Exact ground states and topological order in interacting Kitaev/Majorana chains

Hosho Katsura,[1] Dirk Schuricht,[2] and Masahiro Takahashi[3]

[1]*Department of Physics, Graduate School of Science, The University of Tokyo, Hongo, Tokyo 113-0033, Japan*

[2]*Institute for Theoretical Physics, Center for Extreme Matter and Emergent Phenomena, Utrecht University, Leuvenlaan 4, 3584 CE Utrecht, The Netherlands*

[3]*Department of Physics, Gakushuin University, Tokyo 171-8588, Japan*

We study a system of interacting spinless fermions in one dimension that, in the absence of interactions, reduces to the Kitaev chain [Kitaev, Phys. Usp. **44**, 131 (2001)]. In the noninteracting case, a signal of topological order appears as zero-energy modes localized near the edges. We show that the exact ground states can be obtained analytically even in the presence of nearest-neighbor repulsive interactions when the on-site (chemical) potential is tuned to a particular function of the other parameters. As with the noninteracting case, the obtained ground states are twofold degenerate and differ in fermionic parity. We prove the uniqueness of the obtained ground states and show that they can be continuously deformed to the ground states of the noninteracting Kitaev chain without gap closing. We also demonstrate explicitly that there exists a set of operators each of which maps one of the ground states to the other with opposite fermionic parity. These operators can be thought of as an interacting generalization of Majorana edge zero modes.

I. INTRODUCTION

In recent years, Majorana zero modes have been in the focus of research in condensed matter physics [1–4]. Experimental signatures of Majorana zero modes have been observed in the tunneling conductance in hybrid superconductor-semiconductor nanowire systems [5–8]. However, an unambiguous identification of Majorana particles remains open [9,10]. Very recently, an important step [11] toward such an identification was taken by scanning tunneling microscopy experiments on ferromagnetic atomic chains on a superconductor, which allowed us to obtain spatially resolved signatures showing localization of the zero modes at the edges, as predicted by theory.

The prototypical toy model possessing Majorana zero modes is the Kitaev chain with open boundaries [12], a one-dimensional tight-binding model for spinless fermions in the presence of *p*-wave superconducting pairing. This model possesses a topological phase with twofold-degenerate ground states that cannot be distinguished by any local order parameter. There exist zero-energy modes that commute with the Hamiltonian and anticommute with the fermionic parity; these are the Majorana zero modes. They are exponentially localized near the boundaries. As is well known (see, e.g., Ref. [13]), the Kitaev chain can be mapped to the one-dimensional transverse-field Ising chain via a nonlocal Jordan-Wigner transformation. In the resulting spin variables, the topological phase reduces to the ordered phase showing spontaneous magnetization while the fermionic parity maps to a string of spin operators.

The recent surge of interest in condensed-matter realizations of Majorana fermions includes the investigation of the effects of disorder [14–18] or dimerization [19] on the topological phase, the study of nanostructures possessing Majorana zero modes [20–24], and strongly correlated systems showing Kondo physics [25–30]. Furthermore, generalizations to higher symmetries including super-

symmetry [31–35] and parafermion modes were analyzed [13,36–40].

Here we focus on interaction effects. Motivated by the proposal [41,42] to realize systems hosting Majorana zero modes in semiconducting nanowires and the subsequent experimental works [5–9], most previous studies of interaction effects have focused on nanowire setups. This included the original spinful nanowire system with spin-orbit coupling, a Zeeman field and proximity to a superconductor [43–45], similar multiband nanowires [46], and effectively spinless systems [15,18] including helical wires [47] and two-chain ladders [48]. Furthermore, interactions directly in the Kitaev chain were studied [34,35,43,49–52], which were also shown to be implementable in an array of Josephson junctions [49], a realization allowing to reach strong interaction strengths as well as good control of the parameters. From all these works, a physical picture emerged revealing two main effects of interactions: On the one hand, interactions suppress the bulk gap and thus decrease the stability of the topological phase, while on the other hand, repulsive interactions broaden the chemical-potential window over which Majorana zero modes exist. Which of these two effects dominates or is more relevant depends on the specific realization at hand. General aspects of interaction effects on topological phases were also analyzed in Refs. [53–59].

In this article, we further investigate the Kitaev chain in the presence of repulsive interactions between the spinless fermions. Employing the Jordan-Wigner transformation, this model can be mapped to an XYZ Heisenberg chain, allowing us to obtain a wealth of results on the phase diagram [47,49]. Here, however, we are not interested in the phase diagram but directly in the twofold-degenerate ground states in the topological phase. Using classic results in the spin chain literature [60–62], we find that the exact ground states of the model can be obtained when the chemical potential is tuned to a particular function of the other parameters: the hopping amplitude, the *p*-wave pairing gap, and the strength of the

repulsive interaction. In this special case, the ground states can be written in a simple product form. The ground-state degeneracy is a necessary but not sufficient condition for topological order and the existence of Majorana zero modes. To prove the existence of topological order, we show that there exists a smooth path that interpolates between the solvable Hamiltonian and the noninteracting Kitaev chain in the topological phase, along which the ground states remain unchanged. We also prove rigorously that the spectral gap above the ground states does not vanish along the entire path. Then, thanks to the results of Fidkowski and Kitaev [54], it follows that the topological order of the noninteracting Kitaev chain persists along the path. To provide further evidence for the topological order, we prove the existence of a set of operators each of which maps one of the ground states to the other with the opposite fermionic parity. The operators obtained (i) are Hermitian, (ii) anticommute with the fermionic parity, and (iii) are localized near the edges. Therefore, they can be regarded as an interacting generalization of the Majorana zero modes. In fact, in the absence of the interactions, they exactly commute with the Hamiltonian and reduce to the standard Majorana zero modes in the original Kitaev chain.

The paper is organized as follows. In Sec. II, we give a precise definition of the model and list the symmetries of its Hamiltonian. In Sec. III, we first introduce the notion of frustration-free Hamiltonians. Then we show how one can find the ground states of a chain of arbitrary length from the exact results of the two-site problem. The expressions for the ground states are also given explicitly. In Sec. IV, we discuss the topological order of the present system. We present a theorem about the continuous deformation between the interacting and noninteracting Kitaev chains that share the same ground states. We show how the theorem follows from a lemma about the auxiliary free-fermion problem. We also present the proof of the lemma and derive explicit expressions for the Majorana zero modes in the auxiliary free-fermion problem. Concluding remarks are presented in Sec. V. In Appendix A, we summarize the relation between Majorana and complex fermions. The explicit expression for the Hamiltonian in terms of Majorana fermions is also given. In Appendix B, we show the mapping between the fermionic Hamiltonian and the XYZ chain in a magnetic field. We also show how the frustration-free condition and the ground states obtained translate into the spin language. In Appendix C, we present a derivation of the inequality used in Sec. III. A few examples of correlation functions in the ground states are also provided. In Appendix D, we extend the theorem to the case in which the couplings are spatially inhomogeneous. In Appendix E, we present a detailed exposition of the eigenvalue problem related to the auxiliary free-fermion problem.

II. MODEL

We consider a system of spinless fermions on a chain of length L with open boundaries. For each site $j = 1,2,\ldots,L$, we denote by $c_j^\dagger$ and c_j the creation and the annihilation operators, respectively. As usual, the number operators are defined by $n_j := c_j^\dagger c_j$.

A. Hamiltonian

We consider the Hamiltonian of interacting spinless fermions described by

$$H = \sum_{j=1}^{L-1}[-t(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) + \Delta(c_j c_{j+1} + c_{j+1}^\dagger c_j^\dagger)]$$
$$-\frac{1}{2}\sum_{j=1}^{L}\mu_j(2n_j - 1) + U\sum_{j=1}^{L-1}(2n_j - 1)(2n_{j+1} - 1),$$
(1)

where t is the hopping amplitude and Δ is the p-wave paring gap, which is assumed to be real. Without loss of generality, we can assume that $t \geqslant 0$ because the case with $t \leqslant 0$ can be achieved by local unitary transformations: $c_j \to -i(-1)^j c_j$. We can further assume that $\Delta \geqslant 0$ because the case with $\Delta \leqslant 0$ can be achieved by $c_j \to ic_j$ for all j. Here, μ_j is the on-site (chemical) potential and $U \geqslant 0$ is the strength of the nearest-neighbor repulsive interaction. It is convenient, for later purposes, to keep a site-dependent on-site potential μ_j. In the absence of the interaction ($U = 0$), the model reduces to the Kitaev chain [12], in which Majorana edge zero modes occur provided that it is in the topological phase. Thus one can think of the interacting system as Majorana fermions with a quartic interaction. An explicit expression for the Hamiltonian in terms of Majorana fermions is given in Appendix A. One can also express the Hamiltonian in terms of spin-1/2 operators via a Jordan-Wigner transformation. The corresponding model turns out to be the XYZ spin chain in a magnetic field (see Appendix B for more details).

B. Symmetries

Let us first consider the symmetries of the Hamiltonian H. Because of the presence of the pairing term, H does not conserve the total fermion number $F := \sum_{j=1}^{L} n_j$, i.e., $[H,F]\neq 0$. However, the fermion number modulo 2 is conserved, $[H,(-1)^F] = 0$. The Hamiltonian respects time-reversal symmetry, i.e., it is invariant under complex conjugation.

In addition, when $\mu_j = 0$ for all j, the Hamiltonian is invariant under the charge conjugation operation $c_j \to (-1)^j c_j^\dagger$. More precisely, H commutes with the following unitary operator:

$$P = \prod_{j=1}^{L}[c_j + (-1)^j c_j^\dagger].$$
(2)

We also note that this particular case is integrable, because it can be mapped to the XYZ spin chain (without magnetic fields), which was solved by Baxter [63]. For the case with nonvanishing μ_j's, the Hamiltonian H is not invariant under the above charge conjugation. However, we can at least say that H with $\{\mu_j\}_{j=1}^{L}$ and that with $\{-\mu_j\}_{j=1}^{L}$ have the same spectrum. In the following, for simplicity, we assume that $\mu_j \geqslant 0$ for all j.

III. EXACT GROUND STATES

In this section, we show that the Hamiltonian H is *frustration-free* when the μ_j's are tuned to a particular function

of the other parameters (t,Δ,U), in which case the exact ground states are easy to obtain analytically.

Consider the case in which $\mu_1 = \mu_L = \mu/2$ and $\mu_j = \mu$ ($j = 2,3,\ldots,L-1$), i.e., the on-site potentials at the boundary sites are half the bulk ones. In this case, the Hamiltonian takes the form

$$H = \sum_{j=1}^{L-1} h_j, \tag{3}$$

where

$$h_j = -t(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) + \Delta(c_j c_{j+1} + c_{j+1}^\dagger c_j^\dagger)$$
$$- \frac{\mu}{2}(n_j + n_{j+1} - 1) + U(2n_j - 1)(2n_{j+1} - 1). \tag{4}$$

Since $[h_j,h_k] \neq 0$ if $|j - k| = 1$, the local Hamiltonians cannot be diagonalized simultaneously. However, there may exist some special values of μ at which the projection onto the ground-state subspace P_0 satisfies $h_j P_0 = \epsilon_0 P_0$ for all j, where ϵ_0 is the smallest eigenvalue of h_j. In other words, the ground states of H minimize each h_j independently. In such cases, the Hamiltonian H is said to be frustration-free. Many known exactly solvable models fall into this category. Examples include the Affleck-Kennedy-Lieb-Tasaki model [64], the Rokhsar-Kivelson model [65], and the Kitaev toric code model [66].

To search for the condition under which the Hamiltonian (3) is frustration-free, we shall first consider the two-site problem, i.e., deriving the spectrum and the eigenstates of h_j. Let us ignore, for the moment, the region outside the bond $(j, j + 1)$. Then the Hilbert space for the problem is spanned by the four states: $|\circ\circ\rangle := |\text{vac}\rangle$, $|\bullet\circ\rangle := c_j^\dagger|\text{vac}\rangle$, $|\circ\bullet\rangle := c_{j+1}^\dagger|\text{vac}\rangle$, and $|\bullet\bullet\rangle := c_j^\dagger c_{j+1}^\dagger|\text{vac}\rangle$, where $|\text{vac}\rangle$ is the vacuum state such that $c_k|\text{vac}\rangle = 0$ for any k. Since the local Hamiltonian h_j commutes with the fermionic parity $(-1)^F$, we can deal with even and odd sectors separately. In the even sector spanned by $|\circ\circ\rangle$ and $|\bullet\bullet\rangle$, h_j is expressed as

$$
\begin{array}{cc}
& |\circ\circ\rangle \quad |\bullet\bullet\rangle \\
h_j^{(\text{even})} = & \begin{pmatrix} U + \mu/2 & -\Delta \\ -\Delta & U - \mu/2 \end{pmatrix}.
\end{array}
\tag{5}
$$

An unnormalized ground state of $h_j^{(\text{even})}$ and the corresponding eigenvalue are

$$\left|\psi_0^{(\text{even})}\right\rangle = |\circ\circ\rangle + \cot\frac{\theta}{2}|\bullet\bullet\rangle, \tag{6}$$

$$\epsilon_0^{(\text{even})} = U - \sqrt{\Delta^2 + (\mu/2)^2}, \tag{7}$$

respectively, where $\theta = \arctan(2\Delta/\mu)$, which is assumed to be in $[0,\pi]$. Note that $|\psi_0^{(\text{even})}\rangle$ is nondegenerate as long as either Δ or μ is nonvanishing. Similarly, in the odd sector spanned by $|\bullet\circ\rangle$ and $|\circ\bullet\rangle$, h_j is expressed as

$$
\begin{array}{cc}
& |\bullet\circ\rangle \, |\circ\bullet\rangle \\
h_j^{(\text{odd})} = & \begin{pmatrix} -U & -t \\ -t & -U \end{pmatrix}.
\end{array}
\tag{8}
$$

An unnormalized ground state of $h_j^{(\text{odd})}$ and the corresponding eigenvalue are

$$\left|\psi_0^{(\text{odd})}\right\rangle = |\bullet\circ\rangle + |\circ\bullet\rangle, \tag{9}$$

$$\epsilon_0^{(\text{odd})} = -(U + t), \tag{10}$$

which is nondegenerate except when $t = 0$.

The absolute ground state of h_j can be obtained by comparing $\epsilon_0^{(\text{even})}$ and $\epsilon_0^{(\text{odd})}$, which become degenerate if

$$\mu = \mu^* := 4\sqrt{U^2 + tU + \frac{t^2 - \Delta^2}{4}}. \tag{11}$$

Note that though $\mu = -\mu^*$ is also allowed, we have chosen the positive μ for simplicity (see the discussion in Sec. II B). At $\mu = \mu^*$, any linear combination of $|\psi_0^{(\text{even})}\rangle$ and $|\psi_0^{(\text{odd})}\rangle$ is also a ground state of h_j. If we take the following particular combinations, we see that they are *disentangled*:

$$\left|\psi_0^{(\pm)}\right\rangle = |\psi^{(\text{even})}\rangle \pm \sqrt{\cot\frac{\theta^*}{2}}|\psi^{(\text{odd})}\rangle$$
$$= (1 \pm \alpha c_j^\dagger)(1 \pm \alpha c_{j+1}^\dagger)|\text{vac}\rangle$$
$$= \exp(\pm\alpha c_j^\dagger)\exp(\pm\alpha c_{j+1}^\dagger)|\text{vac}\rangle, \tag{12}$$

where $\theta^* = \arctan(2\Delta/\mu^*)$ and $\alpha = \sqrt{\cot(\theta^*/2)}$. We recall that $\theta \in [0,\pi]$ has been assumed. In the entire Hilbert space, the ground states of h_j at $\mu = \mu^*$ are highly degenerate, because any state of the form

$$f(c_1^\dagger,\ldots,c_{j-1}^\dagger)\,e^{\pm\alpha c_j^\dagger}e^{\pm\alpha c_{j+1}^\dagger}\,g(c_{j+2}^\dagger,\ldots,c_L^\dagger)|\text{vac}\rangle \tag{13}$$

is a ground state of h_j, where f and g are arbitrary polynomials in $c_1^\dagger,\ldots,c_{j-1}^\dagger$ and $c_{j+2}^\dagger,\ldots,c_L^\dagger$, respectively.

In this way, we find that the ground states of h_j at $\mu = \mu^*$ can be factorized into the product of two states. This observation hints at the possibility that we may obtain exact ground states of H with $\mu = \mu^*$ for any L. In fact, H is frustration-free at $\mu = \mu^*$. To see this, consider the states of the form

$$\left|\Psi_0^{(\pm)}\right\rangle = \frac{1}{(1 + \alpha^2)^{L/2}}e^{\pm\alpha c_1^\dagger}e^{\pm\alpha c_2^\dagger}\cdots e^{\pm\alpha c_L^\dagger}|\text{vac}\rangle. \tag{14}$$

Here the coefficient has been introduced so that $\langle\Psi_0^{(\pm)}|\Psi_0^{(\pm)}\rangle = 1$. Since $\exp(\pm\alpha c_k^\dagger)$ commutes with h_j unless $k \neq j, j + 1$, one can easily verify that $|\Psi_0^{(\pm)}\rangle$ minimizes each h_j independently at $\mu = \mu^*$. Therefore, H is frustration-free at $\mu = \mu^*$, and $|\Psi_0^{(\pm)}\rangle$ are its exact ground states with energy [67],

$$E_0 = -(L - 1)(U + t). \tag{15}$$

A similar factorization of the ground states has been found in spin-1/2 chains in a magnetic field [60–62]. In fact, the condition under which the Hamiltonian is frustration-free is the same as the exactly solvable condition of the XYZ spin chain in a magnetic field (see Appendix B for more details).

It should be noted that $|\Psi_0^{(\pm)}\rangle$ are not orthogonal and are not eigenstates of the fermionic parity $(-1)^F$, because each $|\Psi_0^{(\pm)}\rangle$ contains both even and odd numbers of fermions. The ground states that are eigenstates of $(-1)^F$ can be obtained

by taking appropriate linear combinations of $|\Psi_0^{(\pm)}\rangle$. Let us decompose the entire Hilbert space $\mathcal{H}$ as $\mathcal{H} = \mathcal{H}^{(\text{even})} \oplus \mathcal{H}^{(\text{odd})}$, where $\mathcal{H}^{(\text{even/odd})}$ consists of the states with even/odd numbers of fermions. Then, noting that $[F, c_j^\dagger] = c_j^\dagger$ and hence $(-1)^F \exp(\pm \alpha c_j^\dagger) = \exp(\mp \alpha c_j^\dagger)(-1)^F$, we find that the states

$$\left|\Psi_0^{(\text{even})}\right\rangle = \left|\Psi_0^{(+)}\right\rangle + \left|\Psi_0^{(-)}\right\rangle \in \mathcal{H}^{(\text{even})}, \qquad (16)$$

$$\left|\Psi_0^{(\text{odd})}\right\rangle = \left|\Psi_0^{(+)}\right\rangle - \left|\Psi_0^{(-)}\right\rangle \in \mathcal{H}^{(\text{odd})}, \qquad (17)$$

are orthogonal to each other. As we will prove in the next section, there is no other ground state of H with $\mu = \mu^*$ except for these two states. Furthermore, for large L, they cannot be distinguished by any local measurement. To illustrate this, let us consider the expectation values of local operators in $|\Psi_0^{(\text{even/odd})}\rangle$. For notational convenience, we write

$$\langle \cdots \rangle_{\text{par}} := \frac{\left\langle \Psi_0^{(\text{par})}\right| \cdots \left|\Psi_0^{(\text{par})}\right\rangle}{\left\langle \Psi_0^{(\text{par})}\middle|\Psi_0^{(\text{par})}\right\rangle}, \qquad (18)$$

where par is either even or odd. Let O_e/O_o be a local operator consisting of an even/odd number of creation and annihilation operators. We assume that O_e/O_o is supported on a set of lattice sites $j_1 < j_2 < \cdots < j_k$ with $j_k - j_1 = \ell - 1$. Noting that $\{O_o, (-1)^F\} = 0$ and using the relations $(-1)^F |\Psi_0^{(\text{even})}\rangle = |\Psi_0^{(\text{even})}\rangle$ and $(-1)^F |\Psi_0^{(\text{odd})}\rangle = -|\Psi_0^{(\text{odd})}\rangle$, one finds

$$\langle O_o \rangle_{\text{even}} = \langle O_o \rangle_{\text{odd}} = 0 \qquad (19)$$

for any O_o. On the other hand, for O_e, one finds that the difference between $\langle O_e \rangle_{\text{even}}$ and $\langle O_e \rangle_{\text{odd}}$ is bounded from above as follows:

$$|\langle O_e \rangle_{\text{even}} - \langle O_e \rangle_{\text{odd}}| \leqslant \mathcal{C} \|O_e\| e^{-L/\xi}, \qquad (20)$$

where $\|O_e\|$ denotes the operator norm of O_e,

$$\xi = \frac{1}{\ln|\eta|} \quad \text{with} \quad \eta = \frac{1 + \cot(\theta^*/2)}{1 - \cot(\theta^*/2)}, \qquad (21)$$

and the coefficient $\mathcal{C}$ is given by

$$\mathcal{C} = 2 \frac{1 + |\eta|^\ell}{1 - 1/\eta^2}. \qquad (22)$$

Clearly, the result shows that the states $|\Psi_0^{(\text{even})}\rangle$ and $|\Psi_0^{(\text{odd})}\rangle$ cannot be distinguished by any local measurement for large L. A detailed proof of the inequality Eq. (20) as well as some explicit examples are given in Appendix C.

Let us finally comment on the location of the solvable case Eq. (11) in the phase diagram. The phase diagram of the model described by the Hamiltonian Eq. (1) with $\Delta = t$ and $\mu_j = \mu$ ($j = 1, 2, \ldots, L$) has been worked out previously [47,49]; Fig. 1 shows the phase diagram for this particular case. Since the spectrum of the Hamiltonian is invariant under sending $\mu \to -\mu$, we only show the region where $\mu/t \geqslant 0$. The origin $(U/t, \mu/t) = (0,0)$ corresponds to the original noninteracting Kitaev chain, which is in a topological phase. When $\Delta = t$, the frustration-free condition reads

$$\mu = 4\sqrt{U^2 + tU}, \qquad (23)$$

which is shown by the red (dashed) line in Fig. 1. Clearly, this line is in the topological phase. We note that both the
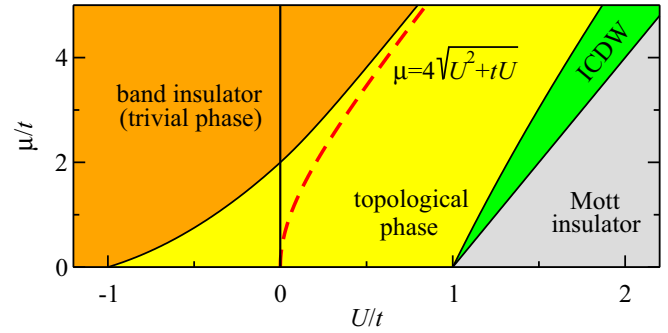

FIG. 1. (Color online) Phase diagram of the interacting Kitaev chain Eq. (1) with $\Delta = t$ and $\mu_j = \mu$. The region $\mu/t < 0$ is not shown since the phase diagram is invariant under sending $\mu \to -\mu$. The red (dashed) line indicates the frustration-free line Eq. (23), along which the exact ground states can be obtained in closed form. ICDW stands for the phase in which an incommensurate charge-density wave state is a ground state.

frustration-free line and the phase boundary between the trivial and topological phases approach $\mu = 4U$ in the infinite-U limit. In this limit, the model can be mapped to a model of hard-core spinless fermions, which is solvable using the free-fermion method [68].

IV. TOPOLOGICAL ORDER

In this section, we discuss the topological order of the ground states obtained in the previous section. We explicitly construct a one-parameter family of Hamiltonians that interpolates between interacting and noninteracting Kitaev chains, the latter of which exhibits topological order. The spectral gap above the ground state and the existence of Majorana zero modes that are localized around the edges are proven rigorously.

A. Smooth path connecting interacting and noninteracting Hamiltonians

Let us introduce the Hamiltonian that interpolates continuously between the interacting and noninteracting model. As discussed in the preceding section, the ground states $|\Psi_0^{(\pm)}\rangle$ depend only on $\theta^* = \arctan(2\Delta/\mu^*)$ at $\mu = \mu^*$ (frustration-free case). Therefore, they are also ground states of the following one-parameter family of the Hamiltonians:

$$H(s) = \sum_{j=1}^{L-1} h_j(s) \quad \text{with } s \geqslant 0, \qquad (24)$$

where the local Hamiltonians are given by

$$\begin{aligned} h_j(s) = &-(c_j^\dagger c_{j+1} + \text{H.c.}) + (1+s)\sin\theta^*(c_j c_{j+1} + \text{H.c.}) \\ &- (1+s)\cos\theta^*(n_j + n_{j+1} - 1) \\ &+ \frac{s}{2}(2n_j - 1)(2n_{j+1} - 1) + \left(1 + \frac{s}{2}\right). \end{aligned} \qquad (25)$$

Here we have set $t = 1$. The last constant was added so that the ground-state energy is equal to zero. Since we have assumed non-negative Δ and μ^*, θ is in $[0, \pi/2]$. When

$\mu = \mu^*$, the original Hamiltonian H in Eq. (3) is the same as $H(2U)$ up to a trivial constant, which can be verified by direct calculation. On the other hand, since the penultimate term in Eq. (25) vanishes when $s = 0$, $H(0)$ reduces to the noninteracting Kitaev Hamiltonian with pairing $\Delta/(1 + 2U)$ and on-site potential $\mu^*/(1 + 2U)$, which we denote by H_0 in the following. Thus we see that the Hamiltonians H and H_0 are smoothly connected to each other. Along the entire path $0 \leqslant s \leqslant 2U$, the ground states remain unchanged [69]. This suggests that the two Hamiltonians H and H_0 are in the same topological phase as long as the energy gap does not close along the path.

Now we show the existence of an energy gap of $H(s)$. Toward that end, we prove the following:

Theorem. For all $s \geqslant 0$ and $\theta^* \neq 0$, (i) the ground state of the Hamiltonian $H(s)$ is unique up to double degeneracy, and (ii) $H(s)$ has a uniform (independent of the length of the chain) spectral gap above the ground state.

The proof of the theorem relies on the following:

Lemma. For $\theta^* \neq 0$, (i) the ground state of the noninteracting Hamiltonian $H(0)$ is unique up to double degeneracy, and (ii) $H(0)$ has a uniform (independent of the length of the chain) spectral gap above the ground state.

This Lemma can be proven by directly computing the spectrum of $H(0)$, which will be given in the next subsection. We shall now show how the theorem follows from the Lemma.

Proof of Theorem. It is convenient to introduce the following operator:

$$Q_j = \frac{1}{2}\cos\frac{\theta^*}{2}(c_j + c_{j+1})(c_j^\dagger - c_{j+1}^\dagger)(c_j^\dagger + c_{j+1}^\dagger)$$
$$+ \frac{1}{2}\sin\frac{\theta^*}{2}(c_j^\dagger - c_{j+1}^\dagger)(c_j + c_{j+1})(c_j - c_{j+1}), \quad (26)$$

which is a sum of triple products of fermion operators [70]. In terms of Q_j, $h_j(s)$ in Eq. (25) is rewritten as

$$h_j(s) = Q_j Q_j^\dagger + (1 + s)Q_j^\dagger Q_j, \quad (27)$$

which can be verified by a tedious but straightforward calculation. The above form of $h_j(s)$ is manifestly positive-semidefinite for all $s \geqslant -1$. It is then clear that $h_j(s) \geqslant h_j(0)$ for $s \geqslant 0$. Here, we write $A \geqslant B$ to denote that $A - B$ is positive-semidefinite [71]. Since the inequalities for the local Hamiltonians hold for all $j = 1, 2, \ldots, L - 1$, we arrive at the inequality

$$H(s) \geqslant H(0) \quad \text{for } s \geqslant 0. \quad (28)$$

The states $|\Psi_0^{(\pm)}\rangle$ are annihilated by both Q_j and $Q_j^\dagger$ for all j, and hence they are ground states of $H(s)$.

We shall next compare the energy eigenvalues of $H(s)$ and $H(0)$. Let $E_n(s)$ be the nth eigenvalue of $H(s)$. Here the energy eigenvalues are arranged in nondecreasing order: $0 = E_1(s) = E_2(s) \leqslant E_3(s) \leqslant \cdots \leqslant E_n(s) \leqslant \cdots$. Note that $H(s)$ has at least two zero-energy ground states. It follows from the min-max principle [72] that the inequality Eq. (28) implies $E_n(s) \geqslant E_n(0)$ for all n. Therefore, the spectral gap is a nondecreasing function of s, and the eigenstates of $H(s)$ continuously connected to those of $H(0)$ with positive energy never join the zero-energy manifold with increasing s. This, together with the Lemma, proves the theorem. ∎

Several comments are in order. Our theorem implies that the interacting Hamiltonian $H(2U)$ and the noninteracting one $H(0)$ are adiabatically connected to each other without gap closing. Thanks to the work of Fidkowski and Kitaev [54], this suffices to show that $H(2U)$ is topologically nontrivial, because the noninteracting $H(0)$ is in a topological phase. However, some care is needed here. As shown in Refs. [53,54], the classification of free-fermion topological phases breaks down in one dimension in the presence of interactions. The noninteracting Kitaev chain with time-reversal symmetry belongs to the BDI symmetry class, different phases of which are characterized by $\mathbb{Z}$ integers. This $\mathbb{Z}$ classification is broken down to $\mathbb{Z}_8$ in the presence of interactions. One might think that there is a possibility that a topologically nontrivial phase is adiabatically connected to the trivial phase with no Majorana edge zero modes via interactions. This is, however, not the case here, because the topological index is basically the number of Majorana edge modes localized at one end of the chain. For $H(0)$, the number of Majorana edge zero modes at each edge is 1, which is apparently different from 0 modulo 8. Therefore, the one-parameter family of $H(s)$ with $s \geqslant 0$ is in a topologically nontrivial phase, which is adiabatically connected to the one in the noninteracting classification.

We also comment on the stability of the spectral gap against small perturbations. Let us first consider the effect of the inhomogeneity arising from the on-site potential term. As noted in Sec. III, the on-site potentials at the boundary sites are half the bulk one in the Hamiltonian Eq. (3). To see whether the spectral gap is robust against the perturbation that restores the homogeneity to some extent, consider the perturbed Hamiltonian $H' = H + V$ with $\mu = \mu^*$ and

$$V = -\frac{\delta\mu}{2}(n_1 + n_L), \quad (29)$$

where $\delta\mu$ is assumed to be positive. Let $E_n(2U, \delta\mu)$ be the nth eigenvalue of H'. The condition for the existence of the spectral gap reads $E_3(2U, \delta\mu) > E_2(2U, \delta\mu)$. From Weyl's theorem (see Theorem 4.3.1 in Ref. [72]), one finds $E_n(2U) - \delta\mu \leqslant E_n(2U, \delta\mu) \leqslant E_n(2U)$ for all n. This implies that H' has a spectral gap if $\delta\mu < E_3(2U)$. Since our Theorem ensures that $E_3(2U)$ is strictly positive, we conclude that the spectral gap is robust against V for small enough $\delta\mu$. A more quantitative estimate is possible, because the lower bound for $E_3(2U)$ is obtained as $E_3(2U) \geqslant E_3(0) = 2(1 - \cos\theta^*)$ in the next subsection. Using this value, we find that the spectral gap of H' is nonzero provided

$$\delta\mu < 2\left(1 - \frac{\sqrt{(1 + 2U)^2 - \Delta^2}}{1 + 2U}\right). \quad (30)$$

A more interesting question is whether the spectral gap of $H(s)$ is robust against small but global perturbations. For quantum spin systems, rigorous perturbation theories for the stability of gapped ground states have been developed [73,74]. Though the existing methods usually assume translation invariant Hamiltonians and do not immediately apply to the present system, we expect that some modification of them is likely to apply particularly to the region around the point $(\Delta, U) = (0, 0)$ corresponding to the classical Ising chain. A thorough analysis is, however, beyond the scope of the present study and is left for future work.

Finally, we comment on the generalization of our model to the case in which the couplings are allowed to vary spatially. One can, in fact, construct such a model without changing the ground states of the Hamiltonian $H(s)$. The key to the construction of such a model is to note that a sum of operators each of which is a polynomial in Q_j and $Q_j^\dagger$ ($j = 1, 2, \ldots, N-1$) annihilates the states $|\Psi_0^{(\pm)}\rangle$. If this sum is positive-semidefinite, then these states are the lowest eigenstates of the sum with zero eigenvalue. One such example is the following Hamiltonian:

$$H^{\text{inh}} = \sum_{j=1}^{L-1} (\alpha_j Q_j Q_j^\dagger + \beta_j Q_j^\dagger Q_j), \quad (31)$$

where α_j and β_j are arbitrary positive numbers. Each local interaction can be either repulsive or attractive, depending on β_j/α_j. More precisely, it is repulsive (attractive) if $\beta_j/\alpha_j > 1$ ($0 < \beta_j/\alpha_j < 1$). For the above Hamiltonian, we can prove a result similar to the theorem above, and hence the presence of topological order. A precise statement and its proof are provided in Appendix D. We emphasize that the above example demonstrates that the couplings in the Hamiltonian need not necessarily be uniform to obtain topological order.

B. Proof of Lemma

Let us prove the Lemma by explicitly computing the spectrum of $H(0)$. Though $H(0)$ is quadratic in c and $c^\dagger$, it is not quite straightforward to diagonalize it because of the boundaries. Remarkably, however, we find that the entire spectrum of $H(0)$ can be obtained analytically. To show this, we first rewrite $H(0)$ in terms of Majorana operators $a_j = c_j + c_j^\dagger$ and $b_j = (c_j - c_j^\dagger)/i$, the properties of which are summarized in Appendix A. The Hamiltonian reads

$$H(0) = \frac{i}{2} \sum_{j,k=1}^{L} B_{j,k} a_j b_k + (L-1), \quad (32)$$

where the real matrix B is given by

$$B = - \begin{pmatrix} \mathfrak{c} & 1-\mathfrak{s} & & & \\ 1+\mathfrak{s} & 2\mathfrak{c} & 1-\mathfrak{s} & & \\ & \ddots & \ddots & \ddots & \\ & & 1+\mathfrak{s} & 2\mathfrak{c} & 1-\mathfrak{s} \\ & & & 1+\mathfrak{s} & \mathfrak{c} \end{pmatrix} \quad (33)$$

with $\mathfrak{s} = \sin\theta^*$ and $\mathfrak{c} = \cos\theta^*$. Here, matrix elements that are zero are left empty. Since the matrix B is non-Hermitian, it may not be diagonalizable, but it can be written in the singular-value decomposition (SVD) form $B = U\Lambda V^T$ with $\Lambda = \text{diag}(\epsilon_1, \epsilon_2, \ldots, \epsilon_L)$, where orthogonal matrices U and V diagonalize BB^T and B^TB, respectively. The singular values ϵ_n ($n = 1, 2, \ldots, L$) can be obtained as non-negative square roots of the eigenvalues of BB^T.

To see the relation between these singular values and single-particle energies of $H(0)$, we introduce the following set of Majorana operators:

$$a_n' = \sum_{j=1}^{L} U_{j,n} a_j, \quad b_n' = \sum_{j=1}^{L} V_{j,n} b_j, \quad (34)$$

which satisfy $(a')_j^\dagger = a_j'$, $(b')_j^\dagger = b_j'$, $\{a_j', a_k'\} = 2\delta_{j,k}$, and $\{a_j', b_k'\} = 0$ for all $j, k = 1, 2, \ldots, L$. It then follows from the SVD of B that $H(0)$ is rewritten as

$$H(0) = \sum_{n=1}^{L} \epsilon_n a_n' b_n' + (L-1) \quad (35)$$

$$= \sum_{n=1}^{L} \epsilon_n \left(f_n^\dagger f_n - \frac{1}{2} \right) + (L-1), \quad (36)$$

where $f_n := (a_n' + ib_n')/2$ are complex fermions satisfying $\{f_m, f_n^\dagger\} = \delta_{m,n}$. Therefore, the set of singular values of B is exactly the same as the single-particle spectrum of $H(0)$. To get ϵ_n, we need to diagonalize BB^T. Since this matrix is pentadiagonal, a direct diagonalization by analytical means is not feasible. In the present case, however, something special happens. In fact, we have

$$BB^T = C^2, \quad (37)$$

where C is a real symmetric and tridiagonal matrix given by

$$C = \begin{pmatrix} 1-\mathfrak{s} & \mathfrak{c} & & & \\ \mathfrak{c} & 2 & \mathfrak{c} & & \\ & \ddots & \ddots & \ddots & \\ & & \mathfrak{c} & 2 & \mathfrak{c} \\ & & & \mathfrak{c} & 1+\mathfrak{s} \end{pmatrix}. \quad (38)$$

A similar factorization was found by Truong and Peschel in the study of the Hamiltonian limit of the corner transfer matrix of the same model [75]. Because of the presence of the boundary terms, the diagonalization of C is still nontrivial, but it can be done analytically. The details are given in Appendix E. The exact eigenvalues of C are as follows:

$$\epsilon_n = \begin{cases} 2 + 2\mathfrak{c}\cos\left(\frac{n\pi}{L}\right), & n = 1, 2, \ldots, L-1, \\ 0, & n = L. \end{cases} \quad (39)$$

The presence of the zero-energy mode clearly signals the topological order in $H(0)$. The system indeed possesses Majorana edge zero modes, as we will see in the next subsection.

Let us now prove the uniqueness of the ground state and the existence of the spectral gap, which does not vanish as $L \to \infty$. From the fact that $\text{tr}\, C = 2(L-1)$, one finds that $\sum_{n=1}^{L} \epsilon_n = 2(L-1)$, which together with $\epsilon_L = 0$ yields $H(0) = \sum_{n=1}^{L-1} \epsilon_n f_n^\dagger f_n$. Since $H(0)$ is positive-semidefinite, a state annihilated by f_n for all $n = 1, 2, \ldots, L-1$ is a many-body ground state with zero energy. The states $|\Psi_0^{(\text{even})}\rangle$ and $|\Psi_0^{(\text{odd})}\rangle$ in Eqs. (16) and (17) are such states, because we

have already shown that they are zero-energy states of $H(0)$. Then by acting with $f_n^\dagger$ ($n = 1, 2, \ldots, L - 1$) on these two states, one can construct many-body eigenstates of $H(0)$ as follows:

$$(f_1^\dagger)^{m_1}(f_2^\dagger)^{m_2}\cdots(f_{L-1}^\dagger)^{m_{L-1}}\big|\Psi_0^{(\text{even})}\big\rangle, \tag{40}$$

$$(f_1^\dagger)^{m_1}(f_2^\dagger)^{m_2}\cdots(f_{L-1}^\dagger)^{m_{L-1}}\big|\Psi_0^{(\text{odd})}\big\rangle, \tag{41}$$

where $m_j = 0$ or 1 for all $j = 1, 2, \ldots, L - 1$. Note that the energy of the above two states is $E = \sum_{j=1}^{L-1} m_j \epsilon_j$, which implies that every energy level is at least twofold-degenerate. In this way, we have obtained 2^L energy eigenstates forming a complete set of orthogonal states, which can be verified by noting $\langle \Psi_0^{(\text{even})} | \Psi_0^{(\text{odd})} \rangle = 0$. Because of the positivity of ϵ_n for $n = 1, 2, \ldots, L - 1$, we find that $|\Psi_0^{(\text{even})}\rangle$ and $|\Psi_0^{(\text{odd})}\rangle$ are the only zero-energy ground states of $H(0)$. This proves the uniqueness part of the Lemma. We now turn to prove the existence of the spectral gap. The lowest-energy excited states are obtained by filling the lowest positive ϵ_n. More explicitly, they are given by $f_{L-1}^\dagger|\Psi_0^{(\text{even})}\rangle$ and $f_{L-1}^\dagger|\Psi_0^{(\text{odd})}\rangle$. We thus obtain the spectral gap as

$$E_3(0) = \epsilon_{L-1} = 2\left[1 - \mathfrak{c}\cos\left(\frac{\pi}{L}\right)\right], \tag{42}$$

which is a decreasing function in L when $L \geqslant 0$. In the infinite-L limit, we have $E_3(0) \to 2(1 - \cos\theta^*)$, which is strictly positive unless $\theta^* = 0$. This completes the proof of the Lemma. ∎

There is an alternative proof of the uniqueness of the twofold-degenerate ground states using the spin-chain Hamiltonian [see Eq. (B4) in Appendix B]. From Eq. (B5), one finds that all the off-diagonal elements of the Hamiltonian in the standard Ising basis are nonpositive when $\Delta \geqslant 0$. Then, noting that the space of eigenstates is separated into two disconnected sectors with opposite parities ($\prod_{j=1}^L \sigma_j^z$), we find that in each sector the Hamiltonian satisfies the connectivity condition when $\Delta > 0$, i.e., $\theta^* \neq 0, \pi$. It then follows from the Perron-Frobenius theorem [72] that the ground state of each sector is unique. Therefore, the absolute ground state is at most twofold-degenerate. But since we have already found two ground states, this proves that the ground-state degeneracy is exactly 2.

C. Majorana edge zero modes

The presence of a zero-energy single-particle state is usually a signal of topological order, but it does not necessarily mean the existence of Majorana zero modes that are localized at the boundaries. Here we show that, for $H(0)$, the explicit expressions for the Majorana operators can be obtained analytically. Let us first define *Majorana edge zero modes* [13]. A Majorana edge zero mode Γ is an operator such that:

(i) $\Gamma^\dagger = \Gamma$,

(ii) $[H(0), \Gamma] = 0$,

(iii) $\{(-1)^F, \Gamma\} = 0$, and

(iv) localized near the edge and normalizable as $\Gamma^2 = 1$ even in the infinite-L limit.

We next show that one can construct such edge modes from the left and right null vectors of B in Eq. (33). To see this, let

$\boldsymbol{u} = (u_1, u_2, \ldots, u_L)$ be a left null vector of B. An elementary calculation shows that

$$\boldsymbol{u} = (1, r, r^2, \ldots, r^{L-1}) \quad \text{with } r = -\frac{\mathfrak{c}}{1 + \mathfrak{s}} \tag{43}$$

satisfies $\boldsymbol{u}B = (0, 0, \ldots, 0)$. Then from the anticommutation relations of a and b operators, one finds that the following operator commutes with $H(0)$:

$$\Gamma_{\text{L}} = \mathcal{N} \sum_{j=1}^L r^{j-1} a_j, \tag{44}$$

where the normalization factor $\mathcal{N}$ has been introduced so that $(\Gamma_{\text{L}})^2 = 1$. Similarly, from the right null vector of B, one finds that

$$\Gamma_{\text{R}} = \mathcal{N} \sum_{j=1}^L r^{L-j} b_j \tag{45}$$

commutes with $H(0)$. Since both Γ_{L} and Γ_{R} are linear in a and b operators, they are manifestly Hermitian and anticommute with the fermionic parity $(-1)^F$. They are localized near the edges and normalizable, because their amplitudes decay exponentially with distance from the boundary unless $\theta^* = 0$. Therefore, the operators Γ_{L} and Γ_{R} satisfy all the criteria, and hence they are Majorana edge zero modes. A linear combination of Γ_{L} and Γ_{R} gives a nonlocal complex fermion that is merely $f_L = (\Gamma_{\text{L}} + i\Gamma_{\text{R}})/2$, corresponding to the zero-energy state of the single-particle Hamiltonian. Then from the properties of Majorana edge zero modes, together with the uniqueness of the ground states of $H(0)$, it follows that either Γ_{L} or Γ_{R} maps one ground state to the other with different fermionic parity.

A special feature of $H(0)$ that is noninteracting *and* frustration-free is that the obtained Majorana edge zero modes exactly commute with $H(0)$ even in finite-size systems. This nice property no longer holds for $H(s)$ with $s > 0$. Nevertheless, they still map one of the ground states of $H(s)$ to the other since the ground states remain unchanged with the introduction of s. Thus, they can be regarded as an *interacting generalization of Majorana edge zero modes*. This is one of the main results of our paper. We note that one can in principle construct an operator that exactly commutes with $H(s)$ from the adiabatic continuation of Γ_{L} or Γ_{R} [76,77]. Though this requires a full diagonalization of $H(s)$ and is not feasible, we expect that the resulting operator, which is no longer linear in a and b, has a significant overlap with Γ_{L} or Γ_{R}. We leave a quantitative analysis for future work.

V. CONCLUDING REMARKS

In this paper, we have studied the one-dimensional Kitaev chain with nearest-neighbor repulsive interactions. We have shown that the Hamiltonian of the model is frustration-free when the on-site (chemical) potential is tuned to a particular function of the other parameters. Under this condition, the

exact ground states can be obtained in closed form. We have also introduced a smooth path between the interacting and noninteracting Kitaev chains, along which the ground states remain unchanged. We proved the following theorem about this one-parameter family of Hamiltonians along this path: (i) the ground state is unique up to double degeneracy, and (ii) there exists a uniform (independent of the chain length) spectral gap above the ground state. The theorem implies that the interacting Kitaev chain in the frustration-free case is topologically nontrivial, because it is adiabatically connected to the noninteracting Kitaev chain in the topological phase. The stability of the spectral gap against perturbations at the boundary sites was also discussed. Furthermore, we have demonstrated explicitly that there exists a set of Majorana edge zero modes each of which maps one of the ground states to the other with the opposite fermionic parity. It would be interesting to explore the implications of this interacting generalization of Majorana edge zero modes in transport properties such as the tunneling conductance. Though the dynamical Green functions cannot be obtained analytically even for the frustration-free case, one might be able to develop an efficient computational method using the information of the exact ground states. It would also be interesting to see how the results obtained translate into the language of continuum field theories. As is well known, an appropriate scaling limit of the XYZ spin chain is described by the sine-Gordon quantum field theory [78]. We thus expect that the continuum limit of the interacting Kitaev chain is also described by a sine-Gordon type model with boundaries. We also expect that boundary bound states [79] arising in integrable quantum field theories are natural candidates for the interacting generalization of Majorana edge zero modes in the continuum limit. The precise correspondence is, however, beyond the scope of the present study and is left for future work.

APPENDIX A: MAJORANA HAMILTONIAN

For each site j, we define the Majorana fermions by

$$a_j := c_j + c_j^\dagger, \quad b_j := (c_j - c_j^\dagger)/i. \tag{A1}$$

One can easily see that they satisfy the defining relations:

$$a_j^\dagger = a_j, \quad b_j^\dagger = b_j, \quad \{a_j, b_k\} = 0,$$
$$\{a_j, a_k\} = \{b_j, b_k\} = 2\delta_{j,k}$$

for all $j, k = 1, 2, \ldots, L$. When written in terms of a_j and b_j, the Hamiltonian H in Eq. (1) becomes

$$H = \frac{i}{2} \sum_{j=1}^{L-1} [(t + \Delta)b_j a_{j+1} - (t - \Delta)a_j b_{j+1}]$$
$$- \frac{i}{2} \sum_{j=1}^{L} \mu_j a_j b_j - U \sum_{j=1}^{L-1} a_j b_j a_{j+1} b_{j+1} + \text{const.} \tag{A2}$$

Clearly, the last term represents a quartic interaction.

APPENDIX B: SPIN-CHAIN HAMILTONIAN AND THE GROUND STATES

In this Appendix, we discuss the relation between the Hamiltonian H in Eq. (1) and the spin-chain Hamiltonian. From the standard Jordan-Wigner transformation [43],

$$c_1 = \frac{\sigma_1^x + i\sigma_1^y}{2}, \quad c_j = \left(\prod_{k=1}^{j-1} \sigma_k^z\right)\frac{\sigma_j^x + i\sigma_j^y}{2} \quad \text{for } 2 \leqslant j \leqslant L,$$

we obtain explicit expressions of Majorana fermions in terms of spin operators as

$$a_1 = \sigma_1^x, \quad a_j = \left(\prod_{k=1}^{j-1} \sigma_k^z\right)\sigma_j^x \quad \text{for } 2 \leqslant j \leqslant L, \tag{B1}$$

$$b_1 = \sigma_1^y, \quad b_j = \left(\prod_{k=1}^{j-1} \sigma_k^z\right)\sigma_j^y \quad \text{for } 2 \leqslant j \leqslant L, \tag{B2}$$

where $(\sigma_j^x, \sigma_j^y, \sigma_j^z)$ are the Pauli matrices at site j. We note here that the fermionic parity in spin variables is written as

$$(-1)^F = \prod_{j=1}^{L} \sigma_j^z. \tag{B3}$$

Substituting Eqs. (B1) and (B2) into Eq. (A2), we have

$$H = \sum_{j=1}^{L-1} \left(-J_x \sigma_j^x \sigma_{j+1}^x - J_y \sigma_j^y \sigma_{j+1}^y + J_z \sigma_j^z \sigma_{j+1}^z\right)$$
$$- \sum_{j=1}^{L} B_j \sigma_j^z + \text{const}, \tag{B4}$$

where the parameters are given by

$$J_x = \frac{t+\Delta}{2}, \quad J_y = \frac{t-\Delta}{2}, \quad J_z = U, \quad B_j = -\frac{\mu_j}{2}. \tag{B5}$$

Recent theoretical proposals to realize a spin-1/2 XYZ chain in a magnetic field using optical lattice systems can be found in Refs. [81–83].

In the spin-chain language, the frustration-free condition Eq. (11) reads $B_1 = B_L = B^*/2$ and $B_j = B^*$ ($j = 2, 3, \ldots, L-1$), where

$$B^* = -2\sqrt{(J_z + J_x)(J_z + J_y)}. \tag{B6}$$

This is exactly what has been found in Refs. [61,62]. When written in terms of spin states, the ground states Eq. (14)

become simple direct product states:

$$|\Psi_0^{(\pm)}\rangle = \bigotimes_{j=1}^{L} (|\uparrow\rangle_j \pm \alpha |\downarrow\rangle_j), \quad (B7)$$

which can be distinguished by the expectation value of the local operator σ_j^x. We note, however, that σ_j^x for general j is nonlocal in the fermions, because it involves a string of fermion operators. The ground states with fixed (fermionic) parities, i.e., Eqs. (16) and (17), are no longer direct products, but they can still be written in the form of matrix product states. Their explicit expressions can be found in Ref. [84].

APPENDIX C: EXPECTATION VALUES OF O_e

In this Appendix, we present a derivation of the inequality Eq. (20). We also provide explicit expressions for several kinds of correlation functions in the infinite-L limit.

1. Derivation of Eq. (20)

Let us first prove the inequality Eq. (20). Noting that $[O_e,(-1)^F] = 0$ and using the relation $(-1)^F|\Psi_0^{(\pm)}\rangle = |\Psi_0^{(\mp)}\rangle$, we find

$$\langle\Psi_0^{(+)}|O_e|\Psi_0^{(+)}\rangle = \langle\Psi_0^{(-)}|O_e|\Psi_0^{(-)}\rangle, \quad (C1)$$

$$\langle\Psi_0^{(+)}|O_e|\Psi_0^{(-)}\rangle = \langle\Psi_0^{(-)}|O_e|\Psi_0^{(+)}\rangle. \quad (C2)$$

This yields

$$\langle O_e\rangle_{\text{even}} = \frac{\langle\Psi_0^{(+)}|O_e|\Psi_0^{(+)}\rangle + \langle\Psi_0^{(+)}|O_e|\Psi_0^{(-)}\rangle}{1 + 1/\eta^L}, \quad (C3)$$

$$\langle O_e\rangle_{\text{odd}} = \frac{\langle\Psi_0^{(+)}|O_e|\Psi_0^{(+)}\rangle - \langle\Psi_0^{(+)}|O_e|\Psi_0^{(-)}\rangle}{1 - 1/\eta^L}, \quad (C4)$$

where we have used the fact that $\langle\Psi_0^{(\pm)}|\Psi_0^{(\pm)}\rangle = 1$ and $\langle\Psi_0^{(\pm)}|\Psi_0^{(\mp)}\rangle = 1/\eta^L$ [see Eq. (21) for the definition of η]. Then we find

$$|\langle O_e\rangle_{\text{even}} - \langle O_e\rangle_{\text{odd}}|$$
$$\leqslant 2\frac{(1/|\eta|^L)\|O_e\| + |\langle\Psi_0^{(+)}|O_e|\Psi_0^{(-)}\rangle|}{1 - 1/\eta^{2L}}, \quad (C5)$$

where the following inequality has been used:

$$|\langle\Psi_0^{(+)}|O_e|\Psi_0^{(+)}\rangle| \leqslant \|O_e\|. \quad (C6)$$

In the following, we focus on $0 < \theta^* < \pi$, which implies $|\eta| > 1$. To evaluate $|\langle\Psi_0^{(+)}|O_e|\Psi_0^{(-)}\rangle|$, we recall that O_e is a local operator supported on a finite number of sites. Since c_j and $c_j^\dagger$ commute with O_e unless $j = j_1, \ldots, j_k$, one finds

$$\langle\Psi_0^{(+)}|O_e|\Psi_0^{(-)}\rangle = \frac{1}{\eta^{L-\ell}}\langle\widetilde{\Psi}_0^{(+)}|O_e|\widetilde{\Psi}_0^{(-)}\rangle, \quad (C7)$$

where

$$|\widetilde{\Psi}_0^{(\pm)}\rangle := \frac{1}{(1+\alpha^2)^{\ell/2}}\left(\prod_{j=j_1}^{j_k} e^{\pm\alpha c_j^\dagger}\right)|\text{vac}\rangle \quad (C8)$$

are the ground states restricted to the ℓ consecutive lattice sites $j = j_1, j_1 + 1, \ldots, j_k$. Then using the Schwartz inequality, one finds

$$|\langle\widetilde{\Psi}_0^{(+)}|O_e|\widetilde{\Psi}_0^{(-)}\rangle|^2 \leqslant \langle\widetilde{\Psi}_0^{(+)}|\widetilde{\Psi}_0^{(+)}\rangle \langle\widetilde{\Psi}_0^{(-)}|O_e^\dagger O_e|\widetilde{\Psi}_0^{(-)}\rangle$$
$$\leqslant \|O_e\|^2. \quad (C9)$$

Substituting this into Eq. (C5), one finds

$$|\langle O_e\rangle_{\text{even}} - \langle O_e\rangle_{\text{odd}}| \leqslant 2\frac{1/|\eta|^L + 1/|\eta|^{L-\ell}}{1 - 1/\eta^{2L}}\|O_e\|$$
$$\leqslant \frac{2}{|\eta|^L}\times\frac{1+|\eta|^\ell}{1 - 1/\eta^2}\|O_e\|, \quad (C10)$$

where the second line follows from

$$\frac{1}{1-x^L} \leqslant \frac{1}{1-x} \quad \text{for } 0 \leqslant x < 1. \quad (C11)$$

Then it is obvious that the desired inequality Eq. (20) follows from Eq. (C10).

2. Correlation functions

As we have shown above, any local O_e has the same expectation value in $|\Psi_0^{(\text{even})}\rangle$ and $|\Psi_0^{(\text{odd})}\rangle$ in the infinite-L limit; for completeness, we finally give explicit expressions for some correlation functions. For density operators n_j, we have

$$\lim_{L\to\infty}\langle n_j\rangle_{\text{even/odd}} = \frac{1}{1 + \tan(\theta^*/2)}, \quad (C12)$$

$$\lim_{L\to\infty}\langle n_j n_k\rangle_{\text{even/odd}} = \left[\frac{1}{1 + \tan(\theta^*/2)}\right]^2. \quad (C13)$$

Similarly one can compute equal-time Green functions. For $j < k$ with $|j - k| < \infty$, we have

$$\lim_{L\to\infty}\langle c_j c_k^\dagger\rangle_{\text{even/odd}}$$
$$= -\frac{\sin\theta^*}{2(1 + \sin\theta^*)}\left[\frac{1 - \cot(\theta^*/2)}{1 + \cot(\theta^*/2)}\right]^{k-j-1}, \quad (C14)$$

which decays exponentially in the distance $|j - k|$.

APPENDIX D: INHOMOGENEOUS MODEL

In this Appendix, we show that the model described by the Hamiltonian H^{inh} [Eq. (31)] exhibits topological order. Toward that end, we prove the following:

Proposition. For all $\alpha_j > 0$, $\beta_j > 0$ ($j = 1, 2, \ldots, L - 1$) and $\theta^* \neq 0$, (i) the ground state of the Hamiltonian H^{inh} is unique up to double degeneracy, and (ii) H^{inh} has a uniform (independent of the length of the chain) spectral gap above the ground state.

This can be proven along the same lines as the Theorem in the main text.

Proof of Proposition. We first introduce the minimum value of the inhomogeneous parameters as

$$\gamma = \min\{\alpha_j\}_{j=1}^{L-1} \cup \{\beta_j\}_{j=1}^{L-1}, \quad (D1)$$

which is, by definition, strictly positive ($\gamma > 0$). Then from the fact that $Q_j Q_j^\dagger \geqslant 0$ and $Q_j^\dagger Q_j \geqslant 0$, we find

$$H^{\text{inh}} \geqslant \gamma \sum_{j=1}^{L-1} (Q_j Q_j^\dagger + Q_j^\dagger Q_j) = \gamma H(0), \qquad \text{(D2)}$$

where $H(0)$ is the Hamiltonian of the noninteracting Kitaev chain. The Hamiltonians $H(0)$ and H^{inh} share the same ground states $|\Psi_0^{(\pm)}\rangle$, because they are annihilated by both Q_j and $Q_j^\dagger$ for all j.

Let us next compare the energy eigenvalues of H^{inh} and $H(0)$. Let E_n^{inh} be the nth eigenvalue of H^{inh}. Here the energy eigenvalues are arranged in nondecreasing order: $0 = E_1^{\text{inh}} = E_2^{\text{inh}} \leqslant E_3^{\text{inh}} \leqslant \cdots \leqslant E_n^{\text{inh}} \leqslant \cdots$. It then follows from the min-max principle [72] that the inequality Eq. (D2) implies $E_n^{\text{inh}} \geqslant \gamma E_n(0)$ for all n. This immediately implies that $E_3^{\text{inh}} > 0$ since we have already shown that $E_3(0)$ is strictly positive unless $\theta^* \neq 0$ [see Eq. (42)]. This completes the proof of property (ii). Property (i) also follows from the min-max principle and the Lemma in Sec. IV A. ∎

Let us next show the existence of topological order in the system described by H^{inh}. We can construct the Hamiltonian that interpolates between H^{inh} and $H(0)$ as follows:

$$H^{\text{inh}}(x) = x H^{\text{inh}} + (1-x)\gamma H(0), \qquad \text{(D3)}$$

where $0 \leqslant x \leqslant 1$. Using the inequality Eq. (D2), one can check that $H^{\text{inh}}(x) \geqslant \gamma H(0)$ for $0 \leqslant x \leqslant 1$. Therefore, by repeating the proof of proposition, one finds that the spectral gap of $H^{\text{inh}}(x)$ does not close along the path connecting $\gamma H(0)$ and H^{inh}. This, together with the result of Fidkowski and Kitaev [54], implies the existence of topological order in the system.

Let us remark that a local interaction between the neighboring sites can be either repulsive or attractive in H^{inh}, depending on β_j/α_j. To see this, we rewrite the Hamiltonian as

$$H^{\text{inh}} = \sum_{j=1}^{L-1} \alpha_j h_j \left(\frac{\beta_j}{\alpha_j} - 1 \right), \qquad \text{(D4)}$$

where $h_j(s)$ is given by Eq. (25). It is then clear that the local interaction between n_j and n_{j+1} is repulsive if $\beta_j/\alpha_j > 1$, while it is attractive if $0 < \beta_j/\alpha_j < 1$. The interactions can be made purely attractive by taking a set of α_j and β_j that satisfy $0 < \beta_j/\alpha_j < 1$ for all j. For instance, if the parameters are chosen as

$$\alpha_j = 1, \quad \beta_j = 1 + 2U \quad (j = 1, 2, \ldots, L-1), \qquad \text{(D5)}$$

the Hamiltonian H^{inh} reduces to $H(2U)$, which describes the Kitaev chain with attractive interactions when $U < 0$. Note that the condition $U > -1/2$ is required so that $H(2U)$ is positive-semidefinite.

APPENDIX E: EIGENVALUES OF THE MATRIX C

In this Appendix, we present a detailed exposition of the calculation of the eigenvalues of C in Eq. (38). We first make an ansatz for the eigenvectors. Let $\boldsymbol{v}(q) = (v_1(q), v_2(q), \ldots, v_L(q))$ be an eigenvector of C. For each component, we choose the following ansatz:

$$v_j(q) = \alpha e^{iqj} + \beta e^{-iqj}, \qquad \text{(E1)}$$

where the coefficients α, β and the "wave number" q will be determined from the matching conditions at the boundaries. The eigenvalue equation $C\boldsymbol{v}(q) = \epsilon(q)\boldsymbol{v}(q)$ can be written in components as

$$(1 - \mathfrak{s})v_1(q) + \mathfrak{c} v_2(q) = \epsilon(q)v_1(q), \qquad \text{(E2)}$$

$$\mathfrak{c} v_{j-1}(q) + 2 v_j(q) + \mathfrak{c} v_{j+1}(q) = v_j(q), \quad 1 < j < L, \qquad \text{(E3)}$$

$$\mathfrak{c} v_{L-1}(q) + (1 + \mathfrak{s})v_L(q) = \epsilon(q)v_L. \qquad \text{(E4)}$$

Substituting Eq. (E1) into Eq. (E3), one finds that the eigenvalue must be of the form

$$\epsilon(q) = 2 + 2\mathfrak{c} \cos q. \qquad \text{(E5)}$$

Then, from the matching conditions at the boundaries, i.e., Eqs. (E2) and (E4), the consistency condition for q is obtained as

$$\begin{pmatrix} -\mathfrak{c} - (1+\mathfrak{s})e^{iq} & -\mathfrak{c} - (1+\mathfrak{s})e^{-iq} \\ (\mathfrak{s} - 1 - \mathfrak{c}e^{iq})e^{iqL} & (\mathfrak{s} - 1 - \mathfrak{c}e^{-iq})e^{-iqL} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The above equation has a nontrivial solution if the determinant of the matrix in the left-hand side vanishes. This condition reads

$$-4i\mathfrak{c}(1 + \mathfrak{c} \cos q)\sin(qL) = 0. \qquad \text{(E6)}$$

The real wave numbers are then obtained as

$$q = \frac{n\pi}{L}, \quad n = 1, 2, \ldots, L-1. \qquad \text{(E7)}$$

Note that $q = 0$ and π are not allowed because they result in $\boldsymbol{v}(q) = \boldsymbol{0}$. Equation (E6) also has a complex solution,

$$q = q_0 = \pi + i \ln \left(\frac{\mathfrak{c}}{1 + \mathfrak{s}} \right), \qquad \text{(E8)}$$

which is a solution of $\cos q_0 = -1/\mathfrak{c}$. One might think that the set of solutions obtained is overcomplete because $q = -q_0$ is a solution as well. However, this is not the case since both q_0 and $-q_0$ yield the same $\boldsymbol{v}(q)$. We finally obtain the desired eigenvalues Eq. (39) by substituting the obtained q into Eq. (E5).

[1] F. Wilczek, Nat. Phys. **5**, 614 (2009).

[2] J. Alicea, Rep. Prog. Phys. **75**, 076501 (2012).

[3] M. Leijnse and K. Flensberg, Semicond. Sci. Technol. **27**, 124003 (2012).

[4] C. W. J. Beenakker, Annu. Rev. Condens. Matter Phys. **4**, 113 (2013).

[5] V. Mourik, K. Zuo, S. M. Frolov, S. R. Plissard, E. P. A. M. Bakkers, and L. P. Kouwenhoven, Science **336**, 1003 (2012).

[6] L. P. Rokhinson, X. Liu, and J. K. Furdyna, Nat. Phys. **8**, 795 (2012).

[7] A. Das, Y. Ronen, Y. Most, Y. Oreg, M. Heiblum, and H. Shtrikman, Nat. Phys. **8**, 887 (2012).

 [8] M. T. Deng, C. L. Yu, G. Y. Huang, M. Larsson, P. Caroff, and H. Q. Xu, Nano Lett. **12**, 6414 (2012).

 [9] H. O. H. Churchill, V. Fatemi, K. Grove-Rasmussen, M. T. Deng, P. Caroff, H. Q. Xu, and C. M. Marcus, Phys. Rev. B **87**, 241401 (2013).

[10] E. J. H. Lee, X. Jiang, M. Houzet, R. Aguado, C. M. Lieber, and S. De Franceschi, Nat. Nanotechnol. **9**, 79 (2014).

[11] S. Nadj-Perge, I. K. Drozdov, J. Li, H. Chen, S. Jeon, J. Seo, A. H. MacDonald, B. A. Bernevig, and A. Yazdani, Science **346**, 602 (2014).

[12] A. Yu. Kitaev, Phys. Usp. **44**, 131 (2001).

[13] P. Fendley, J. Stat. Mech. (2012) P11020.

[14] P. W. Brouwer, M. Duckheim, A. Romito, and F. von Oppen, Phys. Rev. Lett. **107**, 196804 (2011).

[15] A. M. Lobos, R. M. Lutchyn, and S. Das Sarma, Phys. Rev. Lett. **109**, 146403 (2012).

[16] W. DeGottardi, D. Sen, and S. Vishveshwara, Phys. Rev. Lett. **110**, 146404 (2013).

[17] A. Altland, D. Bagrets, L. Fritz, A. Kamenev, and H. Schmiedt, Phys. Rev. Lett. **112**, 206602 (2014).

[18] F. Crépin, G. Zaránd, and P. Simon, Phys. Rev. B **90**, 121407 (2014).

[19] R. Wakatsuki, M. Ezawa, Y. Tanaka, and N. Nagaosa, Phys. Rev. B **90**, 014505 (2014).

[20] M. Leijnse and K. Flensberg, Phys. Rev. B **84**, 140501 (2011).

[21] S. Walter, T. L. Schmidt, K. Børkje, and B. Trauzettel, Phys. Rev. B **84**, 224510 (2011).

[22] R. Hützen, A. Zazunov, B. Braunecker, A. L. Yeyati, and R. Egger, Phys. Rev. Lett. **109**, 166403 (2012).

[23] M. Leijnse, New J. Phys. **16**, 015029 (2014).

[24] R. López, M. Lee, L. Serra, and J. S. Lim, Phys. Rev. B **89**, 205418 (2014).

[25] B. Béri and N. R. Cooper, Phys. Rev. Lett. **109**, 156803 (2012).

[26] A. M. Tsvelik, Phys. Rev. Lett. **110**, 147202 (2013).

[27] A. Altland, B. Béri, R. Egger, and A. M. Tsvelik, Phys. Rev. Lett. **113**, 076401 (2014).

[28] M. Cheng, M. Becker, B. Bauer, and R. M. Lutchyn, Phys. Rev. X **4**, 031051 (2014).

[29] M. R. Galpin, A. K. Mitchell, J. Temaismithi, D. E. Logan, B. Béri, and N. R. Cooper, Phys. Rev. B **89**, 045143 (2014).

[30] E. Eriksson, A. Nava, C. Mora, and R. Egger, Phys. Rev. B **90**, 245417 (2014).

[31] A. M. Tsvelik, Europhys. Lett. **97**, 17011 (2012).

[32] T. Grover, D. N. Sheng, and A. Vishwanath, Science **344**, 280 (2014).

[33] J. Ulrich, İ. Adagideli, D. Schuricht, and F. Hassler, Phys. Rev. B **90**, 075408 (2014).

[34] A. Rahmani, X. Zhu, M. Franz, and I. Affleck, arXiv:1504.05192.

[35] A. Rahmani, X. Zhu, M. Franz, and I. Affleck, arXiv:1505.03966.

[36] D. J. Clarke, J. Alicea, and K. Shtengel, Nat. Phys. **10**, 877 (2014).

[37] J. Klinovaja and D. Loss, Phys. Rev. Lett. **112**, 246403 (2014).

[38] R. S. K. Mong *et al.*, Phys. Rev. X **4**, 011036 (2014).

[39] A. S. Jermyn, R. S. K. Mong, J. Alicea, and P. Fendley, Phys. Rev. B **90**, 165106 (2014).

[40] A. Alexandradinata, N. Regnault, C. Fang, M. J. Gilbert, and B. A. Bernevig, arXiv:1506.03455.

[41] R. M. Lutchyn, J. D. Sau, and S. Das Sarma, Phys. Rev. Lett. **105**, 077001 (2010).

[42] Y. Oreg, G. Refael, and F. von Oppen, Phys. Rev. Lett. **105**, 177002 (2010).

[43] S. Gangadharaiah, B. Braunecker, P. Simon, and D. Loss, Phys. Rev. Lett. **107**, 036801 (2011).

[44] E. M. Stoudenmire, J. Alicea, O. A. Starykh, and M. P. A. Fisher, Phys. Rev. B **84**, 014503 (2011).

[45] A. Manolescu, D. C. Marinescu, and T. D. Stanescu, J. Phys. Condens. Matter **26**, 172203 (2014).

[46] R. M. Lutchyn and M. P. A. Fisher, Phys. Rev. B **84**, 214528 (2011).

[47] E. Sela, A. Altland, and A. Rosch, Phys. Rev. B **84**, 085114 (2011).

[48] M. Cheng and H.-H. Tu, Phys. Rev. B **84**, 094503 (2011).

[49] F. Hassler and D. Schuricht, New J. Phys. **14**, 125018 (2012).

[50] R. Thomale, S. Rachel, and P. Schmitteckert, Phys. Rev. B **88**, 161103 (2013).

[51] A. Milsted, L. Seabra, I. C. Fulga, C. W. J. Beenakker, and E. Cobanera, Phys. Rev. B **92**, 085139 (2015).

[52] Y.-H. Chan, C.-K. Chiu, and K. Sun, arXiv:1506.07860.

[53] L. Fidkowski and A. Kitaev, Phys. Rev. B **81**, 134509 (2010).

[54] L. Fidkowski and A. Kitaev, Phys. Rev. B **83**, 075103 (2011).

[55] A. M. Turner, F. Pollmann, and E. Berg, Phys. Rev. B **83**, 075102 (2011).

[56] V. Gurarie, Phys. Rev. B **83**, 085426 (2011).

[57] G. Goldstein and C. Chamon, Phys. Rev. B **86**, 115122 (2012).

[58] G. Kells, Phys. Rev. B **92**, 081401 (2015).

[59] C.-K. Chiu, D. I. Pikulin, and M. Franz, Phys. Rev. B **91**, 165402 (2015).

[60] I. Peschel and V. J. Emery, Z. Phys. B **43**, 241 (1981).

[61] J. Kurmann, H. Thomas, and G. Müller, Physica A **112**, 235 (1982).

[62] G. Müller and R. E. Shrock, Phys. Rev. B **32**, 5845 (1985).

[63] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic, London, 1982).

[64] I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Phys. Rev. Lett. **59**, 799 (1987); Commun. Math. Phys. **115**, 477 (1987).

[65] D. S. Rokhsar and S. A. Kivelson, Phys. Rev. Lett. **61**, 2376 (1988).

[66] A. Kitaev, Ann. Phys. (NY) **303**, 2 (2003).

[67] This can also be seen by noting that each local Hamiltonian (+const) can be cast in the form Eq. (27) at $\mu = \mu^*$, and $|\Psi_0^{(\pm)}\rangle$ are annihilated by both Q_j and $Q_j^\dagger$ for all j. For the definition of Q_j, see Eq. (26).

[68] G. Gómez-Santos, Phys. Rev. Lett. **70**, 3780 (1993).

[69] This path has no relation to the exactly solvable line shown in Fig. 1.

[70] We remark that each of Q_j and $Q_j^\dagger$ squares to zero.

[71] For positive semidefinite operators and their properties, see, e.g., Appendix C in H. Tasaki, Prog. Theor. Phys. **99**, 489 (1998).

[72] R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge University Press, Cambridge, 2012).

[73] T. Kennedy and H. Tasaki, Commun. Math. Phys. **147**, 431 (1992).

[74] D. A. Yarotsky, Commun. Math. Phys. **261**, 799 (2006).

[75] T. T. Truong and I. Peschel, Int. J. Mod. Phys. B **04**, 895 (1990).

[76] M. B. Hastings and X.-G. Wen, Phys. Rev. B **72**, 045141 (2005).

[77] This program has been carried out for a slightly different model of interacting Majorana fermions, corresponding to the XYZ spin chain without a magnetic field [40].

[78] A. Luther, Phys. Rev. B **14**, 2153 (1976).

[79] S. Ghoshal and A. B. Zamolodchikov, Int. J. Mod. Phys. A **09**, 3841 (1994); **09**, 4353 (1994).

[80] A. Tanaka, arXiv:1503.08998.

[81] F. Pinheiro, G. M. Bruun, J. P. Martikainen, and J. Larson, Phys. Rev. Lett. **111**, 205302 (2013).

[82] M. Piraud, Z. Cai, I. P. McCulloch, and U. Schollwöck, Phys. Rev. A **89**, 063618 (2014).

[83] S. Peotta, L. Mazza, E. Vicari, M. Polini, R. Fazio, and D. Rossini, J. Stat. Mech. (2014) P09005.

[84] M. Asoudeh, V. Karimipour, and A. Sadrolashrafi, Phys. Rev. A **76**, 012320 (2007).