Resonant and nonlocal properties of phononic metasolids

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We derive a general theory of effective properties in metasolids based on phononic crystals with low frequency resonances. We demonstrate that in general these structures need to be described by means of a frequency-dependent and nonlocal anisotropic mass density, stiffness tensor and a third-rank coupling tensor, which shows that they behave like a nonlocal Willis medium. The effect of nonlocality and coupling tensor manifest themselves for some particular resonances, whereas they become negligible for other resonances. Considering the example of a two-dimensional phononic crystal, consisting of triangular arrangements of cylindrical shells in an elastic matrix, we show that its mass density tensor is strongly resonant and anisotropic presenting both positive and negative divergent values, while becoming scalar in the quasistatic limit. Moreover, it is found that the negative value of transverse component of the mass density is induced by a dipolar resonance, while that of the vertical component is induced by a monopolar one. Finally, the dispersion relation obtained by the effective parameters of the crystal is compared with the band structure, showing good agreement for the low-wave-number region, although the nonlocal effects are important given the existence of some resonant values of the wave number.

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I. INTRODUCTION

Metamaterials are artificial structures with unusual constitutive parameters not found in natural materials [1], such as negative compressibility [2], refractive index [3–5], or anisotropic mass density [6]. These properties offer new insights into the propagation of classical waves, and a wide variety of effects and applications have been found, such as cloaking shells [7,8], superlenses [9], optical and acoustical black holes [10–12], or gradient index lenses [13,14].

Metamaterials for acoustic or elastic waves, also named metafluids or metasolids, respectively, have been mainly implemented by means of sonic and phononic crystals, which consist of periodic arrangement of inclusions in a fluid (sonic crystal) or elastic (phononic crystal) matrix [15]. If the inclusion is properly chosen so that it presents low-frequency resonances, these structures behave like effective materials with resonantlike constitutive parameters which can be either positive, zero, or negative [16].

Phononic and sonic crystals are anisotropic structures in general, therefore they present anisotropic constitutive parameters. Then, it was demonstrated that metafluids present anisotropic mass density not only near a local resonance, but also in the quasistatic limit [6,17], although for the case of metasolids it has been assumed in general that the mass density is a scalar [18–20]. Recently, some works have shown that elastic composites have to be described by means of the so-called "Willis form" of the constitutive parameters [21,22], which include a tensorial mass density and an additional coupling tensor, and the dynamic homogenization of phononic crystals has also shown that this general description applies to these structures [23].

In this work the low-frequency limit of phononic crystals is analyzed, and it is shown analytically that low-frequency resonances actually induce an anisotropic mass density, which, however, becomes scalar in the static limit. It is also shown that the usual assumption that the negative mass density is induced by dipolar resonances is not necessarily true, in a similar way as was previously demonstrated for plate metamaterials for flexural waves by the authors in a recent publication [24]. Finally, the dispersion relation of the full phononic crystal is compared with that of a homogeneous material with the obtained effective parameters, and good agreement is found in general, although it is also demonstrated that nonlocal parameters have to be considered.

The paper is organized as follows: After this Introduction, Sec. II explains the homogenization method employed here. Following, Sec. III explains how to apply perturbation theory to derive some important properties of metasolids in the low-frequency limit. Finally, Sec. IV describes a phononic crystal as a locally resonant metamaterial and Sec. V shows a numerical example of application of the theory. Section VI summarizes the work.

II. HOMOGENIZATION OF THE PERIODIC MEDIUM FROM THE BAND STRUCTURE

The equation of motion of an inhomogeneous solid, assuming harmonic time dependence with frequency ω, is given by the classical elastodynamic equation [25]

$$-\rho(\boldsymbol{r})\omega^2 u_i = \partial_j C_{ijkl}(\boldsymbol{r})\partial_k u_l \tag{1}$$

with u_i being the components of the displacement field and C_{ijkl} the components of the stiffness tensor. Hereafter we employ the summation convention in which two repeated indexes implies summation over their possible values, to simplify notation. If the medium is homogeneous the dispersion relation is obtained by assuming plane-wave propagation with wave vector $\boldsymbol{k} = k\boldsymbol{n}$, with k being the wave number and $\boldsymbol{n}$ a unit vector parallel to the propagation direction. Under this assumption, the equation of motion becomes the well-known

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secular equation for elastic waves [25]

$$\rho\omega^2 u_i = k^2 n_{iI} C_{IJ} n_{Jj} u_j, \tag{2}$$

with u_i being the components of the displacement field and C_{IJ} the components of the stiffness tensor in Voigt notation (see Ref. [25] and Appendix A). The solution for the dispersion relation $\omega = \omega(k,\boldsymbol{n})$ is therefore given by the roots of the determinant of the matrix $\bar{\Gamma}$ defined as

$$\bar{\Gamma}_{ij} = \rho\omega^2\delta_{ij} - k^2 n_{iI} C_{IJ} n_{Jj}. \tag{3}$$

In a phononic crystal both $\rho(\boldsymbol{r})$ and $C_{ijkl}(\boldsymbol{r})$ are periodic functions of the spatial coordinates; then the Bloch theorem is applied and the plane-wave expansion method [26] can be used to obtain the dispersion relation $\omega = \omega(k,\boldsymbol{n})$ as the solution of the following eigenvalue equation:

$$\omega^2 \rho_{\boldsymbol{G}-\boldsymbol{G}'}(u_{\boldsymbol{G}'})_i = (k+G)_{iI} C_{IJ}^{\boldsymbol{G}-\boldsymbol{G}'} (k+G')_{Jj} (u_{\boldsymbol{G}'})_j, \tag{4}$$

where $\rho_{\boldsymbol{G}}$, $C_{IJ}^{\boldsymbol{G}}$, and $(u_{\boldsymbol{G}})_i$ stand for the Fourier components of the mass density, stiffness tensor, and displacement field, respectively, and summation over repeated indexes is assumed including the Fourier indexes defined by the reciprocal lattice vector $\boldsymbol{G}$. The matrix elements $(k+G)_{iI}$ are defined in Appendix A. In matrix form the above eigenvalue equation is expressed as

$$\omega^2 N_{\boldsymbol{G}\boldsymbol{G}'} \boldsymbol{u}_{\boldsymbol{G}'} = M_{\boldsymbol{G}\boldsymbol{G}'} \boldsymbol{u}_{\boldsymbol{G}'}, \tag{5}$$

where

$$(N_{\boldsymbol{G}\boldsymbol{G}'})_{ij} = \rho_{\boldsymbol{G}-\boldsymbol{G}'}\delta_{ij}, \tag{6}$$

$$(M_{\boldsymbol{G}\boldsymbol{G}'})_{ij} = (k+G)_{iI} C_{IJ}^{\boldsymbol{G}-\boldsymbol{G}'} (k+G')_{Jj}. \tag{7}$$

This equation solves for the dispersion relation inside the phononic crystal; however, in its current form it is difficult to figure out any property of the crystal as a composite. The description of the crystal as a material can be obtained by averaging the components of the displacement vector in the unit cell and finding in this way an equation similar to Eq. (2), in which the coefficients of the different terms multiplying the wave vector and the frequency define the effective parameters. The average of the displacement vector is given by the $\boldsymbol{G} = 0$ component of $\boldsymbol{u}_{\boldsymbol{G}}$, so that it can be obtained by expressing Eq. (5) as

$$\omega^2 N_{00}\boldsymbol{u}_0 + \omega^2 N_{0\boldsymbol{G}'}\boldsymbol{u}_{\boldsymbol{G}'} = M_{00}\boldsymbol{u}_0 + M_{0\boldsymbol{G}'}\boldsymbol{u}_{\boldsymbol{G}'}, \tag{8a}$$

$$\omega^2 N_{\boldsymbol{G}0}\boldsymbol{u}_0 + \omega^2 N_{\boldsymbol{G}\boldsymbol{G}'}\boldsymbol{u}_{\boldsymbol{G}'} = M_{\boldsymbol{G}0}\boldsymbol{u}_0 + M_{\boldsymbol{G}\boldsymbol{G}'}\boldsymbol{u}_{\boldsymbol{G}'}. \tag{8b}$$

It must be recalled that repeated indexes means summation over all their possible values, and that hereafter it is considered that matrix elements labeled with $\boldsymbol{G}$ do not include the term $\boldsymbol{G} = 0$, which is extracted from the above decomposition. We can now solve from the second equation for $\boldsymbol{u}_{\boldsymbol{G}}$,

$$\boldsymbol{u}_{\boldsymbol{G}'} = -(M_{\boldsymbol{G}'\boldsymbol{G}} - \omega^2 N_{\boldsymbol{G}'\boldsymbol{G}})^{-1}(M_{\boldsymbol{G}0} - \omega^2 N_{\boldsymbol{G}0})\boldsymbol{u}_0 \tag{9}$$

and insert it into the first one, obtaining the following equation:

$$[\omega^2 N_{00} - \omega^2 N_{0\boldsymbol{G}'}\chi_{\boldsymbol{G}'\boldsymbol{G}}(M_{\boldsymbol{G}0} - \omega^2 N_{\boldsymbol{G}0}) - M_{00} + M_{0\boldsymbol{G}'}\chi_{\boldsymbol{G}'\boldsymbol{G}}(M_{\boldsymbol{G}0} - \omega^2 N_{\boldsymbol{G}0})]\boldsymbol{u}_0 = 0, \tag{10}$$

where we have defined

$$\chi_{\ell m}^{\boldsymbol{G}'\boldsymbol{G}}(\omega,\boldsymbol{k}) \equiv (M_{\boldsymbol{G}'\boldsymbol{G}} - \omega^2 N_{\boldsymbol{G}'\boldsymbol{G}})^{-1}_{\ell m}. \tag{11}$$

Equation (10) is formally the same as Eq. (5); however, it is not an eigenvalue equation, but a secular equation for $\boldsymbol{u}_0$ similar to Eq. (2), where the solutions $\omega = \omega(k,\boldsymbol{n})$ are obtained from the zeros of the determinant of the matrix Γ defined as

$$\Gamma = \omega^2 N_{00} - \omega^2 N_{0\boldsymbol{G}'}\chi_{\boldsymbol{G}'\boldsymbol{G}}(M_{\boldsymbol{G}0} - \omega^2 N_{\boldsymbol{G}0}) - M_{00} + M_{0\boldsymbol{G}'}\chi_{\boldsymbol{G}'\boldsymbol{G}}(M_{\boldsymbol{G}0} - \omega^2 N_{\boldsymbol{G}0}). \tag{12}$$

The matrix Γ is actually a 3×3 matrix, in which coefficients are in general functions of both ω and $\boldsymbol{k}$, which makes it less suitable for band structure calculation than Eq. (5) but more suitable for the description of the phononic crystal as a composite. Effectively, we can see that the elements of the N_{00}, $N_{0\boldsymbol{G}'}$, and $N_{\boldsymbol{G}0}$ do not depend explicitly on the wave vector $\boldsymbol{k}$,

$$(N_{00})_{ij} = \bar{\rho}\delta_{ij}, \tag{13}$$

$$(N_{0\boldsymbol{G}'})_{ij} = \rho_{-\boldsymbol{G}'}\delta_{ij}, \tag{14}$$

$$(N_{\boldsymbol{G}0})_{ij} = \rho_{\boldsymbol{G}}\delta_{ij}, \tag{15}$$

with $\bar{\rho} = \rho_{\boldsymbol{G}=0}$ the mass density average in the unit cell. Contrarily, the M_{00}, $M_{0\boldsymbol{G}'}$, and $M_{\boldsymbol{G}0}$ contains this dependence with the wave vector, since

$$(M_{00})_{ij} = k_{iI}\bar{C}_{IJ}k_{Jj}, \tag{16}$$

$$(M_{0\boldsymbol{G}'})_{ij} = k_{iI}C_{IJ}^{-\boldsymbol{G}'}(k+G')_{Jj}, \tag{17}$$

$$(M_{\boldsymbol{G}0})_{ij} = (k+G)_{iI}C_{IJ}^{\boldsymbol{G}}k_{Jj}, \tag{18}$$

with $\bar{C}_{IJ} = C_{IJ}^{\boldsymbol{G}=0}$ the average in the unit cell of the components of the stiffness tensor. The dependence with the wave vector and frequency can be reorganized, then the Γ matrix can be cast as

$$\Gamma_{ij} = \omega^2\rho_{ij}^* - k^2 n_{iI}C_{IJ}^* n_{Jj} - \omega k(n_{iI}S_{Ij} + S_{iJ}^\dagger n_{Jj}), \tag{19}$$

where the coefficients ρ_{ij}^*, C_{IJ}^*, and S_{Ij} are given by

$$\rho_{ij}^*(\omega,\boldsymbol{k}) = \bar{\rho}\delta_{ij} + \omega^2\rho_{-\boldsymbol{G}'}\chi_{ij}^{\boldsymbol{G}'\boldsymbol{G}}(\omega,\boldsymbol{k})\rho_{\boldsymbol{G}}, \tag{20a}$$

$$C_{IJ}^*(\omega,\boldsymbol{k}) = \bar{C}_{IJ} - C_{IL}^{-\boldsymbol{G}'}(k+G')_{L\ell}\chi_{\ell m}^{\boldsymbol{G}'\boldsymbol{G}}(\omega,\boldsymbol{k})(k+G)_{mM}C_{MJ}^{\boldsymbol{G}}, \tag{20b}$$

$$S_{Ij}(\omega,\boldsymbol{k}) = \omega C_{IL}^{-\boldsymbol{G}'}(k+G')_{L\ell}\chi_{\ell j}^{\boldsymbol{G}'\boldsymbol{G}}(\omega,\boldsymbol{k})\rho_{\boldsymbol{G}}. \tag{20c}$$

Equation (19) is similar to Eq. (2), but the constitutive parameters required to describe the phononic solid are more complex. Equations (20) were derived in [23] using a different approach, and they show that the phononic crystal is a nonlocal Willis medium [22], in which the mass density is a tensorial quantity and with the presence of the coupling field S_{Ij}. The above expressions are valid at any frequency and wave number; however, in this work we are especially interested in the low-frequency limit, that is, the limit in which the wavelength of the field in the background is larger than the typical periodicity of the crystal and it is described as a homogeneous material. It will be shown that even in the low-frequency limit these systems

can have some special resonances in which the crystal behaves like a Willis medium with resonant and nonlocal parameters.

If in Eqs. (20) the limit $\omega \to 0$ and $k \to 0$ is taken it is found that the coupling field $S_{Ij} = 0$, since it is directly proportional to ω. Also, the mass density becomes a scalar which is simply the volume average $\rho_{ij} = \bar{\rho}\delta_{ij}$, as is well known from the theory of composites. Finally, the effective stiffness tensor is given by

$$C^*_{IJ} = \bar{C}_{IJ} - C^{-\boldsymbol{G}'}_{IL} G'_{L\ell} \left(M^{-1}_{\boldsymbol{G}'\boldsymbol{G}}\right)_{\ell m} G_{mM} C^{\boldsymbol{G}}_{MJ} \tag{21}$$

and the medium behaves like an effective homogeneous medium with local and frequency-independent parameters. The mass density is a scalar, therefore all the information about the microstructure of the composite is contained in the C^*_{IJ} tensor, whose symmetry will depend on the background, inclusions, and lattice symmetry. In the above expression the limit in which the frequency and the wave number tend to zero has been assumed; however, in practice this limit will be valid from zero to some cut-off frequency in which it will not be possible to neglect some terms containing the frequency or the wave number; the medium then begins to be dispersive and the constitutive parameters will depend on both frequency and wave number. It can happen, however, that these parameters are frequency dependent even in the low-frequency limit, under the condition called a "local resonance." This happens when the parameter $\chi_{\boldsymbol{G}\boldsymbol{G}'}$ is singular, and then it is found that all the constitutive parameters can become resonant and locally singular, with the remarkable result that it presents, depending on the lattice symmetry, anisotropic mass density.

III. PERTURBATION THEORY IN THE LOW-FREQUENCY LIMIT

In this section we develop a perturbation theory for the computation of the effective parameters. The main objective of this approach is to provide an explanation of the resonant and nonlocal properties that the effective parameters can present and, as we will see later, many interesting properties can be deduced from this approach. However, although this methodology can provide good numerical results under certain conditions, for numerical accuracy it is better to employ the full expressions given by Eqs. (20). The origin of the resonant parameters of these structures is the χ matrix defined in general as

$$\chi = (M - \omega^2 N)^{-1}. \tag{22}$$

It must be pointed out that, in the static limit this χ matrix simply is the reciprocal of the matrix M; however, the term $\omega^2 N$ can make that the determinant of the matrix $M - \omega^2 N$ be zero for some specific values of ω, which we call resonances because then the χ matrix is singular. Let us try to understand the nature of these resonances.

Let us assume that we know the eigenvalues λ_n and eigenvectors $\boldsymbol{v}_n$ of the matrix $M - \omega^2 N$. We know then that the reciprocal of this matrix can be expanded by means of the eigendecomposition theorem, thus we have that

$$\chi = (M - \omega^2 N)^{-1} = \sum_n \frac{\boldsymbol{v}^\dagger_n \otimes \boldsymbol{v}_n}{\lambda_n} \tag{23}$$

given that $M - \omega^2 N$ is actually a Hermitian matrix. The matrix M is defined in Eq. (7), and it can be expressed as

$$M = M_0 + kM_1 + k^2 M_2, \tag{24}$$

where

$$M_0 = G_{iI} C^{\boldsymbol{G}-\boldsymbol{G}'}_{IJ} G'_{Jj}, \tag{25}$$

$$M_1 = n_{iI} C^{\boldsymbol{G}-\boldsymbol{G}'}_{IJ} G'_{Jj} + G_{iI} C^{\boldsymbol{G}-\boldsymbol{G}'}_{IJ} n_{Jj}, \tag{26}$$

$$M_2 = n_{iI} C^{\boldsymbol{G}-\boldsymbol{G}'}_{IJ} n_{Jj}. \tag{27}$$

For low frequencies and wave numbers, the matrix M can be considered a perturbation of the M_0 matrix, so that we can apply perturbation theory to relate the eigenvalues λ_k with frequency. Let us define $\boldsymbol{u}_n$ and C^0_n/a^2 (with a being a quantity with units of length, for convenience in the units) as the eigenvectors and eigenvalues of the M_0 matrix, respectively, thus

$$M_0 \boldsymbol{u}_n = C^0_n/a^2 \boldsymbol{u}_n. \tag{28}$$

If we assume that $kM_1 + k^2 M_2 - \omega^2 N$ is a perturbation of the matrix M_0, the eigenvalues and eigenvectors λ_n and $\boldsymbol{v}_n$ will be given, up to first order in perturbation theory, by

$$\lambda_n = C^0_n/a^2 + kC^{(1)}_n/a + k^2 C^{(2)}_n - \omega^2 \rho_n, \tag{29}$$

$$\boldsymbol{v}_n = \boldsymbol{u}_n + k \sum_\ell b^{(1)}_{n\ell} \boldsymbol{u}_\ell + k^2 \sum_\ell b^{(2)}_{n\ell} \boldsymbol{u}_\ell - \omega^2 \sum_\ell a_{n\ell} \boldsymbol{u}_\ell, \tag{30}$$

where (assuming $\boldsymbol{u}_n \cdot \boldsymbol{u}_n = 1$)

$$\rho_n = \boldsymbol{u}^\dagger_n N \boldsymbol{u}_n, \tag{31}$$

$$C^{(1)}_n/a = \boldsymbol{u}^\dagger_n M_1 \boldsymbol{u}_n, \tag{32}$$

$$C^{(2)}_n = \boldsymbol{u}^\dagger_n M_2 \boldsymbol{u}_n, \tag{33}$$

and, for $n \neq \ell$,

$$a_{n\ell} = \frac{\boldsymbol{u}^\dagger_n N \boldsymbol{u}_\ell}{C^0_n/a^2 - C^0_\ell/a^2}, \tag{34}$$

$$b^{(1)}_{n\ell} = \frac{\boldsymbol{u}^\dagger_n M_1 \boldsymbol{u}_\ell}{C^0_n/a^2 - C^0_\ell/a^2}, \tag{35}$$

$$b^{(2)}_{n\ell} = \frac{\boldsymbol{u}^\dagger_n M_2 \boldsymbol{u}_\ell}{C^0_n/a^2 - C^0_\ell/a^2}, \tag{36}$$

which ensures as well that $\boldsymbol{v}_n \cdot \boldsymbol{v}_n = 1$.

The dependence in k of the eigenvalues λ_n implies that the effective parameters will be nonlocal in general. However, as will be shown later, metamaterials are in general designed by means of "soft" scatterers, that is, it is required that the velocity of the waves inside the scatterers be much smaller than that of the background; in other words, the stiffness matrix elements display a much higher contrast between matrix and scatterer than the mass density components. Therefore, as a first approximation, we can neglect the coefficients multiplying

the wave number and approximate λ_n as

$$\lambda_n \approx C_n^0/a^2 - \omega^2\rho_n, \tag{37}$$

allowing one to express the χ matrix as (neglecting the perturbative terms in $\boldsymbol{u}$)

$$\chi_{ij}^{\boldsymbol{G}'\boldsymbol{G}}(\omega) = \sum_n \frac{(u_n^*)_{\boldsymbol{G}'i}(u_n)_{\boldsymbol{G}j}}{C_n^0/a^2 - \omega^2\rho_n}. \tag{38}$$

This interesting result shows that at the resonant frequencies $\omega_n^2 a^2 = C_n^0/\rho_n$ the effective parameters become singular, and in the neighborhood of this frequency they can have positive, negative, or zero values. These resonances are determined by the ratio of the eigenvalues C_n of the matrix M_0 and by the perturbation term ρ_n. Also, the coupling of these resonances with the different constitutive parameters is defined by the symmetry of the eigenvectors $\boldsymbol{u}_n$ of the matrix M_0, as will be explained in the following section.

It must be mentioned that the eigenvectors $\boldsymbol{u}_n$ correspond to the eigenvectors of the matrix M_0, which actually is the matrix $M_{\boldsymbol{G}\boldsymbol{G}'}$ given by Eq. (7) but for $k = 0$ and removing the terms corresponding to $\boldsymbol{G} = 0$. These eigenvectors correspond to a physical system which is easy to understand: Imagine a phononic crystal in which the stiffness tensor is periodic while the mass density is equal to that of the background. It is easy to see that the eigenvalue equation of this system at the Γ point, that is, for $k = 0$, is according to Eqs. (8),

$$\omega^2\bar{\rho}\boldsymbol{u}_0 = 0, \tag{39a}$$

$$\omega^2\bar{\rho}\boldsymbol{u}_{\boldsymbol{G}'} = M_{\boldsymbol{G}\boldsymbol{G}'}\boldsymbol{u}_{\boldsymbol{G}'}. \tag{39b}$$

The above equation for $\omega \neq 0$ has the only solution $\boldsymbol{u}_{\boldsymbol{G}} = (0,\boldsymbol{u}_n)$, with $\boldsymbol{u}_n$ being the eigenvectors of the matrix M_0. This relationship between the eigenvalues and eigenvectors of a physical system with those required to compute the effective parameters suggest that other numerical methods different than the plane-wave expansion method, like the finite element method, could be used. Then, computing the eigenvalues of this virtual system, in which the mass density is equal to that of the background and the stiffness tensor has the desired periodicity, allows computation of all the $\boldsymbol{u}_n$ and, after proper Fourier transform solutions and elastic constants' distribution, we can efficiently compute the effective parameters by means of the expansion (38). Therefore, on the basis of the present theory, this work opens the door to a more efficient calculation of the effective parameters, whose discussion is beyond the objective of the present work. Therefore, the presented effective medium theory, together with the above connection with other numerical tools, suggest that this method is more general than other homogenization schemes found in the literature, since it allows the inclusion of anisotropic, nonlocal effects and arbitrary shapes of the inclusions.

IV. EFFECTIVE PARAMETERS IN THE LOCAL APPROXIMATION

The study of a local resonance is made in a regime in which the wavelength of the propagating field is larger than the typical periodicity of the composite. This hypothesis implies that in Eqs. (20) we can make the approximation $k + G \approx G$, so that in these equations the dependence on k disappears and the parameters, although frequency dependent, are "local" and given by

$$\rho_{ij}^*(\omega) = \bar{\rho}\delta_{ij} + \omega^2\rho_{-\boldsymbol{G}'}\chi_{\ell m}^{\boldsymbol{G}'\boldsymbol{G}}(\omega)\rho_{\boldsymbol{G}}, \tag{40a}$$

$$C_{IJ}^*(\omega) = \bar{C}_{IJ} - C_{IL}^{-\boldsymbol{G}'}G'_{L\ell}\chi_{\ell m}^{\boldsymbol{G}'\boldsymbol{G}}(\omega)G_{mM}C_{MJ}^{\boldsymbol{G}}, \tag{40b}$$

$$S_{Ij}(\omega) = \omega C_{IL}^{-\boldsymbol{G}'}G'_{L\ell}\chi_{\ell j}^{\boldsymbol{G}'\boldsymbol{G}}(\omega)\rho_{\boldsymbol{G}}, \tag{40c}$$

where $\chi_{ij}^{\boldsymbol{G}'\boldsymbol{G}}(\omega)$ is computed from Eq. (11) as $\chi_{ij}^{\boldsymbol{G}'\boldsymbol{G}}(\omega,\boldsymbol{k}=0)$. Better insight into the properties of these parameters can be provided by including the expansion of χ given by Eq. (38); then we have

$$\rho_{ij}^*(\omega) = \bar{\rho}\delta_{ij} + \sum_n \omega^2\rho_{-\boldsymbol{G}'}\frac{(u_n^*)_{\boldsymbol{G}'i}(u_n)_{\boldsymbol{G}j}}{C_n^0/a^2 - \omega^2\rho_n}\rho_{\boldsymbol{G}}, \tag{41a}$$

$$C_{IJ}^*(\omega) = \bar{C}_{IJ} - \sum_n C_{IL}^{-\boldsymbol{G}'}G'_{L\ell}\frac{(u_n^*)_{\boldsymbol{G}'\ell}(u_n)_{\boldsymbol{G}m}}{C_n^0/a^2 - \omega^2\rho_n}G_{mM}C_{MJ}^{\boldsymbol{G}}, \tag{41b}$$

$$S_{Ij}(\omega) = \sum_n \omega C_{IL}^{-\boldsymbol{G}'}G'_{L\ell}\frac{(u_n^*)_{\boldsymbol{G}'\ell}(u_n)_{\boldsymbol{G}j}}{C_n^0/a^2 - \omega^2\rho_n}\rho_{\boldsymbol{G}}. \tag{41c}$$

We can now define the quantities

$$(A_n)_i = (u_n)_{\boldsymbol{G}i}\rho_{\boldsymbol{G}}, \tag{42}$$

$$(B_n)_I = (u_n)_{\boldsymbol{G}m}G_{mM}C_{MI}^{\boldsymbol{G}}, \tag{43}$$

and given that any Fourier coefficient satisfies $F_{-\boldsymbol{G}} = F_{\boldsymbol{G}}^*$ we get for the effective parameters the following expressions:

$$\rho_{ij}^*(\omega) = \bar{\rho}\delta_{ij} + \omega^2\sum_n \frac{(A_n^*)_i(A_n)_j}{C_n^0/a^2 - \omega^2\rho_n}, \tag{44a}$$

$$C_{IJ}^*(\omega) = \bar{C}_{IJ} - \sum_n \frac{(B_n^*)_I(B_n)_J}{C_n^0/a^2 - \omega^2\rho_n}, \tag{44b}$$

$$S_{Ij}(\omega) = \omega\sum_n \frac{(B_n^*)_I(A_n)_j}{C_n^0/a^2 - \omega^2\rho_n}. \tag{44c}$$

The above equations relate the effective parameters with the properties of the eigenvectors and eigenvalues of the matrix M_0, as well as with their perturbations. Let us note that, for a symmetric unit cell, that is, when the position-dependent parameters inside the unit cell are described by functions such that $f(-\boldsymbol{r}) = f(\boldsymbol{r})$, we will have that $F(-\boldsymbol{G}) = F(\boldsymbol{G})$, and then we have (see Appendix B) that for under these conditions we also have that

$$\boldsymbol{u}_{-\boldsymbol{G}} = \pm\boldsymbol{u}_{\boldsymbol{G}}, \tag{45}$$

which implies two types of solutions: $(A_n)_i \neq 0$ and $(B_n)_I = 0$ when $\boldsymbol{u}_{-\boldsymbol{G}} = \boldsymbol{u}_{\boldsymbol{G}}$, and $(A_n)_i = 0$ and $(B_I)_n \neq 0$ when $\boldsymbol{u}_{-\boldsymbol{G}} = -\boldsymbol{u}_{\boldsymbol{G}}$. The former induces a resonant mass density, while the latter induces a resonant stiffness tensor. The two cases imply that the Willis tensor S_{Ij} is equal to zero, and this also implies that a symmetric system cannot simultaneously excite a resonance in the stiffness tensor and the mass density; that is, we cannot have double negative materials in this way, unless the different resonances are too close to each other.

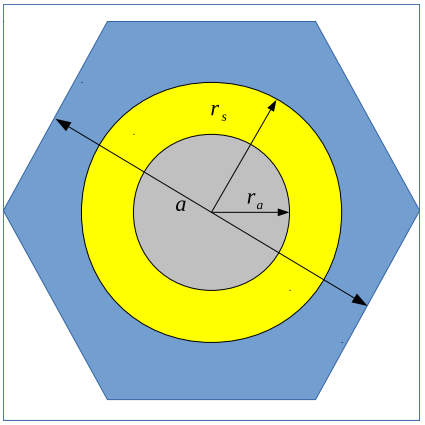

FIG. 1. (Color online) Phononic crystal studied in the present work. The system consists of a triangular arrangement of coated cylinders in an epoxy background. The cylinders consist of a lead core of radius r_a and a rubber shell of radius r_s (see text for numerical values).

A deeper insight into the properties of symmetry and nonsymmetry of these resonances is beyond the objective of the present work, in which we want to focus attention on the properties of the resonant mass density, however, a future work concerning a nonsymmetric lattice will be prepared and published elsewhere.

V. NUMERICAL EXAMPLE: RESONANT AND NONLOCAL ANISOTROPIC MASS DENSITY

Figure 1 shows the system to be studied in the present work. It consists of a periodic arrangement of coated cylinders in an epoxy background ($\rho_b = 1.18\,\mathrm{Kg/dm^3}$, $E_b = 4.35\,\mathrm{GPa}$, and $\nu_b = 0.37$). The cylinders are made of a lead core ($\rho_a = 11.34\,\mathrm{Kg/dm^3}$, $E_a = 16$ GPa, and $\nu_a = 0.44$) of radius $r_a = 0.16a$ and a rubber shell ($\rho_s = 1.3\,\mathrm{Kg/dm^3}$, $E_s = 2.7E-4$ GPa, and $\nu_b = 0.499$) of radius $r_s = 0.4a$, since this combination of soft-hard coatings is known to present low-frequency resonances. The cylinders are arranged in a triangular lattice; in this way we expect the effective material to be transversely isotropic.

Figure 2 shows the effective mass density tensor relative to that of the background ρ_b as a function of frequency as computed by using Eq. (40a), which has been numerically found to be identical to (44a) but, although less efficient for computational use, it is more exact than the mentioned approximation. Because of the symmetry of the lattice, any second rank tensor will have only two components, one for the xy plane and another one for the z plane. It is seen how in the low-frequency limit the two components are identical and equal to the normalized average mass density $\bar{\rho}/\rho_b$, as expected; however, it can also be seen how they split as a function of frequency and present two different resonances, so that the system behaves like an elastic medium with anisotropic mass density. Moreover, it can also be seen how these components are negative in different frequency regions. It is found that the effective stiffness tensor is nearly constant in frequency in this region, and that the coupling field S_{Ij} is zero, as expected from the discussion in the previous section.

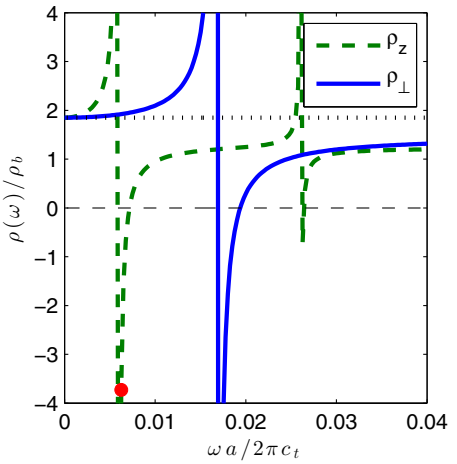

FIG. 2. (Color online) Effective mass density tensor for the proposed phononic crystal. Both the transversal component (blue continuous line) and the z component (green dashed line) present a local resonance. In the vicinity of this resonance both components of the mass density become negative. The red dot indicates the frequency and density of the example used in Fig. 5 (see text for further details).

Figure 3 shows the field distribution of the lower-frequency resonances of the mass density tensor depicted in Fig. 2: the upper panel for the z component and the lower panel for the xy one; the left panels show the real part of the mode, while the right panels show the absolute value. It is interesting to note that the xy mode has a dipolar symmetry, as is commonly assumed in the literature [19], while the z mode has a monopolar symmetry. The fact that a monopolar symmetry could induce a negative mass density behavior was already found by the authors in a recent paper [24] in the study of

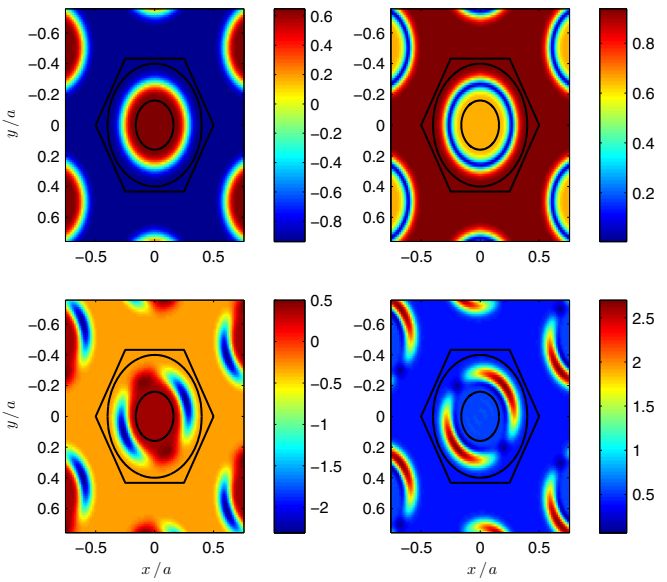

FIG. 3. (Color online) Resonant modes inducing an effective negative mass density. Upper panels for $\omega_k a/2\pi c_t = 0.0072$, corresponding to a resonance in ρ_z, and lower panels for $\omega_k a/2\pi c_t = 0.0194$, corresponding to a resonance in $\rho_\perp$ (see text for further discussion).

flexural waves in thin plates. This result is consistent with the theory of elastic waves in plates, given that a plate with a periodic arrangement of inclusions is indeed a finite slide of the two-dimensional phononic crystal studied here, and this result suggests that the propagation of flexural waves is mainly dominated by the z component of the mass density. This important result should be taken into account in the homogenization theory of plate metamaterials, although a deep insight into it is beyond the objective of the present work.

Equations (40) show then that the phononic crystal can be described by means of locally resonant constitutive parameters, whose frequency dependence can be easily computed. The description of a phononic crystal as a frequency-dependent homogeneous material will not be valid for every wave number and frequency, and to determine these limits the dispersion relation obtained by means of the constitutive parameters is compared with the band structure obtained from the eigenvalue equation (5). As mentioned before, in the local approximation and due to the symmetry of the unit cell, S_{Ij} is zero for this example, and it is also found that C^*_{IJ} is nearly constant in this frequency range, thus we have that, along the ΓX direction, the dispersion relation for the effective material is

$$\omega^2\rho_\perp(\omega)u_x = k_x^2 C^*_{11}u_x, \tag{46}$$

$$\omega^2\rho_\perp(\omega)u_y = k_x^2 C^*_{66}u_y, \tag{47}$$

$$\omega^2\rho_z(\omega)u_z = k_x^2 C^*_{44}u_z, \tag{48}$$

while along the ΓA direction, that is, along the z axis, the dispersion relation is (notice that in this case $C^*_{55} = C^*_{44}$)

$$\omega^2\rho_\perp(\omega)u_x = k_z^2 C^*_{44}u_x, \tag{49}$$

$$\omega^2\rho_\perp(\omega)u_y = k_z^2 C^*_{55}u_y, \tag{50}$$

$$\omega^2\rho_z(\omega)u_z = k_z^2 C^*_{33}u_z. \tag{51}$$

Figure 4, left panel, shows the dispersion relation along the ΓX direction (x axis) computed by means of the eigenvalue equation (5) (black lines) compared with the dispersion relation obtained by means of the constitutive parameters. Red and blue dots show the results for the xy modes, and it is seen that there is good agreement between the eigenvalue equation and the effective material dispersion relation. The dispersion relation for the z mode (green crosses) is, however, different from the eigenvalue equation and the effective material, and there is agreement only for very low wave numbers. As will be seen later, the reason for this disagreement is that the local description of the metamaterial is not accurate here, and the inclusion of the nonlocal components is required, that is, the dependence on the wave number in the constitutive parameters.

Figure 4, right panel, shows similar results for propagation along the ΓA direction (z axis). It is shown here that the xy modes, which are degenerate given that the crystal is transversely isotropic, are perfectly described by means of the effective material; however, the z modes, corresponding to green crosses, agree only for very low wave numbers. There is also a set of flat bands that can be fairly difficult to predict by means of the effective material parameters. The reason for that is that these modes occur only at a given frequency

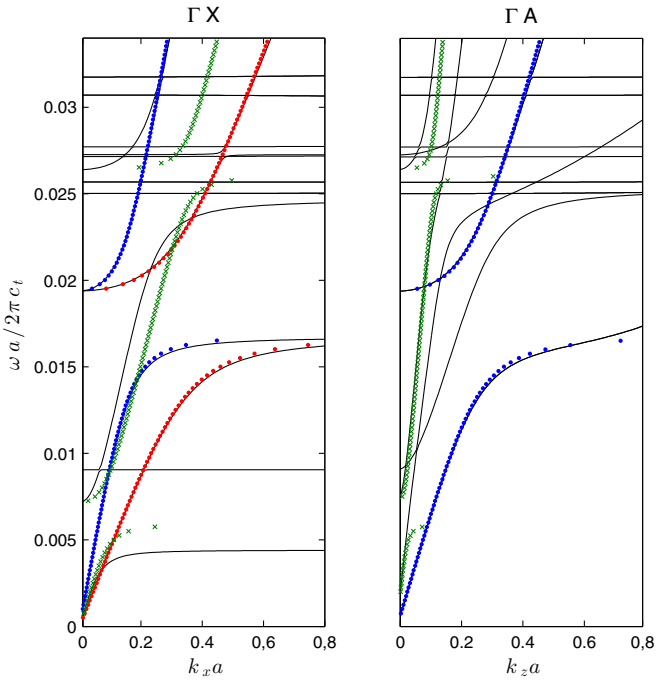

FIG. 4. (Color online) Left panel: Dispersion relation of the phononic crystal along the ΓX direction (black lines) compared with those obtained from the effective local constitutive parameters (green crosses and blue and red dots). Right panel: Dispersion relation of the phononic crystal along the ΓA direction (black lines) compared with those obtained from the effective local constitutive parameters (green crosses and blue dots).

and correspond to very sharp modes, and although they are properly predicted by the theory as a resonant frequency ω_n, their effect is difficult to see in the constitutive parameters.

The spatial dispersion of the z mode can be understood by means of the calculation of the nonlocal constitutive parameters using Eqs. (20). Figure 5 shows these parameters at a frequency $\omega a/2\pi c_t = 0.0063$, corresponding to a frequency in which the z component of the local mass density is negative (red dot in Fig. 2). The upper panel shows the nonlocal ρ_z as a function of the wave number along the ΓX and ΓA directions. It is clear that the origin of the nonlocality is a wave number resonance, for which the major contribution will have its origin in the χ matrix. It is also seen that the C_{33} component, responsible for the propagation of the mode along the ΓA direction, also becomes nonlocal, while the C_{44} component remains constant. Additionally, the S_{53} and S_{33} elements, which are zero for $k = 0$, appear as resonant components. The contribution of these spatial resonances is essentially to displace the opening of the band gaps, as can be seen from Fig. 4, for which their influence is important before considering only the local theory.

VI. SUMMARY

In summary, it has been analytically and numerically demonstrated that phononic crystals behave as elastic metasolids with anisotropic, resonant, and nonlocal effective parameters, with the remarkable result that the mass density is also anisotropic in general, although in the static limit

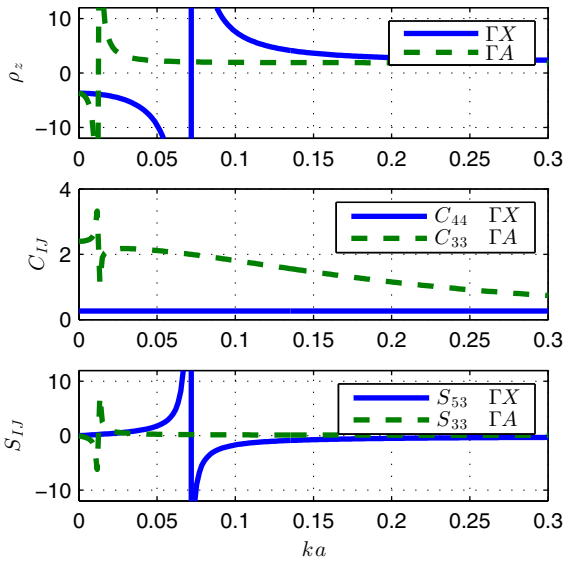

FIG. 5. (Color online) Nonlocal constitutive parameters related with the propagation of the z mode in the phononic crystal. Upper panel: z component of the mass density. Middle panel: C_{44} and C_{33} components of the stiffness tensor. Lower panel: S_{53} and S_{33} components of the coupling field (see text for details).

this quantity recovers its scalar nature. Also, it has been demonstrated that the symmetry of the resonance inducing this behavior is not necessarily dipolar, as it is commonly assumed; it can also be monopolar. The nonlocal and anisotropic nature of the mass density has important implications especially for the study of plate metamaterials, since these structures are essentially finite slides of phononic crystals. It must be pointed out that the generality of the equations derived can be used for the homogenization of phononic crystals with more complex unit cells, with the objective of achieving double negative metasolids. Finally, the theory can be extended to phononic crystals with piezoelectric inclusions, where resonant piezoelectric constants are expected.

ACKNOWLEDGMENTS

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APPENDIX A: MATRIX NOTATION

Throughout the paper Voigt notation for the indexes is used, in such a way that lowercase indexes run from 1 to 3 and upper case indexes run from 1 to 6. Also, the wave vector is defined in terms of the V_{iI} matrix defined as

$$\boldsymbol{V} = \begin{pmatrix} V_x & 0 & 0 & 0 & V_z & V_y \\ 0 & V_y & 0 & V_z & 0 & V_x \\ 0 & 0 & V_z & V_y & V_x & 0 \end{pmatrix}. \tag{A1}$$

Therefore the matrix elements $(k+G)_{iI}$ are

$$\boldsymbol{k+G} = \begin{pmatrix} k_x+G_x & 0 & 0 & 0 & k_z+G_z & k_y+G_y \\ 0 & k_y+G_y & 0 & k_z+G_z & 0 & k_x+G_x \\ 0 & 0 & k_z+G_z & k_y+G_y & k_x+G_x & 0 \end{pmatrix} \tag{A2}$$

with $(k+G)_{Jj}$ therefore being the transpose of the above matrix. Similarly, the same matrix n_{iI} is defined for the normal vector $\boldsymbol{n}$,

$$\boldsymbol{n} = \begin{pmatrix} n_x & 0 & 0 & 0 & n_z & n_y \\ 0 & n_y & 0 & n_z & 0 & n_x \\ 0 & 0 & n_z & n_y & n_x & 0 \end{pmatrix}. \tag{A3}$$

APPENDIX B: PROPERTIES OF THE EIGENVECTORS OF M

The matrix M is a Hermitian matrix, therefore its eigenvalues are real. Also, it is known that the Fourier components satisfy $F_{-\boldsymbol{G}} = F^*_{\boldsymbol{G}}$, which means that

$$M_{-\boldsymbol{G}-\boldsymbol{G}'} = M^*_{\boldsymbol{G}\boldsymbol{G}'}, \tag{B1}$$

$$M_{-\boldsymbol{G}\boldsymbol{G}'} = M^*_{\boldsymbol{G}-\boldsymbol{G}'}. \tag{B2}$$

We can now express the eigenvalue equation for M as

$$\begin{pmatrix} M_{\boldsymbol{G}\boldsymbol{G}'} & M_{\boldsymbol{G}-\boldsymbol{G}'} \\ M^*_{\boldsymbol{G}-\boldsymbol{G}'} & M^*_{\boldsymbol{G}\boldsymbol{G}'} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_{\boldsymbol{G}'} \\ \boldsymbol{u}_{-\boldsymbol{G}'} \end{pmatrix} = \lambda \begin{pmatrix} \boldsymbol{u}_{\boldsymbol{G}'} \\ \boldsymbol{u}_{-\boldsymbol{G}'} \end{pmatrix} \tag{B3}$$

or, to simplify the notation,

$$\begin{pmatrix} M_{aa} & M_{ab} \\ M^*_{ab} & M^*_{aa} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}. \tag{B4}$$

The above eigenvalue equation can be expressed as

$$M_{aa}u + M_{ab}v = \lambda u, \tag{B5}$$

$$M^*_{ab}u + M^*_{aa}v = \lambda v. \tag{B6}$$

Taking the complex conjugate of the above and exchanging the order of the equations, we get

$$M_{aa}v^* + M_{ab}u^* = \lambda v^*, \tag{B7}$$

$$M^*_{ab}v^* + M^*_{aa}u^* = \lambda u^*, \tag{B8}$$

which shows that the eigenvector (v^*,u^*) has the same eigenvalue as the eigenvector (u,v), so that they differ only in phase factor $e^{i\phi}$; then we have

$$v = e^{i\phi}u^* \tag{B9}$$

or

$$\boldsymbol{u}_{-\boldsymbol{G}} = e^{i\phi}\boldsymbol{u}^*_{\boldsymbol{G}}. \tag{B10}$$

For the specific case of a symmetric lattice, that is, if we can find a unit cell such that $F_{-G} = F_G$, we have that matrix M becomes real symmetric, so that it is possible to always find real eigenvectors, so that the phase factor should be π or 0; in other words, in this case we have that

$$\boldsymbol{u}_{-G} = \pm \boldsymbol{u}_G. \tag{B11}$$

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