Breakdown of the topological classification $\mathbb{Z}$ for gapped phases of noninteracting fermions by quartic interactions

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The conditions for both the stability and the breakdown of the topological classification of gapped ground states of noninteracting fermions, the tenfold way, in the presence of quartic fermion-fermion interactions are given for any dimension of space. This is achieved by encoding the effects of interactions on the boundary gapless modes in terms of boundary dynamical masses. Breakdown of the noninteracting topological classification occurs when the quantum nonlinear σ models for the boundary dynamical masses favor quantum disordered phases. For the tenfold way, we find that (i) the noninteracting topological classification $\mathbb{Z}_2$ is always stable, (ii) the noninteracting topological classification $\mathbb{Z}$ in even dimensions is always stable, and (iii) the noninteracting topological classification $\mathbb{Z}$ in odd dimensions is unstable and reduces to $\mathbb{Z}_N$ that can be identified explicitly for any dimension and any defining symmetries. We also apply our method to the three-dimensional topological crystalline insulator SnTe from the symmetry class AII + R, for which we establish the reduction $\mathbb{Z} \to \mathbb{Z}_8$ of the noninteracting topological classification.

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I. INTRODUCTION

Topological insulators (TIs) and topological superconductors (TSs) of noninteracting fermions are characterized by topological numbers ($\mathbb{Z}$ or $\mathbb{Z}_2$) that encode the nontrivial topology of the occupied single-particle wave functions and are accompanied by gapless excitations that are localized along any boundary [1,2]. The integer quantum Hall effect (IQHE) is characterized by the Hall conductivity quantized by the integer $\nu = 1,2,\ldots$, in units of e^2/h. The topological integer ν counts the number of extended chiral edge modes propagating at the boundary of the sample. The $\mathbb{Z}_2$ topological insulator is characterized by the parity of the number of Kramers' doublets of extended boundary modes. Together with polyacetylene and a two-dimensional $p + ip$ superconductor [3,4], both instances are now understood to be nontrivial entries in the periodic table (i.e., the tenfold way) for noninteracting topological insulators and superconductors [5–7].

The gapless modes appearing at the boundary in the IQHE are robust to both elastic and inelastic scattering resulting from one-body impurity potentials and many-body electron-electron interactions [8,9]. Similarly, the gapless modes in the $\mathbb{Z}_2$ TIs are immune to *both* backscattering resulting from one-body impurity potentials and many-body electron-electron interactions, provided time-reversal symmetry (TRS) is neither explicitly nor spontaneously broken [10–13].

Given the robustness to many-body fermion-fermion interactions of the edge states in the IQHE, it was a remarkable observation made by Fidkowski and Kitaev in 2010 that it is possible to gap out eight Majorana zero modes localized at the end of a one-dimensional topological superconducting wire through many-body interactions without closing the spectral gap in the bulk [14,15]. In the terminology of the 10-fold way [5–7], it was demonstrated in Refs. [14,15] that the $\mathbb{Z}$ topological classification for the noninteracting one-dimensional symmetry class BDI, when interpreted as a superconductor, is (i) unstable to quartic contact interactions that neither break explicitly nor spontaneously the TRS and (ii) this instability reduces the noninteracting topological classification $\mathbb{Z}$ to $\mathbb{Z}_8$.

Subsequently, noninteracting two-dimensional topological crystalline superconductors (TCSs) from the symmetry class DIII + R (where "+R" indicates the presence of an additional reflection symmetry) and three-dimensional topological superconductors from the symmetry class DIII were shown in Refs. [16,17] and Refs. [18–22] to display the reduction patterns $\mathbb{Z} \to \mathbb{Z}_8$ and $\mathbb{Z} \to \mathbb{Z}_{16}$, respectively, when perturbed by quartic contact interactions that break neither explicitly nor spontaneously the defining symmetries [23]. The reductions $\mathbb{Z} \to \mathbb{Z}_4$ and $\mathbb{Z} \to \mathbb{Z}_8$ for the three-dimensional symmetry classes CI and AIII were obtained in Ref. [21].

We present in Sec. II a method that allows us to derive the reduction pattern of all noninteracting topological insulators and superconductors without and with reflection symmetries for any dimensionality d of space in the presence of quartic contact interactions that neither break explicitly nor spontaneously the defining symmetries. This method relies on the topology of the classifying spaces from K-theory. It extends the applicability of K-theory for obtaining the tenfold way of noninteracting fermions [6,24], to obtaining the breakdown of the tenfold way induced by interactions.

This method is applied first to the breakdown of the tenfold way in Sec. III [25]. In doing so, we prove the following properties that we report in Table I:

(1) All $\mathbb{Z}_2$ entries of the periodic table irrespectively of the dimensionality of space are stable to quartic contact interactions.

(2) All $\mathbb{Z}$ entries of the periodic table when the dimensionality of space is even are stable to quartic contact interactions.

(3) Only the $\mathbb{Z}$ entries of the periodic table when the dimensionality of space is odd are unstable to quartic contact interactions with a reduction pattern that is computed explicitly and shown to break the Bott periodicity of two for the complex symmetry classes and of eight for the real symmetry classes.

TABLE I. (Color online) The 10 Altland-Zirnbauer (AZ) symmetry classes and their topological classification when (i) fermion-fermion interactions break their defining symmetries neither explicitly nor spontaneously and (ii) the many-body ground state is short-ranged entangled. Two complex and eight real symmetry classes are characterized by the presence or the absence of time-reversal symmetry (T), particle-hole symmetry (C), and chiral symmetry (Γ_5). Their presence is complemented by the sign multiplying the identity in $T^2 = \pm 1$ or $C^2 = \pm 1$ and by 1 for Γ_5. Their absence is indicated by 0. For each symmetry class and for any dimension $d = 0,1,2,\ldots,$ of space, the classifying space V_d, the space of normalized Dirac masses allowed by symmetry, is given in the fifth column. Explicit forms of the classifying spaces C_q and R_q and their stable homotopy groups are found in Table XVI from Appendix B. The reduction, if any, that arises from the effects of interactions on the topological classification of noninteracting fermions for $d = 1,\ldots,8$ is given in the last eight columns. Each entry with a nontrivial Abelian group defines equivalence classes of interacting topological insulators (superconductors) with a short-ranged entangled many-body ground state. We show in blue the entries corresponding to a given symmetry class and a given column of odd dimensionality d to indicate that this entry is a quotient group $\mathfrak{G}_{\text{int}}$ of $\mathfrak{G} = \mathbb{Z}$. The reduction $\mathbb{Z} \to \mathfrak{G}_{\text{int}}$ results from an instability of the noninteracting topological classification to fermion-fermion interactions. The four entries corresponding to the symmetry classes BDI and CII and the dimensions $d = 1$ and $d = 5$ occur in pairs depending on whether these two classes are interpreted as describing superconductors (i.e., interacting Majorana fermions) or insulators (i.e., interacting complex fermions), respectively.

Class	T	C	Γ_5	V_d	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
A	0	0	0	C_{0+d}	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	C_{1+d}	$\mathbb{Z}_4$	0	$\mathbb{Z}_8$	0	$\mathbb{Z}_{16}$	0	$\mathbb{Z}_{32}$	0
AI	+1	0	0	R_{0-d}	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	+1	+1	1	R_{1-d}	$\mathbb{Z}_8,\mathbb{Z}_4$	0	0	0	$\mathbb{Z}_{16},\mathbb{Z}_8$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	+1	0	R_{2-d}	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	−1	+1	1	R_{3-d}	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{16}$	0	0	0	$\mathbb{Z}_{32}$	0
AII	−1	0	0	R_{4-d}	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	−1	−1	1	R_{5-d}	$\mathbb{Z}_2,\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{16},\mathbb{Z}_{16}$	0	0	0
C	0	−1	0	R_{6-d}	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	+1	−1	1	R_{7-d}	0	0	$\mathbb{Z}_4$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{32}$	0

This method is then applied to the three-dimensional topological crystalline insulators (TCIs) from the symmetry class AII + R, which are of relevance to SnTe, in Sec. IV. We show the reduction $\mathbb{Z} \to \mathbb{Z}_8$ in the presence of quartic local fermion-fermion interactions.

The strategy that we use to study the robustness of ν boundary modes to quartic contact fermion-fermion interactions is inspired by the (unpublished) approach pioneered by Kitaev in Refs. [18,26], see also Ref. [22]. It consists of three steps. First, a noninteracting topological phase is represented by the many-body ground state of a massive Dirac Hamiltonian with a matrix dimension that depends on ν. Second, a Hubbard-Stratonovich transformation is used to trade a generic quartic contact interaction in favor of dynamical Dirac masslike bilinears coupled to their conjugate fields (that will be called "Dirac masses"). These dynamical Dirac masses may violate any symmetry constraint other than the particle-hole symmetry (PHS) [27]. Third, the ν boundary modes that are coupled with a suitably chosen subset of dynamical masses are integrated over. The resulting dynamical theory on the $(d-1)$-dimensional boundary is a bosonic one, a quantum nonlinear σ model (QNLSM) in $[(d-1)+1]$-dimensional space and time with a target space that depends on ν. The reduction pattern is then obtained by identifying the smallest value of ν for which this QNLSM cannot be augmented by a topological term. The presence or absence of topological terms in the relevant QNLSM is determined by the topology of the spaces of boundary dynamical Dirac masses, i.e., the topology of classifying spaces. Now, K-theory provides a systematic way to study the topology of the classifying space. Hence, this is why the same approach that was used to obtain the tenfold way of noninteracting fermions can be relied on to deduce a classification of topological short-range entangled (SRE) phases [also known as symmetry-protected topological (SPT) phases] for interacting fermions [28].

Other topological phases are also interesting on their own right. For example, bosonic SPT (SRE) phases show many novel topological phases driven by strong interactions. They have been reviewed in Ref. [29]. The classification of bosonic SPT (SRE) states has been obtained by diverse approaches that include group cohomology [30,31], the K-matrix approach [32], enumerating surface topological order [33], wire constructions [34,35], and so on. Topological order with long-range entanglement (LRE) is also a subject of intensive study, which has relied on parton constructions [36–39], topological field theories [40–42], exactly soluble models [43–48], and wire constructions [49–52].

II. STRATEGY

In this section, we present our strategy to obtain a topological classification for interacting fermions with gapped ground states as an application of K-theory to certain *dynamical Dirac masses for boundary fermions*.

Noninteracting fermions always belong to 1 of the 10 Altland-Zirnbauer (AZ) symmetry classes defined by the presence or absence of the following three symmetries: TRS, PHS, and chiral symmetry (CHS) (see Appendix A). Within any one of these 10 symmetry classes, the defining topological attributes of noninteracting topological insulators and superconductors are shared by equivalence classes of Hamiltonians. Any two members within a topological class can be deformed into each other by a smooth (adiabatic) deformation of the matrix elements of these Hamiltonians without closing the bulk energy gap. These equivalence classes are endowed with an Abelian group structure $\mathfrak{G}$. For any given dimensionality d

of space, topological invariants $\mathfrak{G}$ are nontrivial for 5 of the 10 AZ classes. Specifically, 3 of the 10 AZ classes support Abelian groups $\mathfrak{G}=\mathbb{Z}$, while 2 of the 10 AZ classes support Abelian groups $\mathfrak{G}=\mathbb{Z}_2$. The TRS, PHS, and CHS can be augmented by crystalline symmetries. Noninteracting fermions obeying crystalline symmetries can also be understood as realizing topologically distinct equivalence classes, i.e., TCIs [53].

The topological classification with the Abelian group $\mathfrak{G}$ for noninteracting TIs, TSs, or TCIs can break down in the presence of many-body interactions. Namely, an Abelian group $\mathfrak{G}_{\mathrm{int}}$ that encodes the topological equivalence classes of gapped ground states for interacting fermions can be smaller than $\mathfrak{G}$ as a group (some quotient group of $\mathfrak{G}$).

In order to establish the instability of the noninteracting classification of TIs, TSs, and TCIs, we choose a family of massive Dirac Hamiltonians,

$$\mathcal{H}^{(0)} := -i\sum_{j=1}^{d}\frac{\partial}{\partial x^{j}}\,\widetilde{\alpha}_{j}\otimes\mathbb{1}+m(\boldsymbol{x})\widetilde{\beta}\otimes\mathbb{1}, \tag{2.1}$$

as representative single-particle Hamiltonians. Here Dirac matrices $\widetilde{\boldsymbol{\alpha}}$ and $\widetilde{\beta}$ anticommute with each other and have the minimal dimension (rank) r_{min} under the symmetry constraints, i.e., r_{min} is the minimal rank to realize a Dirac Hamiltonian of the form (2.1). The dimension of the unit matrix $\mathbb{1}$ is $\nu=1,2,\ldots$. The integer $\nu\in\mathfrak{G}$ is then related to the dimension $r(\nu)=r_{\mathrm{min}}\,\nu$ of the Dirac matrices that we choose. The question that we want to address is that of the stability or instability of the boundary states of a noninteracting TI, TS, or TCI in the presence of many-body interactions that do not break the protecting symmetries of the noninteracting limit [54]. Here, whenever $\nu\neq 0$, the extended single-particle boundary states are governed by the massless Dirac Hamiltonian

$$\mathcal{H}^{(0)}_{\mathrm{bd}} := -i\sum_{j=1}^{d-1}\frac{\partial}{\partial x^{j}}\,\alpha_{j}\otimes\mathbb{1}\equiv -i\boldsymbol{\partial}\cdot\boldsymbol{\alpha}\otimes\mathbb{1}, \tag{2.2}$$

which is obtained by introducing a domain wall in the mass $m(\boldsymbol{x})$ along the x^{d} direction that enters Hamiltonian (2.1). The Dirac matrices $\boldsymbol{\alpha}\otimes\mathbb{1}$ have a dimension $r(\nu)/2$ that is half that of the bulk massive Dirac Hamiltonian $\mathcal{H}^{(0)}$. The dimension of the matrices $\boldsymbol{\alpha}$ is $r_{\mathrm{min}}/2$.

The breakdown (reduction) of the topological classification for noninteracting fermions takes place when the boundary states of the TIs, TSs, or TCIs can be gapped by many-body interactions that preserve their defining symmetries. By assumption, we consider many-body interactions that are weak relative to the bulk gap. If so, it is sufficient to treat the effects of many-body interactions for the massless Dirac fermions propagating on the $(d-1)$-dimensional boundary. To establish an instability of the noninteracting topological classification, we need not consider all possible many-body interactions. It suffices to establish that at least one family of strong (on the boundary) interactions implies the instability of the noninteracting classification $\mathfrak{G}$ by gapping out all boundary Dirac fermions. To this end, we limit ourselves to contact interactions.

Contact interactions are constructed from taking squares of local bilinears in the Dirac fermions. We have two options for these bilinears. The bilinear under consideration either commutes or anticommutes with the kinetic contribution to the Dirac Hamiltonian. We shall call the latter option a Dirac mass. In this paper, we only consider the contact interactions obtained from taking squares of those bilinears built of Dirac mass matrices, for only these can gap the noninteracting massless boundary Dirac fermions in a mean-field approximation. Because we assume that the protecting symmetries forbid the presence of Dirac masses on the boundary that are consistent with the protecting symmetries, the only possible Dirac masses induced by a mean-field treatment of a symmetry-preserving quartic interaction on the boundary must be odd under at least one of the protecting symmetries. We shall call such a boundary Dirac mass a boundary dynamical mass and label it with the Greek letter β.

We are thus led to consider the many-body interacting Dirac boundary Hamiltonian

$$\widehat{H}_{\mathrm{bd}} := \widehat{H}^{(0)}_{\mathrm{bd}}+\widehat{H}^{(\mathrm{int})}_{\mathrm{bd}}, \tag{2.3a}$$

where (the subscript “bd” stands for boundary)

$$\widehat{H}^{(0)}_{\mathrm{bd}} := \int d^{d-1}\boldsymbol{x}\;\hat{\Psi}^{\dagger}(t,\boldsymbol{x})\,\mathcal{H}^{(0)}_{\mathrm{bd}}\,\hat{\Psi}(t,\boldsymbol{x}) \tag{2.3b}$$

and

$$\widehat{H}^{(\mathrm{int})}_{\mathrm{bd}} := \lambda\sum_{\{\beta\}}\int d^{d-1}\boldsymbol{x}\;[\hat{\Psi}^{\dagger}(t,\boldsymbol{x})\,\beta\,\hat{\Psi}(t,\boldsymbol{x})]^{2}. \tag{2.3c}$$

We have chosen the real-valued coupling λ with the dimension of (length)$^{d-2}$ to be independent of β for simplicity. This coupling constant is marginal in $d=2$ and irrelevant when $d>2$. (Of course, it can very well be that the set $\{\beta\}$ is empty. If so, we anticipate that $\mathfrak{G}=\mathfrak{G}_{\mathrm{int}}$ must hold. This is what happens for the strong topological insulators in the symmetry classes A, D, and C when $d=2$.) At this stage, it is convenient to treat the many-body Hamiltonian (2.3) with the help of the path integral

$$Z_{\mathrm{bd}} := \int \mathcal{D}[\Psi,\Psi^{\dagger}]\,e^{-S_{\mathrm{bd}}}, \tag{2.4a}$$

where the action in Euclidean time τ is

$$S_{\mathrm{bd}} := \int d\tau\int d^{d-1}\boldsymbol{x}\,\mathcal{L}_{\mathrm{bd}}, \tag{2.4b}$$

with the Lagrangian density

$$\mathcal{L}_{\mathrm{bd}} := \Psi^{\dagger}\big[\partial_{\tau}+\mathcal{H}^{(0)}_{\mathrm{bd}}\big]\Psi+\lambda\sum_{\{\beta\}}(\Psi^{\dagger}\beta\,\Psi)^{2}. \tag{2.4c}$$

The path integral is over Grassmann-valued Dirac spinors.

We rewrite the quartic interaction terms by performing a Hubbard-Stratonovich transformation with respect to the bosonic fields ϕ_{β} conjugate to $\Psi^{\dagger}\beta\,\Psi$,

$$Z_{\mathrm{bd}} \propto \int \mathcal{D}[\Psi,\Psi^{\dagger},\phi_{\beta}]\,e^{-S'_{\mathrm{bd}}}. \tag{2.5a}$$

Here the action in Euclidean time τ is

$$S'_{\mathrm{bd}} := \int d\tau\int d^{d-1}\boldsymbol{x}\,\mathcal{L}'_{\mathrm{bd}}, \tag{2.5b}$$

with the Lagrangian density

$$\mathcal{L}'_{\mathrm{bd}} := \Psi^{\dagger}\big[\partial_\tau + \mathcal{H}^{(\mathrm{dyn})}_{\mathrm{bd}}\big]\Psi + \frac{1}{\lambda}\sum_{\{\beta\}}\phi^2_\beta, \tag{2.5c}$$

where we have introduced the dynamical one-body single-particle Hamiltonian

$$\mathcal{H}^{(\mathrm{dyn})}_{\mathrm{bd}}(\tau,\boldsymbol{x}) := \mathcal{H}^{(0)}_{\mathrm{bd}}(\boldsymbol{x}) + \sum_{\{\beta\}} 2i\,\beta\,\phi_\beta(\tau,\boldsymbol{x}), \tag{2.5d}$$

under the assumption that the sign $\lambda > 0$ corresponds to a repulsive interaction. In a saddle-point approximation, the magnitude of the vector $\boldsymbol{\phi}$ with the components ϕ_β can be frozen both in imaginary time and in $(d-1)$-dimensional space. Fluctuations that change this frozen magnitude are suppressed by the second term on the right-hand side of Eq. (2.5c). We will restrict the set $\{\beta\}$ to pairwise anticommuting Dirac mass matrices. If so, the direction in which the vector $\boldsymbol{\phi}$ with the components ϕ_β freezes in the saddle-point approximation is arbitrary [55]. Since fluctuations about this direction are soft, these are the Goldstone modes associated with the spontaneous breaking of a continuous symmetry.

The effective low-energy theory governing the fluctuations of these Goldstone modes is obtained from a gradient expansion of the fermion determinant

$$\mathrm{Det}\big[\partial_\tau + \mathcal{H}^{(\mathrm{dyn})}_{\mathrm{bd}}\big] := \int \mathcal{D}[\Psi,\Psi^\dagger]\, e^{-\int d\tau \int d^{d-1}\boldsymbol{x}\, \Psi^\dagger[\partial_\tau + \mathcal{H}^{(\mathrm{dyn})}_{\mathrm{bd}}]\Psi}. \tag{2.6}$$

It is captured by the partition function

$$Z_{\mathrm{bd}} \approx \int \mathcal{D}[\boldsymbol{\phi}]\,\delta(\boldsymbol{\phi}^2 - 1)\, e^{-S_{\mathrm{QNLSM}} - S_{\mathrm{top}}}, \tag{2.7}$$

after we have rescaled the vector $\boldsymbol{\phi}$ so it squares to 1. The Euclidean action

$$S_{\mathrm{QNLSM}} = \frac{1}{2\,g}\int d\tau \int d^{d-1}\boldsymbol{x}\,(\partial_i\boldsymbol{\phi})^2 \tag{2.8}$$

is the action of the quantum nonlinear sigma model (QNLSM) with the base space $\mathbb{R}^{(d-1)+1}$ in space and time and the target space

$$S^{N(\nu)-1} \tag{2.9}$$

with the integer $N(\nu)$ counting the pairwise anticommuting Dirac masses that have been retained in the set $\{\beta\}$. The effective coupling constant g is positive. The topological term S_{top} is present whenever any one of the homotopy groups

$$\begin{aligned} &\pi_0[S^{N(\nu)-1}],\\ &\pi_1[S^{N(\nu)-1}],\\ &\ddots\\ &\pi_d[S^{N(\nu)-1}],\\ &\pi_{d+1}[S^{N(\nu)-1}], \end{aligned} \tag{2.10}$$

is nonvanishing [56]. (The reason why we ignore all topological terms associated with nonvanishing homotopy group of order larger than $d+1$ is that such topological terms would modify the local equations of motion derived from S_{QNLSM} in a nonlocal way.) It signals the existence of zero modes of the Dirac Hamiltonian (2.5d) in the presence of topological defects in the order parameter $\boldsymbol{\phi}$. We expect that these zero modes prevent the gapping of the boundary Dirac fermions. We define the smallest value ν_{min} for the dimension ν of the unit matrix $\mathbb{1}$ in Eq. (2.2) for which

$$\begin{aligned} &\pi_0[S^{N(\nu_{\mathrm{min}})-1}] = 0,\\ &\pi_1[S^{N(\nu_{\mathrm{min}})-1}] = 0,\\ &\ddots\\ &\pi_d[S^{N(\nu_{\mathrm{min}})-1}] = 0,\\ &\pi_{d+1}[S^{N(\nu_{\mathrm{min}})-1}] = 0. \end{aligned} \tag{2.11}$$

As all homotopy groups of the spheres are known, one may verify that

$$d+1 < N(\nu_{\mathrm{min}}) - 1. \tag{2.12}$$

When Eq. (2.11) holds, the topological term S_{top} is absent, and the effective action in the partition function is simply the action (2.8) for a QNLSM on a sphere [57]. In this case, the quantum-disordered phase at the strong-coupling fixed point $g \to \infty$ is stable. In this strongly interacting phase and when $\mathfrak{G} = \mathbb{Z}$, quantum fluctuations restore dynamically and nonperturbatively all the symmetries broken by the saddle point, including any protecting symmetries. If so, all boundary Dirac fermions are gapped out. We then conclude that

$$\mathfrak{G}_{\mathrm{int}} = \mathbb{Z}_{\nu_{\mathrm{min}}}. \tag{2.13}$$

The stability

$$\mathfrak{G}_{\mathrm{int}} = \mathfrak{G} \tag{2.14}$$

when $\mathfrak{G} = \mathbb{Z}_2$ follows from the fact that one of the homotopy groups $\pi_D[S^{N(\nu=1)-1}]$ with $D \leqslant d+1$ is always nontrivial when $\mathfrak{G} = \mathbb{Z}_2$ (see Sec. III D).

As an illustration of this method, we give in Table I the equivalence classes of topological insulators and superconductors belonging to the 10 AZ symmetry classes in the presence of interactions that select a short-ranged entangled many-body ground state. It becomes apparent that the Bott periodicity of the tenfold way, i.e., the periodicity of the (zeroth) homotopy groups of the classifying spaces with respect to d, is lost. It also becomes apparent that the reduction of the topologically distinct equivalence classes of noninteracting fermions for any given AZ symmetry class occurs only in odd dimensions of space. Finally, two of the AZ symmetry classes, namely the chiral symmetry classes BDI and CII, have the particularity that they may be interpreted either as a superconductor or an insulator. Correspondingly, the reduction of their classification $\mathbb{Z} \to \mathbb{Z}_m$ for the superconductor interpretation and $\mathbb{Z} \to \mathbb{Z}_n$ for the insulator interpretation of these symmetry classes obeys

$$m = 2n, \quad \text{(class BDI)} \tag{2.15a}$$

$$m = n, \quad \text{(class CII)} \tag{2.15b}$$

when the dimensionality of space is $d = 1$ mod 4.

III. REDUCTION OF THE PERIODIC TABLE FOR STRONG TI AND TS

In this section, we apply the strategy explained above to study the breakdown of the tenfold way in the presence of quartic contact interactions in the ascending order of the spatial dimension d, i.e., $d = 1,2,3$, and higher dimensions.

We will use the following conventions. The operation of complex conjugation will be denoted by K. Linear maps of two-dimensional vector space $\mathbb{C}^2$ shall be represented by 2×2 matrices that we expand in terms of the unit matrix τ_0 and the three Pauli matrices τ_1, τ_2, and τ_3. Linear maps of the four-dimensional vector space $\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$ will be represented by 4×4 matrices that we expand in terms of the 16 Hermitian matrices

$$X_{\mu\mu'} \equiv \tau_\mu \otimes \sigma_{\mu'}, \quad \mu,\mu' = 0,1,2,3, \tag{3.1}$$

where σ_ν is a second set comprised of the unit matrix and the three Pauli matrices. Linear maps of the 2^n-dimensional vector space $\mathbb{C}^{2^n} = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ will be represented by $2^n \times 2^n$ matrices that we expand in terms of the 4^n Hermitian matrices

$$X_{\mu_1\cdots\mu_n} \equiv \tau_{\mu_1} \otimes \tau_{\mu_2} \otimes \cdots \otimes \tau_{\mu_n}, \tag{3.2}$$

where $\mu_1,\ldots,\mu_n = 0,1,2,3$.

A. The case of one-dimensional space

Fidkowski and Kitaev showed in Ref. [14] that, in one spatial dimension, any pair of Hamiltonian in the symmetry class BDI whose noninteracting topological indices differ by eight can be transformed into each other adiabatically (i.e., without closing the spectral gap) in the presence of a quartic contact interaction that preserves TRS. This work was followed up in Refs. [15,58] with the construction of a topological invariant for interacting fermions from the matrix product representation of ground states. This topological invariant establishes that the reduction $\mathbb{Z} \to \mathbb{Z}_8$ is exhaustive. The same approach with matrix product states was used to obtain an exhaustive classification of one-dimensional gapped spin systems in Ref. [30].

Here we focus on the three chiral symmetry classes that support the $\mathbb{Z}$ topological classification in the noninteracting limit. We shall reproduce the reduction $\mathbb{Z} \to \mathbb{Z}_8$ and $\mathbb{Z} \to \mathbb{Z}_2$ when the symmetry classes BDI and CII are interpreted as chains of Majorana fermions, respectively.

The one-dimensional chiral symmetry classes can be also realized as chains of complex fermions with sublattice symmetry and fermion-number conservation, e.g., polyacetylene. For example, polyacetylene-like chains realize the symmetry class AIII when TRS is broken, the symmetry class BDI when both TRS and the SU(2) spin-rotation symmetry are present, and the symmetry class CII when TRS holds but not the SU(2) spin-rotation symmetry if spin-orbit coupling is sizable. We show that the reduction of the noninteracting topological classification is $\mathbb{Z} \to \mathbb{Z}_4$ for the symmetry classes AIII and BDI, while it is $\mathbb{Z} \to \mathbb{Z}_2$ for the symmetry class CII, provided conservation of the fermion number holds.

1. The symmetry class BDI when $d = 1$

Consider the one-dimensional bulk single-particle Dirac Hamiltonian (with Dirac matrices of dimension $r = 2 \equiv r_{\min}$),

$$\mathcal{H}^{(0)}(x) := -i\partial_x\,\tau_3 + m(x)\,\tau_2. \tag{3.3a}$$

This single-particle Hamiltonian belongs to the symmetry class BDI, for

$$\mathcal{T}\,\mathcal{H}^{(0)}(x)\,\mathcal{T}^{-1} = +\mathcal{H}^{(0)}(x), \tag{3.3b}$$
$$\mathcal{C}\,\mathcal{H}^{(0)}(x)\,\mathcal{C}^{-1} = -\mathcal{H}^{(0)}(x), \tag{3.3c}$$

where

$$\mathcal{T} := \tau_1\,\mathsf{K}, \quad \mathcal{C} := \tau_0\,\mathsf{K}. \tag{3.3d}$$

The Dirac mass matrix τ_2 is here the only one allowed for dimension-2 Dirac matrices under the constraints (3.3b) and (3.3c). As was shown by Jackiw and Rebbi, if translation symmetry is broken by the mass term supporting the domain wall

$$m(x) = m_\infty\,\mathrm{sgn}(x), \quad m_\infty \in \mathbb{R}, \tag{3.4a}$$

at $x = 0$, then the zero mode

$$e^{-i\tau_3\,\tau_2\int_0^x dx'\,m(x')}\,\chi = e^{-|m_\infty x|}\,\chi, \tag{3.4b}$$

where

$$\tau_1\,\chi = \mathrm{sgn}\,(m_\infty)\,\chi, \tag{3.4c}$$

is the only normalizable state bound to this domain wall. This boundary state is a zero mode. It is an eigenstate of the single-particle boundary Hamiltonian

$$\mathcal{H}^{(0)}_{\mathrm{bd}} = 0. \tag{3.4d}$$

Suppose that we consider $\nu = 1,2,\ldots,$ identical copies of the single-particle Hamiltonian (3.3) by defining

$$\mathcal{H}^{(0)}_\nu(x) := \mathcal{H}^{(0)}(x) \otimes \mathbb{1}, \tag{3.5a}$$

and

$$\mathcal{T} := \tau_1 \otimes \mathbb{1}\,\mathsf{K}, \quad \mathcal{C} := \tau_0 \otimes \mathbb{1}\,\mathsf{K}, \tag{3.5b}$$

where $\mathbb{1}$ is a $\nu \times \nu$ unit matrix. Observe that $\mathcal{T}$ and $\mathcal{C}$ commute with $\tau_1 \otimes \mathbb{1}$ and with each other. The domain wall (3.4a) must then support ν linearly independent boundary zero modes. They are annihilated by the boundary Hamiltonian

$$\mathcal{H}^{(0)}_{\mathrm{bd}\,\nu} = \mathcal{H}^{(0)}_{\mathrm{bd}} \otimes \mathbb{1} = 0. \tag{3.6}$$

The topological sectors for noninteracting Hamiltonians are thus labeled by the integer ν taking values in $\mathbb{Z}$ in the limit $\nu \to \infty$.

A generic local quartic interaction that respects the defining BDI symmetries with the potential to gap out these boundary zero modes reduces to a dynamical Dirac mass (that depends on imaginary time τ in addition to space x) that belongs to the symmetry class D, upon performing a Hubbard-Stratonovich transformation. Hence, we must consider the dynamical bulk single-particle Hamiltonian

$$\mathcal{H}^{(\mathrm{dyn})}_\nu(\tau,x) := [-i\partial_x\,\tau_3 + m(x)\,\tau_2] \otimes \mathbb{1} + \mathcal{V}(\tau,x). \tag{3.7a}$$

The dynamical Dirac mass $\mathcal{V}(\tau,x)$ is here defined by the condition that it anticommutes with $\mathcal{H}^{(0)}(x)\otimes\mathbb{1}$, when independent of x, and obeys the transformation laws dictated by the symmetry class D, i.e., it is of the form

$$\mathcal{V}(\tau,x) := \tau_1\otimes\gamma'(\tau,x), \quad \gamma'(\tau,x) := i M(\tau,x), \tag{3.7b}$$

where

$$M(\tau,x) = M^{*}(\tau,x), \quad M(\tau,x) = -M^{\mathsf{T}}(\tau,x), \tag{3.7c}$$

is a real-valued antisymmetric $\nu\times\nu$ matrix. Consequently, TRS is only retained for a given $\mathcal{V}(\tau,x)$ if

$$M(\tau,x) = -M(-\tau,x). \tag{3.7d}$$

On the boundary, the operations for reversal of time and charge conjugation are now represented by

$$\mathcal{T}_{\mathrm{bd}} := \mathsf{K}, \quad \mathcal{C}_{\mathrm{bd}} := \mathsf{K}. \tag{3.8a}$$

Hence, we must consider the dynamical single-particle boundary Hamiltonian

$$\mathcal{H}^{(\mathrm{dyn})}_{\mathrm{bd}\,\nu}(\tau) \equiv \gamma'(\tau) := i M(\tau), \tag{3.8b}$$

where $M(\tau)$ is a real-valued antisymmetric $\nu\times\nu$ matrix. The space of boundary normalized Dirac mass matrices obtained by demanding that γ' square to the unit $\nu\times\nu$ matrix is topologically equivalent to the space

$$V_{\nu} = O(\nu)/U(\nu/2) \tag{3.9}$$

for the symmetry class D in zero-dimensional space, provided the rank $\nu\geqslant 2$ and ν is even. The limit $\nu\to\infty$ of these spaces is the classifying space R_2. In order to gap out dynamically the boundary zero modes without breaking the defining symmetries of the symmetry class BDI, we need to construct a (0+1)-dimensional QNLSM for the (boundary) dynamical Dirac masses from the zero-dimensional symmetry class D without topological obstructions. We construct explicitly the spaces for the relevant normalized boundary dynamical Dirac masses of dimension $\nu=2^n$ with $n=0,1,2,3$ in the following [59]. The relevant homotopy groups are given in Table II [60].

Case $\nu=1$: No Dirac mass is allowed on the boundary, because the boundary is the end of a one-dimensional $\mathbb{Z}_2$ topological superconductor in the topologically nontrivial phase of the symmetry class D.

Case $\nu=2$: We use the representation $\mathbb{1}=\sigma_0$. There is one dynamical normalized Dirac mass on the boundary that is proportional to the matrix σ_2. A domain wall in imaginary time such as $m_{2\,\infty}\,\mathrm{sgn}(\tau)\,\sigma_2$ prevents the dynamical generation of a spectral gap on the boundary.

Case $\nu=4$: We use the representation $\mathbb{1}=\sigma_0\otimes\rho_0$. A (maximum) set of pairwise anticommuting boundary dynamical Dirac mass matrices follows from the set

$$\{\sigma_2\otimes\rho_0,\sigma_1\otimes\rho_2,\sigma_3\otimes\rho_2\}. \tag{3.10}$$

This set spans the space of normalized boundary dynamical Dirac masses that is homeomorphic to S^2. Even though $\pi_{0+1}(S^2)=0$, it is possible to add a topological term that is nonlocal, yet only modifies the equations of motion of the (0+1)-dimensional QNLSM on the boundary by local terms as a consequence of the fact that $\pi_{0+1+1}(S^2)=\mathbb{Z}$. Such a term is a (0+1)-dimensional example of a Wess-Zumino (WZ) term. In the presence of this WZ term, the boundary theory remains gapless. It is nothing but a bosonic representation of the gapless $S=1/2$ degrees of freedom at the end of a quantum spin-1 antiferromagnetic spin chain in the Haldane phase [22].

TABLE II. Reduction from $\mathbb{Z}$ to $\mathbb{Z}_8$ for the topologically equivalent classes of the one-dimensional SPT phases in the symmetry class BDI that arises from interactions. We denote by V_ν the space of $\nu\times\nu$ normalized Dirac mass matrices in zero-dimensional Hamiltonians belonging to the symmetry class D. The limit $\nu\to\infty$ of these spaces is the classifying space R_2. The second column shows the stable D-th homotopy groups of the classifying space R_2. The third column gives the number ν of copies of boundary (Dirac) fermions for which a topological obstruction is permissible. The fourth column gives the type of topological obstruction that prevents the gapping of the boundary (Dirac) fermions.

D	$\pi_D(R_2)$	ν	Topological obstruction
0	$\mathbb{Z}_2$	2	Domain wall
1	0		
2	$\mathbb{Z}$	4	WZ term
3	0		
4	0		
5	0		
6	$\mathbb{Z}$	8	None
7	$\mathbb{Z}_2$		

Case $\nu=8$: We use the representation $\mathbb{1}=\sigma_0\otimes\rho_0\otimes\lambda_0$. One set of pairwise anticommuting boundary dynamical Dirac mass matrices follows from the set

$$\{\sigma_2\otimes\rho_0\otimes\lambda_0,\sigma_3\otimes\rho_2\otimes\lambda_0,\sigma_3\otimes\rho_3\otimes\lambda_2,\sigma_1\otimes\rho_0\otimes\lambda_2\}. \tag{3.11}$$

This set spans a manifold homeomorphic to S^3 (we may find a set of pairwise anticommuting masses spanning S^6). No topological term is admissible over this target manifold that delivers local equations of motion. The QNLSM over this target space endows dynamically the boundary Hamiltonian with a spectral gap.

We conclude that the effects of interactions on the one-dimensional SPT phases in the symmetry class BDI are to reduce the topological classification $\mathbb{Z}$ in the noninteracting limit down to $\mathbb{Z}_8$ under the assumption that a Hamiltonian from the symmetry class BDI is interpreted as a mean-field description of a superconductor. The logic used to reach this conclusion is summarized by Table II once the line corresponding to $\nu=2$ has been identified. It is given by the smallest D that accommodates a nontrivial entry for the corresponding homotopy group. The line for $\nu=4$ is then identified with the next smallest D with $\pi_D(R_2)\neq 0$, and so on.

2. *The symmetry class CII when $d=1$*

Consider the one-dimensional bulk single-particle Dirac Hamiltonian (with Dirac matrices of dimension $r=4\equiv r_{\min}$),

$$\mathcal{H}^{(0)}(x) := -i\partial_x\, X_{30} + m(x)\, X_{20}. \tag{3.12a}$$

This single-particle Hamiltonian belongs to the symmetry class CII, for

$$\mathcal{T}\,\mathcal{H}^{(0)}(x)\,\mathcal{T}^{-1} = +\mathcal{H}^{(0)}(x), \tag{3.12b}$$

$$\mathcal{C}\,\mathcal{H}^{(0)}(x)\,\mathcal{C}^{-1} = -\mathcal{H}^{(0)}(x), \tag{3.12c}$$

where

$$\mathcal{T} := i\,X_{12}\,\mathsf{K}, \quad \mathcal{C} := i\,X_{02}\,\mathsf{K}. \tag{3.12d}$$

The Dirac mass matrix X_{20} is here the only one allowed for dimension four Dirac matrices in the symmetry class CII. If translation symmetry is broken by the Dirac mass term supporting the domain wall

$$m(x) = m_\infty\,\mathrm{sgn}(x), \quad m_\infty \in \mathbb{R}, \tag{3.13a}$$

at $x = 0$, then the zero mode

$$e^{-i X_{30} X_{20} \int_0^x dx'\, m(x')}\,\chi = e^{-|m_\infty x|}\,\chi, \tag{3.13b}$$

where

$$X_{10}\,\chi = \mathrm{sgn}(m_\infty)\,\chi, \tag{3.13c}$$

is the only normalizable state bound to this domain wall. This boundary state is a zero mode. It is an eigenstate of the single-particle boundary Hamiltonian

$$\mathcal{H}^{(0)}_{\mathrm{bd}} = 0. \tag{3.13d}$$

Suppose that we consider $\nu = 1,2,\ldots,$ identical copies of the single-particle Hamiltonian (3.3) by defining

$$\mathcal{H}^{(0)}_{\nu}(x) := \mathcal{H}^{(0)}(x) \otimes \mathbb{1}, \tag{3.14a}$$

and

$$\mathcal{T} := i\,X_{12} \otimes \mathbb{1}\,\mathsf{K}, \quad \mathcal{C} := i\,X_{02} \otimes \mathbb{1}\,\mathsf{K}, \tag{3.14b}$$

where $\mathbb{1}$ is a $\nu \times \nu$ unit matrix. Observe that $\mathcal{T}$ and $\mathcal{C}$ commute with $X_{10} \otimes \mathbb{1}$ and with each other. The domain wall (3.13a) must then support ν linearly independent boundary zero modes. They are annihilated by the boundary Hamiltonian

$$\mathcal{H}^{(0)}_{\mathrm{bd}\,\nu} = \mathcal{H}^{(0)}_{\mathrm{bd}} \otimes \mathbb{1} = 0. \tag{3.15}$$

The topological sectors for noninteracting Hamiltonians are thus labeled by the integer ν taking values in $\mathbb{Z}$ in the limit $\nu \to \infty$.

A generic local quartic interaction that respects the defining CII symmetries with the potential to gap out the boundary zero modes reduces to a dynamical Dirac mass (that depends on imaginary time τ in addition to space x) that belongs to the symmetry class C, upon performing a Hubbard-Stratonovich transformation. Hence, we must consider the dynamical bulk single-particle Hamiltonian

$$\mathcal{H}^{(\mathrm{dyn})}_{\nu}(\tau,x) := [-i\partial_x\,X_{30} + m(x)\,X_{20}] \otimes \mathbb{1} + \mathcal{V}(\tau,x), \tag{3.16a}$$

where the dynamical Dirac mass $\mathcal{V}(\tau,x)$ is defined by the condition that it anticommutes with $\mathcal{H}^{(0)}(x) \otimes \mathbb{1}$, when independent of x, and obeys the transformation laws dictated by the symmetry class C, i.e., it must obey

$$\mathcal{C}\,\mathcal{V}(\tau,x)\,\mathcal{C}^{-1} = -\mathcal{V}(\tau,x). \tag{3.16b}$$

On the boundary, the operations for reversal of time and charge conjugation are now represented by

$$\mathcal{T}_{\mathrm{bd}} := \sigma_2 \otimes \mathbb{1}\,\mathsf{K}, \quad \mathcal{C}_{\mathrm{bd}} := \sigma_2 \otimes \mathbb{1}\,\mathsf{K}. \tag{3.17a}$$

Hence, we must consider the dynamical single-particle boundary Hamiltonian

$$\mathcal{H}^{(\mathrm{dyn})}_{\mathrm{bd}\,\nu}(\tau) := \gamma'(\tau), \tag{3.17b}$$

where

$$\mathcal{C}_{\mathrm{bd}}\,\gamma'(\tau)\,\mathcal{C}^{-1}_{\mathrm{bd}} = -\gamma'(\tau). \tag{3.17c}$$

The space of normalized Dirac mass matrices obtained by demanding that $\gamma'(\tau)$ squares to the unit $2\nu \times 2\nu$ matrix for all imaginary times is the space

$$V_\nu = Sp(\nu)/U(\nu) \tag{3.18}$$

for the symmetry class C in zero-dimensional space. The limit $\nu \to \infty$ of these spaces is the classifying space R_6. In order to gap out dynamically the boundary zero modes without breaking the defining symmetries of the symmetry class CII, we need to construct a (0+1)-dimensional QNLSM for the (boundary) dynamical Dirac masses from the zero-dimensional symmetry class C without topological obstructions. We construct explicitly the spaces for the relevant normalized boundary dynamical Dirac masses of dimension $\nu = 2^n$ with $n = 0,1$ in the following. The relevant homotopy groups are given in Table III.

Case $\nu = 1$*:* The three Dirac mass matrices σ_1, σ_2, and σ_3, are allowed on the boundary. They all anticommute pairwise. A WZ term is permissible as $\pi_{0+1+1}(S^2) = \mathbb{Z}$. In the presence of this WZ term, the boundary theory remains gapless.

Case $\nu = 2$*:* The minimum number of anticommuting mass matrices is larger than three. Hence, the zeroth, first,

TABLE III. Reduction from $\mathbb{Z}$ to $\mathbb{Z}_2$ for the topologically equivalent classes of the one-dimensional SPT phases in the symmetry class CII that arises from interactions. We denote by V_ν the space of $\nu \times \nu$ normalized Dirac mass matrices in zero-dimensional Hamiltonians belonging to the symmetry class C. The limit $\nu \to \infty$ of these spaces is the classifying spaces R_6. The second column shows the stable D-th homotopy groups of the classifying space R_6. The third column gives the number ν of copies of boundary (Dirac) fermions for which a topological obstruction is permissible. The fourth column gives the type of topological obstruction that prevents the gapping of the boundary (Dirac) fermions.

D	$\pi_D(R_6)$	ν	Topological obstruction
0	0		
1	0		
2	$\mathbb{Z}$	1	WZ term
3	$\mathbb{Z}_2$	2	None
4	$\mathbb{Z}_2$		
5	0		
6	$\mathbb{Z}$		
7	0		

and second homotopy groups over the boundary normalized dynamical Dirac masses all vanish. No topological term is possible. The (0+1)-dimensional QNLSM over this target space endows dynamically the boundary Hamiltonian with a spectral gap.

We conclude that the effects of interactions on the one-dimensional SPT phases in the symmetry class CII are to reduce the topological classification $\mathbb{Z}$ in the noninteracting limit down to $\mathbb{Z}_2$ under the assumption that a Hamiltonian from the symmetry class CII is interpreted as a mean-field description of a superconductor. The logic used to reach this conclusion is summarized by Table III once the line corresponding to $\nu = 1$ has been identified. It is given by the smallest D that accommodates a nontrivial entry for the corresponding homotopy group.

3. The chiral symmetry classes as one-dimensional insulators

So far we have interpreted the symmetry classes BDI and CII as examples of topological superconductors by focusing on the fact that their second-quantized Hamiltonian respects a unitary charge-conjugation symmetry, a PHS (see Appendix A). As the symmetry classes BDI and CII also preserve TRS, the composition of reversal of time with charge conjugation delivers a nonunitary symmetry of their second-quantized Hamiltonian, namely a CHS (see Appendix A). The third chiral symmetry class AIII is defined by demanding that it preserves CHS, no more and no less. Hence, any representative gapped Hamiltonian from the chiral symmetry class AIII can always be interpreted as a topological insulator with fermion number conservation.

The CHS can be implemented at the single-particle level by a unitary sublattice spectral symmetry for (complex) electrons hopping between two sublattices. From this point of view, the three chiral symmetry classes AIII, BDI, and CII, when interpreted as metals or as insulators, can be treated on equal footing. The fermion number is conserved in a metal or in an insulator, unlike in the mean-field treatment of a superconductor. This is the case when the symmetry class BDI is interpreted as an effective theory for polyacetylene, in which case the Dirac gap is induced by coupling the electrons to phonons, i.e., it realizes a Peierls or bond-density wave instability [3]. In this interpretation of the chiral classes, it is necessary to introduce an additional particle-hole grading in order to include through dynamical Dirac masses the effects of superconducting fluctuations induced by any quartic interaction. Failure to do so can produce a distinct reduction pattern of the noninteracting topological equivalence classes arising from interactions, for it can matter whether the boundary dynamical masses belong to the classifying space associated with the symmetry classes D or to the symmetry class A.

a. Symmetry class AIII. Consider the one-dimensional bulk single-particle Dirac Hamiltonian in the symmetry class AIII,

$$\mathcal{H}^{(0)}(x) := -i\partial_x\tau_3 + m(x)\tau_2. \tag{3.19a}$$

It anticommutes with the unitary operator

$$\Gamma_5 := \tau_1. \tag{3.19b}$$

It supports the zero mode (3.4) at the boundary where it identically vanishes,

$$\mathcal{H}^{(0)}_{\mathrm{bd}}(x) = 0. \tag{3.19c}$$

Upon tensoring Hamiltonians (3.19a) and (3.19c) together with the Dirac Γ_5 matrix by the $\nu\times\nu$ unit matrix $\mathbb{1}$, there follows $\nu = 1,2,3,\ldots$, boundary zero modes.

The dynamical single-particle Hamiltonian that encodes those nonsuperconducting fluctuations arising from a local quartic interactions after a Hubbard-Stratonovich transformation takes the form (3.7a) with

$$\mathcal{V}(\tau,x) := \tau_1\otimes\gamma'(\tau,x) \tag{3.20}$$

and $\gamma'(\tau,x)$ a $\nu\times\nu$ Hermitian matrix.

On the boundary, we must consider the dynamical single-particle boundary Hamiltonian

$$\mathcal{H}^{(\mathrm{dyn})}_{\mathrm{bd}\,\nu}(\tau) = \gamma'(\tau). \tag{3.21}$$

The $\nu\times\nu$ Hermitian matrix $\gamma'(\tau)$ belongs to the zero-dimensional symmetry class A. Consequently, it is assigned the classifying space C_0 in the limit $\nu\to\infty$ with the zeroth-homotopy group $\pi_0(C_0) = \mathbb{Z}$. When $\nu = 1$, γ' is a real number, and the domain wall $\gamma'(\tau)\propto \mathrm{sgn}(\tau)$ binds a zero mode at $\tau = 0$ [i.e., a normalizable zero mode of the operator $\partial_\tau + \mathcal{H}^{(\mathrm{dyn})}_{\mathrm{bd}\,\nu}(\tau)$]. When $\nu = 2$, we can write $\gamma'(\tau)$ as a linear combination of the matrices σ_μ with the real-valued functions $m_\mu(\tau)$ as coefficients for $\mu = 0,1,2,3$, respectively. Any one of the three Pauli matrices $(\sigma_1,\sigma_2,\sigma_3)$ anticommutes with the other two Pauli matrices. Hence, the space of normalized boundary dynamical Dirac masses that anticommute pairwise is homeomorphic to S^2 with the homotopy group $\pi_{0+1+1}(S^2) = \mathbb{Z}$ when $\nu = 2$. A (0+1)-dimensional QNLSM for the (boundary) dynamical Dirac masses is augmented by a WZ term.

We conclude that the effects of interactions on the one-dimensional SPT phases in the symmetry class AIII are to reduce the topological classification $\mathbb{Z}$ in the noninteracting limit down to $\mathbb{Z}_4$ under the assumption that only fermion-number-conserving dynamical Dirac masses are included in the single-particle Hamiltonian. The logic used to reach this conclusion is summarized by Table IV once the line corresponding to $\nu = 1$ has been identified. It is given by the smallest D that accommodates a nontrivial entry for the corresponding homotopy group.

To include the effects of superconducting fluctuations in the single-particle Hamiltonian after a Hubbard-Stratonovich transformation, we need to consider the direct sum

$$\begin{aligned}\mathcal{H}^{(0)}_{\mathrm{BdG}}(x) &:= [\mathcal{H}^{(0)}(x)\otimes\mathbb{1}]\oplus[-\mathcal{H}^{(0)*}(x)\otimes\mathbb{1}]\\ &\equiv \mathcal{H}^{(0)}(x)\otimes\mathbb{1}\otimes\rho_0.\end{aligned} \tag{3.22a}$$

This Bogoliubov-de-Gennes (BdG) single-particle Hamiltonian anticommutes with the operation for charge conjugation

$$\mathcal{C} := \tau_0\otimes\mathbb{1}\otimes\rho_1\,\mathsf{K}, \tag{3.22b}$$

in addition to anticommuting with $\tau_1\otimes\mathbb{1}\otimes\rho_0$. Hence, it belongs to the symmetry class BDI. The dynamical single-particle Hamiltonian that accounts for superconducting fluctuations (dynamical Dirac masses from the symmetry class D) takes the form

$$\mathcal{V}(\tau,x) := \tau_1\otimes\gamma'(\tau,x), \tag{3.23a}$$

TABLE IV. Reduction from $\mathbb{Z}$ to $\mathbb{Z}_4$ for the topologically equivalent classes of the one-dimensional SPT phases in symmetry classes AIII or BDI that arises from the fermion-number-conserving interacting channels. We denote by V_ν the space of $\nu \times \nu$ normalized Dirac mass matrices in zero-dimensional Hamiltonians belonging to the symmetry class A. The limit $\nu \to \infty$ of these spaces is the classifying spaces C_0. The second column shows the stable D-th homotopy groups of the classifying space C_0. The third column gives the number ν of copies of boundary (Dirac) fermions for which a topological obstruction is permissible. The fourth column gives the type of topological obstruction that prevents the gapping of the boundary (Dirac) fermions.

D	$\pi_D(C_0)$	ν	Topological obstruction
0	$\mathbb{Z}$	1	Domain wall
1	0		
2	$\mathbb{Z}$	2	WZ term
3	0		
4	$\mathbb{Z}$	4	None
5	0		
6	$\mathbb{Z}$		
7	0		

where the $2\nu \times 2\nu$ Hermitian matrix $\gamma'(\tau,x)$ must obey

$$\gamma'(\tau,x) = -(\tau_0 \otimes \mathbb{1} \otimes \rho_1)\, \gamma'^{*}(\tau,x)\, (\tau_0 \otimes \mathbb{1} \otimes \rho_1). \tag{3.23b}$$

The stability analysis of the boundary zero modes is similar to the one performed below Eq. (3.9) except that one must replace ν in Eq. (3.9) by $\nu_{\text{BdG}} = 2\nu$ and that we have a different representation of the PHS. When $\nu = 1$,

$$\gamma'(\tau) = M(\tau)\, \rho_3, \quad M(\tau) \in \mathbb{R}, \tag{3.24}$$

supports a domain wall in imaginary time on the boundary. When $\nu = 2$, we use the representation $\mathbb{1} = \sigma_0$ and introduce the notation $X_{\mu\mu'} \equiv \sigma_\mu \otimes \rho_{\mu'}$ with $\mu,\mu' = 0,1,2,3$. Now,

$$\gamma'(\tau) = \sum_{\{(\mu,\mu')\}} M_{\mu\mu'}(\tau)\, X_{\mu\mu'}, \quad M_{\mu\mu'}(\tau) \in \mathbb{R}, \tag{3.25}$$

where the sum on the right-hand side is to be performed over the six matrices $X_{21}, X_{22}, X_{03}, X_{13}, X_{20}, X_{33}$. This set of six matrices decomposes into two triplets of pairwise anticommuting matrices. The first triplet is given by X_{21}, X_{22}, X_{03}. The second triplet is given by X_{13}, X_{20}, X_{33}. Each triplet defines a two-sphere S^2. Hence, each triplet has the potential to accommodate a WZ term. However, we must make sure that the integrity of any one S^2 entering the decomposition $S^2 \cup S^2$ of the normalized dynamical masses on the boundary is compatible with maintaining the global $U(1)$ symmetry associated with the conservation of the fermion number.

To address the fate of the fermion-number conservation, observe that Hamiltonian (3.19a) is invariant under the global $U(1)$ transformation

$$\mathcal{H}^{(0)} \mapsto \mathcal{U}(\alpha)\, \mathcal{H}^{(0)}\, \mathcal{U}^{-1}(\alpha), \quad \mathcal{U}(\alpha) := e^{+i\alpha\, \tau_0}, \tag{3.26}$$

with $0 \leqslant \alpha < 2\pi$ independent of x. This symmetry is to be preserved when treating superconducting fluctuations. In the BdG representation (3.22a), this symmetry becomes the symmetry under the global $U(1)$ transformation

$$\mathcal{H}^{(0)}_{\text{BdG}} \mapsto \mathcal{U}_{\text{BdG}}(\alpha)\, \mathcal{H}^{(0)}_{\text{BdG}}\, \mathcal{U}^{-1}_{\text{BdG}}(\alpha), \tag{3.27a}$$

where

$$\mathcal{U}_{\text{BdG}}(\alpha) := e^{+i\alpha\, \tau_0 \otimes \mathbb{1} \otimes \rho_3}. \tag{3.27b}$$

When $\nu = 1$, the boundary dynamical mass (3.24) is invariant under multiplication from the left with $\exp(+i\alpha\, \rho_3)$ and multiplication from the right with $\exp(-i\alpha\, \rho_3)$. When $\nu = 2$, if we normalize the boundary dynamical mass (3.25) by demanding that

$$1 = M_{21}^2 + M_{22}^2 + M_{03}^2, \tag{3.28a}$$
$$1 = M_{13}^2 + M_{20}^2 + M_{33}^2, \tag{3.28b}$$

we may then identify these boundary dynamical masses as the union of two two-spheres S^2. The global $U(1)$ transformation defined by multiplication from the left with $\exp(+i\alpha\, X_{03})$ and multiplication from the right with $\exp(-i\alpha\, X_{03})$ leaves the two-sphere (3.28a) invariant as a set, for it is represented by a rotation about the north pole X_{03} that rotates the equator spanned by X_{21} and X_{22} with the angle 2α. The same transformation leaves the two-sphere (3.28b) invariant pointwise. Hence, each S^2 in $S^2 \cup S^2$ is compatible with the conservation of the global fermion number. Because $\pi_{0+1+1}(S^2) = \mathbb{Z}$, a WZ topological term in the QNLSM for the boundary is permissible.

We may then safely conclude that the effects of interactions on the one-dimensional SPT phases in the symmetry class AIII are also to reduce the topological classification $\mathbb{Z}$ in the noninteracting limit down to $\mathbb{Z}_4$ under the assumption that superconducting fluctuation channels are included in the stability analysis. The $\mathbb{Z}_4$ classification for one-dimensional SPT phases in the symmetry class AIII agrees with the one derived using group cohomology in Ref. [61] (with AIII interpreted in Ref. [61] as a time-reversal-symmetric superconductor with the full spin-1/2 rotation symmetry broken down to a $U(1)$ subgroup).

b. Symmetry class BDI. Dirac Hamiltonians in the symmetry class BDI are obtained from those in the symmetry class AIII by imposing the constraint of TRS, Eqs. (3.3b) and (3.3d) (note that $r_{\min} = 2$ for both AIII and BDI classes in $d = 1$). Since the TRS is not relevant for dynamical Dirac masses in the single-particle Dirac Hamiltonian after a Hubbard-Stratonovich transformation, the stability analysis of gapless boundary states in the symmetry class BDI in $d = 1$ follows from that of the symmetry class AIII. Consequently, the effects of interactions in the symmetry class BDI, when interpreted as realizing complex fermions as opposed to Majorana fermions, is to reduce the topological classification $\mathbb{Z}$ in the noninteracting limit down to $\mathbb{Z}_4$ under the assumption that only fermion-number preserving dynamical Dirac masses taken from the symmetry class A are included in the stability analysis. Furthermore, if dynamical superconducting fluctuations are allowed by introducing an additional particle-hole grading and dynamical Dirac masses from the symmetry class D, then the same reduction pattern $\mathbb{Z} \to \mathbb{Z}_4$ follows. The topological classification $\mathbb{Z}_8$ of the symmetry class BDI when interpreted as describing Majorana fermions is thus finer than the classification $\mathbb{Z}_4$ of the symmetry class BDI when interpreted as describing complex fermions.

c. Symmetry class CII. We interpret the single-particle Hamiltonian (3.12a) as describing an insulator, not a superconductor. This is to say that the defining symmetries are TRS (3.12b) and the CHS,

$$\Gamma_5 \mathcal{H}^{(0)}(x) \Gamma_5^{-1} = -\mathcal{H}^{(0)}(x), \quad \Gamma_5 := X_{10}. \tag{3.29}$$

If we are after the dynamical effects of interactions that preserve the (complex) fermion number, we may use Eq. (3.16) with the only caveat that the dynamical mass matrix is now required to belong to the symmetry class A instead of the symmetry class C. The boundary dynamical Hamiltonian is then the same as for the symmetry classes AIII and BDI, i.e., Eq. (3.21), except for its rank being twice as large as compared to the symmetry classes AIII and BDI, for an additional grading (that of the spin-1/2 degrees of freedom) has been accounted for. This larger rank implies that the WZ term is already permissible at $\nu = 1$, i.e., the reduction pattern for the noninteracting topological classification is $\mathbb{Z} \to \mathbb{Z}_2$. It remains to verify that the same reduction pattern is also obtained if superconducting fluctuations are included, as was the case for the symmetry classes AIII and BDI. To this end, we must tensor (3.12a) with ρ_0, in which case the charge conjugation symmetry is realized by $\tau_0 \otimes \sigma_0 \otimes \rho_1 \mathsf{K}$. We can borrow the stability analysis with respect to superconducting interacting channels from the symmetry class D that we performed for the symmetry classes AIII and BDI, again with the caveat that the rank of the boundary dynamical Hamiltonian is twice as large as it was. This larger rank implies again that the WZ term is already permissible for $\nu = 1$, i.e., the reduction pattern for the noninteracting topological classification is again $\mathbb{Z} \to \mathbb{Z}_2$. Thus we obtain the same topological classification $\mathbb{Z}_2$ of the symmetry class CII in $d = 1$ both when interpreted as describing Majorana fermions (superconductors) and when interpreted as describing complex fermions (insulators).

B. The case of two-dimensional space

The notion that the chiral edge modes in the IQHE are immune to local interactions is rather intuitive. Neither backscattering nor umklapp scattering is permissible. An operational and quantitative validation for this intuition goes back to Niu and Thouless in Ref. [9], who proposed to average the Kubo Hall conductivity over all twisted boundary conditions of the many-body ground state as a signature of both the IQHE and FQHE. A mathematically rigorous proof of this can be found in Refs. [62,63]. This readily extends to the symmetry class D and C as they realize quantized thermal Hall effects. The robustness of chiral edge modes in the symmetry classes D, C, and A to quartic contact interactions will be derived using the method shown in Sec. II.

Let $\boldsymbol{x} = (x_1, x_2)$ denote a point in two-dimensional space. The single-particle Dirac Hamiltonian with the smallest rank $r_{\min} = 2$ that admits a Dirac mass can be chosen to be represented by

$$\begin{aligned}\mathcal{H}^{(0)}_{\mathrm{A}}(\boldsymbol{x}) :=& [-i\partial_1 + A_1(\boldsymbol{x})]\sigma_3 + [-i\partial_2 + A_2(\boldsymbol{x})]\sigma_1 \\ &+ A_0(\boldsymbol{x})\,\sigma_0 + m(\boldsymbol{x})\,\sigma_2.\end{aligned} \tag{3.30}$$

It belongs to the symmetry class A for arbitrarily chosen vector potentials $\boldsymbol{A}(\boldsymbol{x})$, scalar potential $A_0(\boldsymbol{x})$, and mass $m(\boldsymbol{x})$. When the gauge fields are vanishing,

$$\begin{aligned}\mathcal{H}^{(0)}_{\mathrm{D}}(\boldsymbol{x}) :=& -i\partial_1\,\sigma_3 - i\partial_2\,\sigma_1 + m(\boldsymbol{x})\,\sigma_2 \\ =& -[\mathcal{H}^{(0)}(\boldsymbol{x})]^*\end{aligned} \tag{3.31}$$

belongs to the symmetry class D. Finally, the single-particle Hamiltonian with the smallest rank $r_{\min} = 4$ that belongs to the symmetry class C can be chosen to be represented by

$$\begin{aligned}\mathcal{H}^{(0)}_{\mathrm{C}}(\boldsymbol{x}) :=& -i\partial_1\,X_{30} + \sum_{j=1}^{3} A_{1j}(\boldsymbol{x})\,X_{3j} \\ &- i\partial_2\,X_{10} + \sum_{j=1}^{3} A_{2j}(\boldsymbol{x})\,X_{1j} \\ &+ \sum_{j=1}^{3} A_{0j}(\boldsymbol{x})\,X_{0j} + m(\boldsymbol{x})\,X_{20} \\ =& -X_{02}[\mathcal{H}^{(0)}(\boldsymbol{x})]^* X_{02}.\end{aligned} \tag{3.32}$$

In two spatial dimensions the symmetry classes A, D, and C realize noninteracting topological insulators and superconductors with the Grassmannian manifolds $C_0 \equiv \lim_{N\to\infty} \cup_{n=0}^{N} U(N)/[U(n)\times U(N-n)]$, $R_0 \equiv \lim_{N\to\infty} \cup_{n=0}^{N} O(N)/[O(n)\times)(N-n)]$, and $R_4 \equiv \lim_{N\to\infty} \cup_{n=0}^{N} Sp(N)/[Sp(n)\times)(N-n)]$ as classifying spaces, respectively. They share the same zeroth-order homotopy group $\mathbb{Z}$. This group also serves as defining the topological attributes of noninteracting topological insulators and superconductors in the symmetry classes A, D, and C.

1. The symmetry class D when $d = 2$

Let $\mathbb{1}$ denote a $\nu \times \nu$ unit matrix with $\nu = 1,2,\ldots$. Consider the two-dimensional bulk single-particle Dirac Hamiltonian

$$\mathcal{H}^{(0)}(\boldsymbol{x}) := -i\partial_1\,\sigma_3 \otimes \mathbb{1} - i\partial_2\,\sigma_1 \otimes \mathbb{1} + m(\boldsymbol{x})\,\sigma_2 \otimes \mathbb{1} \tag{3.33}$$

of rank 2ν. There is no Hermitian $(2\nu)\times(2\nu)$ matrix that anticommutes with $\mathcal{H}^{(0)}_{\mathrm{A}}(\boldsymbol{x})$. If so, the set $\{\beta\}$ in Eq. (2.3) is empty. In other words, no dynamical mass is available to induce a dynamical instability of the ν boundary zero modes.

2. The symmetry class C when $d = 2$

The same reasoning applies to the bulk single-particle Hamiltonian

$$\mathcal{H}^{(0)}(\boldsymbol{x}) := [-i\partial_1\,X_{30} - i\partial_2\,X_{10} + m(\boldsymbol{x})\,X_{20}] \otimes \mathbb{1} \tag{3.34a}$$

of rank 4ν that realizes a topological superconductor in the symmetry class C,

$$\mathcal{H}^{(0)}(\boldsymbol{x}) = -(X_{02}\otimes\mathbb{1})[\mathcal{H}^{(0)}(\boldsymbol{x})]^*(X_{02}\otimes\mathbb{1}). \tag{3.34b}$$

No dynamical mass is available to induce a dynamical instability of the ν boundary zero modes.

3. The symmetry class A when $d = 2$

If the single-particle Dirac Hamiltonian (3.33) is interpreted as describing an insulator with fermion-number conservation, then no dynamical mass that preserves the fermion number and anticommutes with $\sigma_2 \otimes \mathbb{1}$ is permissible. The same remains true if we account for superconducting fluctuations for the BdG extension of (3.33) that is given by Eq. (3.34a), whereby charge conjugation is defined by [and not by Eq. (3.34b)]

$$\mathcal{C} := X_{01} \otimes \mathbb{1}\,\mathsf{K} \tag{3.35}$$

fails to anticommute with any $(4\nu) \times (4\nu)$ Hermitian matrix allowed by the PHS generated by the operation of charge conjugation (3.35).

C. The case of three-dimensional space

The reduction $\mathbb{Z} \to \mathbb{Z}_{16}$ for the three-dimensional interacting topological superconductors belonging to the symmetry class DIII has been understood in the following ways after a conjecture by Kitaev from Ref. [18]. One approach is to enumerate the distinct topological orders at the two-dimensional surface of the three-dimensional bulk that cannot be realized with any bulk two-dimensional Hamiltonian [19–21]. In this approach, the breakdown of $\mathbb{Z}$ takes place when vortices (pointlike defects of a symmetry-broken phase) at the surface proliferate (deconfine) to stabilize a gapped and fully symmetric surface phase. Another approach advocated by You and Xu consists in relating fermionic short-ranged entangled ground states to bosonic short-ranged entangled ground states [22]. They also applied their approach to systems with inversion symmetry. The reductions $\mathbb{Z} \to \mathbb{Z}_4$ and $\mathbb{Z} \to \mathbb{Z}_8$ for the symmetry classes CI and AIII was obtained in Ref. [21]. Finally, Kapustin has proposed classifying symmetry-protected topological phases for interacting bosons or fermions by considering low-energy effective actions that are invariant under cobordism (a certain type of equivalence relation between manifolds) [64–66].

We apply the method of Sec. II to the symmetry classes DIII, CI, and AIII in the presence of quartic contact interactions. We recover the reductions $\mathbb{Z} \to \mathbb{Z}_{16}$, $\mathbb{Z} \to \mathbb{Z}_4$, and $\mathbb{Z} \to \mathbb{Z}_8$ for the symmetry class DIII, CI, and AIII, respectively. We also verify that the topological classification $\mathbb{Z}_2$ of the symmetry class AII is stable to quartic contact interactions.

We shall denote with $\boldsymbol{x} \equiv (x,y,z) \equiv (x_1,x_2,x_3)$ a point in three-dimensional space.

1. The symmetry class DIII when $d = 3$

Let $X_{\mu\mu'} \equiv \tau_\mu \otimes \rho_{\mu'}$ with $\mu,\mu' = 0,1,2,3$. Consider the three-dimensional bulk single-particle Dirac Hamiltonian (with Dirac matrices of dimension $r = 4 \equiv r_{\text{min}}$),

$$\mathcal{H}^{(0)}(\boldsymbol{x}) := -i\partial_1 X_{31} - i\partial_2 X_{02} - i\partial_3 X_{11} + m(\boldsymbol{x}) X_{03}. \tag{3.36a}$$

This single-particle Hamiltonian belongs to the three-dimensional symmetry class DIII, for

$$\mathcal{T}\,\mathcal{H}^{(0)}(\boldsymbol{x})\,\mathcal{T}^{-1} = +\mathcal{H}^{(0)}(\boldsymbol{x}), \tag{3.36b}$$

$$\mathcal{C}\,\mathcal{H}^{(0)}(\boldsymbol{x})\,\mathcal{C}^{-1} = -\mathcal{H}^{(0)}(\boldsymbol{x}), \tag{3.36c}$$

where

$$\mathcal{T} := i X_{20}\,\mathsf{K}, \quad \mathcal{C} := X_{01}\,\mathsf{K}. \tag{3.36d}$$

The multiplicative factor i in the definition of $\mathcal{T}$ is needed for $\mathcal{T}$ to commute with $\mathcal{C}$.

The Dirac mass matrix X_{03} is here the only one allowed for dimension four Dirac matrices under the constraints (3.36b) and (3.36c). Consequently, the domain wall

$$m(\boldsymbol{x}) \equiv m(y) := m_\infty \,\text{sgn}(y), \quad m_\infty \in \mathbb{R}, \tag{3.37a}$$

at $y = 0$, binds the zero mode

$$e^{-iX_{02}X_{03}\int_0^y dy'\,m(y')}\,\chi = e^{-|m_\infty y|}\,\chi, \tag{3.37b}$$

where

$$X_{01}\,\chi = -\text{sgn}\,(m_\infty)\,\chi \tag{3.37c}$$

with χ independent of x and z. The kinetics of the gapless boundary states is governed by the Dirac Hamiltonian

$$\mathcal{H}^{(0)}_{\text{bd}}(x,z) = -i\partial_x\tau_3 - i\partial_z\tau_1, \tag{3.38}$$

where we have chosen $m_\infty < 0$.

On the boundary, the operations for reversal of time and charge conjugation are now represented by

$$\mathcal{T}_{\text{bd}\,\nu} := i\tau_2 \otimes \mathbb{1}\,\mathsf{K}, \quad \mathcal{C}_{\text{bd}\,\nu} := \tau_0 \otimes \mathbb{1}\,\mathsf{K}, \tag{3.39a}$$

where $\mathbb{1}$ is the $\nu \times \nu$ unit matrix. We seek the single-particle Hamiltonian on the boundary that encodes the fluctuations arising from the Hubbard-Stratonovich decoupling of quartic interactions through a generic dynamical mass that respects the PHS on the boundary. It is given by

$$\mathcal{H}^{(\text{dyn})}_{\text{bd}\,\nu}(\tau,x,z) := -i\partial_x\tau_3 \otimes \mathbb{1} - i\partial_z\tau_1 \otimes \mathbb{1} + \tau_2 \otimes M(\tau,x,z) \tag{3.39b}$$

with the $\nu \times \nu$ real-valued and symmetric matrix

$$M(\tau,x,z) = M^*(\tau,x,z) = M^{\mathsf{T}}(\tau,x,z). \tag{3.39c}$$

The space of normalized Dirac mass matrices of the form (3.39c) is topologically equivalent to the space

$$V_\nu := \bigcup_{k=1}^{\nu} O(\nu)/[O(k) \times O(\nu - k)] \tag{3.40}$$

for the symmetry class D in two-dimensional space. The limit $\nu \to \infty$ of these spaces is the classifying space R_0. In order to gap out dynamically the boundary zero modes without breaking the defining symmetries of the symmetry class DIII, we need to construct a (2+1)-dimensional QNLSM for the (boundary) dynamical Dirac masses from the two-dimensional symmetry class D without topological obstructions. We construct explicitly the spaces for the relevant normalized boundary dynamical Dirac mass matrices of dimension $\nu = 2^n$ with $n = 0,1,2,3,4$ in the following. The relevant homotopy groups are given in Table V.

Case $\nu = 1$*:* There is one boundary dynamical Dirac mass matrix $\gamma'(\tau,x,z)$ on the boundary that is proportional to τ_2. A domain wall in imaginary time such as $m_{2\infty}\,\text{sgn}(\tau)\,\tau_2$ prevents the dynamical generation of a spectral gap on the boundary.

TABLE V. Reduction from $\mathbb{Z}$ to $\mathbb{Z}_{16}$ for the topologically equivalent classes of the three-dimensional SPT phases in the symmetry class DIII that arises from interactions. We denote by V_ν the space of $\nu \times \nu$ normalized Dirac mass matrices in boundary ($d = 2$) Dirac Hamiltonians belonging to the symmetry class D. The limit $\nu \to \infty$ of these spaces is the classifying space R_0. The second column shows the stable D-th homotopy groups of the classifying space R_0. The third column gives the number ν of copies of boundary (Dirac) fermions for which a topological obstruction is permissible. The fourth column gives the type of topological obstruction that prevents the gapping of the boundary (Dirac) fermions.

D	$\pi_D(R_0)$	ν	Topological obstruction
0	$\mathbb{Z}$	1	Domain wall
1	$\mathbb{Z}_2$	2	Vortex line
2	$\mathbb{Z}_2$	4	Monopole
3	0		
4	$\mathbb{Z}$	8	WZ term
5	0		
6	0		
7	0		
8	$\mathbb{Z}$	16	None

Case $\nu = 2$*:* We use the representation $\mathbb{1} = \sigma_0$. The 2×2 real-valued and symmetric matrix $M(\tau,x,z)$ is a linear combination with real-valued coefficients of the pair of anticommuting matrices σ_x and σ_z. If $M(\tau,x,z)$ is normalized by demanding that it squares to σ_0, then the set spanned by $M(\tau,x,z)$ is homeomorphic to the one-sphere S^1. As $\pi_1(S^1) = \mathbb{Z}$, it follows that $M(\tau,x,z)$ supports vortex lines that bind zero modes in (2+1)-dimensional space and time and thus prevent the gapping of the boundary states [67].

Case $\nu = 4$*:* We use the representation $\mathbb{1} = \sigma_0 \otimes \sigma'_0$. The 4×4 real-valued and symmetric matrix $M(\tau,x,z)$ is a linear combination with real coefficients of $X_{\sigma_\mu \sigma'_{\mu'}} \equiv \sigma_\mu \otimes \sigma'_{\mu'}$ with $\mu,\mu' = 0,1,2,3$ such that either none or two of μ and μ' equal the number 2. Of these the three matrices X_{13}, X_{33}, and X_{01} anticommute pairwise. If $M(\tau,x,z)$ is a linear combinations with real-valued coefficients of these three matrices and if M is normalized by demanding that it squares to X_{00}, then the set spanned by $M(\tau,x,z)$ is homeomorphic to the two-sphere S^2. As $\pi_2(S^2) = \mathbb{Z}$, $M(\tau,x,z)$ supports pointlike defects of the monopole type that bind zero modes in (2+1)-dimensional space and time and thus prevent the gapping of the boundary states [68].

Case $\nu = 8$*:* We use the representation $\mathbb{1} = \sigma_0 \otimes \sigma'_0 \otimes \sigma''_0$. The 8×8 real-valued and symmetric matrix $M(\tau,x,z)$ is a linear combination with real-valued coefficients of the matrices $X_{\mu\mu'\mu''} \equiv \sigma_\mu \otimes \sigma'_{\mu'} \otimes \sigma''_{\mu''}$ where either none or two of $\mu,\mu',\mu'' = 0,1,2,3$ equal the number 2. Of these, one finds the five pairwise anticommuting matrices X_{333}, X_{133}, X_{013}, X_{001}, and X_{212}. If $M(\tau,x,z)$ is a linear combination with real-valued coefficients of these five matrices and if M is normalized by demanding that it squares to X_{000}, then the set spanned by $M(\tau,x,z)$ is homeomorphic to the four-sphere S^4. As $\pi_4(S^4) = \mathbb{Z}$, it is possible to add a topological term to the QNLSM on the boundary that is of the WZ type. This term is conjectured to prevent the gapping of the boundary states.

Case $\nu = 16$*:* We use the representation $\mathbb{1} = \sigma_0 \otimes \sigma'_0 \otimes \sigma''_0 \otimes \sigma'''_0$. The 16×16 real-valued and symmetric matrix $M(\tau,x,z)$ is a linear combination with real-valued coefficients of the matrices $X_{\mu\mu'\mu''\mu'''} = \sigma_\mu \otimes \sigma'_{\mu'} \otimes \sigma''_{\mu''} \otimes \sigma'''_{\mu'''}$ where none, two, or four of $\mu,\mu',\mu'',\mu''' = 0,1,2,3$ equal the number 2. Of these, one finds the nine pairwise anticommuting matrices X_{2222}, X_{0122}, X_{0322}, X_{2012}, X_{2032}, X_{1202}, X_{3202}, X_{0001}, and X_{0003}. If $M(\tau,x,z)$ is a linear combination with real-valued coefficients of these nine matrices and if $M(\tau,x,z)$ is normalized by demanding that it squares to X_{0000}, then the set spanned by $M(\tau,x,z)$ is homeomorphic to the eight-sphere S^8. It is then impossible to add a topological term to the QNLSM on the boundary. The boundary zero modes can be gapped out.

We conclude that the effects of interactions on the three-dimensional SPT phases in the symmetry class DIII are to reduce the topological classification $\mathbb{Z}$ in the noninteracting limit down to $\mathbb{Z}_{16}$. The logic used to reach this conclusion is summarized by Table V once the line corresponding to $\nu = 1$ has been identified. It is given by the smallest D that accommodates a nontrivial entry for the corresponding homotopy group. The line for $\nu = 2$ is then identified with the next smallest D with $\pi_D(R_2) \neq 0$ and so on.

2. The symmetry class CI when $d = 3$

Let $X_{\mu\nu\lambda} \equiv \tau_\mu \otimes \rho_\nu \otimes \sigma_\lambda$ with $\mu,\nu,\lambda = 0,1,2,3$. Consider the three-dimensional bulk single-particle Dirac Hamiltonian (with Dirac matrices of dimension $r = 8 \equiv r_{\min}$),

$$\mathcal{H}^{(0)}(\boldsymbol{x}) := -i\partial_1 X_{310} - i\partial_2 X_{020} - i\partial_3 X_{110} + m(\boldsymbol{x})\, X_{030}. \tag{3.41a}$$

This single-particle Hamiltonian belongs to the three-dimensional symmetry class CI, for

$$\mathcal{T}\,\mathcal{H}^{(0)}(\boldsymbol{x})\,\mathcal{T}^{-1} = +\mathcal{H}^{(0)}(\boldsymbol{x}), \tag{3.41b}$$

$$\mathcal{C}\,\mathcal{H}^{(0)}(\boldsymbol{x})\,\mathcal{C}^{-1} = -\mathcal{H}^{(0)}(\boldsymbol{x}), \tag{3.41c}$$

where

$$\mathcal{T} := X_{202}\,\mathsf{K}, \quad \mathcal{C} := i\,X_{012}\,\mathsf{K}. \tag{3.41d}$$

The multiplicative factor i in the definition of $\mathcal{C}$ is needed for $\mathcal{T}$ to commute with $\mathcal{C}$.

The single-particle Hamiltonian (3.41a) is the direct product of the single-particle Hamiltonian (3.36a) with the unit 2×2 matrix σ_0. If we interpret the degrees of freedom encoded by σ_0 and the Pauli matrices $\boldsymbol{\sigma}$ as carrying spin-1/2 degrees of freedom, we may then interpret Eqs. (3.41) as defining a spin-singlet superconductor that preserves TRS.

The Dirac mass matrix X_{030} is here the only one allowed for dimension eight Dirac matrices under the constraints (3.41b) and (3.41c). Consequently, the domain wall

$$m(\boldsymbol{x}) \equiv m(y) := m_\infty\, \mathrm{sgn}(y), \quad m_\infty \in \mathbb{R}, \tag{3.42a}$$

at $y = 0$, binds the zero mode

$$e^{-i X_{020} X_{030} \int_0^y dy'\, m(y')}\, \chi = e^{-|m_\infty y|}\chi, \tag{3.42b}$$

where

$$X_{010}\, \chi = -\mathrm{sgn}(m_\infty)\, \chi \tag{3.42c}$$

with χ independent of x and z. The kinetics of the gapless boundary states is governed by the Dirac Hamiltonian

$$\mathcal{H}^{(0)}_{\mathrm{bd}}(x,z) = -i\partial_x\tau_3\otimes\sigma_0 - i\partial_z\tau_1\otimes\sigma_0, \tag{3.43}$$

where we have chosen $m_\infty < 0$.

On the boundary, the operations for reversal of time and charge conjugation are now represented by

$$\mathcal{T}_{\mathrm{bd}\,\nu} := \tau_2\otimes\sigma_2\otimes\mathbb{1}\mathsf{K}, \quad \mathcal{C}_{\mathrm{bd}\,\nu} := i\,\tau_0\otimes\sigma_2\otimes\mathbb{1}\mathsf{K}, \tag{3.44a}$$

where $\mathbb{1}$ is the $\nu\times\nu$ unit matrix. We seek the single-particle Hamiltonian on the boundary that encodes the fluctuations arising from the Hubbard-Stratonovich decoupling of quartic interactions through a generic dynamical mass that respects the PHS on the boundary. It is given by

$$\begin{aligned}\mathcal{H}^{(\mathrm{dyn})}_{\mathrm{bd}\,\nu}(\tau,x,z) :=& -i\partial_x\tau_3\otimes\sigma_0\otimes\mathbb{1} - i\partial_z\tau_1\otimes\sigma_0\otimes\mathbb{1}\\ &+ \tau_2\otimes M(\tau,x,z),\end{aligned} \tag{3.44b}$$

with the $2\nu\times 2\nu$ Hermitian matrix

$$M(\tau,x,z) = +(\sigma_2\otimes\mathbb{1})\,M^{*}(\tau,x,z)\,(\sigma_2\otimes\mathbb{1}). \tag{3.44c}$$

The space of normalized Dirac mass matrices satisfying the condition (3.44c) is topologically equivalent to the space

$$V_\nu := \bigcup_{k=1}^{\nu} Sp(\nu)/[Sp(k)\times Sp(\nu-k)] \tag{3.45}$$

for the symmetry class C in two-dimensional space. The limit $\nu\to\infty$ of these spaces is the classifying space R_4. In order to gap out dynamically the boundary zero modes without breaking the defining symmetries of the symmetry class CI, we need to construct a (2+1)-dimensional QNLSM for the (boundary) dynamical Dirac masses from the two-dimensional symmetry class C without topological obstructions. We construct explicitly the spaces for the relevant normalized boundary dynamical Dirac mass matrices of dimension $\nu = 2^n$ with $n = 0,1,2$ in the following. The relevant homotopy groups are given in Table VI.

Case $\nu = 1$*:* There is one 2×2 Hermitian matrix $M(\tau,x,z)$ on the boundary that is proportional to σ_0. A domain wall in imaginary time such as $m_{2\,\infty}\,\mathrm{sign}(\tau)\,\tau_2\otimes\sigma_0$ prevents the dynamical generation of a spectral gap on the boundary.

Case $\nu = 2$*:* We use the representation $\mathbb{1} = \sigma'_0$. The Hermitian 4×4 matrix $M(\tau,x,z)$ is a linear combination with real-valued coefficients of the matrices $X_{\mu\mu'} \equiv \sigma_\mu\otimes\sigma'_{\mu'}$ with $\mu,\mu' = 0,1,2,3$ such that $X_{20}\,X^{*}_{\mu\mu'}\,X_{20} = +X_{\mu\mu'}$. Of these, one finds the five matrices X_{12}, X_{22}, X_{32}, X_{01}, and X_{03} that anticommute pairwise. If $M(\tau,x,z)$ is a linear combinations with real-valued coefficients of these five matrices and if $M(\tau,x,z)$ is normalized by demanding that it squares to X_{00}, then the set spanned by $M(\tau,x,z)$ is homeomorphic to S^4. As $\pi_{2+1+1}(S^4) = \mathbb{Z}$, it is then possible to add a topological term to the (2+1)-dimensional QNLSM on the boundary that is of the WZ type. This term is conjectured to prevent the gapping of the boundary states.

Case $\nu = 4$*:* We use the representation $\mathbb{1} = \sigma'_0\otimes\sigma''_0$. The Hermitian 8×8 matrix $M(\tau,x,z)$ is a linear combination with real-valued coefficients of the matrices $X_{\mu\mu'\mu''} \equiv \sigma_\mu\otimes\sigma'_{\mu'}\otimes\sigma''_{\mu''}$ with $\mu,\mu',\mu'' = 0,1,2,3$ such that $X_{200}\,X^{*}_{\mu\mu'\mu''}\,X_{200} = +X_{\mu\mu'\mu''}$. Of these, one finds the six matrices X_{120}, X_{220}, X_{320}, X_{010}, X_{031}, and X_{033} that anticommute pairwise. If $M(\tau,x,z)$ is a linear combinations with real-valued coefficients of these six matrices and if $M(\tau,x,z)$ is normalized by demanding that it squares to X_{000}, then the set spanned by $M(\tau,x,z)$ is homeomorphic to S^5. It is then impossible to add a topological term to the (2+1)-dimensional QNLSM on the boundary. The boundary zero modes can be gapped out.

TABLE VI. Reduction from $\mathbb{Z}$ to $\mathbb{Z}_4$ for the topologically equivalent classes of the three-dimensional SPT phases in the symmetry class CI that arises from interactions. We denote by V_ν the space of $\nu\times\nu$ normalized Dirac mass matrices in boundary ($d=2$) Dirac Hamiltonians belonging to the symmetry class C. The limit $\nu\to\infty$ of these spaces is the classifying space R_4. The second column shows the stable D-th homotopy groups of the classifying space R_4. The third column gives the number ν of copies of boundary (Dirac) fermions for which a topological obstruction is permissible. The fourth column gives the type of topological obstruction that prevents the gapping of the boundary (Dirac) fermions.

D	$\pi_D(R_4)$	ν	Topological obstruction
0	$\mathbb{Z}$	1	Domain wall
1	0		
2	0		
3	0		
4	$\mathbb{Z}$	2	WZ term
5	$\mathbb{Z}_2$	4	None
6	$\mathbb{Z}_2$		
7	0		

We conclude that the effects of interactions on the three-dimensional SPT phases in the symmetry class CI are to reduce the topological classification $\mathbb{Z}$ in the noninteracting limit down to $\mathbb{Z}_4$. The logic used to reach this conclusion is summarized by Table VI once the line corresponding to $\nu = 1$ has been identified. It is given by the smallest D that accommodates a nontrivial entry for the corresponding homotopy group.

3. *The symmetry class AIII when* $d = 3$

By omitting the contributions arising from the gauge potentials, the single-particle Hamiltonian (3.36a) does not specify uniquely the symmetry class. For example, the single-particle Hamiltonian (3.36a) can also be interpreted as an insulator belonging to the symmetry class AIII, for it anticommutes with the composition

$$\Gamma_5 := -i\mathcal{T}\mathcal{C} = X_{21} \tag{3.46}$$

of the operations $\mathcal{T}$ and $\mathcal{C}$ for time reversal and charge conjugation, respectively, defined in Eq. (3.36d).

The direct product of the single-particle Hamiltonian (3.36a) with the $\nu\times\nu$ unit matrix $\mathbb{1}$ supports ν zero modes bound to the boundary $y = 0$, for they are annihilated by the boundary Hamiltonian

$$\mathcal{H}^{(0)}_{\mathrm{bd}\,\nu}(x,z) := -i\partial_x\,\tau_3\otimes\mathbb{1} - i\partial_z\,\tau_1\otimes\mathbb{1} \tag{3.47a}$$

that anticommutes with

$$\Gamma^{(\mathrm{bd})}_5 := \tau_2\otimes\mathbb{1}. \tag{3.47b}$$

The fate of these zero modes in the presence of fermion-fermion interactions is investigated in two steps. First, we include the effects of interactions by perturbing the boundary Hamiltonian with all boundary dynamical Dirac masses from the symmetry class A. Second, we introduce a BdG (Nambu) grading to account for the interactions-driven superconducting fluctuations by perturbing the boundary Hamiltonian $\mathcal{H}^{(0)}_{\mathrm{bd}\,\nu}(x,z)\otimes\rho_0$ with all boundary dynamical Dirac masses that anticommute with $\tau_0\otimes\mathbb{1}\otimes\rho_1\,\mathsf{K}$, i.e., with all boundary dynamical Dirac masses from the symmetry class D. In the first step, the boundary dynamical single-particle Hamiltonian is

$$\mathcal{H}^{(\mathrm{dyn})}_{\mathrm{bd}\,\nu}(\tau,x,z):=(-i\partial_x\,\tau_3-i\partial_z\,\tau_1)\otimes\mathbb{1}+\tau_2\otimes M(\tau,x,z), \tag{3.48a}$$

with the $\nu\times\nu$ Hermitian matrix

$$M(\tau,x,z)=M^{\dagger}(\tau,x,z). \tag{3.48b}$$

In the second step, the boundary dynamical single-particle Hamiltonian is

$$\begin{aligned}\mathcal{H}^{(\mathrm{dyn})}_{\mathrm{bd}\,\nu}(\tau,x,z):=&(-i\partial_x\,\tau_3-i\partial_z\,\tau_1)\otimes(\rho_0\otimes\mathbb{1})\\&+\tau_2\otimes M(\tau,x,z),\end{aligned} \tag{3.49a}$$

with the $2\nu\times2\nu$ Hermitian matrix

$$M(\tau,x,z)=+(\rho_1\otimes\mathbb{1})\,M^{*}(\tau,x,z)\,(\rho_1\otimes\mathbb{1}). \tag{3.49b}$$

The space of boundary dynamical Dirac mass matrices of the form (3.48b) that square to the unit matrix is homeomorphic to the classifying space C_0 for the symmetry class A in two-dimensional space and in the limit $\nu\to\infty$. In order to gap out dynamically the boundary zero modes without breaking the defining symmetries of the symmetry class AIII, we need to construct a (2+1)-dimensional QNLSM for the (boundary) dynamical Dirac masses from the two-dimensional symmetry class A without topological obstructions. When $\nu=1$, a domain wall such as $M(\tau,x,z)=M_{\infty}\,\mathrm{sgn}(\tau)$ prevents the gapping of the boundary zero modes. When $\nu=2$, we choose the representation $\mathbb{1}=\sigma_0$. The set spanned by $M(\tau,x,z)=\sum_{j=1}^{3}m_j(\tau,x,z)\,\sigma_j$ with the real-valued functions $m_j(\tau,x,z)$ obeying the normalization condition $\sum_{j=1}^{3}m_j^2(\tau,x,z)=1$ supports a monopole that binds zero modes in (2+1)-dimensional space and time, as $\pi_2(S^2)=\mathbb{Z}$. When $\nu=4$, we choose the representation $\mathbb{1}=X_{00}$ where $X_{\mu\mu'}:=\sigma_\mu\otimes\sigma'_{\mu'}$ for $\mu,\mu'=0,1,2,3$. We may then write $M(\tau,x,z)=\sum_{\mu,\mu'=0}^{3}m_{\mu\mu'}(\tau,x,z)\,X_{\mu\mu'}$. Any $X_{\mu\mu'}$ other than the unit matrix X_{00} belongs to a multiplet of five pairwise anticommuting matrices of the form $X_{\nu\nu'}\neq X_{00}$. Hence, we may always construct a set of normalized $M(\tau,x,z)$ homeomorphic to S^4. Since $\pi_{2+1+1}(S^4)=\mathbb{Z}$, it is possible to augment the corresponding boundary QNLSM in (2+1)-dimensional space and time by a WZ term that modifies the equations of motion in a local way. This term is conjectured to prevent the gapping of the boundary states. When $\nu=2^n$ with $n\geqslant3$, we choose the representation $\mathbb{1}=X_{00\cdots}$ where $X_{\mu\mu'\cdots}:=\sigma_\mu\otimes\sigma'_{\mu'}\otimes\cdots$ for $\mu,\mu',\cdots=0,1,2,3$. Any $X_{\mu\mu'\cdots}$ other than the unit matrix $X_{00\cdots}$ belongs to a multiplet of no less than seven pairwise anticommuting matrices. It is for this reason that the boundary states are then necessarily gapped, for it is not permissible to add a topological term to the action of the boundary QNLSM for a sphere of dimension larger than 4. We conclude that the effects of interactions in the three-dimensional SPT phases in the symmetry class AIII is to reduce the topological classification $\mathbb{Z}$ in the noninteracting limit down to $\mathbb{Z}_8$ under the assumption that only fermion-number-conserving interacting channels are included in the stability analysis. The logic used to reach this conclusion is summarized by Table VII once the line corresponding to $\nu=1$ has been identified. It is given by the smallest D that accommodates a nontrivial entry for the corresponding homotopy group.

TABLE VII. Reduction from $\mathbb{Z}$ to $\mathbb{Z}_8$ for the topologically equivalent classes of the three-dimensional SPT phases in the symmetry classes AIII that arises from the fermion-number-conserving interacting channels. We denote by V_ν the space of $\nu\times\nu$ normalized Dirac mass matrices in boundary ($d=2$) Dirac Hamiltonians belonging to the symmetry class A. The limit $\nu\to\infty$ of these spaces is the classifying spaces C_0. The second column shows the stable D-th homotopy groups of the classifying space C_0. The third column gives the number ν of copies of boundary (Dirac) fermions for which a topological obstruction is permissible. The fourth column gives the type of topological obstruction that prevents the gapping of the boundary (Dirac) fermions.

D	$\pi_D(C_0)$	ν	Topological obstruction
0	$\mathbb{Z}$	1	Domain wall
1	0		
2	$\mathbb{Z}$	2	Monopole
3	0		
4	$\mathbb{Z}$	4	WZ term
5	0		
6	$\mathbb{Z}$	8	None
7	0		

The space of boundary dynamical matrices that satisfy the condition (3.49b) and square to the unit matrix is homeomorphic to the classifying space R_2 for the symmetry class D in two-dimensional space and in the limit $\nu\to\infty$. Because the dimension of the boundary dynamical matrix (3.49b) is twice that of the boundary dynamical matrix (3.48b), one might have guessed that the gapping of the boundary zero modes takes place for a value of ν smaller than eight. This is not so, however, because of two constraints. The first constraint is that of PHS. The second constraint restricts the target space for the boundary QNLSM that is built out of the boundary dynamical Dirac masses. The target space of the QNLSM must be invariant as a set under the action of a global gauge $U(1)$ transformation that is generated by $\tau_0\otimes\rho_3\otimes\mathbb{1}$. This global $U(1)$ symmetry implements conservation of the fermion number. Indeed, one verifies the following facts. When $\nu=1$, the boundary dynamical matrix $M(\tau,x,z)$ is a linear combination of ρ_1 and ρ_2 with real-valued functions as coefficients. Hence, the space of normalized boundary dynamical Dirac mass matrices is homeomorphic to S^1 and invariant as a set under any global gauge $U(1)$ transformation when $\nu=1$.

Because of $\pi_1(S^1)=\mathbb{Z}$, vortex lines bind zero modes in (2+1)-dimensional space and time. When $\nu=2$, we represent the unit 4×4 matrix by $\rho_0\otimes\sigma_0$ and we expand any 4×4 Hermitian matrix as a linear combination with real-valued functions as coefficients of the 16 matrices $X_{\mu\mu'}=\rho_\mu\otimes\sigma_{\mu'}$ with $\mu,\mu'=0,1,2,3$. The boundary dynamical matrix $M(\tau,x,z)$ is a linear combination with real-valued functions as coefficients of the 10 matrices X_{00}, X_{01}, X_{03}, X_{10}, X_{11}, X_{13}, X_{20}, X_{21}, X_{23}, X_{32} that satisfy the constraint $X_{\mu\mu'}=+X_{10}\,X^{*}_{\mu\mu'}\,X_{10}$. Other than the unit matrix X_{00}, any one of these 9 matrices belongs to a triplet of pairwise anticommuting matrices. However, not all such triplets are closed under the global $U(1)$ transformation defined by multiplication from the left and right (conjugation) with X_{30}. However, there exists a triplet that is closed under conjugation by X_{30}. For example, each element from the triplet of pairwise anticommuting matrices X_{01}, X_{03}, X_{32} is invariant under conjugation with X_{30}. Moreover, no other matrix, satisfying the condition $X_{\mu\mu'}=+X_{10}\,X^{*}_{\mu\mu'}\,X_{10}$, anticommutes with this triplet. Hence, this triplet spans a set of normalized boundary dynamical Dirac masses that is homeomorphic to S^2, each point of which is invariant under the global $U(1)$ transformation associated with the conservation of the fermion number. Because of $\pi_2(S^2)=\mathbb{Z}$, monopoles bind zero modes in (2+1)-dimensional space and time that prevent the gapping of the boundary states when $\nu=2$. When $\nu=4$, we represent the unit 8×8 matrix by $\rho_0\otimes\sigma_0\otimes\sigma'_0$ and we expand any 8×8 matrix as a linear combination with real-valued coefficients of the 64 matrices $X_{\mu\mu'\mu''}=\rho_\mu\otimes\sigma_{\mu'}\otimes\sigma'_{\mu''}$ with $\mu,\mu',\mu''=0,1,2,3$. The boundary dynamical matrix $M(\tau,x,z)$ is a linear combination with real-valued functions as coefficients of those matrices $X_{\mu\mu'\mu''}=+X_{100}\,X^{*}_{\mu\mu'\mu''}\,X_{100}$. Other than the unit matrix X_{000}, any one of those matrices belong to a quintuplet of pairwise anticommuting matrices. Among these, each element from the quintuplet X_{001}, X_{003}, X_{312}, X_{022}, X_{332} is invariant under conjugation by X_{300}. Moreover, no other matrix, satisfying the condition $X_{\mu\mu'\mu''}=+X_{100}\,X^{*}_{\mu\mu'\mu''}\,X_{100}$ anticommutes with this quintuplet. Hence, this quintuplet spans a set of normalized boundary dynamical Dirac masses that is homeomorphic to S^4, each point of which is invariant under the global $U(1)$ transformation associated with the conservation of the fermion number. Because of $\pi_{2+1+1}(S^4)=\mathbb{Z}$, it is possible to augment the corresponding boundary QNLSM in (2+1)-dimensional space and time by a WZ term that modifies the equations of motion in a local way. This term is conjectured to prevent the gapping of the boundary states. When $\nu=2^{n-1}$ with $n\geqslant 4$, any permissible matrix $X_{\mu\mu'\mu'''\cdots}=+X_{100\cdots}\,X^{*}_{\mu\mu'\mu'''\cdots}\,X_{100\cdots}$ belongs to a $N(\nu)$-tuplet of pairwise anticommuting permissible matrices with $N(\nu)>5$ [69]. However, not all such $N(\nu)$-tuplets are closed under the global $U(1)$ transformation defined by conjugation with $X_{300\cdots}$. The $N(\nu)$-tuplet that contains the pair of anticommuting matrices $X_{00\cdots01}=+X_{10\cdots00}\,X^{*}_{00\cdots01}\,X_{10\cdots00}$ and $X_{00\cdots03}=+X_{10\cdots00}\,X^{*}_{00\cdots03}\,X_{10\cdots00}$ has the particularity that each of its elements is invariant under conjugation with $X_{300\cdots}$ and cannot be augmented by one more anticommuting $X_{\mu\mu'\mu'''\cdots}=+X_{100\cdots}\,X^{*}_{\mu\mu'\mu'''\cdots}\,X_{100\cdots}$. Hence, this $N(\nu)$-tuplet spans a set of normalized boundary dynamical Dirac masses that is homeomorphic to $S^{N(\nu)-1}$, each point of which is invariant under the global $U(1)$ transformation associated with the conservation of the fermion number. Since $N(\nu)>5$ for $\nu=2^{n-1}$ with $n\geqslant 4$, it follows that all homotopy groups of order less than four for the space of the normalized boundary dynamical Dirac masses that are invariant under the global $U(1)$ transformation are vanishing. The boundary states are then necessarily gapped.

We conclude that the effects of interactions on the three-dimensional SPT phases in the symmetry class AIII are to reduce the topological classification $\mathbb{Z}$ in the noninteracting limit down to $\mathbb{Z}_8$.

4. The symmetry class AII when $d=3$

We close the discussion of the stability to fermion-fermion interactions of strong noninteracting topological insulators or superconductors in three-dimensional space by illustrating why the $\mathbb{Z}_2$ noninteracting classification is stable.

To this end, consider the single-particle bulk Dirac Hamiltonian

$$\mathcal{H}^{(0)}(\boldsymbol{x}) := -i\partial_x\,X_{21}-i\partial_y\,X_{11}-i\partial_z\,X_{02}+m(\boldsymbol{x})\,X_{03}, \tag{3.50a}$$

where $X_{\mu\nu}:=\sigma_\mu\otimes\tau_{\mu'}$ for $\mu,\mu'=0,1,2,3$. Because

$$\mathcal{H}^{(0)}(\boldsymbol{x})=+\mathcal{T}\,\mathcal{H}^{(0)}(\boldsymbol{x})\,\mathcal{T}^{-1},\quad \mathcal{T}:=i\,X_{20}\,\mathsf{K}, \tag{3.50b}$$

we interpret this Hamiltonian as realizing a noninteracting topological insulator in the three-dimensional symmetry class AII. The domain wall in the mass

$$m(x,y,z)=m_\infty\,\mathrm{sgn}(z) \tag{3.51a}$$

binds a zero mode to the boundary $z=0$ that is annihilated by the boundary single-particle Hamiltonian

$$\mathcal{H}^{(0)}_{\mathrm{bd}}(x,y)=\,-i\partial_x\,\sigma_2-i\partial_y\,\sigma_1=\mathcal{T}_{\mathrm{bd}}\,\mathcal{H}^{(0)}_{\mathrm{bd}}(x,y)\,\mathcal{T}^{-1}_{\mathrm{bd}}, \tag{3.51b}$$

where

$$\mathcal{T}_{\mathrm{bd}}:=i\sigma_2\,\mathsf{K}. \tag{3.51c}$$

The boundary dynamical Dirac Hamiltonian

$$\mathcal{H}^{(\mathrm{dyn})}_{\mathrm{bd}}(\tau,x,y)=-i\partial_x\,\sigma_2-i\partial_y\,\sigma_1+M(\tau,x,y)\,\sigma_3 \tag{3.52}$$

belongs to the symmetry class A, as the Dirac mass $M\sigma_3$ breaks TRS unless $M(-\tau,x,y)=-M(\tau,x,y)$. The space of normalized boundary dynamical Dirac mass matrices $\{\pm\sigma_3\}$ is homeomorphic to the space of normalized Dirac mass matrices for the two-dimensional system in the symmetry class A

$$V_{\nu=1}=\bigcup_{k=0}^{1}U(\nu)/[U(k)\times U(\nu-k)]. \tag{3.53}$$

The domain wall in imaginary time $M(\tau,x,y)=M_\infty\,\mathrm{sgn}(\tau)$ prevents the gapping of the boundary zero modes.

We conclude that the noninteracting topological classification $\mathbb{Z}_2$ of three-dimensional insulators in the symmetry class AII is robust to the effects of interactions under the assumption that only fermion-number-conserving interacting channels are included in the stability analysis. The logic used to reach this conclusion is summarized by Table VIII once the line

TABLE VIII. Stability to fermion-fermion interactions of the noninteracting topological classification $\mathbb{Z}_2$ for three-dimensional strong topological insulators belonging to the symmetry classes AII. We denote by V_ν the space of $\nu\times\nu$ normalized Dirac mass matrices in boundary ($d=2$) Dirac Hamiltonians belonging to the symmetry class A. The limit $\nu\to\infty$ of these spaces is the classifying space C_0. The second column shows the stable D-th homotopy groups of the classifying space C_0. The third column gives the number ν of copies of boundary (Dirac) fermions for which a topological obstruction is permissible. The fourth column gives the type of topological obstruction that prevents the gapping of the boundary (Dirac) fermions.

D	$\pi_D(C_0)$	ν	Topological obstruction
0	$\mathbb{Z}$	1	Domain wall
1	0		

corresponding to $\nu=1$ has been identified. It is given by the smallest D that accommodates a nontrivial entry for the corresponding homotopy group. Moreover, one verifies by introducing a BdG (Nambu) grading that this robustness extends to interaction-driven dynamical superconducting fluctuations.

D. Higher dimensions

By working out explicitly the effects of fermion-fermion interactions on the boundary states supported by single-particle Dirac Hamiltonians representing strong topological insulators and superconductors when the dimensionality of space ranges from $d=1$ to $d=8$, the following rules can be deduced [70].

Rule 1: The $\mathbb{Z}_2$ topological classification of strong topological insulators and superconductors is robust to interactions in all dimensions.

Rule 2: The $\mathbb{Z}$ topological classification of strong topological insulators and superconductors is robust to interactions in all even dimensions.

Rule 3: The $\mathbb{Z}$ topological classification of strong topological insulators and superconductors is unstable to interactions in all odd dimensions.

We prove Rules 1 and 2 in Secs. III D 1 and III D 2, respectively. Finally, we work out explicitly the reduction pattern of the noninteracting $\mathbb{Z}$ topological classification for any odd dimension in Sec. III D 3.

1. The case of $\mathbb{Z}_2$ classification

The proof of Rule 1 follows the same logic as in the example of the three-dimensional symmetry class AII in Sec. III C 4. When $d=1$, there are two symmetry classes with $\pi_0(V)=\mathbb{Z}_2$, the symmetry classes D and DIII (see Table I). No dynamical Dirac mass is allowed in class D, since there is no protecting symmetry to break. For the symmetry class DIII, the normalized boundary dynamical Dirac masses are taken from Dirac masses in the symmetry class D and belong to the classifying space R_2, according to Table I. According to Table XVI in Appendix B, $\pi_0(R_2)=\mathbb{Z}_2$. Hence, the two noninteracting $\mathbb{Z}_2$ topological classification are stable in one dimension. To treat the case of $d\geqslant 2$, let V denote any one of the eight real classifying spaces V and observe that, according to Table XVI, at least one of the homotopy groups $\pi_D(V)$ with $D=0,1,2,3$ is nontrivial. We specialize to any one of the two symmetry classes in d dimensions with the classifying space (the space of normalized bulk Dirac masses) V such that $\pi_0(V)=\mathbb{Z}_2$. By assumption $d+1\geqslant 3$. Let V_{bd} denote the space of the boundary dynamical Dirac masses. If this space is empty, then the $\mathbb{Z}_2$ classification is stable. If this space is not empty, then we know that at least one of $\pi_D(V_{\mathrm{bd}})$ with $D=0,1,\ldots,d+1$ is nonvanishing. In turn, this implies that at least one of the homotopy groups from Eq. (2.10) is nontrivial. As the sphere $S^{N(1)-1}$ entering Eq. (2.10) is the target space for the QNLSM in $(d-1)+1$ space and time dimensions obtained from integrating the $\nu=1$ boundary Dirac fermions subjected to dynamical masses, the QNLSM accommodates a topological term that prevents the gapping of the $\nu=1$ boundary zero mode. Hence, the two noninteracting $\mathbb{Z}_2$ topological classification are stable in any spatial dimension.

2. The case of even dimensions

Because of Rule 1, we only need to consider the symmetry classes in even dimensions which have $\mathbb{Z}$ topological classification for gapped noninteracting fermions. According to Table I and the Bott periodicity of two (eight) for the complex (real) symmetry classes, these are the symmetry classes (i) A for $d=0$ mod 2, (ii) AI and AII for $d=4,8$ mod 8, and (iii) D and C for $d=2,6$ mod 8.

Proof for case (i): We start with the complex symmetry class A in even dimensions. We proceed in two steps. First we rule out dynamical superconducting fluctuations. We then show that the inclusion of dynamical superconducting fluctuations is harmless.

Without dynamical superconducting fluctuations, the classifying space for the normalized dynamical Dirac masses is that for the complex symmetry class A. Because there is no symmetry that is violated by such dynamical Dirac masses, dynamical Dirac masses are forbidden altogether.

With dynamical superconducting fluctuations, normalized dynamical Dirac masses form the space of Dirac masses in the symmetry class D. The original single-particle Hamiltonian $\mathcal{H}^{(0)}_\nu$ that annihilates ν zero modes is extended to a BdG (Nambu) single-particle Hamiltonian $\mathcal{H}^{(0)}_{\mathrm{BdG}\,\nu}$ that commutes with ρ_3 and anticommutes with $\rho_1\mathsf{K}$. Here ρ_0 is the unit $2{\times}2$ matrix and $\boldsymbol{\rho}$ are the Pauli matrices acting on the auxiliary particle-hole degrees of freedom. Boundary dynamical Dirac masses may then exist. However, they must anticommute with ρ_3, since no boundary Dirac mass that commutes with ρ_3 is allowed after restricting $\mathcal{H}^{(0)}_{\mathrm{BdG}\,\nu}$ to the boundary. Upon integrating the boundary Dirac fermions, a QNLSM in $(d-1)+1$ space and time dimensions ensues. The target space of this QNLSM has to be closed under the action of a global $U(1)$ gauge transformation. This is to say that a generic boundary dynamical Dirac mass must be of the form

$$\gamma'=[\cos(\theta)\rho_1+\sin(\theta)\rho_2]\otimes M, \tag{3.54a}$$

with the $(r_{\mathrm{min}}\,\nu/2)\times(r_{\mathrm{min}}\,\nu/2)$ matrix M satisfying

$$M=M^\dagger=-M^*. \tag{3.54b}$$

We recall that r_{min} is the minimal rank of the BdG Hamiltonian. The action on γ' of a global $U(1)$ gauge transformation parametrized by the global phase α is simply the shift $\theta \mapsto \theta+\alpha$. If so, the target space of the QNLSM is homeomorphic to $S^1 \times V_{\text{BdG}\,\nu}$ whereby S^1 is parametrized by θ and $V_{\text{BdG}\,\nu}$ is parametrized by M squaring to the unit matrix. For such a target space, we can always assign a topological term to account for the vortices supported by the parameter θ for the S^1 factor, as $\pi_1(S^1)=\mathbb{Z}$. These vortices bind ν zero modes.

We conclude that the noninteracting topological classification $\mathbb{Z}$ for the symmetry class A in even dimensions survives strong interactions on the boundary provided the fermion-number conservation is neither explicitly nor spontaneously broken.

Proof for case (ii): First, we show the statement for: (a) cases with dynamical Dirac masses that preserve the fermion-number $U(1)$ symmetry. We then proceed to: (b) cases with $U(1)$-breaking dynamical Dirac masses.

(a) We consider the massive Dirac Hamiltonian

$$\mathcal{H}^{(0)}(\boldsymbol{x}) = \sum_{j=1}^{d}(-i\partial_j)\,\gamma_j + m(\boldsymbol{x})\,\gamma_0, \quad m(\boldsymbol{x})\in\mathbb{R}, \tag{3.55}$$

obeying the TRS represented by $\mathcal{T}$ for classes AI and AII in $d=4n$ for $n=1,2,\ldots$. Here the Dirac matrices are of dimension $r \geqslant r_{\text{min}}$. They obey the Clifford algebra $\{\gamma_\mu,\gamma_{\mu'}\}=2\,\delta_{\mu\mu'}$ with $\mu,\mu'=0,\ldots,d$. The Dirac matrices entering $\mathcal{H}^{(0)}(\boldsymbol{x})$ and the operator $\mathcal{T}$ that represents reversal of time can be used to define the following pair of Clifford algebras [24,71].

For the symmetry class AI, reversal of time is represented by an element of the Clifford algebra $\mathcal{T}$ that squares to the unit matrix. It and $i\mathcal{T}$, together with the gamma matrices $\gamma_1,\ldots,\gamma_d$ satisfying the conditions $\gamma_1=-\mathcal{T}\,\gamma_1\,\mathcal{T}^{-1},\ldots,\gamma_d=-\mathcal{T}\,\gamma_d\,\mathcal{T}^{-1}$, are generators of the Clifford algebra. On the other hand, the Dirac mass matrix $i\gamma_0$ is chosen as the generator that squares to minus the unit matrix, for $\gamma_0=+\mathcal{T}\,\gamma_0\,\mathcal{T}^{-1}$. We arrive at the Clifford algebra

$$Cl_{1,2+d} := \{J\gamma_0;\mathcal{T},J\mathcal{T},\gamma_1,\ldots,\gamma_d\} \tag{3.56}$$

for the symmetry class AI, where J is the generator that represents the imaginary unit "i" and satisfies the relations $J^2=-1$ and $\{\mathcal{T},J\}=0$ [72].

For the symmetry class AII, reversal of time is represented by an element of the Clifford algebra $\mathcal{T}$ that squares to minus the unit matrix. It and $i\mathcal{T}$ enter on equal footing with $i\gamma_0$ as the generators that square to minus the unit matrix, for $\gamma_0=+\mathcal{T}\,\gamma_0\,\mathcal{T}^{-1}$. On the other hand, γ matrices $\gamma_1,\ldots,\gamma_d$, satisfying the conditions $\gamma_1=-\mathcal{T}\,\gamma_1\,\mathcal{T}^{-1},\ldots,\gamma_d=-\mathcal{T}\,\gamma_d\,\mathcal{T}^{-1}$, are the generators that square to the unit matrix in the Clifford algebra. We arrive at the Clifford algebra

$$Cl_{3,d} := \{J\gamma_0,\mathcal{T},J\mathcal{T};\gamma_1,\ldots,\gamma_d\} \tag{3.57}$$

for the symmetry class AII.

In both symmetry classes the choice of γ_0 is unique, up to a sign, as a consequence of the fact that the zeroth homotopy groups of the classifying spaces for the symmetry classes AI and AII is $\mathbb{Z}$ in $4n$ dimensions. In other words, no other Dirac mass matrix that is invariant under reversal of time anticommutes with γ_0 [71]. This leaves open the possibility that the Clifford algebras (3.56) and (3.57) for $d=4n$ could admit the addition of a generator γ_0' that anticommutes with $\mathcal{H}^{(0)}$ and is odd under reversal of time, $\gamma_0'=-\mathcal{T}\,\gamma_0'\,\mathcal{T}^{-1}$. If so, the choice of γ_1 to γ_d in the Clifford algebras (3.56) and (3.57) would not be unique in an uncountable (in a continuous) way. The existence of γ_0' is thus tied to the task of parametrizing in a continuous way the representation of the generator (e.g., γ_d) present in $Cl_{p,q+1}$ but absent in $Cl_{p,q}$ applied to the cases $(p,q)=(1,4n+1)$ and $(p,q)=(3,4n-1)$ for the $4n$-dimensional symmetry classes AI and AII, respectively [73]. Both tasks are denoted by the extension problem of Clifford algebras

$$Cl_{p,q} \to Cl_{p,q+1}, \tag{3.58}$$

with the classifying spaces

$$R_{q-p} = \begin{cases} R_{4n}, & (p,q)=(1,4n+1),\\ R_{4n-4}, & (p,q)=(3,4n-1),\end{cases} \tag{3.59}$$

as solutions for the set of representations of possible γ_0' in the symmetry classes AI and AII, respectively. Hereto, it is the zeroth homotopy group of the classifying space R_{p-q} that seals the fate of the existence of γ_0'. As $\pi_0(R_{4n})=\pi_0(R_{4n-4})=\mathbb{Z}$, it follows that γ_0' does not exist, i.e., no dynamical Dirac mass that breaks the TRS symmetry but preserves the global $U(1)$ gauge symmetry is permissible for the symmetry classes AI and AII when $d=4n$.

(b) After having established the absence of $U(1)$-preserving dynamical Dirac masses, the stability analysis in the presence of dynamical superconducting fluctuations for the symmetry classes AI and AII is the same as that for the symmetry class A. The boundary dynamical Dirac mass takes the form (3.54). The target space of the QNLSM is homeomorphic to $S^1\times V_{\text{BdG}\,\nu}$ since it has to be closed under the action of a global $U(1)$ gauge transformation. Vortices that bind boundary zero modes originate from the S^1 manifold.

Proof for case (iii): The symmetry classes D and C for $d=2,6 \pmod 8$ do not support dynamical Dirac masses along the boundary, because the PHS is kept as a fundamental symmetry. Their noninteracting topological classification $\mathbb{Z}$ survives strong interactions on the boundary provided the PHS is neither explicitly nor spontaneously broken.

3. *The case of odd dimensions*

The topological classification $\mathbb{Z}$ of noninteracting strong topological insulators and superconductors in odd dimensions of space is reduced to the coarser classification $\mathbb{Z}_{\nu_{\text{max}}}$ with ν_{max} an integer:

$$\mathbb{Z}\to\mathbb{Z}_{\nu_{\text{max}}}. \tag{3.60a}$$

The label "max" stands here for maximum. The task at hand is thus to compute the integer ν_{max}. Computing ν_{max} proceeds with the following algorithm (see Tables IX–XII).

Step 1: Choose any one of the 10 AZ symmetry classes from Table I.

Step 2: Choose any odd dimension d for which the zeroth homotopy group of the classifying space of the chosen symmetry class V_d is the set of integers [$\pi_0(V_d)=\mathbb{Z}$]. This step restricts the symmetry classes to the complex symmetry

TABLE IX. Application of the Bott periodicity obeyed by the homotopy groups $\pi_D(V)$ for $D = 0,1,\ldots,$ of a given classifying space V'_{d-1} of dynamical boundary Dirac masses to deduce the reduction pattern $\mathbb{Z} \to \mathbb{Z}_{\nu_{\max}}$ for the symmetry class BDI in dimensions (a) $d = 1$ for which $V'_{d-1} = R_2$ and $\nu_{\max} = 8$, (b) $d = 5$ for which $V'_{d-1} = R_6$ and $\nu_{\max} = 16$, and (c) $d = 9$ for which $V'_{d-1} = R_2$ and $\nu_{\max} = 128$. The column ν fixes the rank $r := r_{\min}\,\nu$ of the Dirac Hamiltonian in the symmetry class BDI. The fourth column gives the target manifold of the QNLSM with the action S_{QNLSM} that encodes the fermion-fermion interactions on the $(d-1)$-dimensional boundary. The fifth column indicates if a topological action S_{top} can be added to S_{QNLSM}.

(a) Symmetry class BDI in $d = 1$

D	$\pi_D(R_2)$	ν	S_{QNLSM}	S_{top}
0	$\mathbb{Z}_2$	2	S^0	✓
1	0			
2	$\mathbb{Z}$	4	S^2	✓
3	0			
4	0			
5	0			
6	$\mathbb{Z}$	8	S^6	–
7	$\mathbb{Z}_2$	16	S^7	–

(b) Symmetry class BDI in $d = 5$

D	$\pi_D(R_6)$	ν	S_{QNLSM}	S_{top}
0	0			
1	0			
2	$\mathbb{Z}$	1	S^2	✓
3	$\mathbb{Z}_2$	2	S^3	✓
4	$\mathbb{Z}_2$	4	S^4	✓
5	0			
6	$\mathbb{Z}$	8	S^6	✓
7	0			
8	0			
9	0			
10	$\mathbb{Z}$	16	S^{10}	–

(c) Symmetry class BDI in $d = 9$

D	$\pi_D(R_2)$	ν	S_{QNLSM}	S_{top}
0	$\mathbb{Z}_2$	2	S^0	✓
1	0			
2	$\mathbb{Z}$	4	S^2	✓
3	0			
4	0			
5	0			
6	$\mathbb{Z}$	8	S^6	✓
7	$\mathbb{Z}_2$	16	S^7	✓
8	$\mathbb{Z}_2$	32	S^8	✓
9	0			
10	$\mathbb{Z}$	64	S^{10}	✓
11	0			
12	0			
13	0			
14	$\mathbb{Z}$	128	S^{14}	–

class AIII and the real symmetry classes BDI, DIII, CII, and CI.

Step 3: Identify the parent symmetry class and its classifying space V'_d that follows if CHS is broken for the complex symmetry class AIII or if TRS is broken for the real symmetry classes. This step restricts the parent symmetry classes to the complex symmetry class A if the symmetry class AIII is interpreted as realizing an insulator, the real symmetry class D if the symmetry classes BDI and DIII are interpreted as superconductors, and the real symmetry

TABLE X. Application of the Bott periodicity obeyed by the homotopy groups $\pi_D(V)$ for $D = 0,1,\ldots,$ of a given classifying space V'_{d-1} of dynamical boundary Dirac masses to deduce the reduction pattern $\mathbb{Z} \to \mathbb{Z}_{\nu_{\max}}$ for the symmetry class DIII in dimensions (a) $d = 3$ for which $V'_{d-1} = R_0$ and $\nu_{\max} = 16$, (b) $d = 7$ for which $V'_{d-1} = R_4$ and $\nu_{\max} = 32$, and (c) $d = 11$ for which $V'_{d-1} = R_0$ and $\nu_{\max} = 256$. The column ν fixes the rank $r := r_{\min}\,\nu$ of the Dirac Hamiltonian in the symmetry class DIII. The fourth column gives the target manifold of the QNLSM with the action S_{QNLSM} that encodes the fermion-fermion interactions on the $(d-1)$-dimensional boundary. The fifth column indicates if a topological action S_{top} can be added to S_{QNLSM}.

(a) Symmetry class DIII in $d = 3$

D	$\pi_D(R_0)$	ν	S_{QNLSM}	S_{top}
0	$\mathbb{Z}$	1	S^0	✓
1	$\mathbb{Z}_2$	2	S^1	✓
2	$\mathbb{Z}_2$	4	S^2	✓
3	0			
4	$\mathbb{Z}$	8	S^4	✓
5	0			
6	0			
7	0			
8	$\mathbb{Z}$	16	S^8	–

(b) Symmetry class DIII in $d = 7$

D	$\pi_D(R_4)$	ν	S_{QNLSM}	S_{top}
0	$\mathbb{Z}$	1	S^0	✓
1	0			
2	0			
3	0			
4	$\mathbb{Z}$	2	S^4	✓
5	$\mathbb{Z}_2$	4	S^5	✓
6	$\mathbb{Z}_2$	8	S^6	✓
7	0			
8	$\mathbb{Z}$	16	S^8	✓
9	0			
10	0			
11	0			
12	$\mathbb{Z}$	32	S^{12}	–

(c) Symmetry class DIII in $d = 11$

D	$\pi_D(R_0)$	ν	S_{QNLSM}	S_{top}
0	$\mathbb{Z}$	1	S^0	✓
1	$\mathbb{Z}_2$	2	S^1	✓
2	$\mathbb{Z}_2$	4	S^2	✓
3	0			
4	$\mathbb{Z}$	8	S^4	✓
5	0			
6	0			
7	0			
8	$\mathbb{Z}$	16	S^8	✓
9	$\mathbb{Z}_2$	32	S^9	✓
10	$\mathbb{Z}_2$	64	S^{10}	✓
11	0			
12	$\mathbb{Z}$	128	S^{12}	✓
13	0			
14	0			
15	0			
16	$\mathbb{Z}$	256	S^{16}	–

TABLE XI. Application of the Bott periodicity obeyed by the homotopy groups $\pi_D(V)$ for $D = 0,1,\ldots,$ of a given classifying space V'_{d-1} of dynamical boundary Dirac masses to deduce the reduction pattern $\mathbb{Z} \to \mathbb{Z}_{\nu_{\max}}$ for the symmetry class CII in dimensions (a) $d = 1$ for which $V'_{d-1} = R_6$ and $\nu_{\max} = 2$, (b) $d = 5$ for which $V'_{d-1} = R_2$ and $\nu_{\max} = 16$, and (c) $d = 9$ for which $V'_{d-1} = R_6$ and $\nu_{\max} = 32$. The column ν fixes the rank $r := r_{\min}\,\nu$ of the Dirac Hamiltonian in the symmetry class CII. The fourth column gives the target manifold of the QNLSM with the action S_{QNLSM} that encodes the fermion-fermion interactions on the $(d-1)$-dimensional boundary. The fifth column indicates if a topological action S_{top} can be added to S_{QNLSM}.

(a) Symmetry class CII in $d = 1$

D	$\pi_D(R_6)$	ν	S_{QNLSM}	S_{top}
0	0			
1	0			
2	$\mathbb{Z}$	1	S^2	✓
3	$\mathbb{Z}_2$	2	S^3	–
4	$\mathbb{Z}_2$	4	S^4	–
5	0			
6	$\mathbb{Z}$	8	S^6	–
7	0			

(b) Symmetry class CII in $d = 5$

D	$\pi_D(R_2)$	ν	S_{QNLSM}	S_{top}
0	$\mathbb{Z}_2$	2	S^0	✓
1	0			
2	$\mathbb{Z}$	4	S^2	✓
3	0			
4	0			
5	0			
6	$\mathbb{Z}$	8	S^6	✓
7	$\mathbb{Z}_2$	16	S^7	–

(c) Symmetry class CII in $d = 9$

D	$\pi_D(R_6)$	ν	S_{QNLSM}	S_{top}
0	0			
1	0			
2	$\mathbb{Z}$	1	S^2	✓
3	$\mathbb{Z}_2$	2	S^3	✓
4	$\mathbb{Z}_2$	4	S^4	✓
5	0			
6	$\mathbb{Z}$	8	S^6	✓
7	0			
8	0			
9	0			
10	$\mathbb{Z}$	16	S^{10}	✓
11	$\mathbb{Z}_2$	32	S^{11}	–

class C if the symmetry classes CII and CI are interpreted as superconductors.

Step 4: Assign the minimal value

$$\nu_{\min} := \begin{cases} 1, & \pi_0(V'_d) = 0, \\ 2, & \pi_0(V'_d) \neq 0, \end{cases} \tag{3.60b}$$

if the zeroth homotopy group of V'_d is trivial or nontrivial, respectively.

Step 5: Identify the classifying space V'_{d-1} that determines the dynamical Dirac mass matrices induced by the fermion-fermion interactions on the boundary.

Step 6: Construct a table with lines labeled by the integer $D = 0,1,2,\ldots$. The first column gives the order D of the homotopy group $\pi_D(V'_{d-1})$ given in the second column. The third column is the number ν of boundary zero modes in the selected symmetry class. Enter the value $\nu_{\min}$ in the third column for the smallest value of D for which $\pi_D(V'_{d-1})$ is

TABLE XII. Application of the Bott periodicity obeyed by the homotopy groups $\pi_D(V)$ for $D = 0,1,\ldots,$ of a given classifying space V'_{d-1} of dynamical boundary Dirac masses to deduce the reduction pattern $\mathbb{Z} \to \mathbb{Z}_{\nu_{\max}}$ for the symmetry class CI in dimensions (a) $d = 3$ for which $V'_{d-1} = R_4$ and $\nu_{\max} = 4$, (b) $d = 7$ for which $V'_{d-1} = R_0$ and $\nu_{\max} = 32$, and (c) $d = 11$ for which $V'_{d-1} = R_4$ and $\nu_{\max} = 64$. The column ν fixes the rank $r := r_{\min}\,\nu$ of the Dirac Hamiltonian in the symmetry class CI. The fourth column gives the target manifold of the QNLSM with the action S_{QNLSM} that encodes the fermion-fermion interactions on the $(d-1)$-dimensional boundary. The fifth column indicates if a topological action S_{top} can be added to S_{QNLSM}.

(a) Symmetry class CI in $d = 3$

D	$\pi_D(R_4)$	ν	S_{QNLSM}	S_{top}
0	$\mathbb{Z}$	1	S^0	✓
1	0			
2	0			
3	0			
4	$\mathbb{Z}$	2	S^4	✓
5	$\mathbb{Z}_2$	4	S^5	–
6	$\mathbb{Z}_2$	8	S^6	–
7	0			

(b) Symmetry class CI in $d = 7$

D	$\pi_D(R_0)$	ν	S_{QNLSM}	S_{top}
0	$\mathbb{Z}$	1	S^0	✓
1	$\mathbb{Z}_2$	2	S^1	✓
2	$\mathbb{Z}_2$	4	S^2	✓
3	0			
4	$\mathbb{Z}$	8	S^4	✓
5	0			
6	0			
7	0			
8	$\mathbb{Z}$	16	S^8	✓
9	$\mathbb{Z}_2$	32	S^9	–

(c) Symmetry class CI in $d = 11$

D	$\pi_D(R_4)$	ν	S_{QNLSM}	S_{top}
0	$\mathbb{Z}$	1	S^0	✓
1	0			
2	0			
3	0			
4	$\mathbb{Z}$	2	S^4	✓
5	$\mathbb{Z}_2$	4	S^5	✓
6	$\mathbb{Z}_2$	8	S^6	✓
7	0			
8	$\mathbb{Z}$	16	S^8	✓
9	0			
10	0			
11	0			
12	$\mathbb{Z}$	32	S^{12}	✓
13	$\mathbb{Z}_2$	64	S^{13}	–

nontrivial. The value of ν is then doubled for each successive line with $\pi_D(V'_{d-1})$ nontrivial. The fourth column denotes the target space of the QNLSM with the action S_{QNLSM} defined by integrating out all the boundary Dirac fermions when coupled to $(D+1)$ real-valued bosonic fields, each one of which couples to a Dirac mass matrix from a $(D+1)$-tuplet of pairwise anticommuting Dirac mass matrices allowed on the boundary by the parent symmetry class. The fifth column indicates when a topological term S_{top} can be added to the action S_{QNLSM} [74].

Step 7: Let n_{WZ} be the number of lines with $\pi_D(V'_{d-1})$ nontrivial when D takes the values $D = 0,1,\ldots,d+1$. It then follows that

$$\nu_{\max} = \nu_{\min} \times 2^{n_{\text{WZ}}}. \tag{3.60c}$$

For the complex symmetry class AIII in dimension $d = 2n+1$ with $n = 0,1,2,\ldots$, the reduction pattern induced by the fermion-fermion interactions is

$$\mathbb{Z} \to \mathbb{Z}_{2^{n+2}}. \tag{3.61}$$

By making use of the eightfold Bott periodicity, one verifies that the reduction patterns are

	$d=8n+1$	$d=8n+3$	$d=8n+5$	$d=8n+7$
BDI	$\mathbb{Z}\to\mathbb{Z}_{2^{4n+3}}$	–	$\mathbb{Z}\to\mathbb{Z}_{2^{4n+4}}$	–
DIII	–	$\mathbb{Z}\to\mathbb{Z}_{2^{4n+4}}$	–	$\mathbb{Z}\to\mathbb{Z}_{2^{4n+5}}$
CII	$\mathbb{Z}\to\mathbb{Z}_{2^{4n+1}}$	–	$\mathbb{Z}\to\mathbb{Z}_{2^{4n+4}}$	–
CI	–	$\mathbb{Z}\to\mathbb{Z}_{2^{4n+2}}$	–	$\mathbb{Z}\to\mathbb{Z}_{2^{4n+5}}$

(3.62)

if we interpret the symmetry classes BDI, DIII, CII, and CI as superconductors or

	$d=8n+1$	$d=8n+3$	$d=8n+5$	$d=8n+7$
BDI	$\mathbb{Z}\to\mathbb{Z}_{2^{4n+2}}$	–	$\mathbb{Z}\to\mathbb{Z}_{2^{4n+3}}$	–
CII	$\mathbb{Z}\to\mathbb{Z}_{2^{4n+1}}$	–	$\mathbb{Z}\to\mathbb{Z}_{2^{4n+4}}$	–

(3.63)

if we interpret the symmetry classes BDI and CII as insulators. We have thus found two different patterns for the reduction of the topological classification of the symmetry class BDI depending on these two interpretations, as we have observed for $d=1$ in Sec. III A 3.

IV. REDUCTION FOR TCS AND TCI

The addition of discrete symmetries (such as spatial ones) to the PHS, TRS, and CHS enriches the classification of noninteracting fermions. A general method to account for additional symmetries that square to the unity has been proposed in Ref. [24]. Hereto, local fermion-fermion interactions can reduce the noninteracting topological classification over $\mathbb{Z}$, as was first observed in Refs. [16,17,75,76] by way of explicit examples. We are interested in the robustness to local fermion-fermion interactions of noninteracting topological phases with reflection symmetry. We treat two-dimensional topological superconductors in the symmetry class DIII with an additional reflection symmetry (the Yao-Ryu model from Ref. [16]) and three-dimensional topological insulators in the symmetry class AII with an additional reflection symmetry (a topological crystalline insulator is realized in SnTe, as was shown in Ref. [77]). We show that the reduction $\mathbb{Z}\to\mathbb{Z}_8$ holds in both cases by applying the approach detailed in Sec. II.

The notation $X_{\mu\mu'\mu''\cdots} := \tau_\mu \otimes \tau'_{\mu'} \otimes \tau''_{\mu''} \otimes \cdots$ for $\mu,\mu',\mu'',\ldots = 0,1,2,3$ where τ_0, τ'_0, $\tau''_0,\ldots$ are unit 2×2 matrices and the other τ_μ, $\tau'_{\mu'}$, $\tau''_{\mu''},\ldots$ are the corresponding Pauli matrices.

A. Two-dimensional superconductors with time-reversal and reflection symmetries (DIII + *R*)

Let $\boldsymbol{x} \equiv (x,y) \equiv (x_1,x_2)$ denote a point in two-dimensional space. Let $X_{\mu\mu'} := \sigma_\mu \otimes \tau_{\mu'}$ with $\mu,\mu' = 0,1,2,3$ denote a basis for the vector space of all Hermitian 4×4 matrices. Following Yao and Ryu [16], consider the two-dimensional bulk single-particle Dirac Hamiltonian

$$\mathcal{H}^{(0)}(x,y) := -i\partial_x X_{31} - i\partial_y X_{02} + m(x,y) X_{03}. \tag{4.1a}$$

This single-particle Dirac Hamiltonian belongs to the symmetry class DIII, for

$$\mathcal{T}\,\mathcal{H}^{(0)}(x,y)\,\mathcal{T}^{-1} = +\mathcal{H}^{(0)}(x,y), \tag{4.1b}$$

$$\mathcal{C}\,\mathcal{H}^{(0)}(x,y)\,\mathcal{C}^{-1} = -\mathcal{H}^{(0)}(x,y), \tag{4.1c}$$

where

$$\mathcal{T} := i X_{20}\,\mathsf{K}, \quad \mathcal{C} := X_{01}\,\mathsf{K}. \tag{4.1d}$$

In addition, the Dirac Hamiltonian is invariant under reflection in the x direction,

$$\mathcal{R}_x\,\mathcal{H}^{(0)}(-x,y)\,(\mathcal{R}_x)^{-1} = +\mathcal{H}^{(0)}(x,y), \tag{4.1e}$$

where

$$\mathcal{R}_x = i X_{20}. \tag{4.1f}$$

The operators $\mathcal{T}$, $\mathcal{C}$, and $\mathcal{R}_x$ commute pairwise and square to $\mathcal{T}^2 = -1$, $\mathcal{C}^2 = +1$, and $\mathcal{R}_x^2 = -1$.

The Dirac mass matrix X_{03} is here the only one allowed for dimension $r = r_{\min} = 4$ Dirac matrices under the symmetry constraints (4.1b), (4.1c), and (4.1e). The domain wall

$$m(x,y) = m_\infty\,\text{sgn}(y) \tag{4.2a}$$

at $y=0$ binds the zero mode

$$e^{-iX_{02}X_{03}\int_0^y dy\, m(x,y')}\,\chi = e^{-|m_\infty y|}\,\chi, \tag{4.2b}$$

where

$$X_{01}\,\chi = -\mathrm{sgn}(m_\infty)\,\chi \tag{4.2c}$$

with χ independent of x and z, that is annihilated by the boundary Hamiltonian

$$\mathcal{H}^{(0)}_{\mathrm{bd}}(x) := -i\partial_x\sigma_3, \tag{4.2d}$$

where we have chosen $m_\infty < 0$. On the boundary $y = 0$, the symmetries (4.1e) are realized by

$$\mathcal{T}_{\mathrm{bd}} := i\sigma_2\mathsf{K}, \quad \mathcal{C}_{\mathrm{bd}} := \mathsf{K}, \quad \mathcal{R}_{x\,\mathrm{bd}} := i\sigma_2. \tag{4.2e}$$

The boundary single-particle Hamiltonian (4.2d) and the operators (4.2e) are denoted by $\mathcal{H}^{(0)}_{\mathrm{bd}\,\nu}(x)$, $\mathcal{T}_{\mathrm{bd}\,\nu}$, $\mathcal{C}_{\mathrm{bd}\,\nu}$, and $\mathcal{R}_{x\,\mathrm{bd}\,\nu}$, respectively, when tensored with the $\nu\times\nu$ unit matrix $\mathbb{1}$. The single-particle Hamiltonian $\mathcal{H}^{(0)}_{\mathrm{bd}\,\nu}(x)$ supports ν linearly independent zero modes. Their stability to interactions that preserve the symmetries is probed by studying the dynamical single-particle boundary Hamiltonian

$$\mathcal{H}^{(\mathrm{dyn})}_{\mathrm{bd}\,\nu}(\tau,x) := -i\partial_x\sigma_3\otimes\mathbb{1} + \gamma'(\tau,x), \tag{4.3a}$$

where the boundary dynamical Dirac mass matrix $\gamma'(x)$ satisfies the particle-hole symmetry

$$\mathcal{C}_{\mathrm{bd}\,\nu}\gamma'\mathcal{C}^{-1}_{\mathrm{bd}\,\nu} = -\gamma' \tag{4.3b}$$

and takes the form

$$\gamma'(x) = \sigma_2\otimes M_1(\tau,x) + \sigma_1\otimes M_2(\tau,x) \tag{4.3c}$$

with the $\nu\times\nu$ Hermitian matrices

$$M_1(\tau,x) = +M_1^*(\tau,x), \quad M_2(\tau,x) = -M_2^*(\tau,x), \tag{4.3d}$$

i.e., $M_1(\tau,x)$ is a real-valued symmetric matrix while $M_2(\tau,x)$ is an imaginary-valued antisymmetric matrix. This is to say, the normalized boundary dynamical Dirac mass matrix $\gamma'(x)$ belongs to the space

$$V_\nu := O(\nu), \quad R_1 = \lim_{\nu\to\infty} O(\nu), \tag{4.3e}$$

for the Dirac matrices in the boundary ($d = 1$) Dirac Hamiltonians belonging to the symmetry class D [78]. Integrating the boundary Dirac fermions delivers a QNLSM in (1+1)-dimensional space and time. In order to gap out dynamically the boundary zero modes without breaking the time-reversal, particle-hole, and reflection symmetries, this QNLSM must be free of topological obstructions. We construct explicitly the spaces for the relevant normalized boundary dynamical Dirac mass matrices of dimension $\nu = 2^n$ with $n = 0,1,2,3$ in the following. The relevant homotopy groups are given in Table XIII.

Case $\nu = 1$*:* There is a topological obstruction of the domain wall type as the target space is

$$S^0 = \{\pm\sigma_y\} \tag{4.4a}$$

and $\pi_0(S^0) \neq 0$.

Case $\nu = 2$*:* There is a topological obstruction of the vortex type as the target space is

$$S^1 = \left\{c_1X_{21} + c_2X_{23}\middle|c_1^2 + c_2^2 = 1, c_i\in\mathbb{R}\right\} \tag{4.4b}$$

and $\pi_1(S^1) = \mathbb{Z}$.

TABLE XIII. Reduction from $\mathbb{Z}$ to $\mathbb{Z}_8$ due to interactions for the topologically equivalent classes of the two-dimensional topological superconductors protected by time-reversal and reflection symmetries (DIII + R). We denote by V_ν the space of $\nu\times\nu$ normalized Dirac mass matrices in boundary ($d = 1$) Dirac Hamiltonians belonging to the symmetry class D. The limit $\nu\to\infty$ of these spaces is the classifying space R_1. The second column shows the stable D-th homotopy groups of the classifying space R_1. The third column gives the number ν of copies of boundary (Dirac) fermions for which a topological obstruction is permissible. The fourth column gives the type of topological obstruction that prevents the gapping of the boundary (Dirac) fermions.

D	$\pi_D(R_1)$	ν	Topological obstruction
0	$\mathbb{Z}_2$	1	Domain wall
1	$\mathbb{Z}_2$	2	Vortex
2	0		
3	$\mathbb{Z}$	4	WZ term
4	0		
5	0		
6	0		
7	$\mathbb{Z}$	8	None

Case $\nu = 4$*:* There is a topological obstruction of the WZ type as the target space is

$$S^3 = \left\{c_1X_{210} + c_2X_{230} + c_3X_{102} + c_4X_{222}\middle|\sum_{i=1}^{4}c_i^2 = 1, c_i\in\mathbb{R}\right\} \tag{4.4c}$$

and $\pi_3(S^3) = \mathbb{Z}$.

Case $\nu = 8$*:* There is no topological obstruction as one can find more than four pairwise anticommuting matrices such as the set

$$\{X_{2100}, X_{2310}, X_{2331}, X_{2333}, X_{1120}\}. \tag{4.4d}$$

We conclude that the effects of interactions on the two-dimensional topological superconductors in the symmetry class DIII with additional reflection symmetry are to reduce the topological classification $\mathbb{Z}$ in the noninteracting limit down to $\mathbb{Z}_8$.

B. Three-dimensional insulators with time-reversal and reflection symmetries (AII + R)

We consider again the bulk, boundary, and dynamical boundary Hamiltonian defined in Sec. III C 4, i.e., Eqs. (3.50)–(3.53). We observe that the single-particle Hamiltonian (3.50a) has the symmetry

$$\mathcal{R}_x\,\mathcal{H}^{(0)}(-x,y,z)\,(\mathcal{R}_x)^{-1} = +\mathcal{H}^{(0)}(x,y,z), \tag{4.5a}$$

where

$$\mathcal{R}_x := i\,X_{10}, \quad \mathcal{R}_x^2 = -1, \quad [\mathcal{T},\mathcal{R}_x] = 0, \tag{4.5b}$$

in addition to the TRS (3.50b). The presence of the additional reflection symmetry allows one to define a mirror Chern number ($n_+\in\mathbb{Z}$) for the sector with the eigenvalue $\mathcal{R}_x = +i$ on the two-dimensional mirror plane ($k_x = 0$) in the three-dimensional Brillouin zone [77]. (The mirror Chern number

TABLE XIV. Reduction from $\mathbb{Z}$ to $\mathbb{Z}_8$ due to interactions for the topologically equivalent classes of the three-dimensional topological insulators with time-reversal and reflection symmetries (AII + R). We denote by V_ν the space of $\nu \times \nu$ normalized Dirac mass matrices in boundary ($d = 2$) Dirac Hamiltonians belonging to the symmetry class A. The limit $\nu \to \infty$ of these spaces is the classifying space C_0. The second column shows the stable D-th homotopy groups of the classifying space C_0. The third column gives the number ν of copies of boundary (Dirac) fermions for which a topological obstruction is permissible. The fourth column gives the type of topological obstruction that prevents the gapping of the boundary (Dirac) fermions.

D	$\pi_D(C_0)$	ν	Topological obstruction
0	$\mathbb{Z}$	1	Domain wall
1	0		
2	$\mathbb{Z}$	2	Monopole
3	0		
4	$\mathbb{Z}$	4	WZ term
5	0		
6	$\mathbb{Z}$	8	None
7	0		

for the eigensector $\mathcal{R}_x = -i$ is $-n_+$.) Thus, the ν linearly independent zero modes that follow from tensoring the single-particle Hamiltonian (3.50a) with the $\nu \times \nu$ unit matrix $\mathbb{1}$ along the domain wall (3.51a) are stable to strong one-body perturbations on the boundary that preserve the reflection symmetry (4.5). (If we forget the reflection symmetry and keep only the TRS, as we did in Sec. III C 4, it is only the parity of ν that is stable to strong one-body perturbations on the boundary.) If we only consider dynamical masses that preserve the fermion-number $U(1)$ symmetry, the space of normalized boundary dynamical Dirac mass matrices after tensoring the boundary dynamical Dirac Hamiltonian (3.53) with $\mathbb{1}$ is homeomorphic to the space of normalized Dirac masses for the two-dimensional system in the symmetry class A,

$$V_\nu = \bigcup_{k=0}^{\nu} U(\nu)/[U(k) \times U(\nu - k)]. \tag{4.6}$$

The limit $\nu \to \infty$ of these spaces is the classifying space C_0.

Integrating the boundary Dirac fermions delivers a QNLSM in (2+1)-dimensional space and time. In order to gap out dynamically the boundary zero modes without breaking the symmetries, this QNLSM must be free of topological obstructions. We construct explicitly the spaces for the relevant normalized boundary dynamical Dirac mass matrices [$M(\tau,x,y)$ in Eq. (3.52)] of dimension $\nu = 2^n$ with $n = 0,1,2,3$ in the following. The relevant homotopy groups are given in Table XIV.

Case $\nu = 1$*:* There is a topological obstruction of the domain wall type as the target space is

$$S^0 = \{\pm 1\} \tag{4.7a}$$

and $\pi_0(S^0) \neq 0$.

Case $\nu = 2$*:* There is a topological obstruction of the monopole type as the target space is

$$S^2 = \{c_1 X_1 + c_2 X_2 + c_3 X_3 | c_1^2 + c_2^2 + c_3^2 = 1\} \tag{4.7b}$$

and $\pi_2(S^2) = \mathbb{Z}$.

Case $\nu = 4$*:* There is a topological obstruction of the WZ type as the target space is

$$S^4 = \left\{ c_1 X_{13} + c_2 X_{23} + c_3 X_{33} + c_4 X_{01} + c_5 X_{02} \middle| \sum_{i=1}^{5} c_i^2 = 1, c_i \in \mathbb{R} \right\} \tag{4.7c}$$

and $\pi_4(S^4) = \mathbb{Z}$.

Case $\nu = 8$*:* There is no topological obstruction as one can find more than five pairwise anticommuting matrices such as the set

$$\{X_{133}, X_{233}, X_{333}, X_{013}, X_{023}, X_{001}, X_{002}\}. \tag{4.7d}$$

We conclude that the effects of interactions on the three-dimensional topological insulators in the symmetry class AII with an additional symmetry are to reduce the topological classification $\mathbb{Z}$ in the noninteracting limit down to $\mathbb{Z}_8$.

This $\mathbb{Z}_8$ classification is unchanged if all boundary dynamical masses that break the fermion-number $U(1)$ symmetry are accounted for. The corresponding target spaces for the boundary dynamical masses and their topological obstructions are derived as was done in the stability analysis made for the symmetry class AIII in $d = 3$ in Sec. III C 3. Namely, we extend the single-particle Hamiltonian [Eq. (3.52)] to a BdG Hamiltonian

$$\mathcal{H}_{\mathrm{bd}}^{(\mathrm{dyn})} = (-i\partial_x\, \sigma_2 \otimes \rho_3 - i\partial_y\, \sigma_1 \otimes \rho_0) \otimes \mathbb{1} + \gamma'(\tau,x,y), \tag{4.8}$$

where ρ_0 and ρ_μ are unit 2×2 and Pauli matrices, respectively, acting on the particle-hole (Nambu) space and the particle-hole symmetry is given by $\mathcal{C} = \rho_1 \mathsf{K}$. In this case, the target spaces of the QNLSM made of normalized boundary dynamical Dirac mass matrices γ' of dimension $\nu = 2^n$ with $n = 0,1,2,3$ are modified as listed in the following with the notation $X_{\mu\mu'\mu''\mu'''\ldots} = \sigma_\mu \otimes \rho_{\mu'} \otimes \tau_{\mu''} \otimes \tau_{\mu'''} \ldots$. The relevant homotopy groups are given in Table XV. We note that these target spaces are closed under the global $U(1)$ transformation generated by ρ_3.

Case $\nu = 1$*:* There is a topological obstruction of the vortex type as the target space is

$$S^1 = \{c_1 X_{21} + c_2 X_{22} | c_1^2 + c_2^2 = 1\} \tag{4.9a}$$

and $\pi_1(S^1) = \mathbb{Z}$.

Case $\nu = 2$*:* There is a topological obstruction of the monopole type as the target space is

$$S^2 = \{c_1 X_{210} + c_2 X_{220} + c_3 X_{302} | c_1^2 + c_2^2 + c_3^2 = 1\} \tag{4.9b}$$

and $\pi_2(S^2) = \mathbb{Z}$.

TABLE XV. Reduction from $\mathbb{Z}$ to $\mathbb{Z}_8$ due to interactions for the topologically equivalent classes of the three-dimensional topological insulators with time-reversal and reflection symmetries (AII + R) when the superconducting fluctuations are accounted for. We denote by V_ν the space of $\nu \times \nu$ normalized Dirac mass matrices in boundary ($d = 2$) Dirac Hamiltonians belonging to the symmetry class D. The limit $\nu \to \infty$ of these spaces is the classifying space R_0. The second column shows the stable D-th homotopy groups of the classifying space R_0. The third column gives the number ν of copies of boundary (Dirac) fermions for which a topological obstruction is permissible. The fourth column gives the type of topological obstruction that prevents the gapping of the boundary (Dirac) fermions.

D	$\pi_D(R_0)$	ν	Topological obstruction
0	$\mathbb{Z}$		
1	$\mathbb{Z}_2$	1	Vortex
2	$\mathbb{Z}_2$	2	Monopole
3	0		
4	$\mathbb{Z}$	4	WZ term
5	0		
6	0		
7	0		
8	$\mathbb{Z}$	8	None

Case $\nu = 4$*:* There is a topological obstruction of the WZ type as the target space is

$$S^4 = \Big\{ c_1 X_{2100} + c_2 X_{2200} + c_3 X_{3020} + c_4 X_{3012} + c_5 X_{3032} \Big| \sum_{i=1}^{5} c_i^2 = 1, c_i \in \mathbb{R} \Big\} \tag{4.9c}$$

and $\pi_4(S^4) = \mathbb{Z}$.

Case $\nu = 8$*:* There is no topological obstruction as one can find more than five pairwise anticommuting matrices such as the set

$$\{X_{21000}, X_{22000}, X_{30200}, X_{30120}, X_{30312}, X_{30332}\}. \tag{4.9d}$$

Therefore, the topological classification $\mathbb{Z}_8$ for three-dimensional TCIs in the symmetry class AII with an additional reflection symmetry is unchanged when the superconducting fluctuations are accounted for. This $\mathbb{Z}_8$ classification is consistent with the results obtained recently in Refs. [79,80]. We note that the classifying space R_0 for the dynamical masses is the same as that in the case of three-dimensional TSs in the symmetry class DIII. There is an important difference, however. Namely, the line corresponding to $\nu = 1$ is moved to $D = 1$ in Table XV from $D = 0$ in Table V. This change originates from the fact that the minimum matrix dimension of the BdG Hamiltonian $\mathcal{H}^{(\mathrm{dyn})}_{\mathrm{bd}}$ for three-dimensional TCIs in the symmetry class AII+R is four, while that for three-dimensional TSs in the symmetry class DIII is two. Hence, the breakdown of the topological classification $\mathbb{Z}$ for three-dimensional TCIs in the symmetry class AII+R takes place at $\nu = 8$, which is the half of $\nu = 16$ for three-dimensional TSs in the symmetry class DIII.

C. Massless Dirac fermions on the surfaces of SnTe

The crystal SnTe is a three-dimensional topological crystalline insulator protected by time-reversal and reflection symmetries (AII + R). SnTe supports four Dirac cones on the [001] surface and six Dirac cones on the [111] surface. If strong interaction effects are present, we expect that the $\nu = 4$ phase described by a QNLSM with a WZ term should be realized on the [001] surface. At the [111] surface, we expect that the $\nu = 6 = 4 + 2$ phase be realized, whereby the effective field theory is that of a QNLSM with a WZ term for 4 out of the six surface Dirac cones and that of a QNLSM with a topological term arising from a gas of monopoles of the remaining two surface Dirac cones.

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APPENDIX A: DEFINING SYMMETRIES OF STRONG TOPOLOGICAL INSULATORS (SUPERCONDUCTORS)

Define the many-body quadratic form

$$\widehat{H} = \int d^d \boldsymbol{x} \int d^d \boldsymbol{y} \sum_{ij} \hat{\psi}^\dagger_i(t,\boldsymbol{x})\, \mathcal{H}_{ij}(\boldsymbol{x},\boldsymbol{y})\, \hat{\psi}_j(t,\boldsymbol{y}), \tag{A1a}$$

where

$$\mathcal{H}_{ij}(\boldsymbol{x},\boldsymbol{y}) = \mathcal{H}^*_{ji}(\boldsymbol{y},\boldsymbol{x}), \tag{A1b}$$

and

$$\{\hat{\psi}_i(t,\boldsymbol{x}), \hat{\psi}^\dagger_j(t,\boldsymbol{y})\} = \delta_{ij}\, \delta(\boldsymbol{x} - \boldsymbol{y}) \tag{A1c}$$

are the only nonvanishing equal-time anticommutators.

1. Time-reversal symmetry

Let K denote complex conjugation. Define the time-reversal transformation by the antiunitary transformation

$$\hat{\Theta} := \hat{\mathcal{T}}\, \mathsf{K} \tag{A2a}$$

that reverses time but leaves space unchanged by demanding that

$$\hat{\mathcal{T}}^{-1} = \hat{\mathcal{T}}^\dagger \tag{A2b}$$

and

$$\hat{\Theta}\, \hat{\psi}_j(t,\boldsymbol{y})\, \hat{\Theta}^{-1} = \sum_{j'} \mathcal{T}^*_{j'j}\, \hat{\psi}_{j'}(-t,\boldsymbol{y}). \tag{A2c}$$

One verifies that

$$\hat{\Theta}\, \widehat{H}\, \hat{\Theta}^{-1} = \widehat{H} \tag{A3a}$$

if and only if

$$\sum_{ij} \mathcal{T}_{i'i}\, \mathcal{H}^*_{ij}(\boldsymbol{x},\boldsymbol{y})\, \mathcal{T}^{-1}_{jj'} = \mathcal{H}_{i'j'}(\boldsymbol{x},\boldsymbol{y}). \tag{A3b}$$

2. Particle-hole (charge-conjugation) symmetry

Assume that

$$\sum_i \mathcal{H}_{ii}(\boldsymbol{x},\boldsymbol{y}) = 0. \tag{A4}$$

Define the particle-hole transformation by the unitary transformation

$$\hat{\mathrm{C}} := \hat{\mathcal{C}} \tag{A5a}$$

that reverses the sign of the fermion number

$$\hat{n}_i(x) - \tfrac{1}{2}\,\delta(\boldsymbol{x}=0) := \hat{\psi}^{\dagger}_i(\boldsymbol{x})\,\hat{\psi}_i(\boldsymbol{x}) - \tfrac{1}{2}\,\delta(\boldsymbol{x}=0) \tag{A5b}$$

measured relative to the background of the fermion density $1/2$ but leaves space unchanged by demanding that

$$\hat{\mathcal{C}}^{-1} = \hat{\mathcal{C}}^{\dagger} \tag{A5c}$$

and

$$\hat{\mathrm{C}}\,\hat{\psi}_j(t,\boldsymbol{y})\,\hat{\mathrm{C}}^{-1} = \sum_{j'} \mathcal{C}_{j'j}\,\hat{\psi}^{\dagger}_{j'}(t,\boldsymbol{y}). \tag{A5d}$$

One verifies that

$$\hat{\mathrm{C}}\,\widehat{H}\,\hat{\mathrm{C}}^{-1} = \widehat{H} \tag{A6a}$$

if and only if

$$\sum_{ij} \mathcal{C}_{i'i}\,\mathcal{H}^{*}_{ij}(\boldsymbol{y},\boldsymbol{x})\,\mathcal{C}^{-1}_{jj'} = -\mathcal{H}_{i'j'}(\boldsymbol{y},\boldsymbol{x}). \tag{A6b}$$

3. Chiral symmetry

Assume that

$$\sum_i \mathcal{H}_{ii}(\boldsymbol{x},\boldsymbol{y}) = 0. \tag{A7}$$

Define the chiral transformation by the antiunitary transformation

$$\hat{\mathrm{S}} := \hat{\mathcal{S}}\,\mathsf{K} \tag{A8a}$$

that reverses the sign of the fermion number

$$\hat{n}_i(x) - \tfrac{1}{2}\,\delta(\boldsymbol{x}=0) := \hat{\psi}^{\dagger}_i(\boldsymbol{x})\,\hat{\psi}_i(\boldsymbol{x}) - \tfrac{1}{2}\,\delta(\boldsymbol{x}=0) \tag{A8b}$$

measured relative to the background of the fermion density $1/2$ but leaves space unchanged by demanding that

$$\hat{\mathcal{S}}^{-1} = \hat{\mathcal{S}}^{\dagger} \tag{A8c}$$

and

$$\hat{\mathrm{S}}\,\hat{\psi}_j(t,\boldsymbol{y})\,\hat{\mathrm{S}}^{-1} = \sum_{j'} \mathcal{S}_{j'j}\,\hat{\psi}^{\dagger}_{j'}(t,\boldsymbol{y}). \tag{A8d}$$

One verifies that

$$\hat{\mathrm{S}}\,\widehat{H}\,\hat{\mathrm{S}}^{-1} = \widehat{H} \tag{A9a}$$

if and only if

$$\sum_{ij} \mathcal{S}_{i'i}\,\mathcal{H}_{ij}(\boldsymbol{y},\boldsymbol{x})\,\mathcal{S}^{-1}_{jj'} = -\mathcal{H}_{i'j'}(\boldsymbol{y},\boldsymbol{x}). \tag{A9b}$$

The unitary symmetry under $\hat{\mathrm{C}}$ is called charge conjugation symmetry or PHS. The antiunitary symmetry under $\hat{\mathrm{S}}$ is called the CHS. The antiunitary symmetry under $\hat{\mathrm{T}}$ is called TRS.

APPENDIX B: TENFOLD WAY AND CLASSIFYING SPACES

In this Appendix, we summarize the classification of gapped phases of noninteracting fermions in terms of the tenfold way. We also define the classifying spaces of normalized Dirac masses. The 10 Altland-Zirnbauer (AZ) symmetry classes for Hermitian matrices are shown in Table I. There, two complex and eight real symmetry classes are characterized by the presence or the absence of time-reversal symmetry (T), particle-hole symmetry (C), and chiral symmetry (Γ). Their presence is indicated by the sign entering the squared operators, $T^2 = \pm 1$ or $C^2 = \pm 1$, and by 1 for Γ. Their absence is indicated by 0. For each symmetry class and for any dimension $d = 0,1,2,\ldots$, of space, the classifying space V_d, which is the space of normalized Dirac masses, is given in the last column by labels to symmetric spaces. We list the 10 relevant symmetric spaces and their homotopy groups in the stable homotopy regime in Table XVI. The number N is related to the dimension $r = r_{\min}N$ of the Dirac matrices, i.e., $N = 1,2,\ldots$ is the number of copies of the minimal massive Dirac Hamiltonian of rank $r_{\min}$. The stable homotopy

TABLE XVI. Complex and real classifying spaces and their stable homotopy groups. Homotopy groups $\pi_D(V)$ for complex and real classifying spaces are periodic in D with periods of 2 and 8, respectively.

Label	Classifying space V	$\pi_0(V)$	$\pi_1(V)$	$\pi_2(V)$	$\pi_3(V)$	$\pi_4(V)$	$\pi_5(V)$	$\pi_6(V)$	$\pi_7(V)$
C_0	$\cup_{n=0}^{N}\{U(N)/[U(n)\times U(N-n)]\}$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
C_1	$U(N)$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
R_0	$\cup_{n=0}^{N}\{O(N)/[O(n)\times O(N-n)]\}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0
R_1	$O(N)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
R_2	$O(2N)/U(N)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$
R_3	$U(2N)/Sp(N)$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
R_4	$\cup_{n=0}^{N}\{Sp(N)/[Sp(n)\times Sp(N-n)]\}$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
R_5	$Sp(N)$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
R_6	$Sp(N)/U(N)$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0
R_7	$U(N)/O(N)$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0

regime refers to the limit $N \to \infty$. According to the Bott periodicity, the complex classifying spaces obey the periodicity condition

$$\pi_D(C_q) = \pi_{D+2}(C_q), \quad (q = 0,1), \tag{B1}$$

and the real classifying spaces obey the periodicity condition

$$\pi_D(R_q) = \pi_{D+8}(R_q), \quad (q = 0,\dots,7). \tag{B2}$$

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[54] We consider interactions that do not break the protecting symmetries of the noninteracting limit, that are strong on the boundary, yet are not-too-strong as measured by the single-particle gap for the bulk states of insulators.
[55] The saddle-point equation for $\boldsymbol{\phi}$ is given as follows. Integrating the fermionic degrees of freedom leads to the effective

Lagrangian,

$$\mathcal{S}_{\rm eff}[\boldsymbol{\phi}] := (-1)\mathrm{Tr}\log\left[\partial_\tau + \sum_{j=1}^{d-1}(-i\partial_j)\alpha_j + \sum_{\{\beta\}} 2i\beta\,\phi_\beta\right] + \frac{1}{\lambda\, r}\sum_{\{\beta\}}\mathrm{Tr}\left(\phi_\beta^2\right).$$

The symbol Tr represents tracing over the single-particle Hilbert space of the Dirac Hamiltonian with the Dirac matrices $\boldsymbol{\alpha}$ and β of dimension r. The saddle point equations $\delta\mathcal{S}_{\rm eff}[\boldsymbol{\phi}]/\delta\boldsymbol{\phi}|_{\boldsymbol{\phi}=\bar{\boldsymbol{\phi}}} = 0$ are

$$\int d\omega \int d^{d-1}\boldsymbol{k}\left(\frac{2\bar{\phi}_\beta}{\omega^2 + |\boldsymbol{k}|^2 - 4\,\bar{\boldsymbol{\phi}}^2}\right) = \frac{1}{\lambda\, r}\,\bar{\phi}_\beta.$$

We denote with Ω_{d-1} the area of the unit sphere S^{d-1}, with k the length of the vector $(\omega,\boldsymbol{k})$, and with Λ the ultraviolet cutoff in $(\omega,\boldsymbol{k})$ space. The saddle-point equations reduce to the equation

$$\Omega_{d-1}\int_0^\Lambda dk k^d\,\frac{2}{k^2 - 4\bar{\boldsymbol{\phi}}^2} = \frac{1}{\lambda\, r}.$$

It has the solution

$$|\bar{\phi}_\beta| = i\phi_0(\lambda\, r), \quad \phi_0(\lambda\, r) > 0.$$

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[60] The homotopy groups for the space of $\nu\times\nu$ normalized Dirac mass matrices V_ν for finite ν can be different from those for the space R_2 (i.e., the limit $\nu\to\infty$). In fact, the latter obey the Bott periodicity, while the former do not. However, we find by an explicit enumeration of the Dirac mass matrices in the following that the non-trivial entries of the relevant homotopy groups $\pi_D(V_\nu)$ appear when $\pi_D(R_2)$ is non-trivial. It turns out that this correspondence between homotopy groups at finite ν and infinite ν always holds for any example that we worked out later. While we do not rely on this fact for the analysis in one, two, and three dimensions, the analysis in higher dimensions made in Sec. III D assumes this correspondence.

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[67] Observe that $\pi_1(S^1) = \mathbb{Z}$ whereas $\pi_1(R_0) = \mathbb{Z}_2$. This discrepancy arises because we enter the stable homotopy group $\pi_D(R_0) = \mathbb{Z}_2$ by taking the limit $R_0 := \lim_{\nu\to\infty} V_\nu$ in the second column of Table V.

[68] Observe that $\pi_2(S^2) = \mathbb{Z}$ whereas $\pi_2(R_0) = \mathbb{Z}_2$. This discrepancy arises because we enter the stable homotopy group $\pi_D(R_0) = \mathbb{Z}_2$ by taking the limit $R_0 := \lim_{\nu\to\infty} V_\nu$ in the second column of Table V.

[69] In general, $N(\nu) - 1$ is determined from Table X(a) by shifting the entries of ν downward by one non-trivial homotopy group entry. For example, $N(4) - 1 = 4$, $N(8) - 1 = 8$, $N(16) - 1 = 9$, $N(32) - 1 = 10$, and so on.

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[73] These tasks correspond to the following classification problem. How does one parametrize the generators of $Cl_{p,q+1}$ that enter the kinetic contribution to the Dirac Hamiltonian? This classification problem is thus distinct from the one in which one seeks to parametrize the generators that enter the Dirac Hamiltonian as a Dirac mass.

[74] When identifying non-trivial homotopy groups and topological terms, we assume that the homotopy groups $\pi_D(V_\nu)$ for the space V_ν of $\nu\times\nu$ normalized Dirac mass matrices with the relevant finite ν are nontrivial whenever $\pi_D(R_q)$ is nontrivial. This is valid when ν is larger than a certain value determined by D. Here, R_q is the space of normalized Dirac mass matrices in the limit $\nu\to\infty$, and $\pi_D(R_q)$ obeys the Bott periodicity and are known from the mathematic literature. We are not able to prove that this assumption is true for all dimensions, but we have observed that it always holds in one, two, and three dimensions.

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$$i\gamma' = \begin{pmatrix} 0 & +M \\ -M^\mathsf{T} & 0 \end{pmatrix}$$

Now, demand that γ' squares to the unit matrix $\mathbb{1}$. This implies that $M\,M^\mathsf{T} = M^\mathsf{T}\,M = \mathbb{1}$, i.e., $M \in O(\nu)$. Hence, the classifying space is homeomorphic to $O(\nu)$.

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