Transport in thin polarized Fermi-liquid films

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We calculate expressions for the state-dependent quasiparticle lifetime τ_σ, the thermal conductivity κ, the shear viscosity η, and discuss the spin diffusion coefficient D for Fermi-liquid films in two dimensions. The expressions are valid for low temperatures and arbitrary polarization. In two dimensions, as in three dimensions, the integrals over the transition rates factor into energy and angular parts. However, the angular integrations contain a weak divergence. This problem is addressed using the method of K. Miyake and W. J. Mullin [Phys. Rev. Lett. **50**, 197 (1983); J. Low Temp. Phys. **56**, 499 (1984)]. The low-temperature expressions for the transport coefficients are essentially exact. We find that $\kappa^{-1} \sim T \ln T$, and $\eta^{-1} \sim T^2$ for arbitrary polarizations $0 \leqslant \mathcal{P} \leqslant 1$. These results are in agreement with earlier zero-polarization results of H. H. Fu and C. Ebner [Phys. Rev. A **10**, 338 (1974).], but differ from the temperature dependence of the shear viscosity found by D. S. Novikov (arXiv:cond-mat/0603184). They also differ from the discontinuous change of temperature dependence in D from zero to nonzero polarization that was discovered by Miyake and Mullin. We note that in two dimensions the shear viscosity requires a unique analysis. We obtain predictions for the density, temperature, and polarization dependence of κ, η, and D for second-layer ^{3}He films on graphite, and thin $^3\mathrm{He}-{}^4\mathrm{He}$ superfluid mixtures. For ^{3}He on graphite, we find roughly an order of magnitude increase in magnitude for κ and η as the polarization is increased from 0 to 1. For D a similar large increase is predicted from zero polarization to the polarization where D is a maximum ($\sim$0.74). We discuss the applicability of ^{3}He thin films to the question of the existence of a universal lower bound for the ratio of the shear viscosity to the entropy density.

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I. INTRODUCTION

Fermi-liquid theory, developed by Landau [1,2] in the mid-1950s, showed how low-temperature collective excitations and thermodynamic properties of strongly interacting normal many-fermion systems could be encoded in a few parameters, the Landau parameters, and that these parameters were related to a certain limiting value of the microscopic scattering function [3]. In Ref. [2], Landau also introduced a kinetic equation to describe the nonequilibrium properties of a Fermi liquid. The kinetic equation is of the same form as the classical Boltzmann equation with the local quasiparticle energy $\tilde{\epsilon}_{\mathbf{p}\sigma}(\mathbf{r},t)$ playing the role of a Hamiltonian. The application of the linearized Landau kinetic equation to the calculation of transport coefficients for bulk ^{3}He has been very successful. In this paper we shall apply this approach to a strongly interacting many-fermion system in two dimensions. Reviews of the bulk calculations at zero polarization can be found in the works of Abrikosov and Khalatnikov [4], Pines and Nozières [5], and Baym and Pethick [6]. The calculation of transport coefficients for Fermi liquids in three dimensions with arbitrary polarization can be found in Anderson, Pethick, and Quader [7], and Meyerovich [8]. The former set of authors used a slick general notation that emphasized the similarities in the calculations of the various coefficients. There exist some measurements of transport coefficients as a function of polarization for bulk ^{3}He. Buu, Forbes, Puech, and Wolf [9], and also Akimoto, Xia, Adams, Candela, Mullin, and Sullivan [10] studied the shear viscosity. Sawkey, Puech, and Wolf [11] studied the thermal conductivity.

Abrikosov and Khalatnikov [4] (AK) in particular showed that the integrals involved in the collision integral factor neatly into a product of integrals involving angular variables and those involving energy variables. The resulting expression for the kinetic equation could then be brought into the form of a linear integral eigenvalue problem for essentially the nonequilibrium part of the fermion distribution function. The exact solutions of these integral eigenvalue problems are derived, and discussed in detail by Sykes and Brooker [12] and also Jensen, Smith, and Wilkins [13].

In recent work, we have utilized the kinetic equation approach to study the transition between collective excitations in the ballistic regime (zero sound) and the hydrodynamic regime (first sound) in thin, arbitrarily polarized Fermi-liquid films [14]. For sound, the kinetic equation is usually solved by rewriting the integral equation as an (infinite) set of algebraic equations by using a Fourier expansion, and then taking moments with respect to the angular functions. This procedure is not unique, and we have compared and discussed in detail the predicted propagation speeds and attenuation for two different approaches [15]. In the above cited works we have utilized previously calculated [14,16] density and polarization dependent Landau parameters in order to obtain numerical predictions for thermodynamic and collective excitations for the specific case of ^{3}He films. In this paper we shall use these same Landau parameters to calculate predicted values for the density and polarization dependent transport coefficients in thin ^{3}He films.

The calculation of transport coefficients for two-dimensional (2D) Fermi liquids has been considered by Fu and Ebner [17], Miyake and Mullin [18], and Novikov [19]. Fu and

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Ebner applied the variational approach that was developed by Baym and Ebner [20] in order to calculate transport coefficients for ^{3}He in superfluid ^{4}He bulk solutions. The variational approach of Fu and Ebner as applied in two dimensions does not lend itself to analytic solution; nevertheless, they were able to extract the lowest order temperature dependencies together with numerically determined coefficients for the thermal conductivity κ, the first (or shear) viscosity η, and the spin diffusion coefficient D all at zero polarization. Fu and Ebner obtained $\ln T$ behavior for two of the coefficients, and pointed out that the source was a weak divergence in the momentum space integrals. Novikov calculated the shear viscosity of an unpolarized two-dimensional Fermi liquid. Most importantly, he pointed out the critical role played by "head-on" quasiparticle collisions (see Fig. 3) in determining the zero-polarization shear viscosity.

Miyake and Mullin [18] (MM) derived an exact expression for the spin diffusion coefficient for two-dimensional fermions with arbitrary polarization. They indicated that in two dimensions one obtains a logarithmic divergence at *finite* temperature in the integrand of the angular integrals if one proceeds by strictly following the three-dimensional approach developed by AK. They identified the source of the divergence at finite temperature as an artifice of using zero-temperature values for the Fermi momenta in an integrand of one of the angular integrals in the kinetic equation. In a very clever analysis, by generalizing the analysis to low but finite temperature they were able to extract an expression that yielded a logarithmic divergence only in the zero-temperature limit.

In Sec. II we first apply the MM method to calculate the state-dependent quasiparticle lifetime at arbitrary polarization. This is convenient because the lifetime is needed in the calculation of transport coefficients in later sections. We shall compare the present result for the quasiparticle lifetime to a previous one [14] that was obtained using a method that is completely independent of MM. In Secs. III A and III B we calculate the thermal conductivity and the shear viscosity, respectively. In Sec. III C we include a brief discussion of the zero and finite polarization behavior of the spin diffusion coefficient. We note that as in three dimensions (3D) the calculations of the thermal conductivity and the spin diffusion are very similar. However, unlike 3D, for 2D we find that the analysis for the shear viscosity needs significant modification. The root of the problem in the shear viscosity calculation is identified as being due once more to the incorrect use of the zero-temperature limit in the integrands of the angular integrals. In Sec. IV we utilize Landau parameters that were previously determined for second-layer ^{3}He films on a graphite substrate, and also for thin-film ^{3}He - ^{4}He mixtures to compute density, temperature, and polarization dependencies for the transport coefficients. Our results for the shear viscosity are used to calculate the ratio of the shear viscosity to the entropy density. Sec. V is the conclusion.

II. QUASIPARTICLE LIFETIME

We examine a system of $N = N_\uparrow + N_\downarrow$, spin-$\frac{1}{2}$ fermions in a box of area L^2. The particles have bare mass m, and interact with two-body potential $V(r)$ that is assumed to depend only on the scalar distance between the particles. The particles fill two Fermi seas up to Fermi momenta $k_\uparrow$ and $k_\downarrow$, and we introduce the convention that the spin-down Fermi sea will always be the minority Fermi sea in the case of nonzero polarization. The term *polarization* denotes the magnetization per particle which will be denoted by $\mathcal{P}$; thus $\mathcal{P} \equiv M/N = (N_\uparrow - N_\downarrow)/N$. The terms coverage and areal density (N/L^2) are used interchangeably. The system is assumed to be at some finite but low temperature T in the sense that $T \ll T_{F\downarrow}$.

The quasiparticle lifetime τ_σ due to quasiparticle-quasiparticle interactions in two-dimensional Fermi liquids was calculated in Ref. [14]. The method used in that reference was borrowed from two-dimensional electron theory, and took advantage of the similarity in structure between the collision integral and the free fermion dynamic structure function. The fluctuation-dissipation theorem together with Stern's analytic expression [21] for the two-dimensional susceptibility yielded an analytic expression for the low-temperature lifetime. In this section we calculate the quasiparticle lifetime by using a completely different approach. This approach uses a method introduced by MM to treat the divergence in the angular integrals in two dimensions. The lifetime calculated here is needed in the calculation of each of the transport coefficients. It is thus convenient to separate it into an independent section, and simply refer to the result in the later transport section. In addition, the technique we use to treat the divergence in the integrals is similar to that in calculating the transport coefficients, and since the lifetime calculation is simpler, this allows us to present this technique in a clearer context.

After some simplification [6], the quasiparticle collision frequency is given by

$$\frac{1}{\tau_{\sigma_1}(\mathbf{p}_1)} = \sum_{\mathbf{p}_2,\sigma_2}\sum_{\mathbf{p}_3,\sigma_3}\sum_{\mathbf{p}_4,\sigma_4} W(1,2;3,4)\delta(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4) \times \delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_3+\mathbf{p}_4}\delta_{\sigma_1+\sigma_2,\sigma_3+\sigma_4} n_2\overline{n}_3\overline{n}_4, \tag{2.1}$$

where $n_{\mathbf{p}\sigma} \equiv 1/\{\exp[\beta(\epsilon_{\mathbf{p}\sigma} - \mathbf{p}\cdot\mathbf{u} - \mu_\sigma)] + 1\}$ is the Fermi distribution function, $\beta \equiv 1/k_B T$, μ_σ is the chemical potential for the σth Fermi sea, and $\mathbf{u}$ is the fluid velocity. In this section we can set $\mathbf{u} = 0$. The W's are transition probabilities, and we have defined

$$\overline{n}_{\mathbf{p}\sigma} \equiv 1 - n_{\mathbf{p}\sigma} = \frac{1}{1 + e^{-\beta(\epsilon_{\mathbf{p}\sigma} - \mathbf{p}\cdot\mathbf{u} - \mu_\sigma)}}. \tag{2.2}$$

The standard treatment in three-dimensions follows Abrikosov and Khalatnikov [22], and introduces new integration variables in terms of energies and angles. These integrations are independent of one another, and in lowest order in temperature one can find a closed form expression for $1/\tau$ in terms of an angular average of the transition probabilities. Label the incoming quasiparticles as $\mathbf{p}_1, \mathbf{p}_2$ and the outgoing quasiparticles $\mathbf{p}_3, \mathbf{p}_4$. The standard angular variables, θ and φ, are defined as follows: θ is the angle of $\mathbf{p}_2$ measured relative to the direction of $\mathbf{p}_1$, and φ is the angle between the planes formed by the pairs of vectors $\{\mathbf{p}_1, \mathbf{p}_2\}$ and $\{\mathbf{p}_3, \mathbf{p}_4\}$. As discussed by MM, in two dimensions φ can only take on two values: $0, \pi$. We illustrate these two possibilities in Figs. 1 and 2. Along with MM we shall refer to these two processes as forward and backward scattering, respectively. We note that the forward and backward scattering

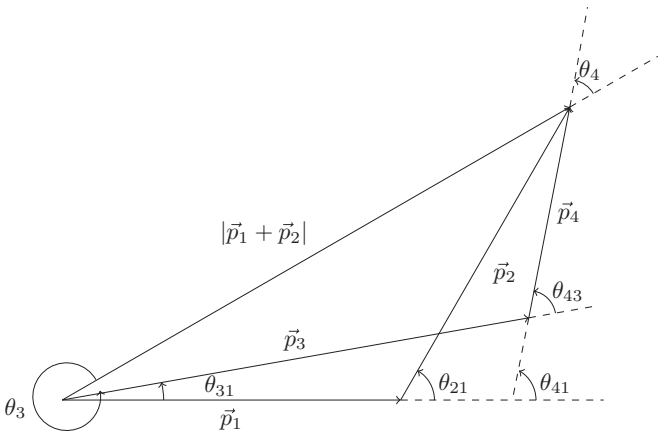

FIG. 1. The momentum space diagram for the forward scattering process, $\pi < \theta_3 \leqslant 2\pi$. From momentum conservation $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4$. The angle θ is the angle of $\mathbf{p}_2$ as measured from the direction of $\mathbf{p}_1$; θ_3 is the angle of $\mathbf{p}_3$ as measured from the direction of $\mathbf{p}_1 + \mathbf{p}_2$. We shall also need the angles $\Phi_3 \equiv \theta_{31}$ and $\Phi_4 \equiv \theta_{41}$ which are the angles of $\mathbf{p}_3$ and $\mathbf{p}_4$ as measured from the direction of $\mathbf{p}_1$, respectively.

processes have a direct and exchange relationship since Fig. 2 is obtained from Fig. 1 by exchanging $\mathbf{p}_3$ and $\mathbf{p}_4$.

Figures 1 and 2 define the conventions that we shall use throughout this paper to label the angles associated with the quasiparticle momenta. All angles are measured counterclockwise. The angles θ_i with $i = 1,2,3,4$ are the angles of $\mathbf{p}_i$ as measured from $\mathbf{p}_1 + \mathbf{p}_2$. The angle θ_{ij} is the angle of $\mathbf{p}_i$ as measured from the direction of $\mathbf{p}_j$. In the discussion below, we shall follow MM and use the notation $\Phi_i^{f/b} \equiv \theta_{i1}$, and as noted above $\theta \equiv \Phi_2$. For further reference, by inspection of Fig. 2, we note the following useful expressions:

$$\cos\Phi_3 = 1 - \frac{2p_2^2\sin^2\theta}{\ell^2}, \quad \sin\Phi_3 = \sin\theta\frac{2p_2(p_1+p_2\cos\theta)}{\ell^2},$$
$$\cos\Phi_4 = \cos\theta + \frac{2p_1p_2\sin^2\theta}{\ell^2}, \quad \sin\Phi_4 = \sin\theta\frac{\left(p_2^2-p_1^2\right)}{\ell^2}, \tag{2.3}$$

where following MM we utilize the variable $\boldsymbol{\ell} \equiv \mathbf{p}_1 + \mathbf{p}_2$.

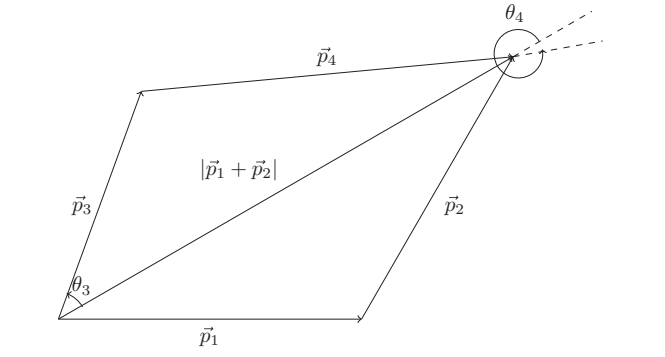

FIG. 2. The momentum space diagram for the backwards scattering process, $0 < \theta_3 \leqslant \pi$. We note that this figure can be obtained from Fig. 1 by exchanging $\mathbf{p}_3$ and $\mathbf{p}_4$. From momentum conservation $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4$.

Since the interaction is assumed spin independent, total spin is conserved in the collisions as indicated by the Kronecker delta in (2.1). Thus, we must have $\sigma_3 = \sigma_1$ and $\sigma_4 = \sigma_2$. In the spin-parallel case $\sigma_2 = \sigma_1$ the exchange diagram is identical to the direct diagram, and therefore they must be counted only once in the phase space integrations. In the spin-antiparallel case $\sigma_1 = -\sigma_2$ the direct and exchange diagrams give different contributions to the total transition probability. Performing the spins sums, replacing the momentum sums by integrations, and performing an integration over $\mathbf{p}_4$ yields

$$\begin{aligned}\frac{1}{\tau_{\sigma_1}(\mathbf{p}_1)} &= \frac{1}{h^4}\int d\mathbf{p}_2 d\mathbf{p}_3\left(\frac{1}{2}W^{\sigma_1\sigma_1} + W^{\sigma_1-\sigma_1}\right)\\ &\quad\times\delta(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4)n_2\overline{n}_3\overline{n}_4,\\ &\equiv \frac{1}{\tau_{\sigma_1\sigma_1}} + \frac{1}{\tau_{\sigma_1-\sigma_1}},\end{aligned} \tag{2.4}$$

where for later use we have introduced spin-parallel and spin-antiparallel collision frequencies. In this expression we have set $A = 1$. Thus, the units of the $W^{\sigma\sigma'}$'s are (energy) (time)$^{-1}$ (area)^2. As usual, the factor of one-half appearing with the spin-parallel transition probability prevents overcounting as discussed above [6].

We first consider the spin-parallel lifetime, and separate out the angular integrals:

$$\begin{aligned}\frac{1}{\tau_{\sigma_1\sigma_1}} &= \frac{2}{h^4}\int_0^\infty p_2dp_2\int_0^\infty p_3dp_3\int_0^\pi d\theta\int_\pi^{2\pi}d\theta_3\,W_f^{\sigma_1\sigma_1}(\theta)\\ &\quad\times\delta(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4)n_2\overline{n}_3\overline{n}_4,\end{aligned} \tag{2.5}$$

where we have taken advantage of the symmetry in θ about π, θ_3 is defined as the angle of $\mathbf{p}_3$ measured with respect to $\mathbf{p}_1 + \mathbf{p}_2$, see Fig. 1, and the subscript f or b on W identifies the transition probability as that for forward scattering ($\pi < \theta_3 \leqslant 2\pi$) or backward scattering ($0 < \theta_3 \leqslant \pi$), respectively. We now rewrite the angular variable θ_3 in a more useful form. With an eye on Fig. 1 the law of cosines yields

$$p_4^2 = p_3^2 + \ell^2 - 2p_3\ell\cos\theta_3, \tag{2.6}$$

$$d\theta_3 = \frac{dp_4^2}{2p_3\ell\sin\theta_3}. \tag{2.7}$$

By inspection of Fig. 1:

$$-p_3\ell\sin\theta_3 = p_3p_4\sin\theta_{43}. \tag{2.8}$$

From the law of cosines again:

$$p_1^2 + p_2^2 + 2p_1p_2\cos\theta = p_3^2 + p_4^2 + 2p_3p_4\cos\theta_{43}. \tag{2.9}$$

Equation (2.9) can be simplified using energy conservation: $\varepsilon_1 + \varepsilon_2 = \varepsilon_3 + \varepsilon_4$, where we have defined $\varepsilon_i \equiv p_i^2/2m_i^*$. The quasiparticle label on the effective masses is needed since at finite polarization the effective masses are state dependent. We find

$$p_3p_4\cos\theta_{43} = (m_2^* - m_1^*)\xi_3 + p_1p_2\cos\theta, \tag{2.10}$$

where we have defined the important quantity ξ_3:

$$\xi_3 \equiv \frac{p_3^2 - p_1^2}{2m_1^*} \sim O(k_BT). \tag{2.11}$$

We use (2.10) to eliminate θ_{43} in (2.8), and then with (2.7) we find

$$d\theta_3 = -\frac{dp_4^2}{2p_1p_2\sqrt{\sin^2\theta - \frac{1}{\epsilon_{12}}\xi_3 - \frac{(m_1^*+m_2^*)^2}{p_1^2p_2^2}\xi_3^2}} \quad (\pi < \theta_3 \leqslant 2\pi), \tag{2.12}$$

where

$$\frac{1}{\epsilon_{12}} \equiv \left(1+\frac{p_2}{p_1}\cos\theta\right)\frac{1}{\epsilon_2} - \left(1+\frac{p_1}{p_2}\cos\theta\right)\frac{1}{\epsilon_1} \approx \left(1+\frac{p_{F2}}{p_{F1}}\cos\theta\right)\frac{1}{\epsilon_{F2}} - \left(1+\frac{p_{F1}}{p_{F2}}\cos\theta\right)\frac{1}{\epsilon_{F1}}. \tag{2.13}$$

The F subscripts on the Fermi energies and Fermi momenta indicate that we only need the zero-temperature limit for ϵ_{12} since ξ_3 itself is $O(k_BT)$.

We note that Eq. (2.12) is exact. For the spin-parallel case the linear term in ξ_3 vanishes making the thermal correction term $\sim O((k_BT)^2)$. The thermal correction changes from linear order in ξ_3 for antiparallel spin scattering to quadratic order in ξ_3 for parallel spin scattering. With Eq. (2.12), Eq. (2.5) becomes

$$\frac{1}{\tau_{\sigma_1\sigma_1}} = \frac{1}{h^4}\int_0^\infty dp_2\,dp_3\,dp_4^2\int_0^\pi \frac{d\theta}{\sqrt{\sin^2\theta - \left(\frac{\xi_3}{\epsilon_1}\right)^2}}\,W_f^{\sigma_1\sigma_1} \times \delta(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4)n_2\bar{n}_3\bar{n}_4. \tag{2.14}$$

The integrals are brought to their final form by introducing dimensionless variables $x_i \equiv \beta(\epsilon_i - \mu)$, and retaining the contribution from the angular integral with the lowest order temperature dependence:

$$\frac{1}{\tau_{\sigma_1\sigma_1}} \approx \frac{2\left(m_{\sigma_1}^*\right)^3(k_BT)^2}{h^4p_{\sigma_1}^2}\int_{-\infty}^\infty dx_2dx_3dx_4 \times \delta(x_1+x_2-x_3-x_4)n_2\bar{n}_3\bar{n}_4\int_\Delta^{\pi-\Delta} d\theta\,\frac{W_f^{\sigma_1\sigma_1}}{\sin\theta}, \tag{2.15}$$

where here $\Delta = k_BT/\epsilon_1$. The energy integrals are evaluated by Morel and Nozières [23] and the final result is $\pi^2/4$.

The transition rates can be Fourier analyzed as usual [14] yielding

$$W_{f,b}^{\sigma\sigma'}(\theta) = \sum_{\ell=0}^\infty \alpha_\ell T_\ell(\cos\theta)W_{\ell\,f,b}^{\sigma\sigma'}, \tag{2.16}$$

where the $T_\ell(\cos\theta) = \cos(\ell\theta)$ are Chebyshev polynomials of the first kind [24], and the parameters $\alpha_0 = 1$ and $\alpha_\ell = 2$ for $\ell \geqslant 1$. We can now introduce the lowest order "$\ell = 0$" approximation by replacing the full transition rate by its $\ell = 0$ value $W_f^{\sigma_1\sigma_1} \approx W_{f,0}^{\sigma_1\sigma_1}$. The remaining angular integral can now be evaluated, and the final result is

$$\frac{1}{\tau_{\sigma_1\sigma_1}} = \frac{\pi^2}{2}\frac{\left(m_{\sigma_1}^*\right)^2}{h^4}W_{f,0}^{\sigma_1\sigma_1}\frac{(k_BT)^2}{\epsilon_1}\ln\left(\frac{2\epsilon_1}{k_BT}\right). \tag{2.17}$$

The calculation for antiparallel spins proceeds analogously. From Eq. (2.4):

$$\frac{1}{\tau_{\sigma_1-\sigma_1}} = \frac{2m_{\sigma_1}^*m_{-\sigma_1}^*}{h^4}\int_0^\infty d\epsilon_2\,d\epsilon_3\int_0^\pi d\theta \times\left(\int_0^\pi d\theta_3\,W_b^{\sigma_1-\sigma_1} + \int_\pi^{2\pi} d\theta_3\,W_f^{\sigma_1-\sigma_1}\right) \times \delta(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4)n_2\bar{n}_3\bar{n}_4. \tag{2.18}$$

We now use (2.12) for antiparallel spins:

$$d\theta_3 = \frac{\pm dp_4^2}{2p_1p_2\sqrt{\sin^2\theta - \frac{1}{\epsilon_{12}}\xi_3}}, \tag{2.19}$$

where $-$ is for forward scattering $\pi < \theta_3 \leqslant 2\pi$, and $+$ is for backward scattering $0 < \theta_3 \leqslant \pi$. Performing the energy integrals yields

$$\frac{1}{\tau_{\sigma_1-\sigma_1}} = \frac{\pi^2m_{\sigma_1}^*\left(m_{-\sigma_1}^*\right)^2}{2h^4p_{\sigma_1}p_{-\sigma_1}}(k_BT)^2 \times\int_\Delta^{\pi-\Delta}\left[W_f^{\sigma_1-\sigma_1}+W_b^{\sigma_1-\sigma_1}\right]\frac{d\theta}{\sin\theta}, \tag{2.20}$$

where in this case $\Delta = \sqrt{k_BT/|\epsilon_{12}|}$. Utilizing the $\ell = 0$ approximation we obtain

$$\frac{1}{\tau_{\sigma_1-\sigma_1}} = \frac{\pi^2}{2h^4}\frac{m_{\sigma_1}^*\left(m_{-\sigma_1}^*\right)^2}{p_{\sigma_1}p_{-\sigma_1}}\left(W_{f,0}^{\sigma_1-\sigma_1}+W_{b,0}^{\sigma_1-\sigma_1}\right)(k_BT)^2 \times\ln\left(\frac{4|\epsilon_{12}|}{k_BT}\right). \tag{2.21}$$

By combining (2.17) and (2.21) we obtain the total quasiparticle-quasiparticle collision frequency at finite polarization:

$$\frac{1}{\tau_{\sigma_1}} = \frac{\pi^2}{2h^4}\left[\left(m_{\sigma_1}^*\right)^2W_{f,0}^{\sigma_1\sigma_1}\ln\left(\frac{2\epsilon_1}{k_BT}\right) + \left(m_{-\sigma_1}^*\right)^2\frac{p_{\sigma_1}}{p_{-\sigma_1}}\frac{1}{2}\left(W_{f,0}^{\sigma_1-\sigma_1}+W_{b,0}^{\sigma_1-\sigma_1}\right)\ln\left(\frac{4|\epsilon_{12}|}{k_BT}\right)\right] \times\frac{(k_BT)^2}{\epsilon_1}. \tag{2.22}$$

In this expression the momenta and energies are zero-temperature Fermi momenta and Fermi energies [see also Eq. (2.13)].

1. Zero polarization

At zero polarization, Eq. (2.17) for the spin-parallel collision frequency is still valid. However for the spin-antiparallel collision frequency $1/\epsilon_{12} = 0$ in the zero-polarization limit, and thus the leading order correction in the denominator of $d\theta_3$ is quadratic. For zero polarization Eq. (2.21) becomes

$$\frac{1}{\tau_{\sigma-\sigma}} = \frac{\pi^2(m^*)^2}{2h^4}\left(W_{f,0}^{\sigma-\sigma}+W_{b,0}^{\sigma-\sigma}\right)\frac{(k_BT)^2}{\epsilon_F}\ln\left(\frac{2\epsilon_F}{k_BT}\right). \tag{2.23}$$

By combining this result with (2.17) we obtain the total zero-polarization collision frequency:

$$\frac{1}{\tau_0} \equiv \frac{1}{\tau_\sigma}(\mathcal{P}=0) = \frac{\pi^2(m^*)^2}{2h^4}\left[W_{f,0}^{\sigma\,\sigma} + W_{f,0}^{\sigma\,-\sigma} + W_{b,0}^{\sigma\,-\sigma}\right] \times \frac{(k_BT)^2}{\epsilon_F}\ln\left(\frac{2\epsilon_F}{k_BT}\right), \tag{2.24}$$

where the τ_0 notation will be used below. The same quantity in Ref. [14], Eq. (3.24), differs by the appearance of a factor of $3/8$ in the leading coefficient instead of $1/2$. This is not a problem since in that derivation the coefficient of the log term is uncertain with regard to factors of $O(1)$ because of the vagaries of the low-temperature limiting process. We note that one cannot obtain (2.24) by taking the zero-polarization limit of (2.22).

2. *Full polarization*

In the full-polarization limit one simply omits the contribution from the antiparallel spins in (2.22):

$$\frac{1}{\tau_1} \equiv \frac{1}{\tau_\uparrow}(\mathcal{P}=1) = \frac{\pi^2(m_\uparrow^*)^2}{2h^4}W_{f,0}^{\uparrow\uparrow}\frac{(k_BT)^2}{\epsilon_{F\uparrow}}\ln\left(\frac{2\epsilon_{F\uparrow}}{k_BT}\right). \tag{2.25}$$

III. TRANSPORT

The derivation of the transport coefficients in a two-dimensional Fermi liquid proceeds in a very similar way to that in three dimensions. Thus, this and the following sections on transport coefficients will necessarily be brief. For the details we refer the reader to Baym and Pethick [6] for example. We shall concentrate on those aspects that are specific to finite polarization and two dimensions. The general transport equation can be written

$$\frac{\partial\tilde{n}_{\mathbf{p}\sigma}}{\partial t} + \boldsymbol{\nabla}\tilde{n}_{\mathbf{p}\sigma}\cdot\boldsymbol{\nabla}_{\mathbf{p}}\tilde{\epsilon}_{\mathbf{p}\sigma} - \boldsymbol{\nabla}_{\mathbf{p}}\tilde{n}_{\mathbf{p}\sigma}\cdot\boldsymbol{\nabla}\tilde{\epsilon}_{\mathbf{p}\sigma} = I_{\mathbf{p}\sigma}, \tag{3.1}$$

where $\tilde{n}_{\mathbf{p}\sigma}(\mathbf{r})$ is the local quasiparticle distribution function defined with the local quasiparticle energies $\tilde{\epsilon}_{\mathbf{p}\sigma}(\mathbf{r})$. As usual the local quasiparticle distribution function is expanded around *local equilibrium* [5,6]:

$$\tilde{n}_{\mathbf{p}\sigma}(\mathbf{r}) = n_{\mathbf{p}\sigma}[\tilde{\epsilon}_{\mathbf{p}\sigma}(\mathbf{r})] + \delta n_{\mathbf{p}\sigma}(\mathbf{r}), \tag{3.2}$$

where $n_{\mathbf{p}\sigma}$ is the Fermi distribution function. Equivalently, we can expand the local quasiparticle energies around a set of local equilibrium energies $\epsilon_{\mathbf{p}\sigma}(\mathbf{r})$:

$$\tilde{\epsilon}_{\mathbf{p}\sigma}(\mathbf{r}) = \epsilon_{\mathbf{p}\sigma} + \sum_{\mathbf{p}'} f_{\mathbf{p}\mathbf{p}'}^{\sigma\sigma'}\delta n_{\mathbf{p}'\sigma'}(\mathbf{r}). \tag{3.3}$$

The collision integral on the right-hand side of the transport equation can be written

$$I(n_1) = -\sum_{\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4} W(1,2;3,4)\delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_3+\mathbf{p}_4}\delta_{\sigma_1+\sigma_2,\sigma_3+\sigma_4} \times \delta(\tilde{\epsilon}_1+\tilde{\epsilon}_2-\tilde{\epsilon}_3-\tilde{\epsilon}_4)[\tilde{n}_1\tilde{n}_2(1-\tilde{n}_3)(1-\tilde{n}_4) - (1-\tilde{n}_1)(1-\tilde{n}_2)\tilde{n}_3\tilde{n}_4]. \tag{3.4}$$

As discussed in Sec. II, the sums over $\mathbf{p}_3$ and $\mathbf{p}_4$ include only distinguishable final states. We now expand the collision integral (3.4) to first order in the $\delta n_{\mathbf{p}\sigma}(\mathbf{r})$. Consider the products of distribution functions in the square brackets of (3.4), and substitute (3.2). This yields

$$[\ldots] = -\beta(\zeta_1+\zeta_2-\zeta_3-\zeta_4)n_1n_2\overline{n}_3\overline{n}_4, \tag{3.5}$$

where we have defined

$$\delta n_i \equiv \frac{\partial n_i}{\partial\epsilon_i}\zeta_i = -\beta n_i\overline{n}_i\zeta_i, \tag{3.6}$$

and we have made use of the identity

$$[n_1n_2\overline{n}_3\overline{n}_4 - \overline{n}_1\overline{n}_2n_3n_4]\delta(\tilde{\epsilon}_1+\tilde{\epsilon}_2-\tilde{\epsilon}_3-\tilde{\epsilon}_4) = 0. \tag{3.7}$$

Performing the spin sums the collision integral becomes

$$I(n_1) = \beta\sum_{\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4}\left[\frac{1}{2}W^{\sigma_1\sigma_1}(\theta) + W^{\sigma_1-\sigma_1}(\theta)\right] \times \delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_3+\mathbf{p}_4}\delta(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4) \times (\zeta_1+\zeta_2-\zeta_3-\zeta_4)n_1n_2\overline{n}_3\overline{n}_4, \tag{3.8}$$

where to lowest order we have now replaced all of the remaining local energies $\tilde{\epsilon}$ by local equilibrium energies ϵ. As discussed in Sec. II the value of θ_3 determines whether the transition probability is given by the forward or backward scattering diagrams in Figs. 1 and 2, respectively. Equation (3.8) is the starting point for the collision integral for the calculation of the transport coefficients as discussed below.

A. Thermal conductivity

The thermal current is given by

$$\boldsymbol{j}_{th} = \sum_\sigma\int\frac{d\mathbf{k}}{h^2}\,\delta n_{\mathbf{k}\sigma}(\epsilon_{\mathbf{k}\sigma}-\mu_\sigma)\mathbf{v}_{\mathbf{k}\sigma}, \tag{3.9}$$

where

$$\delta n_{\mathbf{k}\sigma} = \frac{\partial n_{\mathbf{k}\sigma}}{\partial\epsilon_{\mathbf{k}\sigma}}\mathbf{v}_{\mathbf{k}\sigma}\cdot\boldsymbol{\nabla}(k_BT)q_\sigma(\epsilon_{\mathbf{k}\sigma}), \tag{3.10}$$

and Eq. (3.10) defines the function q_σ. By definition the thermal conductivity is given by $\boldsymbol{j}_{th} = -\kappa\boldsymbol{\nabla}T$; thus we obtain

$$\kappa = -\frac{\pi k_B^2}{h^2}T\sum_\sigma m_\sigma^*v_{F\sigma}^2\int_{-\infty}^{\infty}dx\frac{\partial n}{\partial x}\,q_\sigma(x)\,x. \tag{3.11}$$

The integral over q_σ can be evaluated exactly from the kinetic equation by using the results of Sykes and Brooker [12]. In Eq. (3.8) we have expanded the collision integral (3.4) to linear order in the nonequilibrium distribution functions $\delta n_{\mathbf{p}\sigma}(\mathbf{r})$. Now we need to proceed to do the same for the left-hand side of (3.1). Since the system is assumed to be in steady state we can immediately set $\frac{\partial\tilde{n}_{\mathbf{p}\sigma}}{\partial t} = 0$. We systematically ignore the gradients of $\delta n_{\mathbf{p}\sigma}(\mathbf{r})$. Then with $\mathbf{u} = 0$ and no polarization gradient we have

$$\boldsymbol{\nabla}\tilde{n}_{\mathbf{p}\sigma} \simeq \frac{\partial n_{\mathbf{p}\sigma}}{\partial\epsilon_{\mathbf{p}\sigma}}\boldsymbol{\nabla}\tilde{\epsilon}_{\mathbf{p}\sigma} + \frac{\partial n_{\mathbf{p}\sigma}}{\partial T}\boldsymbol{\nabla}T(\mathbf{r}), \tag{3.12}$$

$$\boldsymbol{\nabla}_{\mathbf{p}}\tilde{n}_{\mathbf{p}\sigma} \simeq \frac{\partial n_{\mathbf{p}\sigma}}{\partial\epsilon_{\mathbf{p}\sigma}}\boldsymbol{\nabla}_{\mathbf{p}}\tilde{\epsilon}_{\mathbf{p}\sigma}, \tag{3.13}$$

and thus the left-hand side of (3.8) becomes $\frac{\partial n_{\mathbf{p}\sigma}}{\partial T}\boldsymbol{\nabla}T(\mathbf{r})\cdot\mathbf{v}_{\mathbf{p}\sigma} = \beta(\epsilon_1-\mu_1)\frac{n_1\overline{n}_1}{T}(\boldsymbol{\nabla}T)\cdot\mathbf{v}_1$. To lowest order $\boldsymbol{\nabla}_{\mathbf{p}}\tilde{\epsilon}_{\mathbf{p}\sigma} \approx \boldsymbol{\nabla}_{\mathbf{p}}\epsilon_{\mathbf{p}\sigma} = \mathbf{v}_{\mathbf{p}\sigma}$ is the Fermi velocity.

We note that we have omitted a $\nabla\mu$ contribution in the expansion. This term, proportional to the entropy, was shown by Sykes and Brooker to contribute only to the even part of q, and they showed that this is of higher order in temperature than the odd part. It will be pointed out below that our $q_\sigma(x)$'s have odd symmetry.

Using (3.6) and (3.10) we find

$$\zeta_i = \mathbf{v}_i \cdot \nabla(k_B T) q_\sigma(\epsilon_i). \tag{3.14}$$

The kinetic equation can then be written

$$\beta(\epsilon_1-\mu_1)n_1\overline{n}_1(\mathbf{v}_1\cdot\nabla T) = \sum_{\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4,\sigma_1'} \delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_3+\mathbf{p}_4}\delta(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4)n_1n_2\overline{n}_3\overline{n}_4$$
$$\times W^{\sigma_1\sigma_1'}(\theta)\big[(\mathbf{v}_1\cdot\nabla T)q_{\sigma_1}(\epsilon_1)+(\mathbf{v}_2\cdot\nabla T)q_{\sigma_1'}(\epsilon_2)-(\mathbf{v}_3\cdot\nabla T)q_{\sigma_1}(\epsilon_3)-(\mathbf{v}_4\cdot\nabla T)q_{\sigma_1'}(\epsilon_4)\big], \tag{3.15}$$

where it is understood that in the case of spin-parallel scattering the range of the θ_3 integration is restricted from π to 2π. It is straightforward to show that the angle between $\mathbf{v}_1$ and ∇T is arbitrary. Define γ such that $\mathbf{v}_1\cdot\nabla T = v_1|\nabla T|\cos\gamma$. Then similarly $\mathbf{v}_i\cdot\nabla T = v_i|\nabla T|\cos(\gamma+\theta_{i1})$, where $i=2,3,4$. After expanding the trigonometric functions and using the symmetry under transformation $\theta\to 2\pi-\theta$, the $\cos\gamma$ cancels out on both sides. We then divide through by v_1, and transform the momentum integrations into energy and angular integrals as was done for the collision time in the previous section:

$$\begin{aligned}
&\beta(\epsilon_1-\mu_1)n_1\overline{n}_1 \\
&= \frac{2}{h^4}\int_0^\infty d\epsilon_2 d\epsilon_3 d\epsilon_4\,\delta(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4)n_1n_2\overline{n}_3\overline{n}_4 \\
&\times\Bigg\{\frac{(m^*_{\sigma_1})^3}{p^2_{\sigma_1}}\int_0^\pi d\theta\frac{W_f^{\sigma_1\sigma_1}(\theta)}{\sqrt{\sin^2\theta-\left(\frac{\xi_3}{\epsilon_{F\sigma_1}}\right)^2}}\big[q_{\sigma_1}(\epsilon_1)+q_{\sigma_1}(\epsilon_2)\cos\theta-q_{\sigma_1}(\epsilon_3)-q_{\sigma_1}(\epsilon_4)\cos\theta\big] \\
&+\frac{m^*_{\sigma_1}(m^*_{-\sigma_1})^2}{p_{\sigma_1}p_{-\sigma_1}}\int_0^\pi d\theta\frac{W_f^{\sigma_1-\sigma_1}(\theta)}{\sqrt{\sin^2\theta-\left(\frac{\xi_3}{\epsilon_{12}}\right)}}\left[q_{\sigma_1}(\epsilon_1)+\frac{v_{-\sigma_1}}{v_{\sigma_1}}q_{-\sigma_1}(\epsilon_2)\cos\theta-q_{\sigma_1}(\epsilon_3)-\frac{v_{-\sigma_1}}{v_{\sigma_1}}q_{-\sigma_1}(\epsilon_4)\cos\theta\right] \\
&+\frac{m^*_{\sigma_1}(m^*_{-\sigma_1})^2}{p_{\sigma_1}p_{-\sigma_1}}\int_0^\pi d\theta\frac{W_b^{\sigma_1-\sigma_1}(\theta)}{\sqrt{\sin^2\theta-\left(\frac{\xi_3}{\epsilon_{12}}\right)}}\left[q_{\sigma_1}(\epsilon_1)+\frac{v_{-\sigma_1}}{v_{\sigma_1}}q_{-\sigma_1}(\epsilon_2)\cos\theta-q_{\sigma_1}(\epsilon_3)\cos\Phi_3^b-\frac{v_{-\sigma_1}}{v_{\sigma_1}}q_{-\sigma_1}(\epsilon_4)\cos\Phi_4^b\right]\Bigg\},
\end{aligned} \tag{3.16}$$

where we have used the symmetry in the integrals about $\theta=\pi$, and set $\Phi_3^f=0$ and $\Phi_4^f=\theta$; ξ_3 and ϵ_{12} are defined in Eqs. (2.11) and (2.13), respectively. At this point we note that to lowest order in temperature the terms in (3.16) that are proportional to $\cos\theta$ can be omitted.

Following the same line of argument as in three dimensions, we introduce dimensionless variables, $x_i\equiv\beta(\epsilon_i-\mu_i)$, and then let $x_i\to -x_i$. We now see that $q_\sigma(x_i)$ is an *odd* function of its argument, $q_\sigma(x_i)=-q_\sigma(-x_i)$, in lowest order of temperature for the thermal conductivity. We now relabel the variables $x_3,x_4\to x_2$ due to symmetry under the integral, and we obtain

$$x_1n_1\overline{n}_1 = \int_{-\infty}^\infty dx_2dx_3dx_4\delta(x_1+x_2+x_3+x_4)n_1n_2n_3n_4\left\{\frac{4}{\pi^2}\frac{1}{\tau_\sigma}[q_\sigma(x_1)+q_\sigma(x_2)]-\frac{4}{\pi^2}\nu_\sigma q_\sigma(x_2)+\frac{4}{\pi^2}\left(\frac{\overline{n}_{-\sigma}}{\overline{n}_\sigma}\right)\nu_{-\sigma}q_{-\sigma}(x_2)\right\}, \tag{3.17}$$

where for simplicity in notation we have switched from σ_1 to σ. The collision time is taken from Eq. (2.22):

$$\frac{4}{\pi^2}\frac{1}{\tau_\sigma} = \frac{2}{h^4}(k_BT)^2\left[\frac{(m^*_\sigma)^3}{p^2_\sigma}\int_0^\pi d\theta\frac{W_f^{\sigma\sigma}(\theta)}{\sqrt{\sin^2\theta-(\frac{\xi_3}{\epsilon_{F\sigma}})^2}}+\frac{m^*_\sigma(m^*_{-\sigma})^2}{p_\sigma p_{-\sigma}}\int_0^\pi d\theta\frac{W_f^{\sigma-\sigma}(\theta)+W_b^{\sigma-\sigma}(\theta)}{\sqrt{\sin^2\theta-(\frac{\xi_3}{\epsilon_{12}})}}\right], \tag{3.18}$$

and following MM we have introduced generalized frequencies:

$$\frac{4}{\pi^2}\nu_\sigma \equiv \frac{2}{h^4}(k_BT)^2\frac{m^*_\sigma(m^*_{-\sigma})^2}{p_\sigma p_{-\sigma}}\int_0^\pi d\theta\frac{W_b^{\sigma-\sigma}(\theta)}{\sqrt{\sin^2\theta-\left(\frac{\xi_3}{\epsilon_{12}}\right)}}\frac{2p^2_{-\sigma}\sin^2\theta}{\ell^2}, \tag{3.19a}$$

$$\frac{4}{\pi^2}\left(\frac{p_{-\sigma}}{p_\sigma}\right)^2\nu_{-\sigma} \equiv \frac{2}{h^4}(k_BT)^2\frac{m^*_\sigma(m^*_{-\sigma})^2}{p_\sigma p_{-\sigma}}\int_0^\pi d\theta\frac{W_b^{\sigma-\sigma}(\theta)}{\sqrt{\sin^2\theta-\left(\frac{\xi_3}{\epsilon_{12}}\right)}}\left[\frac{v_{-\sigma}}{v_\sigma}\frac{2p_\sigma p_{-\sigma}\sin^2\theta}{\ell^2}\right], \tag{3.19b}$$

in which we have replaced Φ_3^b and Φ_4^b in Eq. (3.16) with the use of Eq. (2.3).

In the third term in curly brackets in (3.17) we have introduced the notation $\overline{n}_\sigma$ for the *areal density* of the σth Fermi sea, and this should not be confused with the similar-looking Fermi distribution function (2.2) whose subscript is a momentum label and not a Fermi sea label; thus

$$\overline{n}_\sigma \equiv \frac{N_\sigma}{A}. \tag{3.20}$$

The energy integral can be found in Appendix A of Baym and Pethick [6]:

$$\int_{-\infty}^{\infty} dx_2\, dx_3\, dx_4\, \delta(x_1+x_2+x_3+x_4)\, n_2 n_3 n_4 = \frac{1}{2}\frac{x_1^2+\pi^2}{1+e^{-x_1}}. \tag{3.21}$$

Thus, the kinetic equation can be brought into nondiagonal Sykes-Brooker form:

$$\left(\frac{\pi^2}{4}\right)\tau_\sigma x_1 = \int_{-\infty}^{\infty} dx_2\, K(x_1,x_2)\Big[q_\sigma(x_1) - (1-\nu_\sigma\tau_\sigma)q_\sigma(x_2) - \left(\frac{\overline{n}_{-\sigma}}{\overline{n}_\sigma}\nu_{-\sigma}\tau_\sigma\right)q_{-\sigma}(x_2)\Big], \tag{3.22}$$

where the Sykes-Brooker kernel is defined as

$$K(x_1,x_2) = \frac{(1+e^{-x_1})(x_2-x_1)}{(1+e^{-x_2})(e^{x_2-x_1}-1)}. \tag{3.23}$$

This kinetic equation mixes the two components of q_σ. It is in very similar form as the kinetic equation for spin diffusion as derived by MM. We rewrite (3.22) with a matrix representation of the coefficients of $q_\sigma(x_2)$:

$$\left(\frac{\pi^2}{4}\right)\tau_\sigma x_1 = \int_{-\infty}^{\infty} dx_2\, K(x_1,x_2)\Big[q_\sigma(x_1) - \sum_{\sigma'}\lambda_{\sigma\sigma'}q_{\sigma'}(x_2)\Big]. \tag{3.24}$$

The coefficient matrix is given by

$$\lambda = \begin{pmatrix} 1-\nu_\uparrow\tau_\uparrow & \frac{\overline{n}_\downarrow}{\overline{n}_\uparrow}\nu_\downarrow\tau_\uparrow \\ \frac{\overline{n}_\uparrow}{\overline{n}_\downarrow}\nu_\uparrow\tau_\downarrow & 1-\nu_\downarrow\tau_\downarrow \end{pmatrix}. \tag{3.25}$$

The matrix diagonalization proceeds by using the general method described in Anderson, Pethick, and Quader [7]. We note that in this case λ is not symmetric. The eigenvalues of λ are

$$\lambda_+ = 1, \quad \lambda_- = 1-(\nu_\uparrow\tau_\uparrow+\nu_\downarrow\tau_\downarrow). \tag{3.26}$$

The $\pm$ subscripts on the λ's refer to the plus and minus roots of the quadratic equation generated by diagonalizing (3.25). Thus, the plus and minus subscripts label the top and bottom rotated spin states, respectively.

We introduce transformed variables $\tilde{\tau} = S\tau$ and $\tilde{q} = Sq$ where the transformation matrix S is given by

$$S = \frac{1}{\nu_\uparrow\tau_\uparrow+\nu_\downarrow\tau_\downarrow}\begin{pmatrix} \frac{\tau_\downarrow}{\overline{n}_\downarrow} & \frac{\tau_\uparrow}{\overline{n}_\uparrow} \\ \frac{\nu_\uparrow}{\overline{n}_\downarrow} & -\frac{\nu_\downarrow}{\overline{n}_\uparrow} \end{pmatrix}. \tag{3.27}$$

In terms of these variables the diagonalized pair of kinetic equations are

$$\frac{\pi^2}{4}\tilde{\tau}_\sigma x_1 = \int_{-\infty}^{\infty} dx_2\, K(x_1,x_2)[\tilde{q}_\sigma(x_1) - \lambda_\sigma\tilde{q}_\sigma(x_2)]. \tag{3.28}$$

As shown in Eq. (3.11), the important quantity is not $\tilde{q}_\sigma$ itself but rather the integrated quantity:

$$\tilde{Q}_\sigma = -\int_{-\infty}^{\infty} dx\frac{\partial n}{\partial x}\tilde{q}_\sigma(x)x = \frac{\pi^2\tilde{\tau}_\sigma}{2(3-\lambda_\sigma)}H(\lambda_\sigma), \tag{3.29}$$

where from Sykes and Brooker (60) $H(\lambda)$ is an infinite series involving the eigenvalues $\lambda_\pm$ that will be explicitly written below. We now inverse-transform the $\tilde{Q}_\sigma$ to obtain

$$Q_\sigma = \frac{(\pi^2/2)}{\nu_\sigma\tau_\sigma+\nu_{-\sigma}\tau_{-\sigma}}\left[\nu_{-\sigma}\tau_{-\sigma}\left(1+\frac{\overline{n}_{-\sigma}}{\overline{n}_\sigma}\right)\frac{H(\lambda_+)}{3-\lambda_+} + \frac{\overline{n}_{-\sigma}}{\overline{n}_\sigma}\nu_{-\sigma}\tau_{-\sigma}\frac{H(\lambda_-)}{3-\lambda_-}\right]\tau_\sigma. \tag{3.30}$$

The exact solution for the low-temperature thermal conductivity in two dimensions can be written

$$\kappa = \frac{\pi k_B^2}{h^2}T\sum_\sigma m_\sigma^* v_{F\sigma}^2 Q_\sigma, \tag{3.31}$$

where the explicit expression for $H(\lambda)$ is given by

$$H(\lambda) = \frac{3-\lambda}{4}\sum_{n=0}^{\infty}\frac{4n+5}{(n+1)(2n+3)[(n+1)(2n+3)-\lambda]}. \tag{3.32}$$

According to Eqs. (2.22) and (3.19) we find $\nu_\sigma\tau_\sigma \sim 1/\ln T$. Thus, at very low temperatures we can set $\lambda_- \approx \lambda_+ = 1$. In this limit then Eq. (3.30) simplifies to $Q_\sigma = \frac{\pi^2}{4}H(1)\tau_\sigma$, and the thermal conductivity becomes

$$\kappa = \frac{\pi^3}{2h^2}k_B^2 T H(1)\sum_\sigma \epsilon_{F\sigma}\tau_\sigma. \tag{3.33}$$

1. Zero polarization

At zero polarization the eigenvalues are $\lambda_+ = 1$ and $\lambda_- = 1-2\nu\tau_0$. By inspection of (3.30) we have $Q_\uparrow = Q_\downarrow = (\pi^2/4)H(1)\tau_0$ and therefore

$$\kappa(\mathcal{P}=0) = \frac{\pi^3}{h^2}k_B^2 T\epsilon_F H(1)\tau_0, \tag{3.34}$$

in agreement with (3.33). In the $\ell = 0$ approximation this becomes

$$\kappa(\mathcal{P}=0) = \pi h^2 k_B\frac{v_F^2}{m^*}\frac{H(1)}{[W_{f,0}^{\uparrow\uparrow}+W_{f,0}^{\uparrow\downarrow}+W_{b,0}^{\uparrow\downarrow}]}\frac{(\epsilon_F/k_BT)}{\ln(2\epsilon_F/k_BT)}. \tag{3.35}$$

2. Full polarization

At full polarization all quasiparticles are in the $\uparrow$ Fermi sea, and thus the terms with the spin-antiparallel contributions $\nu_\uparrow$ and $\nu_\downarrow$ do not appear. By inspection of the kinetic equation (3.24) the eigenvalue $\lambda = 1$. From (3.30) we have

$Q_\uparrow = [\pi^2 H(\lambda)/2(3-\lambda)]\tau_1$ and therefore

$$\kappa(\mathcal{P}=1) = \frac{\pi^3}{2h^2} k_B^2 T \epsilon_{F\uparrow} H(1)\tau_1, \tag{3.36}$$

in agreement with (3.33).

Summary

We find that the temperature dependence for the thermal conductivity at arbitrary polarization $0 \leqslant \mathcal{P} \leqslant 1$ is $\kappa^{-1} \sim T \ln T$. This is in agreement with the zero-polarization results of Fu and Ebner [17].

B. Shear viscosity

We consider a Fermi-liquid film flowing with speed u_x in the x direction. The flow is not uniform. There exists a small nonzero y gradient of the velocity $\partial u_x/\partial y$ that will drive a transverse momentum flux σ_{xy}. The coefficient of proportionality η is the first or shear viscosity:

$$\sigma_{xy} = \eta \frac{\partial u_x}{\partial y}. \tag{3.37}$$

The stress tensor σ_{xy} can be written in terms of the nonequilibrium part of the distribution function [6]:

$$\sigma_{xy} = -\sum_\sigma \int \frac{d\mathbf{p}}{h^2}\, p_x \left(\frac{\partial \epsilon_{\mathbf{p}\sigma}}{\partial p_y}\right) \delta n_{\mathbf{p}\sigma}; \tag{3.38}$$

we note that $v_{\mathbf{p}\sigma,y} = \partial\epsilon_{\mathbf{p}\sigma}/\partial p_y$. Then using (3.6) we can write $\delta n_{\mathbf{p}\sigma}$ in terms of the driving field:

$$\delta n_i \equiv \frac{\partial n_i}{\partial \epsilon_i}\zeta_i = \frac{\partial n_i}{\partial \epsilon_i}\frac{1}{2}(p_{ix}v_{iy} + p_{iy}v_{ix})\frac{\partial u_x}{\partial y} q_\sigma(\epsilon_i), \tag{3.39}$$

where from symmetry, $p_{ix}v_{iy} = p_{iy}v_{ix} = m_i^* v_{ix}v_{iy}$. For the shear viscosity we will need the first two terms of (3.39) in powers of $(\epsilon_{\mathbf{p}\sigma} - \mu)$. Following Sykes and Brooker [12] we obtain

$$\sigma_{xy} = -\left[\sum_\sigma \int \frac{d\mathbf{p}}{h^2}(p_x v_{\mathbf{p}\sigma,y})\frac{\partial n_{\mathbf{p}\sigma}}{\partial \epsilon_{\mathbf{p}\sigma}}\left(1 + \frac{x_\sigma}{\beta\epsilon_{F\sigma}}\right) \times m_\sigma^* v_{\sigma,x} v_{\sigma,y} q_\sigma(\epsilon_{\mathbf{p}\sigma})\right]\frac{\partial u_x}{\partial y}, \tag{3.40}$$

where $x_\sigma = \beta(\epsilon_{\mathbf{p}\sigma} - \mu)$. The shear viscosity is then

$$\eta = -\frac{\pi}{4h^2}\sum_\sigma \frac{(p_{F\sigma})^4}{m_\sigma^*}\int_{-\infty}^{\infty} dx \left(1 + \frac{x_\sigma}{\beta\epsilon_{F\sigma}}\right)\frac{\partial n}{\partial x} q_\sigma(x). \tag{3.41}$$

We can resolve $q_\sigma(x)$ into symmetric and antisymmetric components: $q_\sigma(x) = q_\sigma^{(s)}(x) + q_\sigma^{(a)}(x)$. Because of the even and odd symmetry of the two components of the integrand, we can write $\eta = \eta^{(s)} + \eta^{(a)}$, where

$$\eta^{(s)} = -\frac{\pi}{4h^2}\sum_\sigma \frac{(p_{F\sigma})^4}{m_\sigma^*}\int_{-\infty}^{\infty} dx\, \frac{\partial n}{\partial x} q_\sigma^{(s)}(x), \tag{3.42a}$$

$$\eta^{(a)} = -\frac{\pi}{4h^2}\sum_\sigma \frac{(p_{F\sigma})^4}{m_\sigma^*}\frac{1}{\beta\epsilon_{F\sigma}}\int_{-\infty}^{\infty} dx\, x\frac{\partial n}{\partial x} q_\sigma^{(a)}(x). \tag{3.42b}$$

As in the previous section, the integrals over q_σ can be evaluated exactly [12]. It is straightforward to show that $\eta^{(a)}/\eta^{(s)} \sim O(T^2)$, the same as in three dimensions, and thus we can ignore the contributions of the antisymmetric part of q_σ in the remainder of this discussion.

In the absence of thermal or polarization gradients, the left-hand side of the kinetic equation (3.1) in leading order reduces to [12] $-(\frac{\partial n_{\mathbf{p}\sigma}}{\partial \epsilon_{\mathbf{p}\sigma}})\frac{1}{2}[p_x v_{\mathbf{p}\sigma,y} + p_y v_{\mathbf{p}\sigma,x}]\frac{\partial u_x}{\partial y}$. Then with (3.39), the kinetic equation becomes

$$\begin{aligned} n_1\overline{n}_1 v_{1y} p_{1x} = \sum_{\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4} & \delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_3+\mathbf{p}_4}\delta(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4) n_1 n_2 \overline{n}_3 \overline{n}_4 \\ & \times \big\{ W_f^{\sigma_1\sigma_1}(\theta)[v_{1y}p_{1x}q_{\sigma_1}(\epsilon_1) + v_{2y}p_{2x}q_{\sigma_1}(\epsilon_2) - v_{3y}p_{3x}q_{\sigma_1}(\epsilon_3) - v_{4y}p_{4x}q_{\sigma_1}(\epsilon_4)] \\ & + W_f^{\sigma_1-\sigma_1}(\theta)[v_{1y}p_{1x}q_{\sigma_1}(\epsilon_1) + v_{2y}p_{2x}q_{-\sigma_1}(\epsilon_2) - v_{3y}p_{3x}q_{\sigma_1}(\epsilon_3) - v_{4y}p_{4x}q_{-\sigma_1}(\epsilon_4)] \\ & + W_b^{\sigma_1-\sigma_1}(\theta)[v_{1y}p_{1x}q_{\sigma_1}(\epsilon_1) + v_{2y}p_{2x}q_{-\sigma_1}(\epsilon_2) - v_{3y}p_{3x}q_{\sigma_1}(\epsilon_3) - v_{4y}p_{4x}q_{-\sigma_1}(\epsilon_4)]\big\}, \end{aligned} \tag{3.43}$$

where we have canceled out a common factor of $\partial u_x/\partial y$. This is similar to the kinetic equation for the thermal conductivity (3.15) except that in this system both the x and y directions play special roles. Thus we need to include information as to the directions of the momenta with respect to the x direction, say. Angle γ is the angle between $\mathbf{p}_1$ and the x-axis: $p_{1x} = p_1\cos\gamma$, and $v_{1y} = v_1\sin\gamma$. Similarly $v_{iy}p_{ix} = v_i p_i \sin(\gamma+\theta_{i1})\cos(\gamma+\theta_{i1})$ for $i = 2,3,4$. Like in the case of thermal conductivity, after expanding the trigonometric functions the factors $\sin\gamma\cos\gamma$ will cancel out from both sides of the equation. The right-hand side of the equation can be cast into a product of energy and angular integrals as was done in the previous section, and (3.43) becomes

$$\begin{aligned} n_1\overline{n}_1 = (k_B T)^2 \frac{2}{h^4}\int_{-\infty}^{\infty} dx_2 dx_3 dx_4\, \delta(x_1+x_2-x_3-x_4) n_1 n_2 \overline{n}_3 \overline{n}_4 \\ \times \left\{ \frac{(m_{\sigma_1}^*)^3}{p_{\sigma_1}^2}\int_0^\pi d\theta \frac{W_f^{\sigma_1\sigma_1}(\theta)}{\sqrt{\sin^2\theta - \left(\frac{\xi_3}{\epsilon_{F\sigma_1}}\right)^2}}\left[q_{\sigma_1}(x_1) + q_{\sigma_1}(x_2)\cos(2\theta) - q_{\sigma_1}(x_3)\cos\left(2\Phi_3^f\right) - q_{\sigma_1}(x_4)\cos\left(2\Phi_4^f\right)\right] \end{aligned}$$

$$+\frac{m^*_{\sigma_1}\left(m^*_{-\sigma_1}\right)^2}{p_{\sigma_1}p_{-\sigma_1}}\int_0^\pi d\theta\,\frac{W_f^{\sigma_1-\sigma_1}(\theta)}{\sqrt{\sin^2\theta-\left(\frac{\xi_3}{\epsilon_{12}}\right)}}\left[q_{\sigma_1}(x_1)+D_{\sigma_1}q_{-\sigma_1}(x_2)\cos(2\theta)-q_{\sigma_1}(x_3)\cos\left(2\Phi_3^f\right)-D_{\sigma_1}q_{-\sigma_1}(x_4)\cos\left(2\Phi_4^f\right)\right]$$

$$+\frac{m^*_{\sigma_1}\left(m^*_{-\sigma_1}\right)^2}{p_{\sigma_1}p_{-\sigma_1}}\int_0^\pi d\theta\,\frac{W_b^{\sigma_1-\sigma_1}(\theta)}{\sqrt{\sin^2\theta-\left(\frac{\xi_3}{\epsilon_{12}}\right)}}\left[q_{\sigma_1}(x_1)+D_{\sigma_1}q_{-\sigma_1}(x_2)\cos(2\theta)-q_{\sigma_1}(x_3)\cos\left(2\Phi_3^b\right)-D_{\sigma_1}q_{-\sigma_1}(x_4)\cos\left(2\Phi_4^b\right)\right]\Bigg\}, \tag{3.44}$$

where for convenience we have defined $D_\sigma \equiv (v_{-\sigma}p_{-\sigma})/(v_\sigma p_\sigma)$.

If we proceed as we did for the thermal conductivity, we would now set the four momenta involved in the quasiparticle scattering equal to their zero-temperature values. Then we would have $\Phi_3^f = 0$ and $\Phi_4^f = \theta$, and from Fig. 2 and Eqs. (2.3), $\Phi_3^b = \theta$ and $\Phi_4^b = 0$ at zero polarization. If we substitute these values into (3.44) then at zero polarization we will obtain an integral eigenvalue equation in the same form as Eq. (41) in Ref. [12] with $\lambda = 1$. However, $\lambda = 1$ is an eigenvalue of the homogeneous version of Eq. (41), and since (3.44) is clearly inhomogeneous there would be no solution.

The root of the problem lies with using Fermi momenta fixed at their zero-temperature values. If we allow the momenta to differ from the zero-temperature p_F's by $O(k_BT)$, then a new configuration of the momenta will become available (illustrated in Fig. 3). This happens in the head-on collision process ($\theta = \pi$), and turns out to give the dominant contribution to the shear viscosity at zero polarization. The fact that the viscosity of an unpolarized 2D Fermi liquid is fully determined by head-on quasiparticle scattering was identified by Novikov [19].

In the following discussion of this process, we shall fix $p_1 = p_{F1}$ and $p_2 = p_{F2}$, and only permit p_3 and p_4 to differ from their zero-temperature values. This simplifies the calculation while maintaining the essential features of the effect of finite temperature, and also maintains consistency

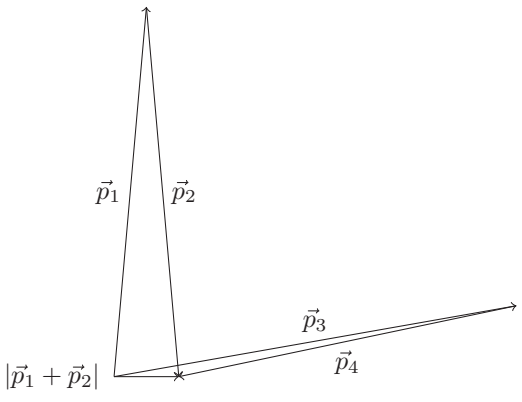

FIG. 3. Momentum space diagram illustrating the effects of letting the quasiparticle momenta move slightly off of the zero-temperature value p_F. In this zero-polarization example we set $p_1 = p_2 = p_F$ as discussed in the text. It is clear that Φ_3 and Φ_4 can be quite different from 0 or π in both forward and backward scattering processes.

with our previous treatment of the divergence in the angular integrals as discussed in Sec. II. We will show how this process makes a dominant contribution to the viscosity of an unpolarized system, and also (though to a lesser extent) to that of a polarized system.

The second and third integrals in Eq. (3.44) describe forward and backward scattering between spin-antiparallel quasiparticles, and their exact forms depend on polarization. Appropriate expressions will be derived below where we will discuss three polarization ranges separately. At this point we can evaluate the contribution from spin-parallel forward scattering. The calculation of $\cos\Phi_3^f$ and $\cos\Phi_4^f$ begins by determining $\cos\theta_3$ and $\cos\theta_4$. From Eq. (2.6),

$$\cos\theta_3 = \frac{|\mathbf{p}_1+\mathbf{p}_2|}{2p_3}+\frac{p_3^2-p_4^2}{2p_3|\mathbf{p}_1+\mathbf{p}_2|}. \tag{3.45}$$

We note that the factor of $|\mathbf{p}_1+\mathbf{p}_2|$ in the denominator of (3.45) is the source of the enhancement of the role of head-on scattering as will be discussed below.

Noting $|\mathbf{p}_1+\mathbf{p}_2| = 2p_F\cos\left(\frac{\theta}{2}\right)$, we can write to lowest order in T

$$\cos(\theta_3) = \cos\left(\tfrac{\theta}{2}\right)+\frac{\Delta_3}{2\cos\left(\frac{\theta}{2}\right)}, \tag{3.46}$$

where we have defined $\Delta_3 \equiv \xi_3/\epsilon_F$, and ξ_3 is defined in (2.11). Then exchanging p_3 and p_4 in (3.45) we have

$$\cos(\theta_4) = \cos\left(\tfrac{\theta}{2}\right)-\frac{\Delta_3}{2\cos\left(\frac{\theta}{2}\right)}. \tag{3.47}$$

From Eqs. (3.46) and (3.47) it is clear that θ_3 and θ_4 can be significantly different from $\theta/2$ when $\theta \approx \pi$. Noting that $\theta_1 = 2\pi - \theta/2$, we then have

$$\sin\Phi_3^f = \sin(\theta_3-\theta_1) \approx \sin\theta_1\cos\theta_1-\sin\theta_1\cos\theta_3 = \frac{\tan\left(\frac{\theta}{2}\right)\Delta_3}{2},$$

and similarly

$$\sin\Phi_4^f \approx \sin\theta-\frac{\tan\left(\frac{\theta}{2}\right)\Delta_3}{2}. \tag{3.48}$$

Therefore

$$\cos\left(2\Phi_3^f\right) \approx 1-\frac{\tan^2\left(\frac{\theta}{2}\right)\Delta_3^2}{2}, \tag{3.49}$$

and

$$\cos\left(2\Phi_4^f\right) \approx \cos(2\theta)-\frac{\tan^2\left(\frac{\theta}{2}\right)\Delta_3^2}{2}. \tag{3.50}$$

It is clear now that Φ_3^f and Φ_4^f can be significantly different from 0 and π at $\theta \approx \pi$. Substitute Eqs. (3.49) and (3.50) into the first integral in (3.44), and one obtains

$$\frac{\left(m^*_{\sigma_1}\right)^3}{p^2_{\sigma_1}} \int_\Delta^{\pi-\Delta} d\theta \frac{W_f^{\sigma_1\sigma_1}(\theta)}{\sin\theta} \times \left\{ q_{\sigma_1}(x_1) - \left[1 - \tan^2\left(\frac{\theta}{2}\right)\Delta_3^2\right] q_{\sigma_1}(x_2) \right\}, \quad (3.51)$$

where we have used the even parity of $q_\sigma(x)$, and the usual change of variables has been applied.

1. $\mathcal{P} = 0$ *zero polarization*

At zero polarization, we set $p_{F\uparrow} = p_{F\downarrow}, m^*_\uparrow = m^*_\downarrow$, and $D_\sigma = 1$. The contribution from the forward scattering of spin-antiparallel quasiparticles is identical to that of the forward scattering between spin-parallel quasiparticles as analyzed in (3.51). Thus, the second integral in (3.44) is identical to (3.51) with the substitution $W_f^{\sigma\,\sigma} \to W_f^{\sigma\,-\sigma}$.

For the backward scattering between spin-antiparallel quasiparticles, one notices that $\theta_3 \approx 2\pi - \theta_1$ and $\theta_4 \approx \theta_1$ as θ is not close to π; therefore proceeding as in the previous analysis

$$\sin\Phi_3^b \approx \sin\theta + \frac{\tan\left(\frac{\theta}{2}\right)\Delta_3}{2} \quad (3.52)$$

and

$$\sin\Phi_4^b \approx -\frac{\tan\left(\frac{\theta}{2}\right)\Delta_3}{2}. \quad (3.53)$$

Then

$$\cos\left(2\Phi_3^b\right) \approx \cos(2\theta) - \frac{\tan^2\left(\frac{\theta}{2}\right)\Delta_3^2}{2} \quad (3.54)$$

and

$$\cos\left(2\Phi_4^b\right) \approx 1 - \frac{\tan^2\left(\frac{\theta}{2}\right)\Delta_3^2}{2}. \quad (3.55)$$

Substituting these back into the backward scattering integral together with (3.51), we obtain for (3.44)

$$n_1\bar{n}_1 = (k_BT)^2\frac{2}{h^4}\frac{(m^*)^3}{p_F^2}\int_{-\infty}^{+\infty} dx_2dx_3dx_4\delta(x_1+x_2+x_3+x_4)n_1n_2n_3n_4 \times \int_\Delta^{\pi-\Delta} d\theta \frac{W_f^{\sigma_1\,\sigma_1}(\theta) + W_f^{\sigma_1\,-\sigma_1}(\theta) + W_b^{\sigma_1\,-\sigma_1}(\theta)}{\sin\theta}\left\{ q_{\sigma_1}(x_1) - \left[1 - \tan^2\left(\frac{\theta}{2}\right)\Delta_3^2\right] q_{\sigma_1}(x_2)\right\}. \quad (3.56)$$

As before, the kinetic equation can be written

$$\frac{\pi^2}{4}\tau_0 n_1\bar{n}_1 = \int_{-\infty}^{+\infty} dx_2dx_3dx_4\delta(x_1+x_2+x_3+x_4)n_1n_2n_3n_4[q(x_1) - (1-\nu_0\tau_0)q(x_2)], \quad (3.57)$$

where τ_0 is given by (2.24), and we have defined a generalized frequency

$$\nu_0 = \frac{\pi^2}{2h^4}(k_BT)^2\frac{(m^*)^3}{p_F^2}\int_\Delta^{\pi-\Delta} d\theta \frac{W_f^{\uparrow\uparrow}(\theta) + W_f^{\uparrow\downarrow}(\theta) + W_b^{\uparrow\downarrow}(\theta)}{\sin\theta}\tan^2\left(\frac{\theta}{2}\right)\Delta_3^2. \quad (3.58)$$

Following the same steps as for the thermal conductivity, Eq. (3.57) can be cast into Sykes-Brooker form:

$$\frac{\pi^2}{4}\tau_0 = \int_{-\infty}^{+\infty} dx_2 K(x_1,x_2)[q(x_1) - \lambda q(x_2)]. \quad (3.59)$$

The solution is then

$$Q \triangleq -\int_{-\infty}^{+\infty} dx \frac{\partial n}{\partial x} q(x) = \frac{c(\lambda)}{2\nu_0},$$

with the eigenvalue

$$\lambda = 1 - \nu_0\tau_0, \quad (3.60)$$

and from Ref. [12],

$$\frac{c(\lambda)}{(1-\lambda)} = \frac{1}{4}\sum_{n=0}^{\infty}\frac{(4n+3)}{(n+1)(2n+1)[(n+1)(2n+1)-\lambda]}. \quad (3.61)$$

If we keep only the zeroth-order components of the transition rates, and simply set $\Delta_3 = \Delta$ to extract the correct order of temperature dependence in ν_σ, we obtain

$$\nu_0 \approx (k_BT)^2\frac{\pi^2}{h^4}\frac{(m^*)^3}{p_F^2}(W_{f,0}^{\uparrow\uparrow} + W_{f,0}^{\uparrow\downarrow} + W_{b,0}^{\uparrow\downarrow}). \quad (3.62)$$

The expression for the zero-polarization viscosity follows from (3.42):

$$\eta(\mathcal{P}=0) = \frac{\pi}{4h^2}\frac{p_F^4}{m^*}\tau_0\frac{c(\lambda)}{1-\lambda}. \quad (3.63)$$

Using $1 - \lambda = \nu_0\tau \sim O(1/\ln T)$ and $c(\lambda \approx 1) \approx \frac{3}{4}$, we find

$$\eta(\mathcal{P}=0) = \frac{3}{4}\frac{h^2}{\pi}\frac{v_F^2}{W_{f,0}^{\uparrow\uparrow} + W_{f,0}^{\uparrow\downarrow} + W_{b,0}^{\uparrow\downarrow}}\left(\frac{\epsilon_F}{k_BT}\right)^2. \quad (3.64)$$

An important feature of this expression that needs to be pointed out is that the explicit dependence on the quasiparticle lifetime τ_0 has canceled out, leaving the viscosity solely dependent on ν_0. This contrasts with the result in three dimensions where η is found to be directly proportional to the lifetime. Further, the integrand in Eq. (3.58) is only appreciable near $\theta \approx \pi$,

which means that at zero polarization ν_0, and η in turn, are solely dominated by the head-on collisions between both spin-parallel and spin-antiparallel quasiparticles.

The zero-polarization temperature dependence $\eta^{-1} \approx T^2$ is in agreement with the variational result by Fu and Ebner, but is different from Novikov's main result Eq. (3) in Ref. [19], in which he found the temperature dependence for the inverse viscosity to be $T^2/\ln^2(T_F/T)$. The extra $\ln^2(T_F/T)$ comes from his assumption that the Landau parameters have a logarithmic divergence in their denominators at $\theta = \pi$, which make them vanishingly small in the head-on direction. This behavior was first demonstrated by Abrikosov, Gorkov, and Dzyaloshinski for a 3D system (Ref. [4] in Novikov). However, Engelbrecht, Randeria, and Zhang [25] subsequently proved that in a repulsive 2D system, Landau parameters calculated using a low-density T-matrix expansion (the method used in Ref. [26]) are finite in the head-on direction. Thus in this work we do not include any divergence factors in the denominators of the Landau parameters.

2. $0 < \mathcal{P} < 1$

In this polarization range, we note that the second term in Eq. (3.45) is always negligible compared to the first term. Thus for the forward scattering between spin-antiparallel quasiparticles we set $\Phi_3^f = 0$ and $\Phi_4^f = \theta$. Then the second term in the curly brackets of Eq. (3.44) becomes

$$\frac{m^*_{\sigma_1}\left(m^*_{-\sigma_1}\right)^2}{p_{\sigma_1}p_{-\sigma_1}}\int_0^\pi d\theta\frac{W_f^{\sigma_1\,-\sigma_1}(\theta)}{\sqrt{\sin^2\theta-\frac{\xi_3}{\epsilon_{12}}}}[q_{\sigma_1}(x_1)-q_{\sigma_1}(x_2)]. \quad (3.65)$$

The integral for the spin-antiparallel backward scattering can be transformed into

$$\begin{aligned}&\frac{m^*_{\sigma_1}\left(m^*_{-\sigma_1}\right)^2}{p_{\sigma_1}p_{-\sigma_1}}\int_0^\pi d\theta\frac{W_b^{\sigma_1,-\sigma_1}(\theta)}{\sqrt{\sin^2\theta-\frac{\xi_3}{\epsilon_{12}}}}\\&\quad\times\left[q_{\sigma_1}(x_1)-\left(1-2\sin^2\Phi_3^b\right)q_{\sigma_1}(x_2)\right.\\&\quad\left.+D_{\sigma_1}2\left(\cos^2\theta-\cos^2\Phi_4^b\right)q_{-\sigma_1}(x_2)\right]. \quad (3.66)\end{aligned}$$

The integral over the last term is much smaller than that for the first two terms. Indeed if one sets $W_b^{\sigma_1\,-\sigma_1}(\theta) \approx W_{b,0}^{\sigma_1\,-\sigma_1}$, using Eq. (2.3) one obtains

$$\int_0^\pi d\theta\frac{\cos^2\theta-\cos^2\Phi_4^b}{\sqrt{\sin^2\theta-\frac{\xi_3}{\epsilon_{12}}}}=0. \quad (3.67)$$

Thus we can ignore the last term in the integral, and therefore the kinetic equation at finite polarization becomes spin decoupled.

Combining Eqs. (3.51), (3.65), (3.66), and (3.67), the kinetic equation becomes

$$\begin{aligned}n_1\bar{n}_1=(k_BT)^2\frac{2}{h^4}\int_{-\infty}^{+\infty}&dx_2dx_3dx_4\delta(x_1+x_2+x_3+x_4)n_1n_2n_3n_4\\&\times\left\{\frac{\left(m^*_{\sigma_1}\right)^3}{p^2_{\sigma_1}}\int_\Delta^{\pi-\Delta}d\theta\frac{W_f^{\sigma_1\,\sigma_1}(\theta)}{\sin\theta}\left(q_{\sigma_1}(x_1)-\left[1-\tan^2\left(\frac{\theta}{2}\right)\Delta_3^2\right]q_{\sigma_1}(x_2)\right)\right.\\&+\frac{m^*_{\sigma_1}\left(m^*_{-\sigma_1}\right)^2}{p_{\sigma_1}p_{-\sigma_1}}\int_0^\pi d\theta\frac{W_f^{\sigma_1\,-\sigma_1}(\theta)}{\sqrt{\sin^2\theta-\frac{\xi_3}{\epsilon_{12}}}}\left[q_{\sigma_1}(x_1)-q_{\sigma_1}(x_2)\right]\\&\left.+\frac{m^*_{\sigma_1}\left(m^*_{-\sigma_1}\right)^2}{p_{\sigma_1}p_{-\sigma_1}}\int_0^\pi d\theta\frac{W_b^{\sigma_1\,-\sigma_1}(\theta)}{\sqrt{\sin^2\theta-\frac{\xi_3}{\epsilon_{12}}}}\left[q_{\sigma_1}(x_1)-\left(1-2\sin^2\Phi_3^b\right)q_{\sigma_1}(x_2)\right]\right\}. \quad (3.68)\end{aligned}$$

With the aid of the definition of the quasiparticle lifetime (2.22) we find

$$\begin{aligned}\frac{\pi^2}{4}\tau_{\sigma_1}n_1\bar{n}_1=\int_{-\infty}^{+\infty}&dx_2dx_3dx_4\delta(x_1+x_2+x_3+x_4)n_1n_2n_3n_4\\&\times\left\{q_{\sigma_1}(x_1)-\left[1-\left(\nu^f_{\sigma_1}+\nu^{(3)}_{\sigma_1}\right)\tau_{\sigma_1}\right]q_{\sigma_1}(x_2)\right\}, \quad (3.69)\end{aligned}$$

with the definitions

$$\nu^f_\sigma=\frac{\pi^2}{2h^4}(k_BT)^2\frac{(m^*_\sigma)^3}{p^2_\sigma}\int_\Delta^{\pi-\Delta}d\theta\frac{W_f^{\sigma\,\sigma}(\theta)}{\sin\theta}\tan^2\left(\frac{\theta}{2}\right)\Delta_3^2, \quad (3.70)$$

$$\nu^{(3)}_\sigma=\frac{\pi^2}{h^4}(k_BT)^2\frac{m^*_\sigma(m^*_{-\sigma})^2}{p_\sigma p_{-\sigma}}\int_\Delta^{\pi-\Delta}d\theta\frac{W_b^{\sigma\,-\sigma}(\theta)}{\sin\theta}\sin^2\Phi_3. \quad (3.71)$$

The kinetic equation can now be cast into Sykes-Brooker form:

$$\frac{\pi^2}{4}\tau_\sigma=\int_{-\infty}^{+\infty}dx_2K(x_1,x_2)[q_\sigma(x_1)-\lambda_\sigma q_\sigma(x_2)], \quad (3.72)$$

with

$$\lambda_\sigma=1-\left(\nu^f_\sigma+\nu^{(3)}_\sigma\right)\tau_\sigma. \quad (3.73)$$

The solution is

$$Q_\sigma=\frac{c(\lambda_\sigma)}{2\left(\nu^f_\sigma+\nu^{(3)}_\sigma\right)}, \quad (3.74)$$

where $c(\lambda)$ is given in Eq. (3.61). In lowest order of W, we can obtain approximate analytic expressions for the generalized frequencies:

$$\nu^f_\sigma\approx(k_BT)^2\frac{\pi^2}{h^4}\frac{(m^*_\sigma)^3}{p^2_\sigma}W^{\sigma\,\sigma}_{f,0}, \quad (3.75)$$

$$\nu_\sigma^{(3)} \approx \frac{2\pi^2}{h^4}(k_B T)^2 \frac{(m^*_{-\sigma})^2}{\epsilon_{F\sigma}} W_{b,0}^{\sigma\,-\sigma} \frac{p_{-\sigma}}{p_\sigma} \times \left(1 + \frac{p_\sigma^2 - p_{-\sigma}^2}{2p_\sigma p_{-\sigma}} \ln\left|\frac{p_\sigma + p_{-\sigma}}{p_\sigma - p_{-\sigma}}\right|\right). \quad (3.76)$$

We note they are both on the order of T^2.

For the polarization range $0 < \mathcal{P} < 1$ the shear viscosity becomes

$$\eta = \frac{3\pi}{32h^2} \sum_\sigma \frac{(p_{F\sigma})^4}{m_\sigma^*} \frac{1}{\left(\nu_\sigma^f + \nu_\sigma^{(3)}\right)}, \quad (3.77)$$

where $c(\lambda) \approx 3/4$ since $\nu_\sigma \tau_\sigma \sim O(1/\ln T)$. In lowest order we find

$$\eta = \frac{3}{8}\frac{h^2}{\pi} \sum_\sigma \frac{v_{F\sigma}^2}{W_{f,0}^{\sigma\,\sigma} + 4\left(\frac{v_{F-\sigma}}{v_{F\sigma}}\right)\left(\frac{m^*_{-\sigma}}{m^*_\sigma}\right)^3 W_{b,0}^{\sigma\,-\sigma}\left(1 + \frac{p_\sigma^2 - p_{-\sigma}^2}{2p_\sigma p_{-\sigma}} \ln\left|\frac{p_\sigma + p_{-\sigma}}{p_\sigma - p_{-\sigma}}\right|\right)} \left(\frac{\epsilon_{F\sigma}}{k_B T}\right)^2. \quad (3.78)$$

We note the lifetime has canceled out in the same manner as at zero polarization, and the temperature dependence $\eta^{-1} \approx T^2$ is unchanged at finite polarization. There is an important difference from zero polarization, however. At zero polarization we showed that the shear viscosity is fully determined by head-on collisions between both spin-parallel and spin-antiparallel quasiparticles. At finite polarization the major contribution from scattering is from the backward scattering process depicted in Fig. 2. We note that in such processes scattering events with a wide range of relative angle θ (from 0 to π) have to be taken into account. This backward scattering contributes to the second term in the denominator in Eq. (3.78). At low energies, where the s-wave scattering favors the spin singlet configuration, we expect the second term should always dominate the first term in the denominator, until the polarization hits a certain high value (see Fig. 4).

3. *Full polarization* $\mathcal{P} = 1$

At full polarization we ignore those terms that involve scattering between antiparallel-spin quasiparticles since there are no particles in the minority Fermi sea. Thus we ignore the $\nu^{(3)}$ term in (3.73), and therefore $\lambda_\uparrow = 1 - \nu_\uparrow^f \tau_1$. The solution becomes $Q_\uparrow = c(\lambda_\uparrow)/(2\nu_\uparrow^f) \approx 3/(8\nu_\uparrow^f)$. From Eqs. (3.77) and (3.78) the shear viscosity at full polarization becomes

$$\eta(\mathcal{P} = 1) = \frac{3\pi}{32h^2} \frac{(p_{F\uparrow})^4}{m_\uparrow^*} \frac{1}{\nu_\uparrow^f} \approx \frac{3}{8}\frac{h^2}{\pi} \frac{v_{F\uparrow}^2}{W_{f,0}^{\uparrow\uparrow}} \left(\frac{\epsilon_{F\uparrow}}{k_B T}\right)^2. \quad (3.79)$$

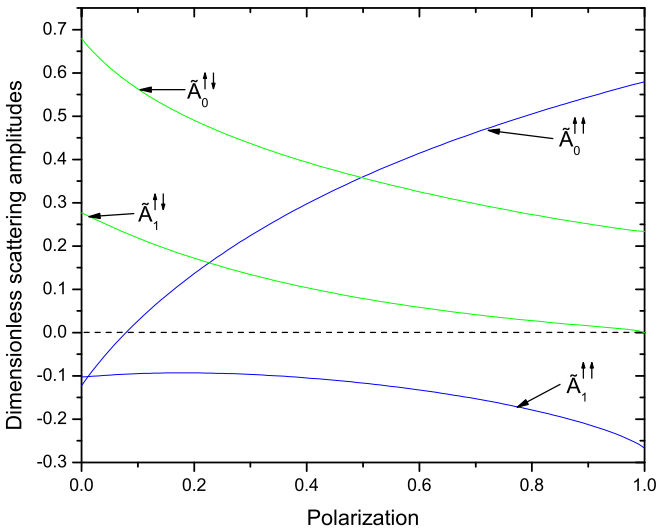

FIG. 4. (Color online) Dimensionless scattering amplitudes (4.1) versus polarization at $\bar{n} = 0.0132$ Å^{-2} on a graphite substrate. Comparison of $\tilde{A}_0^{\uparrow\uparrow}$ with $\tilde{A}_1^{\uparrow\uparrow}$, and $\tilde{A}_0^{\uparrow\downarrow}$ with $\tilde{A}_1^{\uparrow\downarrow}$. We note that at $\mathcal{P} = 0$ we find $\tilde{A}_0^{\uparrow\uparrow} \approx \tilde{A}_1^{\uparrow\uparrow}$. For all polarizations $\tilde{A}_0^{\uparrow\downarrow} > \tilde{A}_1^{\uparrow\downarrow}$.

Summary

We find that the temperature dependence for the shear viscosity at arbitrary polarization $0 \leqslant \mathcal{P} \leqslant 1$ is $\eta^{-1} \sim T^2$. This is in agreement with the zero-polarization results of Fu and Ebner [17].

C. Spin diffusion

The longitudinal spin diffusion coefficient D for a two-dimensional Fermi liquid at arbitrary polarization was calculated by Miyake and Mullin. The derivation of D is very similar to that of κ in Sec. III A. However, unlike κ or η, the temperature dependence of D depends on whether one is at zero polarization or nonzero polarization. For the details of the derivations of the zero and nonzero polarization forms of D we refer the reader to Ref. [18]. At the time MM did their calculation nothing was known of the Fermi liquid parameters for ^{3}He films. Thus, MM applied their general results to the weakly interacting limit. In this section we shall briefly discuss and display D at zero and nonzero polarization and arbitrary density. Further, using our results for the Landau parameters for ^{3}He films we can calculate predicted values for D which will be presented in the following section.

1. *Zero polarization*

The diffusion coefficient at zero polarization in the $\ell = 0$ approximation is

$$D = \frac{1}{2}\left(\frac{h^4}{\pi^2}\right) \frac{\left(1 + F_0^a\right)}{(m^*)^3 W_{b,0}^{\sigma\,-\sigma}} \frac{c(\lambda_-)}{\left(\frac{k_B T}{\epsilon_F}\right)^2 \ln\left(\frac{2\epsilon_F}{k_B T}\right)}, \quad (3.80)$$

where $\lambda_- = 1 - 2[W_{b,0}^{\sigma\,-\sigma}/(W_{f,0}^{\sigma\,\sigma} + W_{f,0}^{\sigma\,-\sigma} + W_{b,0}^{\sigma\,-\sigma})]$; the series $c(\lambda)$ is defined in (3.61). The dependence of D on the spin-parallel transition probability through the eigenvalue mimics the solution for bulk Fermi liquids [12]. In a more general form than the $\ell = 0$ approximation, the eigenvalue λ_- depends on the transition probabilities $W_f^{\sigma\sigma}, W_f^{\sigma-\sigma}$, and $W_b^{\sigma-\sigma}$ through the angular averages that appear in the generalized frequencies and quasiparticle lifetimes.

2. *Nonzero polarization*

The spin diffusion coefficient at $\mathcal{P} \neq 0$ in the $\ell = 0$ approximation is given by

$$D = \frac{3}{2}\left(\frac{h^4}{\pi^2}\right)\frac{\epsilon_{F\uparrow}\epsilon_{F\downarrow}R_\uparrow|R_\downarrow|}{m^*_\uparrow m^*_\downarrow\left[m^*_\uparrow\frac{p^2_\downarrow}{p^2_\uparrow}|R_\downarrow| + m^*_\downarrow\frac{p^2_\uparrow}{p^2_\downarrow}R_\uparrow\right](W^{\uparrow\downarrow}_{b,0})\ln\left|\frac{p_\uparrow+p_\downarrow}{p_\uparrow-p_\downarrow}\right|}(k_BT)^{-2}, \tag{3.81}$$

where the R_σ parameters are the proportionality constants that connect the chemical potential gradients with magnetization gradients. They are given by

$$R_\sigma = \frac{\sigma\overline{n}_{-\sigma}}{N_0^{-\sigma}}\frac{\left(1+F_0^{\sigma\sigma}\right)\left(1+F_0^{-\sigma-\sigma}\right) - F_0^{\sigma-\sigma}F_0^{-\sigma\sigma}}{\overline{n}_\sigma\left(1+F_0^{\sigma\sigma}+F_0^{\sigma-\sigma}\right)/N_0^\sigma + \overline{n}_{-\sigma}\left(1+F_0^{-\sigma-\sigma}+F_0^{-\sigma\sigma}\right)/N_0^{-\sigma}}, \tag{3.82}$$

and we associate $\sigma = \{+1,-1\}$ with $\sigma = \{\uparrow,\downarrow\}$, respectively. The dimensionless Landau parameters are defined by

$$F_0^{\sigma\sigma'} = N_0^\sigma\int_0^{2\pi}\frac{d\theta}{2\pi}f^{\sigma\sigma\sigma'}_{\mathbf{pp}'}, \tag{3.83}$$

where $N_0^\sigma = m^*_\sigma/(2\pi\hbar^2)$ is the single spin-state density of states at the Fermi surface. The absence of the $\ln(T)$ divergence in D at finite polarization was first noticed by MM who pointed out that the change in temperature dependence from zero polarization was due to the inability of the system to conserve momentum in spin-antiparallel collisions at nonzero polarization and low enough temperature.

The limits of small polarization and full polarization of (3.81) are important for the numerical work to be presented in the following section. In the limit of very small polarization $|\mathcal{P}| \ll 1$ (3.81) becomes

$$D(|\mathcal{P}| \ll 1) \approx \frac{3}{8}\left(\frac{h^4}{\pi^2}\right)\frac{\left(1+F_0^a\right)}{(m^*)^3W^{\uparrow\downarrow}_{b,0}}\frac{1}{\ln\left(\frac{1}{2|\mathcal{P}|}\right)}\left(\frac{\epsilon_F}{k_BT}\right)^2, \tag{3.84}$$

where all terms have been set to their $\mathcal{P} = 0$ values except for the term that is singular in that limit. Although this expression appears to vanish in the $\lim\mathcal{P} \to 0$, we note that there exists a lower limit of polarization beyond which these results may not be applied. This lower limit is determined by the requirement that a well-defined polarization requires $\epsilon_{F\uparrow} - \epsilon_{F\downarrow} \gg k_BT$. In the opposite limit of full polarization the spin diffusion coefficient does vanish. From (3.81) it can be seen that $D(\mathcal{P} \to 1) \sim O(p^3_\downarrow)$. Equivalent results can be found in MM in the weak interaction limit.

IV. APPLICATION TO THIN ^{3}He FILMS

In this section we calculate transport coefficients for thin-film ^{3}He systems. The system-specific information is provided by the angular integrals of the transition rates $W^{\sigma\sigma'}(\theta)$ that appear in the quasiparticle lifetimes τ and the generalized frequencies ν. The transition rates can be written in terms of the scattering amplitudes: $W^{\sigma\sigma'}_{f,b}(\theta) = \frac{2\pi}{\hbar}|a^{\sigma\sigma'}_{f,b}(\theta)|^2$. Dimensionless scattering amplitudes can be defined by

$$\tilde{A}^{\sigma\sigma'}_{f,b}(\theta) = \tilde{N}_0a^{\sigma\sigma'}_{f,b}(\theta), \tag{4.1}$$

where $\tilde{N}_0 = m/(2\pi\hbar^2)$. The transition rates then become

$$W^{\sigma\sigma'}_{f,b}(\theta) = \frac{h^3}{m^2}\left|\tilde{A}^{\sigma\sigma'}_{f,b}(\theta)\right|^2. \tag{4.2}$$

In turn, for forward scattering, the Fourier components of the scattering amplitudes can be written in terms of the Landau parameters [26]:

$$a^{\uparrow\uparrow}_{f,\ell} = \frac{f^{\uparrow\uparrow}_\ell(1+N_0^\downarrow f^{\downarrow\downarrow}_\ell) - N_0^\downarrow(f^{\uparrow\downarrow}_\ell)^2}{(1+N_0^\uparrow f^{\uparrow\uparrow}_\ell)(1+N_0^\downarrow f^{\downarrow\downarrow}_\ell) - N_0^\uparrow N_0^\downarrow(f^{\uparrow\downarrow}_\ell)^2}, \tag{4.3a}$$

$$a^{\uparrow\downarrow}_{f,\ell} = \frac{f^{\uparrow\downarrow}_\ell}{(1+N_0^\uparrow f^{\uparrow\uparrow}_\ell)(1+N_0^\downarrow f^{\downarrow\downarrow}_\ell) - N_0^\uparrow N_0^\downarrow(f^{\uparrow\downarrow}_\ell)^2}. \tag{4.3b}$$

We note in passing that in this notation the forward scattering sum rules [27] become $W^{\sigma\sigma}_f(0) = 0$.

At zero polarization one can also write the backward scattering transition probability $W_b^{\sigma-\sigma}$ in terms of the forward scattering amplitudes [18] by utilizing

$$a_b^{\sigma-\sigma}(\theta) = a_f^{\sigma-\sigma}(\theta) - a_f^{\sigma\sigma}(\theta). \tag{4.4}$$

It is not known whether a comparable exact result can be obtained for nonzero polarization in two dimensions. The important point is that at zero polarization in two dimensions D can be written solely in terms of the Landau parameters [28]. This point was made by Miyake and Mullin [18]. In fact this is valid for all of the transport coefficients. In order to compute backward scattering contributions at nonzero polarization, and also head-on transition rates used for the shear viscosity, we shall proceed by making some reasonable assumptions. For backward scattering at nonzero polarization,

$$a_b^{\sigma-\sigma}(\theta) \approx a_f^{\sigma-\sigma}(\theta), \tag{4.5}$$

and for the head-on scattering needed for the shear viscosity,

$$a^{\sigma\sigma'}_{\text{head-on}}(\theta = \pi, \phi = 0 \text{ or } \pi) \approx a_f^{\sigma\sigma'}(\theta = \pi). \tag{4.6}$$

In three dimensions one faces a similar problem because one also needs information concerning the ϕ dependence of the scattering amplitudes where ϕ is the angle between the planes formed by the momenta of the incoming and outgoing quasiparticles. An approximate solution at zero polarization was obtained by Dy and Pethick [29]. Unfortunately the *s-p* approximation does not generalize to nonzero polarization. In two dimensions as noted previously ϕ can only take on the values 0 and π.

For ^{3}He in two dimensions we can calculate Landau parameters to high orders [16] by utilizing effective *s*-wave and *p*-wave T-matrix elements determined by experimental data. Thus in principle we can also calculate the Fourier sum for $W_f^{\sigma\sigma'}(\theta)$ to high orders. For the numerical work to be

TABLE I. Expressions for the transport coefficients at $\mathcal{P}=0$ and $\mathcal{P}=1$ rewritten in forms that are useful for analyzing the density dependence.

Transport coefficient	Polarization	Expression
κ	$\mathcal{P}=0$	$\kappa T\ln(\frac{2T_F}{T})=(\frac{\hbar^3}{8\pi m^2})H(1)\frac{\bar{n}^2}{(m^*/m)^4}\frac{1}{\lvert\tilde{A}^{\uparrow\uparrow}_{f,0}\rvert^2+\lvert\tilde{A}^{\uparrow\downarrow}_{f,0}\rvert^2+\lvert\tilde{A}^{\uparrow\downarrow}_{b,0}\rvert^2}$
κ	$\mathcal{P}=1$	$\kappa T\ln(\frac{2T_F}{T})=(\frac{\hbar^3}{4\pi m^2})H(1)\frac{\bar{n}^2}{(m^*_\uparrow/m)^4}\frac{1}{\lvert\tilde{A}^{\uparrow\uparrow}_{f,0}\rvert^2}$
η	$\mathcal{P}=0$	$\eta T^2=(\frac{3\pi\hbar^5}{4k_B^2m^2})\frac{\bar{n}^3}{(m^*/m)^4}\frac{1}{\lvert\tilde{A}^{\uparrow\uparrow}_{f,0}\rvert^2+\lvert\tilde{A}^{\uparrow\downarrow}_{f,0}\rvert^2+\lvert\tilde{A}^{\uparrow\downarrow}_{b,0}\rvert^2}$
η	$\mathcal{P}=1$	$\eta T^2=(\frac{3\pi\hbar^5}{k_B^2m^2})\frac{\bar{n}^3}{(m^*_\uparrow/m)^4}\frac{1}{\lvert\tilde{A}^{\uparrow\uparrow}_{f,0}\rvert^2}$
D	$\mathcal{P}=0$	$DT^2\ln(\frac{2T_F}{T})=(\frac{\pi\hbar^5}{k_B^2m^3})c(1)\frac{\bar{n}^2}{(m^*/m)^5}\frac{1+F_0^a}{\lvert\tilde{A}^{\uparrow\downarrow}_{b,0}\rvert^2}$

discussed below, however, we shall use the lowest-order $\ell=0$ approximation for the transition rates. In Fig. 4 we compare $\tilde{A}^{\uparrow\uparrow}_\ell$ and $\tilde{A}^{\uparrow\downarrow}_\ell$ for $\ell=0,1$ at $\bar{n}=0.0132$ Å^{-2} on a graphite substrate. We see, at zero polarization, $\tilde{A}^{\uparrow\downarrow}_0$ dominates the other three components: as one expects, the singlet channel dominates the s-wave scattering. As the polarization increases however, $\tilde{A}^{\uparrow\uparrow}_0$ increases rapidly and eventually becomes the dominant component. Therefore we can approximate the transition rates by simply keeping the $\ell=0$ components over the whole polarization range.

We begin by examining the transport coefficients in the forms $\kappa T\ln(2T_F/T)$, ηT^2, and $DT^2\ln(2T_F/T)$ to analyze their density dependence for $\mathcal{P}=0$ and $\mathcal{P}=1$. The results are presented in this manner in order to take advantage of the fact that at zero polarization and full polarization the explicit temperature dependence factors; see Table I. In Table II we show the values for the system of second layer ^{3}He on graphite. We include in the table the Fermi energies and the effective masses. The values of the effective masses at full polarization come from Ref. [16]. We can compare the qualitative behavior of the transport coefficients with their bulk ^{3}He analogs. Figure 3 in Bedell and Pines [30] shows the pressure dependence at zero polarization of κT, ηT^2, and DT^2. In three dimensions each of these quantities appears to be a monotonically decreasing function of pressure. In two dimensions this is not necessarily the case. One can easily extract the explicit density and effective mass dependence of the transport coefficients by examining Eqs. (2.24), (3.34), (3.64), and (3.80). By inspection of Table I, we find $\kappa\sim\bar{n}^2/(m^*)^4$, $\eta\sim\bar{n}^3/(m^*)^4$, and $D\sim\bar{n}^2/(m^*)^5$. Thus, the explicit density dependence tries to increase the transport coefficients with increasing density whereas the explicit effective mass dependence tries to decrease the transport coefficients with increasing effective mass.

At $\mathcal{P}=0$, $\kappa T\ln(2T_F/T)$ decreases monotonically with increasing density. This behavior is dominated by the increase in the effective mass. There is additional density dependence carried by the scattering amplitudes. In Table III we include the most important scattering amplitudes for the ^{3}He system of Table II. The contribution of the scattering amplitudes to κ is shown in the second column of Table III. The quantity $|\tilde{A}^{\uparrow\uparrow}_{f,0}|^2+|\tilde{A}^{\uparrow\downarrow}_{f,0}|^2+|\tilde{A}^{\uparrow\downarrow}_{b,0}|^2$ is nonmonotonic; however, the extent of variation is small compared to that of the effective mass. This is also the case for the spin diffusion coefficient. The important scattering amplitude in this case is $|\tilde{A}^{\uparrow\downarrow}_{b,0}|^2$, which is fairly constant. Thus, the density dependence of the spin diffusion coefficient is dominated by that of the effective mass. The case of ηT^2 is more intriguing, as it seems first to increase, and then to decrease with density. At low density, the cubic density dependence dominates the viscosity. Thus at low densities the viscosity increases with increasing $\bar{n}$. At higher densities the increase in the effective mass eventually takes over, and the viscosity starts to decrease. This "bump" feature is also present in the density-dependent behavior of T_F.

In contrast to zero polarization, at full polarization the transport coefficients exhibit an *increase* with increasing density. This behavior can be understood by referring to the

TABLE II. ^{3}He on a graphite substrate. The zero-polarization $\mathcal{P}=0$ and full-polarization $\mathcal{P}=1$ thermal conductivity κ, shear viscosity η, and spin diffusion coefficient D as functions of areal density $\bar{n}$ with the explicit temperature dependencies factored out. The units are κT (10^{-5} ergs s^{-1}); ηT^2 (10^{-9} g s^{-1} mK^2); DT^2 (cm^2 s^{-1} mK^2). We also include the effective masses, the Fermi energies, and we note that $D(\mathcal{P}=1)=0$ from its definition. The values of κ are obtained from (3.34) and (3.36), for η from (3.64) and (3.79), and for D from (3.80), for zero and full polarization, respectively.

	m^*/m		ϵ_F (K)		$\kappa T\ln(2T_F/T)$		ηT^2		$DT^2\ln(2T_F/T)$
Density (Å^{-2})	$\mathcal{P}=0$	$\mathcal{P}=1$	$\mathcal{P}=0$	$\mathcal{P}=1$	$\mathcal{P}=0$	$\mathcal{P}=1$	$\mathcal{P}=0$	$\mathcal{P}=1$	$\mathcal{P}=0$
0.013	1.29	0.82	0.52	1.64	0.138	5.4	4.88	0.38×10^3	5.42
0.025	1.72	0.81	0.75	3.16	0.123	10.4	8.34	1.41×10^3	2.67
0.037	2.64	0.86	0.72	4.36	0.047	15.2	4.70	3.01×10^3	0.47
0.046	3.66	0.92	0.64	5.10	0.040	18.7	4.91	4.61×10^3	0.29
0.054	4.88	0.95	0.57	5.82	0.014	23.3	2.06	6.79×10^3	0.06

TABLE III. ^{3}He on a graphite substrate. The dimensionless scattering amplitudes $\tilde{A}^{\sigma\sigma'}_{f/b,0}$ and Landau parameter F^a_0 that are the input into calculating the transport coefficients shown in Table II.

Density (Å^{-2})	$\lvert\tilde{A}^{\uparrow\uparrow}_{f,0}\rvert^2+\lvert\tilde{A}^{\uparrow\downarrow}_{f,0}\rvert^2+\lvert\tilde{A}^{\uparrow\downarrow}_{b,0}\rvert^2$ $\mathcal{P}=0$	F^a_0 $\mathcal{P}=0$	$\lvert\tilde{A}^{\uparrow\downarrow}_{b,0}\rvert^2$ $\mathcal{P}=0$	$\lvert\tilde{A}^{\uparrow\uparrow}_{f,0}\rvert^2$ $\mathcal{P}=1$
0.013	1.12	−0.51	0.64	0.35
0.025	1.44	−0.62	0.88	0.68
0.037	1.46	−0.72	0.93	0.79
0.046	0.72	−0.71	0.46	0.78
0.054	0.90	−0.79	0.59	0.76

effective masses at $\mathcal{P}=1$ shown in Table II. In the limit of full polarization the effective mass shows only a slow increase with increasing density. Thus, at full polarization the explicit increases with density dominate the small increases in the effective masses.

In Tables IV, V and VI, VII we show analogous results for ^{3}He adsorbed to 3.14 Å and 4.33 Å superfluid ^{4}He films, respectively. It is important to note that these results are restricted to a much smaller density range than for ^{3}He on the second layer of graphite. The reason for this difference is that in the superfluid ^{4}He environment the ^{3}He undergoes a transition to a transverse excited state at an areal density $\overline{n}=0.036$ Å^{-2} [33]. The data in Tables IV–VII cover a density range less than the first three data points in Tables II, III. Using Tables III, V, and VII, we can compare the Landau parameter F^a_0 and some of the scattering amplitudes for the two substrates. Over the same density range F^a_0 is markedly smaller in magnitude in the mixture film than on graphite. The denominators for κ and η, $\lvert\tilde{A}^{\uparrow\uparrow}_{f,0}\rvert^2+\lvert\tilde{A}^{\uparrow\downarrow}_{f,0}\rvert^2+\lvert\tilde{A}^{\uparrow\downarrow}_{b,0}\rvert^2$, are considerably smaller for the mixture films than for graphite. We also note that the effective masses only increase moderately with increasing density. However, they are fairly constant over the density range of interest. As a consequence, we can identify the density $\overline{n}$ as the major component driving the increases in $\kappa T\ln(2T_F/T)$ and ηT^2 for $\mathcal{P}=0$ and $\mathcal{P}=1$.

On the other hand, for the mixture film in Table IV $DT^2\ln(2T_F/T)$ follows an irregular pattern with increasing density. This is primarily due to the drastic variation of $\lvert\tilde{A}^{\uparrow\downarrow}_{b,0}\rvert^2$ as can be seen in column 4 of Table V. This behavior is due to the fact that $\tilde{A}^{\uparrow\downarrow}_{b,0}$ is calculated from Eq. (4.4): $\tilde{A}^{\uparrow\downarrow}_{b,0}=\tilde{A}^{\uparrow\downarrow}_{f,0}-\tilde{A}^{\uparrow\uparrow}_{f,0}$. Thus a small change in the difference between the two forward scattering amplitudes can result in a significant change in the backward scattering amplitude. We note that in comparing the denominators of the mixture film transport coefficients $\lvert\tilde{A}^{\uparrow\downarrow}_{b,0}\rvert^2\ll\lvert\tilde{A}^{\uparrow\uparrow}_{f,0}\rvert^2+\lvert\tilde{A}^{\uparrow\downarrow}_{f,0}\rvert^2+\lvert\tilde{A}^{\uparrow\downarrow}_{b,0}\rvert^2$, and thus we expect for example that the spin diffusion coefficient for the mixture films may be the most sensitive quantity with regard to our use of the lowest order "$\ell=0$" approximation for numerical calculations.

In Figs. 5, 6, and 7 we show the *polarization* dependence of the thermal conductivity, shear viscosity times temperature squared, and the spin diffusion coefficient times temperature squared, respectively, for ^{3}He on graphite at $\overline{n}=0.0252$ Å^{-2}, and ^{3}He on a 4.33 Å ^{4}He film at $\overline{n}=0.0248$ Å^{-2}. The data for κ were calculated at a temperature $T=5$ mK which was chosen to ensure that the inequality $T<T_{F\downarrow}$ is obeyed at all polarizations. The units for κ are different in the figure than in the table because for this quantity the temperature dependence is not factorable for $0<\mathcal{P}<1$. The major prediction for this section then is that κ and η increase dramatically, by roughly an order of magnitude for ^{3}He on graphite, as $\mathcal{P}$ increases from 0 to 1 for both substrates. The spin diffusion coefficient goes through a similar large increase from its zero-polarization value to its maximum value in the region $\mathcal{P}\approx 0.74$ for both graphite and ^{4}He, and then vanishes in the full-polarization limit. D vanishes like $(1-\mathcal{P})^{3/2}$ in the limit of full polarization (see Sec. III C). Thus we predict an increase in D from zero polarization to its maximum value of $1\sim 2$ orders of magnitude.

In a recent interesting development, Kovtun, Son, and Starinets [34] have conjectured that there is a *universal* lower bound to the ratio of the shear viscosity to entropy density:

$$4\pi\frac{\eta/\hbar}{s/k_B}\geqslant 1, \tag{4.7}$$

where $s=S/A$, and S is the entropy. The authors describe the distance from the lower bound as a way to characterize how close a fluid is to being perfect. They argue that possible systems that may satisfy the lower bound ought to be strongly interacting systems that are normally characterized by a small viscosity (i.e., a small mean-free path). They suggested that quark-gluon plasmas and ultracold gases at the unitarity limit are candidates. There is now evidence that an ultracold Fermi gas nearly satisfies the lower bound [35].

TABLE IV. ^{3}He in a 3.14 layer film of ^{4}He. The zero-polarization $\mathcal{P}=0$ and full-polarization $\mathcal{P}=1$ thermal conductivity κ, shear viscosity η, and spin diffusion coefficient D as functions of areal density $\overline{n}$ with the explicit temperature dependencies factored out. The units are κT (10^{-5} ergs s^{-1}); ηT^2 (10^{-9} g s^{-1} mK^2); DT^2 (cm^2 s^{-1} mK^2). For this mixture film $m_H=1.56m$ is the hydrodynamic effective mass [31]. The values of κ are obtained from (3.34) and (3.36), for η from (3.64) and (3.79), and for D from (3.80), for zero and full polarization, respectively.

	m^*/m_H		ϵ_F (K)		$\kappa T\ln(2T_F/T)$		ηT^2		$DT^2\ln(2T_F/T)$
Density (Å^{-2})	$\mathcal{P}=0$	$\mathcal{P}=1$	$\mathcal{P}=0$	$\mathcal{P}=1$	$\mathcal{P}=0$	$\mathcal{P}=1$	$\mathcal{P}=0$	$\mathcal{P}=1$	$\mathcal{P}=0$
0.013	1.31	0.84	0.32	0.99	0.285	0.92	0.098	0.63	1.34
0.016	1.47	0.87	0.36	1.21	0.341	1.24	0.147	1.07	3.33
0.019	1.60	0.88	0.39	1.43	0.398	1.63	0.206	1.69	5.21
0.024	1.70	0.88	0.46	1.78	0.510	2.52	0.330	3.26	1.99
0.029	1.77	0.88	0.53	2.14	0.615	3.66	0.476	5.67	1.07

TABLE V. ^{3}He in a 3.14 layer film of ^{4}He. The dimensionless scattering amplitudes $\tilde{A}^{\sigma\sigma'}_{f/b,0}$ and Landau parameter F_0^a that are the input into calculating the transport coefficients shown in Table IV.

Density (Å^{-2})	$\lvert\tilde{A}^{\uparrow\uparrow}_{f,0}\rvert^2+\lvert\tilde{A}^{\uparrow\downarrow}_{f,0}\rvert^2+\lvert\tilde{A}^{\uparrow\downarrow}_{b,0}\rvert^2$ $\mathcal{P}=0$	F_0^a $\mathcal{P}=0$	$\lvert\tilde{A}^{\uparrow\downarrow}_{b,0}\rvert^2$ $\mathcal{P}=0$	$\lvert\tilde{A}^{\uparrow\uparrow}_{f,0}\rvert^2$ $\mathcal{P}=1$
0.013	0.20	−0.11	0.0098	0.71
0.016	0.16	−0.08	0.0036	0.75
0.019	0.14	−0.07	0.0022	0.76
0.024	0.14	−0.12	0.0063	0.76
0.029	0.14	−0.17	0.0132	0.77

Using the results from Sec. III B we can estimate the value of this expression for a ^{3}He film. From Fig. 6 we see that the viscosity is a monotonically increasing function of polarization. The polarization-dependent entropy density is given by $s/k_B=(\pi/6\hbar^2)(m^*_\uparrow+m^*_\downarrow)k_BT$ [16]. This entropy is a monotonically decreasing function of polarization. Thus, we need only to concern ourselves with the zero-polarization limit. Using (3.64) we find for the left-hand side of (4.7)

$$4\pi\frac{\eta/\hbar}{s/k_B}=\frac{9}{\pi^2}\frac{(m/m^*)^2}{\left[\lvert\tilde{A}^{\uparrow\uparrow}_{f,0}\rvert^2+\lvert\tilde{A}^{\uparrow\downarrow}_{f,0}\rvert^2+\lvert\tilde{A}^{\uparrow\downarrow}_{b,0}\rvert^2\right]}\left(\frac{T_F}{T}\right)^3$$
$$\approx 0.28\left(\frac{T_F}{T}\right)^3, \tag{4.8}$$

where we have used (4.2) to write the transition rates in terms of the dimensionless scattering amplitudes. The numbers come from Tables II and III, and so they refer to the second layer of ^{3}He on graphite at $\overline{n}=0.025$ Å^{-2} ($T_F=0.74$ K). It is clear from the inverse cubic temperature dependence that deep in the Fermi-liquid regime the system satisfies the lower bound. At higher temperatures this expression passes through 1 when $T\approx 0.5$ K. This is not that high, and suggests that at temperatures on the order of hundreds of mK the ratio may not be very far from 1 for this ^{3}He thin film system.

V. CONCLUSION

We have derived exact expressions for the transport coefficients κ and η utilizing methods developed by numerous groups [6] for application to bulk ^{3}He. We calculated predicted values for the polarization dependence of κ, η, and D for thin, degenerate ^{3}He films using previously determined Landau parameters. The key to performing the principal angular integration in phase space is the procedure suggested by Miyake and Mullin [18] for avoiding a finite-temperature singularity. The Miyake-Mullin approach is discussed in detail in Sec. II. In that section we derive the polarization-dependent expression for the quasiparticle lifetime due to quasiparticle-quasiparticle collisions. We compare that result with that of a previous derivation of the quasiparticle lifetime using completely different techniques, and note that they are identical up to factor of order 1.

The derivation of the transport coefficients in Sec. III follows the methods developed by Abrikosov and Khalatnikov [4] and Sykes and Brooker [12]. The calculation of κ is very similar to that of the spin diffusion coefficient D as described by Miyake and Mullin. The collision integral is reduced to an integral eigenvalue problem whose integrand depends on both spin-up and spin-down fluctuations. The system is diagonalized by standard methods, and is reduced to an independent pair of equations in Sykes-Brooker form. The temperature dependencies for the transport coefficients are in agreement with older work at zero polarization by Fu and Ebner [17]. Further, we find that, unlike spin diffusion, these dependencies ($T\ln T$ for κ and T^2 for η) are not changed by polarization. The solution for the shear viscosity is unlike that of any other fermion transport coefficient. The key physics lies in including the contributions of scattering from quasiparticles whose momenta differ slightly from their zero-temperature values but are still allowed by energy and momentum conservation at nonzero temperature. In Ref. [19]

TABLE VI. ^{3}He in a 4.33 layer film of ^{4}He. The zero-polarization $\mathcal{P}=0$ and full-polarization $\mathcal{P}=1$ thermal conductivity κ, shear viscosity η, and spin diffusion coefficient D as functions of areal density $\overline{n}$ with the explicit temperature dependencies factored out. The units are κT (10^{-5} erg s^{-1}); ηT^2 (10^{-9} g s^{-1} mK^2); DT^2 (cm^2 s^{-1} mK^2). For this mixture film $m_H=1.29m$ is the hydrodynamic effective mass [32]. The values of κ are obtained from (3.34) and (3.36), for η from (3.64) and (3.79), and for D from (3.80), for zero and full polarization respectively.

	m^*/m_H		ϵ_F (K)		$\kappa T\ln(2T_F/T)$		ηT^2		$DT^2\ln(2T_F/T)$
Density (Å^{-2})	$\mathcal{P}=0$	$\mathcal{P}=1$	$\mathcal{P}=0$	$\mathcal{P}=1$	$\mathcal{P}=0$	$\mathcal{P}=1$	$\mathcal{P}=0$	$\mathcal{P}=1$	$\mathcal{P}=0$
0.015	1.22	0.83	0.50	1.46	0.68	2.29	0.28	1.88	1.74
0.019	1.32	0.84	0.56	1.77	0.80	3.03	0.40	3.03	2.05
0.022	1.37	0.84	0.62	2.03	0.93	3.87	0.53	4.46	1.93
0.025	1.40	0.84	0.70	2.34	1.08	5.18	0.72	6.89	1.61
0.028	1.45	0.84	0.76	2.63	1.21	6.40	0.91	9.60	1.59
0.031	1.50	0.84	0.82	2.92	2.17	7.73	1.08	12.9	1.42

TABLE VII. ^{3}He in a 4.33 layer film of ^{4}He. The dimensionless scattering amplitudes $\tilde{A}^{\sigma\sigma'}_{f/b,0}$ and Landau parameter F^a_0 that are the input into calculating the transport coefficients shown in Table VI.

Density (Å^{-2})	$\lvert\tilde{A}^{\uparrow\uparrow}_{f,0}\rvert^2+\lvert\tilde{A}^{\uparrow\downarrow}_{f,0}\rvert^2+\lvert\tilde{A}^{\uparrow\downarrow}_{b,0}\rvert^2$ $\mathcal{P}=0$	F^a_0 $\mathcal{P}=0$	$\lvert\tilde{A}^{\uparrow\downarrow}_{b,0}\rvert^2$ $\mathcal{P}=0$	$\lvert\tilde{A}^{\uparrow\uparrow}_{f,0}\rvert^2$ $\mathcal{P}=1$
0.015	0.23	−0.16	0.03	0.64
0.019	0.21	−0.16	0.02	0.68
0.022	0.21	−0.18	0.03	0.71
0.025	0.22	−0.21	0.04	0.71
0.028	0.22	−0.22	0.04	0.72
0.031	0.22	−0.24	0.04	0.73

Novikov, in the zero-polarization limit, allows all four quasiparticle momenta to drift from their zero-temperature values. We introduced a simplified model in which at arbitrary polarization we fix the incoming quasiparticle momenta at the zero-temperature values, and allow the outgoing momenta to vary (see Fig. 3). We find that in lowest order the viscosity is formally independent of the quasiparticle lifetime [see Eq. (3.64), for example]. We note however that $1/\nu_0$, Eq. (3.62), is very similar to τ_0. We find at zero polarization, in agreement with Novikov, that the head-on collisions between quasiparticles with opposite momenta dominate the scattering processes; at finite polarization, we find that the backward scattering between spin-antiparallel quasiparticles takes over and becomes the dominant process determining the shear viscosity. The temperature dependence in our final result of the shear viscosity at zero polarization differs from that of Novikov because he assumes a logarithmic divergence in the denominator of the Landau parameters at $\theta=\pi$, and this gives an extra factor of $\ln^2(T_F/T)$ in his final result for the viscosity.

In Sec. IV we apply these results to a system of thin ^{3}He films both in the second layer on a graphite substrate, and also in a thin ^{3}He-^{4}He film mixture. In Table VIII we gather together the main results from this paper concerning the temperature dependence of the thermal conductivity and the shear viscosity, and we have also included the spin diffusion coefficient results from Miyake and Mullin [18].

The predicted polarization dependence of the transport coefficients for ^{3}He on the second layer of graphite and also for the 4.33-Å-thick ^{4}He film is shown in Figs. 5, 6, and 7. These results show a dramatic increase in the magnitudes of the coefficients as the polarization increases from zero. We showed in Sec. III A that for the thermal conductivity in two dimensions κ is proportional to the quasiparticle lifetime. Further, we showed in previous work, see Fig. 7 in Ref. [16], that the magnitude of the contribution to the quasiparticle lifetime from the majority spin component *decreases* dramatically as a function of increasing polarization. Thus, for the thermal conductivity a fairly simple qualitative picture emerges of the role of polarization: increasing $\mathcal{P}$ induces an increase in the quasiparticle lifetime, and thus the transport coefficient. For very dilute systems this mechanism is basically understood as the quenching of s-wave scattering with increased $\mathcal{P}$. However for the shear viscosity such a simple picture does

TABLE VIII. The temperature dependencies of the inverse transport coefficients as a function of polarization. The thermal conductivity κ and the shear viscosity η are calculated in Sec. III. The spin diffusion coefficient D is from Ref. [18]. We note that D^{-1} is undefined at $\mathcal{P}=1$.

Coefficient	$\mathcal{P}=0$	$0<\mathcal{P}\leqslant 1$
κ^{-1}	$T\ln T$	$T\ln T$
η^{-1}	T^2	T^2
D^{-1}	$T^2\ln T$	T^2

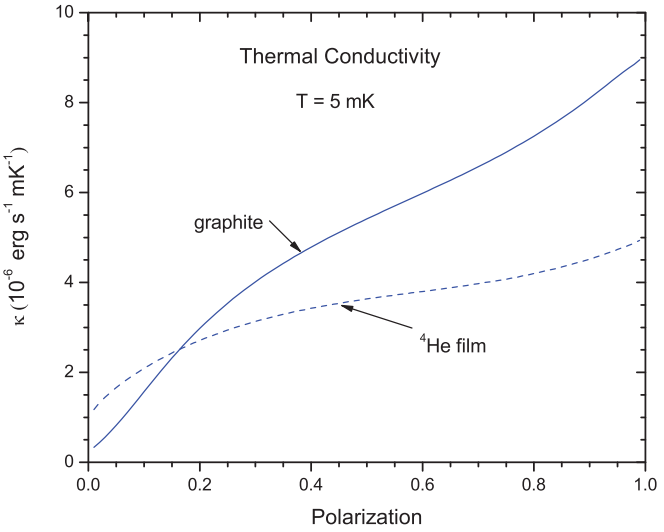

FIG. 5. (Color online) The thermal conductivity κ (3.33) as a function of polarization for ^{3}He on substrates of graphite (solid line), and a 4.33 Å superfluid ^{4}He film (dashed line). Both results are at $T=5$ mK, and the ^{3}He areal densities are 0.0252 Å^{-2} and 0.0248 Å^{-2} on graphite and ^{4}He, respectively.

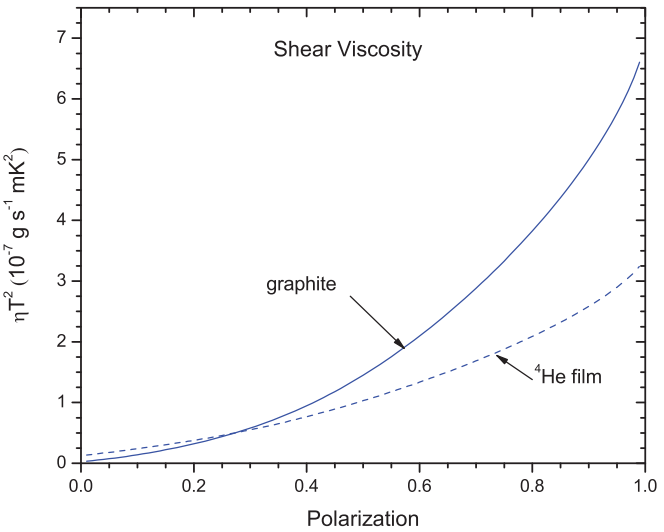

FIG. 6. (Color online) The shear viscosity η (3.78) times temperature squared as a function of polarization for ^{3}He on substrates of graphite (solid line), and a 4.33 Å superfluid ^{4}He film (dashed line). The results are shown for ^{3}He areal densities of 0.0252 Å^{-2} and 0.0248 Å^{-2} on graphite and ^{4}He, respectively.

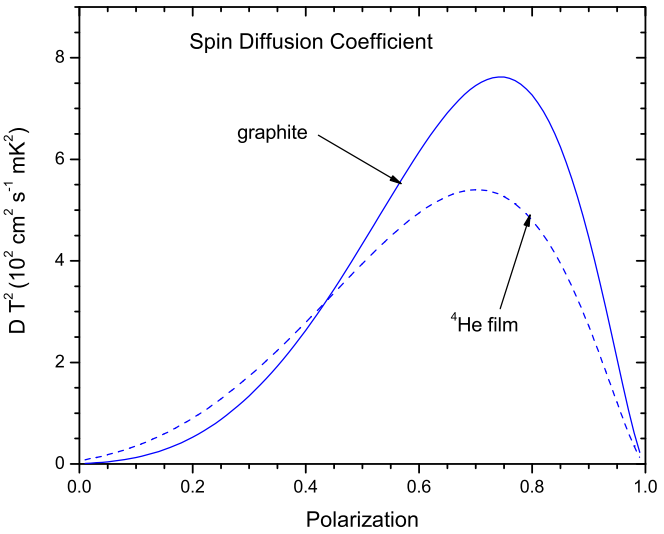

FIG. 7. (Color online) The spin diffusion coefficient D (3.81) times temperature squared as a function of polarization for ^{3}He on substrates of graphite (solid line), and a 4.33 Å superfluid ^{4}He film (dashed line). The results are shown for ^{3}He areal densities of 0.0252 Å^{-2} and 0.0248 Å^{-2} on graphite and ^{4}He, respectively.

not seem to be relevant if for no other reason than because the quasiparticle lifetime does not contribute directly to the transport coefficient. In this case we must consider instead the complicated dynamical question of the relative importance of the spin-antiparallel backward scattering to the spin-parallel head-on scattering as per the discussion in Sec. III B, which itself is related to the balance of s-wave and p-wave scattering.

In lowest order of temperature the derivation of the expressions for the transport coefficients is essentially exact. The calculation of explicit results for ^{3}He films suffers from the use of the $\ell = 0$ approximation for the scattering amplitudes. An improvement in the present results would be the inclusion of additional Fourier components in the expressions for the transition probabilities in terms of the scattering amplitudes; see Eq. (2.16). The approximations used in the determination of the ^{3}He film Landau parameters from experimental measurements of the specific heat effective mass and the spin susceptibility have been discussed in Ref. [16].

At this time to the best of our knowledge there have been no measurements of any transport coefficient in a thin ^{3}He film. In addition there have been no measurements at all in a polarized thin ^{3}He film. These experiments would be very difficult. In fact the first measurement of zero sound in a thin, unpolarized ^{3}He film was only reported in 2010 by Godfrin, Meschke, Lauter, Böhm, Krotscheck, and Panholzer [36]. Our Landau parameters do yield excellent agreement with this zero-sound measurement. For bulk ^{3}He there has been some work on the polarization dependence of transport coefficients. A recent review [37] summarizes the state of the field.

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