c}{2g_{\rm sd}}, \tag{A25}$$

respectively. It is easy to see that $J_c > 0$ by rewriting it as

$$J_c = \frac{a^3}{S_c^2}\frac{\hbar^2}{120\pi^2 m}\frac{(k_{F+} - k_{F-})^2}{k_{F+} + k_{F-}}\left(k_{F+}^2 + 3k_{F+}k_{F-} + k_{F-}^2\right). \tag{A26}$$

We may safely neglect the effect of impurity, because the lifetime τ of the conduction electrons will enter only as $\hbar/(\epsilon_F\tau)$ or $\hbar/(g_{\rm sd}\tau)$, both of which we assume to be much smaller than one.

Adding this $\Delta\mathcal{S}_{\rm eff}[\boldsymbol{n}]$ to the action of localized spin (A3), we obtain the total effective action $\mathcal{S}_{\rm eff}[\boldsymbol{n}]$,

$$\mathcal{S}_{\rm eff}[\boldsymbol{n}] = \int \frac{d^3x}{a^3}dt\left[S\dot{\phi}(\cos\theta - 1) - \frac{JS^2}{2}(\partial_i\boldsymbol{n})^2 + \frac{m_s}{2}\dot{\boldsymbol{n}}^2\right] \tag{A27}$$

with $S \equiv S_c + S_l$ and $J \equiv (J_cS_c^2 + J_lS_l^2)/S^2$.

APPENDIX B: DERIVATION OF THE ANGULAR MOMENTUM (10)

In this appendix, we describe the derivation of the angular momentum current $(\boldsymbol{j}^0,\boldsymbol{j}^i)$ (10) from the effective action $\mathcal{S}_{\rm eff}$ (7), for the readers who are not familiar with such derivation. Consider a SO(3) rotation in the internal space (spin space).[8] Under this rotation, a vector $\boldsymbol{v}$ in the internal space, such as the spin $\boldsymbol{n}$ and the magnetic field $\boldsymbol{B}$, is rotated.[9] When the rotation is infinitesimally small, it is given by $\delta v^a(\boldsymbol{x},t) = \varepsilon^{abc}\theta^b v^c(\boldsymbol{x},t)$, where θ^a are constant infinitesimal transformation parameters. The effective action (7) is invariant under this rotation. The Noether current corresponding to this invariance, i.e., the angular momentum current, can be obtained by the following trick. Make the transformation parameters θ^a be dependent on the position in the real space and time, $\theta^a(\boldsymbol{x},t)$. Then, the variation of the action under this space-time dependent internal rotation does not vanish, but is

[7]The order of $\boldsymbol{q} \to 0,\Omega \to 0$ limits can be determined also by the $U(1)$ gauge symmetry $U \to U\exp(i\chi\sigma_3), a \to \exp(-i\chi\sigma_3)a$ described below Eq. (A4). Under this gauge transformation, the spin gauge field $A_\mu = A_\mu^a\sigma^a$ changes as

$$\begin{pmatrix}A_\mu^1\\A_\mu^2\end{pmatrix} \to \begin{pmatrix}\cos 2\chi & -\sin 2\chi\\ \sin 2\chi & \cos 2\chi\end{pmatrix}\begin{pmatrix}A_\mu^1\\A_\mu^2\end{pmatrix}, \quad A_\mu^3 \to A_\mu^3 + \partial_\mu\chi.$$

The effective action $\Delta\mathcal{S}_{\rm eff}$ (A12) should be invariant under this gauge transformation. Therefore $(A_i^1)^2 + (A_i^2)^2$ term is allowed but $(A_i^3)^2$ term is not.

[8]An element R of SO(3) group and an element U of SU(2) group are related by $R_{ab}\sigma^b = U^\dagger\sigma^a U$ with σ^a the Pauli matrices. We are interested in the classical dynamics of a spin vector, that is, the behavior of $\boldsymbol{n} \equiv \langle\psi|\boldsymbol{\sigma}|\psi\rangle$, where $|\psi\rangle$ is a quantum spin state. An SU(2) rotation on $|\psi\rangle$ causes an SO(3) rotation on $\boldsymbol{n}$, i.e., $\langle\psi|U^\dagger\sigma^a U|\psi\rangle = R_{ab}n^b$.

[9]The magnetic field $\boldsymbol{B}$ forms a scalar product with $\boldsymbol{n}$ in the Zeeman term, which means that $\boldsymbol{B}$ is also a vector in the internal space with the same transformation property as $\boldsymbol{n}$, within the framework presented here.

given in the form

$$\delta\mathcal{S}_{\text{eff}} = \int \frac{d^3x}{a^3}dt(\partial_0\boldsymbol{\theta}\cdot\boldsymbol{j}^0 + \partial_i\boldsymbol{\theta}\cdot\boldsymbol{j}^i)$$
$$= -\int \frac{d^3x}{a^3}dt\boldsymbol{\theta}\cdot(\partial_0\boldsymbol{j}^0 + \partial_i\boldsymbol{j}^i). \tag{B1}$$

Here, terms without any time and spatial derivatives on θ^a are absent, because the rotation would be a symmetry of the action if θ^a were constants. We have performed the partial integration in the last equality in Eq. (B1). When $\boldsymbol{n}$ satisfies its equation of motion, then $\delta\mathcal{S}_{\text{eff}} = 0$ because such $\boldsymbol{n}$ gives a saddle point of $\mathcal{S}_{\text{eff}}$. Since $\delta\mathcal{S}_{\text{eff}} = 0$ for arbitrarily position-dependent $\theta^a(\boldsymbol{x},t)$, it follows that $\partial_0\boldsymbol{j}^0 + \partial_i\boldsymbol{j}^i = 0$ when $\boldsymbol{n}$ satisfies its equation of motion. Thus $(\boldsymbol{j}^0,\boldsymbol{j}^i)$ is the conserved current corresponding to the symmetry under internal rotations, i.e., the angular momentum current. (In fact, since $\boldsymbol{n}$ is a unit vector, an arbitrary variation of $\boldsymbol{n}$ is given by a rotation in the internal space. Therefore the conservation law $\partial_0\boldsymbol{j}^0 + \partial_i\boldsymbol{j}^i = 0$ is the equation of motion itself.)

Concretely, the variation of each term in the effective action (7) is given as follows. For the spin Berry phase term,

$$\delta\int\frac{d^3x}{a^3}dtS\dot{\phi}(\cos\theta - 1)$$
$$= \delta\int\frac{d^3x}{a^3}dt\int_0^1 duS\tilde{\boldsymbol{n}}\cdot(\partial_u\tilde{\boldsymbol{n}}\times\partial_t\tilde{\boldsymbol{n}})$$
$$= \int\frac{d^3x}{a^3}dt\int_0^1 duS[\partial_u(\tilde{\boldsymbol{\theta}}\cdot\partial_t\tilde{\boldsymbol{n}}) - \partial_t(\tilde{\boldsymbol{\theta}}\cdot\partial_u\tilde{\boldsymbol{n}})]$$
$$= \int\frac{d^3x}{a^3}dt\dot{\boldsymbol{\theta}}\cdot S\boldsymbol{n}, \tag{B2}$$

where we have temporarily used $\tilde{\boldsymbol{n}}(\boldsymbol{x},t,u)$, which is defined over an extended space-time $(\boldsymbol{x},t,u)$ with $0 \leqslant u \leqslant 1$ a dummy direction and with boundary conditions $\tilde{\boldsymbol{n}}(\boldsymbol{x},t,u = 0) = \boldsymbol{n}(\boldsymbol{x},t)$ and $\tilde{\boldsymbol{n}}(\boldsymbol{x},t,u = 1) = \text{const}$. The transformation parameters $\boldsymbol{\theta}$ are also extended to $\tilde{\boldsymbol{\theta}}(\boldsymbol{x},t,u)$ similarly. In the last equality in Eq. (B2), we have performed a partial integration with respect to t. For the inertial term and the exchange coupling term,

$$\delta\int\frac{d^3x}{a^3}dt\frac{m_s}{2}\dot{\boldsymbol{n}}^2 = \int\frac{d^3x}{a^3}dt\dot{\boldsymbol{\theta}}\cdot(m_s\boldsymbol{n}\times\dot{\boldsymbol{n}}),$$
$$\delta\int\frac{d^3x}{a^3}dt\frac{-JS^2}{2}(\partial_i\boldsymbol{n})^2 = \int\frac{d^3x}{a^3}dt\partial_i\boldsymbol{\theta}\cdot(JS^2\partial_i\boldsymbol{n}\times\boldsymbol{n}). \tag{B3}$$

The Zeeman term is irrelevant for deriving the angular momentum because it does not involve the derivative on $\boldsymbol{n}$, which means that it does not yield terms proportional to $\partial_0\boldsymbol{\theta}$ or $\partial_i\boldsymbol{\theta}$ in Eq. (B1) under the rotational transformation. In all, from Eq. (B1), the angular momentum current can be read as

$$\boldsymbol{j}^0 = S\boldsymbol{n} + m_s\boldsymbol{n}\times\dot{\boldsymbol{n}}, \quad \boldsymbol{j}^i = JS^2\partial_i\boldsymbol{n}\times\boldsymbol{n}. \tag{B4}$$

APPENDIX C: DERIVATION OF HIGHER-ORDER TERMS IN SPIN EFFECTIVE ACTION

In this appendix, we derive higher-order terms in spin effective action $\Delta\mathcal{S}_{\text{eff}}$ (15) in the way discussed in the last part of Sec. II. We concentrate only on the time-derivative terms, but the derivation of higher-order terms with spatial derivatives is essentially the same. First, all possible terms in the spin effective action to the fourth order are

$$\Delta\mathcal{S}_{\text{eff}} = \int\frac{d^3x}{a^3}dt\bigg[S_c\dot{\phi}(\cos\theta - 1) + \frac{m_s}{2}\dot{\boldsymbol{n}}^2$$
$$+ c_3\boldsymbol{n}\cdot(\dot{\boldsymbol{n}}\times\ddot{\boldsymbol{n}}) + c_4\ddot{\boldsymbol{n}}^2 + \tilde{c}_4(\dot{\boldsymbol{n}}^2)^2\bigg]. \tag{C1}$$

The angular momentum $\Delta\boldsymbol{j}^0$ can be derived from this action in the way described in Appendix B, as

$$\Delta\boldsymbol{j}^0 = S_c\boldsymbol{n} + m_s\boldsymbol{n}\times\dot{\boldsymbol{n}} + c_3(2\ddot{\boldsymbol{n}} + 3\dot{\boldsymbol{n}}^2\boldsymbol{n})$$
$$+ 2c_4(\dot{\boldsymbol{n}}\times\ddot{\boldsymbol{n}} - \boldsymbol{n}\times\dddot{\boldsymbol{n}}) + 4\tilde{c}_4\dot{\boldsymbol{n}}^2\boldsymbol{n}\times\dot{\boldsymbol{n}}. \tag{C2}$$

This $\Delta\boldsymbol{j}^0$ must satisfy $\partial_0\Delta\boldsymbol{j}^0 + (2g_{\text{sd}}/\hbar)\boldsymbol{n}\times\Delta\boldsymbol{j}^0 = 0$. Substitution of Eq. (C2), with $\omega_{\text{sd}} \equiv g_{\text{sd}}/\hbar$, results in

$$\partial_0\Delta\boldsymbol{j}^0 + (2g_{\text{sd}}/\hbar)\boldsymbol{n}\times\Delta\boldsymbol{j}^0$$
$$= (S_c - 2\omega_{\text{sd}}m_s)\dot{\boldsymbol{n}} + (m_s + 4\omega_{\text{sd}}c_3)\boldsymbol{n}\times\ddot{\boldsymbol{n}}$$
$$+ (2c_3 + 4\omega_{\text{sd}}c_4)\dddot{\boldsymbol{n}} + (6c_3 + 12\omega_{\text{sd}}c_4)(\dot{\boldsymbol{n}}\cdot\ddot{\boldsymbol{n}})\boldsymbol{n}$$
$$+ (3c_3 - 4\omega_{\text{sd}}c_4 - 8\omega_{\text{sd}}\tilde{c}_4)\dot{\boldsymbol{n}}^2\dot{\boldsymbol{n}}. \tag{C3}$$

The unique solution in order for each coefficient to vanish is

$$m_s = \frac{S_c}{2\omega_{\text{sd}}}, \quad c_3 = -\frac{m_s}{4\omega_{\text{sd}}}, \quad c_4 = -\frac{c_3}{2\omega_{\text{sd}}}, \quad \tilde{c}_4 = \frac{5c_3}{8\omega_{\text{sd}}}. \tag{C4}$$

[1] A. Altland and B. Simons, *Condensed Matter Field Theory* (Cambridge University Press, Cambridge, 2006), Sec. 3.3.

[2] H. Kohno, G. Tatara, and J. Shibata, J. Phys. Soc. Jpn. **75**, 113706 (2006).

[3] T. L. Gilbert, IEEE Trans. Magn. **40**, 3443 (2004).

[4] H. Suhl, IEEE Trans. Magn. **34**, 1834 (1998).

[5] E. Beaurepaire, J.-C. Merle, A. Daunois, and J.-Y. Bigot, Phys. Rev. Lett. **76**, 4250 (1996).

[6] A. Kirilyuk, A. V. Kimel, and Th. Rasing, Rev. Mod. Phys. **82**, 2731 (2010).

[7] M. C. Ciornei, J. M. Rubi, and J. E. Wegrowe, Phys. Rev. B **83**, 020410 (2011).

[8] E. Olive, Y. Lansac, and J. E. Wegrowe, Appl. Phys. Lett. **100**, 192407 (2012).

[9] W. F. Brown, Jr., Phys. Rev. **130**, 1677 (1963).

[10] J. E. Wegrowe and M. C. Ciornei, Am. J. Phys. **80**, 607 (2012).

[11] M. Fähnle, D. Steiauf, and C. Illg, Phys. Rev. B **84**, 172403 (2011).

[12] S. Bhattacharjee, L. Nordstrom, and J. Fransson, Phys. Rev. Lett. **108**, 057204 (2012).

[13] G. Tatara, H. Kohno, and J. Shibata, Phys. Rep. **468**, 213 (2008).
[14] H. Kawaguchi and G. Tatara, J. Phys. Soc. Jpn. **83**, 074710 (2014).
[15] T. Mori and M. Katsuhara, J. Phys. Soc. Jpn. **71**, 826 (2002).
[16] M. Amusia, K. Popov, V. Shaginyan, and W. Stefanowicz, *Theory of Heavy-Fermion Compounds* (Springer, Berlin, 2014).
[17] A. H. Morrish, *The Physical Principles of Magnetism* (Wiley-IEEE Press, 2001), Sec. 10.4.
[18] L. D. Landau and E. M. Lifshitz, *Mechanics*, 3rd ed. (Butterworth-Heinemann, Oxford, 1982).
[19] J. X. Zhu, Z. Nussinov, A. Shnirman, and A. V. Balatsky, Phys. Rev. Lett. **92**, 107001 (2004).
[20] Z. Nussinov, A. Shnirman, D. P. Arovas, A. V. Balatsky, and J. X. Zhu, Phys. Rev. B **71**, 214520 (2005).
[21] J. X. Zhu and J. Fransson, J. Phys.: Condens. Matter **18**, 9929 (2006).
[22] J. Fransson, Nanotechnol. **19**, 285714 (2008).
[23] J. Fransson and J. X. Zhu, New J. Phys. **10**, 013017 (2008).
[24] J. Fransson, Phys. Rev. B **77**, 205316 (2008).
[25] M. Onoda, A. S. Mishchenko, and N. Nagaosa, J. Phys. Soc. Jpn. **77**, 013702 (2008).
[26] F. W. Hehl and W.-T. Ni, Phys. Rev. D **42**, 2045 (1990).
[27] M. Matsuo, J. Ieda, E. Saitoh, and S. Maekawa, Phys. Rev. Lett. **106**, 076601 (2011).
[28] A. V. Kimel, B. A. Ivanov, R. V. Pisarev, P. A. Usachev, A. Kirilyuk, and Th. Rasing, Nat. Phys. **5**, 727 (2009).
[29] E. G. Tveten, A. Qaiumzadeh, O. A. Tretiakov, and A. Brataas, Phys. Rev. Lett. **110**, 127208 (2013).
[30] Y. Kawaguchi and M. Ueda, Phys. Rep. **520**, 253 (2012).
[31] G. Tatara and H. Fukuyama, J. Phys. Soc. Jpn. **63**, 2538 (1994).