Dynamics of many-body localization in a translation-invariant quantum glass model

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We study the real-time dynamics of a translationally invariant quantum spin chain, based on the East kinetically constrained glass model, in search for evidence of many-body localization in the absence of disorder. Numerical simulations indicate a change, controlled by a coupling parameter, from a regime of fast relaxation-corresponding to thermalization-to a regime of very slow relaxation. This slowly relaxing regime is characterized by dynamical features usually associated with nonergodicity and many-body localization (MBL): memory of initial conditions, logarithmic growth of entanglement entropy, and nonexponential decay of time correlators. We show that slow relaxation is a consequence of sensitivity to spatial fluctuations in the initial state. While numerical results and physical considerations indicate that relaxation time scales grow markedly with size, our finite size results are consistent both with an MBL transition, expected to only occur in disordered systems, and with a pronounced quasi-MBL crossover.

Introduction. The framework of many-body localization (MBL) [1] has been shown to be a powerful new approach in the study of nonequilibrium dynamics in closed interacting quantum systems. Since its initial proposal [2,3], abundant [4–8] numerical and analytical evidence has accumulated for the existence of MBL, and protocols for realizing MBL in particular systems, such as ultracold atomic gases, have been proposed. MBL has recently been observed in an experiment on interacting Fermi gases [9]. While the existence of MBL in systems with explicit quenched disorder is now mostly uncontroversial, it is less clear whether a similar transition can be present in disorder-free transitionally invariant systems [10–18].

Aspects of MBL have been observed in particular models of translation-invariant many-body systems, such as certain mixtures of light and heavy particles [11,13]. Localization in such disorder-free systems occurs due to the interplay between interactions and inhomogeneous initial conditions, leading to nonergodic behavior where relaxation times diverge with system size. In contrast, a general mechanism for delocalization has been suggested in Ref. [16], by which the presence of mobile resonant spots act as carriers of energy or charge. At present it is unclear whether true MBL can occur in a finite-dimensional translation-invariant system, and *quasi*-MBL has been suggested [13] for cases where transient MBL behavior is followed by the restoration of ergodicity on long time scales. (The situation is reminiscent in some sense to that of the classical glass transition problem, where it is debated whether relaxation times diverge at finite temperature with a transition to an ideal glass state, or whether relaxation times are very long but remain finite for all nonzero temperature. See, e.g., the discussion in Ref. [19].)

In this Rapid Communication we argue for a transition between an ergodic (delocalized) phase and an MBL phase in a specific translation-invariant system. Generally, a disorderless delocalized-to-MBL phase transition is conjectured to be related to a quantum glass transition [1]. In particular, in Ref. [12], indications of such a delocalized-to-MBL transition

were found in a one-dimensional quantum spin chain based on the classical glassy Fredrickson-Anderson model [20,21]. Here, we study an even simpler system-and for which MBL-like dynamics is even more striking-a disorderless quantum spin chain based on the (infinite temperature) East glass model [21–23]. We find that in this model a single coupling parameter controls a significant change in the dynamics, from a regime of fast relaxation and thermalization, to one where relaxation times appear to diverge exponentially with system size. We show that in this slowly relaxing regime the system exhibits many of the dynamical features associated with MBL: logarithmic spreading of excitations, lack of thermalization, initial state dependence at long times, and logarithmic growth of entanglement entropy. While these observations are highly suggestive of MBL, we cannot discard (very pronounced) quasi-MBL behavior as an alternative explanation.

Model. We consider a translation-invariant spin-1/2 chain with a Hamiltonian operator,

$$H = \frac{J}{2} \sum_{k=1}^{N} n_k - \frac{J}{2} \sum_{k=1}^{N} e^{-s} \sigma_k^x n_{k+1}. \tag{1}$$

Here, J is an energy scale, which, without loss of generality, we take to be $J = 1$. On the other hand, varying the parameter s, which controls the coupling between sites, drastically changes the dynamical behavior of the model, as can be observed in Fig. 1. This Hamiltonian is closely related to the master operator of the classical East glass model [21–23]. In fact, if used to generate imaginary time dynamics, the $s = 0$ Hamiltonian in Eq. (1) is equivalent to the master operator of the classical East model at infinite temperatures [i.e., $s = 0$ is a Rokhsar-Kivelson point [24,25] of (1)]. At $s \neq 0$ the operator H is a so-called tilted generator [26], whose s-dependent ground state is the cumulant generating function of the "dynamical activity" [27]. The ground state of the Hamiltonian (1) is known to change discontinuously at $s = 0$ [28]. (See also the discussion in Ref. [12] for a related model).

Classical mechanism for relaxation. In the classical stochastic dynamics generated by (1) when $s = 0$, classical configurations are represented by Fock states in the z basis. The
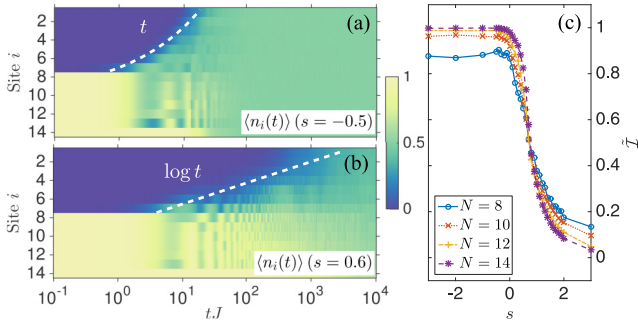
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FIG. 1. (Color online) Time evolution of site densities $\langle n_i(t) \rangle$ starting in the half-filled initial state $|\downarrow \cdots \downarrow \uparrow \cdots \uparrow \rangle$ in the thermal regime, (a) $s = -0.5$, and in the MBL-like regime, (b) $s = 0.6$, showing a linear (ballistic) spreading of the state and a logarithmic spread, respectively. (c) Equilibrium value $\overline{\mathcal{I}}_{\text{eq}}$ of the initial participation ratio $\mathcal{I}(t)$, averaged over all half-filling initial states.

key feature of these dynamics is that changes of configuration occur via individual spin flips only on sites which have a right nearest neighbor with $\sigma^z = 1$. This means that a region of space with contiguous down spins can only be relaxed from its rightmost boundary, and, as a result, a longer region of this type take longer to relax (as the connecting dynamical pathway is required to show more collective features). This "rare region" behavior is what makes the East model glassy at low temperatures (when the "facilitating" up spins are sparse) [21–23]. Furthermore, fluctuations in the initial configuration associated with these down spin domains lead to fluctuations in the spatial relaxation pattern, known as "dynamical heterogeneity," of the classical East model [29]. As we will see below, analogous initial state fluctuations strongly affect the coherent dynamics generated by Eq. (1) for general s.

Dynamical features. Considering pure initial states which lack translational invariance reveals the dynamical features of our model when unitarily evolving the system with the Hamiltonian H. As the system is closed and subject to recurrence, one has to consider time-integrated observables to account for relaxation, or limit the dynamics to a small portion of the recurrence cycle [30,31].

We find two distinct dynamical regimes parametrized by s. For $s < 0$ there is fast relaxation and ballistic spreading of initial excitations (up spins), as illustrated in Fig. 1(a). In contrast, the $s > 0$ regime exhibits slow relaxation and logarithmic spreading of initial excitations, as shown in Fig. 1(b). In both figures the initial state is a half-filling Fock state (a Fock vector with equal number of up and down spins in the z basis) where the first half of the spins are down and the rest is up. We now argue that the two regimes are distinguished by thermalization when $s < 0$ and lack of thermalization for $s > 0$, and the nonthermal regime in particular exhibits many of the features of an MBL system.

To probe the dynamics for the possible occurrence of a thermal-to-MBL transition, we consider the dynamical inverse participation ratio (IPR) [32] $\mathcal{I}(t) = 2^{-N} [\sum_{\mathbf{i}} |\langle \psi(t)|\mathbf{i}\rangle|^4]^{-1}$ associated with the state $|\psi(t)\rangle$ at time $t > 0$. Figure 1(c) shows the equilibrium [33] value of the IPR $\overline{\mathcal{I}}_{\text{eq}}$, averaged over all half-filling initial states. The quantity $\overline{\mathcal{I}}_{\text{eq}}$ acts as an indicator of localization of the equilibrium state in the Fock
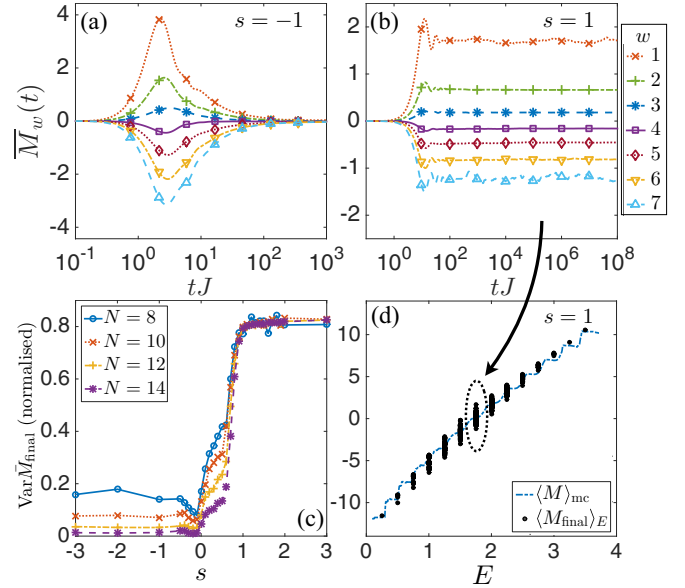

FIG. 2. (Color online) (a), (b) Time-averaged magnetization $\overline{M}_w(t)$ resolved by the largest interval w between excitations in the initial state. (c) Initial state dependence quantified by $\text{var}[M]_{\text{eq}}/\text{var}[M]_{\text{max}}$. (d) Comparison between the equilibrium magnetization $\overline{M}_{\text{eq}}$, and its microcanonical average $\langle M \rangle_{\text{mc}}$. The highlighted points are the equilibrium values $\overline{M}_{\text{eq}}$ obtained in (b), corresponding to initial states with energy $E \approx 1.7$.

basis; in particular, its value changes from approximately unity for $s < 0$ to $O(N^{-1})$ for $s > 0$, suggesting that at the point $s \approx 0$ an MBL transition might occur.

Initial state dependence. We consider the dynamical features of the system starting from a pure initial state picked randomly from the half-filling sector. We characterize the initial state dependence by associating with each initial classical spin configuration a quantity w, defined as the maximal number of consecutive down spins, taking into account periodic boundary conditions (e.g., $w = 3$ for a Fock vector $|\downarrow\uparrow\downarrow\downarrow\uparrow\uparrow\uparrow\downarrow\downarrow\rangle$).

The long-time dependence of single realizations on the initial state indicates a lack of thermalization, and is considered a candidate indicator of many-body localization [3–5,34]. We find that the equilibrium state in the regime $s > 0$ shows nonthermal behavior by retaining a memory of the initial conditions even at long time scales. This can be seen by considering the time-integrated total magnetization in the z direction, $M(t) = t^{-1} \int_0^t m(t')dt'$, where $m(t) = \langle \varphi(t)| \sum_k \sigma_k^z |\varphi(t)\rangle$. In Figs. 2(a) and 2(b) we display this quantity respectively in the thermal and nonergodic regimes resolved according to the maximum distance of excitations present in the initial state Fock vector. We denote with $\overline{M}_w(t)$ the average over realizations restricted to initial states with a maximum distance of w sites between excitations.

In the regime $s < 0$ the system does not retain memory of the initial state in the long-time limit, as $\overline{M}_w(t) \to 0$ for all values of w. When $s > 0$, on the other hand, the long-time values of $\overline{M}_w(t)$ show a clear dependence on w, suggesting an equilibration which depends on the initial conditions, and as such no thermalization occurs. This lack of thermalization is reflected in the time-dependent variance (with respect to initial state realizations) $\text{var}[M(t)]$. In Fig. 2(c) we make

the potential transition apparent by plotting the quantity $\text{var}[M]_{\text{eq}}/\text{var}[M]_{\text{max}}$, where $\text{var}[M]_{\text{max}} = \max_{t>0} \text{var}[M(t)]$ and $\text{var}[M]_{\text{eq}}$ denotes the equilibrium value of $\text{var}[M(t)]$. This scaled variance compares the maximum transient spread of the integrated magnetization (a variation which is expected since at intermediate times different initial states will evolve differently) to the spread at long times (a variation which only occurs in the absence of thermalization).

Further evidence for the absence of thermalization for $s > 1$ is shown in Fig. 2(d), where we plot the equilibrium values of the magnetization $\overline{M}_{\text{eq}}$ along with its microcanonical average expectation value (at infinite temperature) $\langle M \rangle_{\text{mc}}$ as a function of energy [35]. Here, it can be seen that the microcanonical prediction does not reproduce the long-time average when the initial state energy lies in the bulk of the spectrum. In this case $\overline{M}_{\text{eq}}$ strongly depends on the initial conditions; these results suggest a breakdown of the eigenstate thermalization hypothesis leading to the absence of thermalization [36], which in turn is connected to the MBL [4].

Logarithmic growth of entanglement entropy. Slow growth of the entanglement entropy has been associated with an MBL phase in both translation-invariant [13] and disordered systems [5,37]. We consider the entanglement entropy with respect to an equal splitting of the system into two half chains, denoted A and B. Let $S(t) = -\text{Tr}_A \, \rho_A(t) \log \rho_A(t)$ be the time-dependent entanglement entropy, where $\rho_A(t) = \text{Tr}_B |\psi(t)\rangle\langle\psi(t)|$ is the reduced density matrix on the half chain A. We define $\overline{S}(t)$ as the mean entanglement entropy obtained by taking the average over the $S(t)$ associated with all possible initial states in the half-filling sector; additionally, to maintain translation invariance, $\overline{S}(t)$ is also averaged over all possible locations of the cut between half chains A and B.

In Fig. 3(a) we compare the evolution of the average entropy $\overline{S}(t)$ for different values of s, starting from the same initial half-filling ensemble. Clearly evident is a difference in behavior in the thermal ($s < 0$) and the nonthermal ($s > 0$) regimes: The former is characterized by an exponential growth which culminates in the equilibrium entropy $\overline{S}_{\text{eq}}$ which is independent of s. In the nonthermal regime, on the other hand, after an initial fast growth we observe a slow logarithmic increase

which extends over several orders of magnitude of Jt. The equilibrium value $\overline{S}_{\text{eq}}$ in this regime drops significantly for increasing s, as shown in the inset in Fig. 3(a).

The time evolution of $\overline{S}(t)$ can be understood qualitatively as follows: As previously discussed, the typical initial state will contain a largest domain of down spins of length w. The initial linear increase of $\overline{S}(t)$ corresponds to the mixing of the state in the regions away from this empty interval. Once the state is completely mixed over these regions, the only increase in entanglement can come from propagating excitations into the w down spin domain, which, as shown in Figs. 1(a) and 1(b), is linear in time for $s < 0$ and logarithmic for $s > 0$. In the thermal regime, $s < 0$, there is no difference in the growth of the entanglement entropy starting from states with different w, and equilibration is fast. In contrast, in the nonthermal regime, $s > 0$, this growth shows a pronounced dependence on the value of w in the initial state.

The range of logarithmic growth of the entropy in the $s > 0$ regime increases with system size, as shown in Fig. 3(d). We can extract a relaxation time τ as the time the $\bar{S}(t)$ reaches $\bar{S}_{\text{eq}}$, for example, such that $|\bar{S}_{\text{eq}} - \bar{S}(\tau)| = \varepsilon$ with $\varepsilon = 0.01$ (the precise value of this cutoff is unimportant) [38]. The inset to Fig. 3(d) shows the scaling of the relaxation time τ for different values of s. For $s > 0$, τ it appears to grow exponentially with the system size in the range of sizes which we were able to consider. This exponential divergence with size is a hallmark of the loss of ergodicity and of MBL.

Polarization relaxation. Finally, we consider the relaxation of fluctuations at different wavelengths. To do this we consider the (infinite temperature) averaged time correlator of the Fourier transformed longitudinal magnetization, $D(k,t) = \text{Tr}[e^{-iHt} F^*(k) e^{iHt} F(k)]$, where $F(k) = \sum_j \sigma_j^z e^{ikj}$. The relaxation of this polarization correlator can be used to quantify the transport of spin perturbations [4,13] and thus can probe for MBL. Since in a translation-invariant system of finite size $D(k,t)$ has to decay to zero at large enough times, the interesting property to look for is in the (transient) behavior of the relaxation times [13].

In Fig. 4 we show $D(k,t)$ as a function of time for all the wave vectors k of a system of size $N = 12$ (the largest
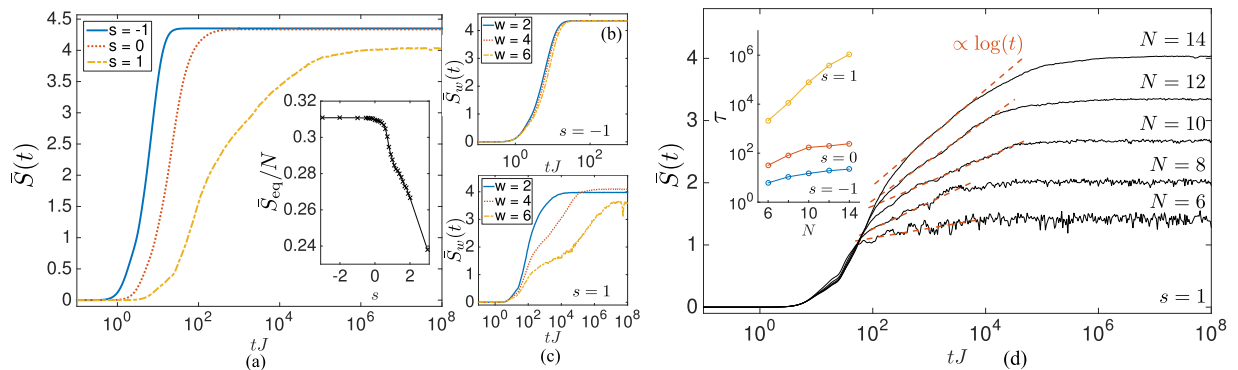

FIG. 3. (Color online) (a) Entanglement entropy $\overline{S}(t)$ (with respect to an equal bipartition of the spin chain) showing qualitative difference as s is varied. Inset: Equilibrium value $\overline{S}_{\text{eq}}$ of entanglement entropy as a function of s and N, suggesting a transition from the thermal regime $s < 0$ to the many-body localized regime $s > 0$. (c), (d) Average entanglement entropies $\overline{S}_w(t)$ restricted to initial states with domain size w, showing that initial state dependence is absent in the thermal phase. (d) Characterization of the logarithmic regime in the growth of the entanglement entropy $\overline{S}(t)$. The dashed lines show the best logarithmic fit $\overline{S}(t) \propto \log t$ for each value of N. Inset: Estimated relaxation time τ associated with $\overline{S}(t)$.
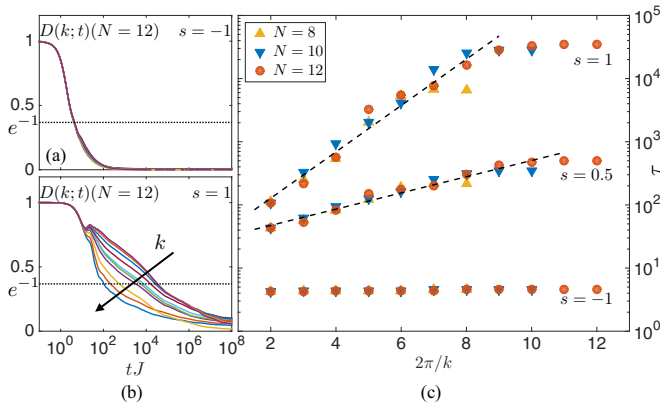
FIG. 4. (Color online) Dynamics of the spin polarization $D(k,t)$ for short $(k = \pi)$, medium $(k = 4\pi/N)$, and long $(k = 2\pi/N)$ wavelengths, respectively, in the thermal $(s = -1$, left panel) and nonergodic phase $(s = 1$, right panel) for system size $N = 12$.

we were able to simulate for this kind of correlator). For the thermalizing case of $s = -1$ [Fig. 4(a)], all correlators decay exponentially and there is no k dependence in their relaxation. In contrast, for the nonthermalizing case of $s = 1$ [Fig. 4(a)], relaxation is nonexponential and time scales are k dependent. Figure 4(c) shows estimates of the relaxation times τ as a function of k [obtained from $D(k,\tau) = e^{-1}$] for three system sizes: For $s < 0$, $\tau(k)$ is independent of k (and of system size), while for $s > 0$, they are strongly k dependent. In fact, in the nonthermalizing phase, $\tau(k) \sim e^{1/k\xi}$ for a large range of wavelengths, a similar behavior as that observed in Ref. [13].

The exponential in wavelength behavior of the relaxation time can be rationalized as follows. $D(k,t)$ measures the relaxation over a wavelength $2\pi/k$. In the MBL-like regime,

the initial states that couple to this wavelength and decay the slowest [and thus dominate $D(k,t)$] will be those with $w = 2\pi/k$, where w is the size of the largest domain of down spins. As shown above, such domains decay logarithmically in time [cf. Fig. 1(b)], that is, the relaxation into an inactive domain goes as $w \approx \xi \log t$. Inverting this relation and replacing w by $1/k$, we get the form of $\tau(k)$ above. In order for this argument to hold, these slow initial states have to be numerous in the infinite temperature average. At the largest wavelength they become rare, and this is the reason why the exponential scaling is not obeyed in Fig. 4(c) for small k.

Conclusions and outlook. In this Rapid Communication we studied a disorderless quantum spin chain closely related to the classical glassy East model in search for an MBL transition in a translationally invariant system. The coherent dynamics of this model displays two regimes: a thermal regime, and a nonergodic regime characterized by a strong dependence on the initial conditions and breaking of thermalization. The nonthermalizing regime displays dynamical features characteristic of an MBL phase, in particular, a logarithmic spreading of excitations and a logarithmic growth of the entanglement entropy. The slow relaxation behind these MBL-like features can be traced back to spatial fluctuations in the initial state corresponding to rare inactive domains. While relaxation times seem to diverge with system size for the entanglement entropy, they remain finite (yet anomalous) for dynamic structure factors. These features would be compatible both with MBL or with quasi-MBL. In either case, our results suggest a close connection between classical mechanisms for glassy arrest and those responsible for MBL dynamics.

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