

Intermediate fixed point in a Luttinger liquid with elastic and dissipative backscatteringAlexander Altland,¹ Yuval Gefen,² and Bernd Rosenow³¹*Institut für Theoretische Physik, Universität zu Köln, Köln, 50937, Germany*²*Department of Condensed Matter Physics, The Weizmann Institute of Science, Rehovot 76100, Israel*³*Institut für Theoretische Physik, Universität Leipzig, D-04103, Leipzig, Germany*

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In a recent work [A. Altland, Y. Gefen, and B. Rosenow, *Phys. Rev. Lett.* **108**, 136401 (2012)], we have addressed the problem of a Luttinger liquid with a scatterer that allows for both coherent and incoherent scattering channels. We have found that the physics associated with this model is qualitatively different from the elastic impurity setup analyzed by Kane and Fisher, and from the inelastic scattering scenario studied by Furusaki and Matveev, thus proposing a paradigmatic picture of Luttinger liquid with an impurity. Here we present an extensive study of the renormalization group flows for this problem, the fixed point landscape, and scaling near those fixed points. Our analysis is nonperturbative in the elastic tunneling amplitudes, employing an instanton calculation in one or two of the available elastic tunneling channels. Our analysis accounts for nontrivial Klein factors, which represent anyonic or fermionic statistics. These Klein factors need to be taken into account due to the fact that higher-order tunneling processes take place. In particular, we find a stable fixed point, where an incoming current is split $\frac{1}{2}$ - $\frac{1}{2}$ between a forward and a backward scattered beams. This intermediate fixed point, between complete backscattering and full forward scattering, is stable for the Luttinger parameter $g < 1$.

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I. INTRODUCTION

The concept of a Luttinger liquid (LL) provides a very general framework to deal with a strongly interacting electron gas confined to one spatial dimension (1D) [1]. Contrary to higher-dimensional situations, where the quasiparticle (qp) concept describes low-energy excitations, in an LL only collective excitations are long lived and stable. Particlelike excitations of an LL have an energy dependent density of states and can relax in energy in the presence of backscattering. Applications of the LL concept include semiconductor [2], metallic [3] and polymer [4] nanowires, carbon nanotubes [5], quantum Hall edges [6], and cold atoms [7].

Due to the geometrical restriction to 1D, electrons may be divided into two sectors, right moving and left moving. The presence of an impurity [8] gives rise to interbranch scattering, i.e., backscattering with a power-law dependence of the scattering probability on energy. Hitherto there have been two paradigmatic models that addressed impurity scattering in the context of LL: a purely elastic impurity as discussed by Kane and Fisher [9] (KF), and totally inelastic scattering described by Furusaki and Matveev [10] (FM).

In a recent publication [11], we have introduced and studied a model which is a hybrid between the two. We have addressed interacting electrons in one dimension described by a Luttinger model, which includes a scatterer that may give rise to elastic coherent tunneling, and at the same time may accommodate inelastic modes. Our analysis has led to predictions that are qualitatively different from the KF and the FM pictures, and in this sense can be considered as a new paradigmatic scheme of impurity scattering in the context of LL. For generalizations to systems comprising more than one LL coupled by tunneling terms see [12–21].

In particular, we have obtained the following results. (i) Due to the presence of both elastic and inelastic scattering channels, there exists a new stable non-Fermi liquid fixed point (FP). Asymptotically, the impurity becomes a symmetric beam splitter, which halves the incoming beam into two equal

outgoing beams. For this reason we termed this limit a $\frac{1}{2}$ - $\frac{1}{2}$ FP. We also identify other non-Fermi liquid FPs, which are stable in a certain direction and unstable in another direction in parameter space. A major facet of our earlier work was that upon renormalizing our model down in temperature (or bias voltage), under generic conditions the model flows to a stable FP. The neighborhood of this FP is marked by the non-Fermi liquid correlations described above. (ii) At equilibrium, there is thermal noise at each incoming and each outgoing terminal, but no cross-correlation in the thermal noise, i.e., no correlations between incoming-incoming and outgoing-outgoing edges—in stark contrast to the Landauer-Buttiker picture. (iii) Out of equilibrium, when the system is voltage biased at one of the incoming terminals, the impurity acts as a $\frac{1}{2}$ - $\frac{1}{2}$ beam splitter with no shot noise component in the outgoing current. Similarly, cross-current correlators do not contain a shot noise component.

Our analysis in Ref. [11] involved bosonization of the interacting fermionic Hamiltonian. It was based on perturbation in the elastic (coherent) scattering amplitudes in the presence of a charging term representing electrostatic correlations in the scattering region. We have concluded that for energies below the charging energy, the bosonic fields affected by the elastic scattering terms become massive. As a result, only one of the four independent bosonic fields in the model remains asymptotically free. The asymptotically Gaussian action has facilitated calculation of noise and current-current correlations.

Naturally, the fact that our analysis has been based on, and the results have been motivated by perturbative analysis, raises some questions. Most important is the fact that various relevant terms of the action have been analyzed within a renormalization group (RG) scheme separately. Evidently, this procedure ceases to be justified when the respective amplitudes of the elastic tunneling terms grow through the RG analysis. One may need to evaluate expectation values of products of noncommuting terms, which a naive perturbative analysis is incapable of doing. Careful analysis is required in this process.

We note that the elastic tunneling channels involve scattering from any of the two incoming channels to any of the outgoing channels. To establish the right language, and to enable a compact description of the RG flow diagram, we divide the elastic scattering terms into back scattering (the terms connecting channels 1 with 3, and 2 with 4, cf. Fig. 2) and forward scattering (the terms connecting 1 with 4, and 2 with 3). We assume that the coefficients of the two back scattering terms are equal to each other, and similarly the coefficients of the two forward scattering terms are equal to each other. Relaxing this assumption does not modify the asymptotics of the problem, but makes the description of the RG flows more cumbersome. Assuming we have renormalized the problem down from high temperature (or voltage), and that certain irrelevant nonuniversal terms have been renormalized out, we are left with flows in a two-dimensional parameter space. We indeed establish the existence of a new stable $\frac{1}{2}$ - $\frac{1}{2}$ FP, which corresponds to strong coupling in both scattering channels (backscattering and forward-scattering respectively). We denote this a strong-strong fixed point (SSFP). In addition, we find and characterize three more FPs: a trivial unstable weak scattering FP (weak-weak fixed point, WWFP), and two other FPs: a weak-strong fixed point (WSFP), and a strong-weak fixed point (SWFP), each being stable in one direction of parameter space and unstable in the other direction. All these points are marked by non-Fermi liquid correlations.

We note that the issue of noncommutativity of the various tunneling operators is central to the analysis of the model beyond the perturbative limit. Here, we will be dealing with operators describing tunneling of Abelian quasiparticles. Such particles (known as anyons) possess fractional statistics, intermediate between bosons and fermions. The exchange of two qps involves a statistical phase which is ill-defined: its sign depends on the “history” of the trajectories employed in the course of the exchange. In order to avoid such a pathology, one may resort to either one of the three following tricks: (i) avoid single qp field operators (qp creation or annihilation operators) and assign statistical Klein factors only to operators which are bilinear in qp field operators, e.g., to qp tunneling operators [22,23], (ii) attach a statistical flux tube to a qp and follow the kinetics by way of a quantum master equation [24,25], (iii) introduce all relevant edges on a single contour, and choose a convention how qp’s are being exchanged [26] (i.e., clockwise or anticlockwise). In the present analysis, we will assign generalized Klein factors to qp operators, following the philosophy of (iii).

The outline of the paper is the following. In Sec. II, we introduce the problem addressed here, and discuss its mapping onto a model setup that allows systematic bosonization and subsequent analysis. The assumptions underlining this modelization and the applicability thereof are outlined. In Sec. III, we position our building blocks (in particular the chiral channels of our model) within a particular geometry, giving rise to a well-defined convention that determines the commutation relations of the anyonic quasiparticles. Performing a sequence of canonical transformations allows us to derive an effective action that captures the symmetries of the problem. Section IV is devoted to the perturbative analysis of the weak scattering fixed point (WWFP), while Sec. V is focused on the study of the scaling near the weak-strong fixed

point WSFP. We note that the physics of the strong-weak fixed point SWFP is trivially obtained by trivial exchange of field indices from the WSFP. Here, we argue that the low-energy dynamics is dominated by a phase slip-instanton mechanism. In Sec. VI, we analyze the scaling near the SSFP. In this limit, an instanton picture within a two-dimensional parameter space is employed. In the limit of noninteracting Luttinger wires, our analysis needs special care: some of the flows become marginal. This is discussed in Sec. VII, where comparison with known results is presented as a consistency check. Finally, in Sec. VIII, we present a short summary of our results, and a few proposals and speculations concerning further directions of our analysis. In Appendix A, we make a few comments concerning experimental verification of our predictions, and in Appendix B, we compare the results obtained within our bosonic theory with an exact refermionization treatment for a specific value of the Luttinger parameter.

II. THE SETUP AND ITS MODELING

Our model is motivated by an experimental setup with a two-dimensional electron gas in a strong magnetic field, such that the fractional quantum Hall (FQH) regime is reached. We focus on a Laughlin filling fraction, such that, in the absence of edge reconstruction, the structure of the edge is simple, consisting of a single chiral channel at each edge. In addition, we imagine the existence of a compressible puddle with somewhat reduced density, e.g., $\nu = 1/3$ in the bulk and a puddle with a compressible $\nu = 1/4$ state of composite fermions. Such a setup can be realized by locally modifying the backgate voltage in a certain region, see Fig. 1.

Due to its small geometric size, the puddle has a charging energy associated with it and constitutes a quantum dot (QD). The edge modes support anyonic qps, which may be backscattered through the incompressible bulk near the entrance and exit of the QD. The setup we thus envision is that of a QD with four semi-infinite chiral edges coupled to it. We assume that the contacts between the leads and the QD are ideal, i.e., reflectionless. In total, one needs to consider a Hamiltonian, which describes the ideal chiral edges, the contact between edges and puddle, the puddle modeled as a metallic quantum dot (QD) with charging energy, and elastic (coherent) tunneling channels that may support qp tunneling (see Fig. 2). In terms of modeling this setup, we need to find a way to describe an ideal, reflectionless contact between

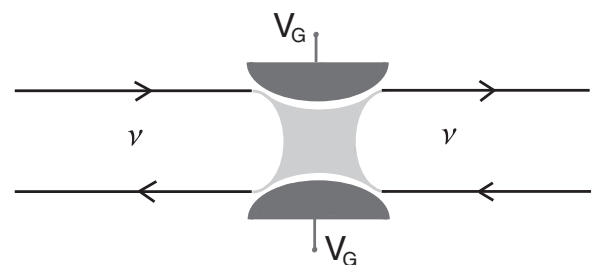


FIG. 1. Sketch of a QH bar with a gate defined QPC. The bulk of the Hall bar is incompressible with filling fraction ν , the density in the QPC region is lower, such that mixing between edge states in a compressible region can occur, both via elastic and inelastic channels.



FIG. 2. (Color online) Schematic model of a quantum wire with two incoming (labeled 1 and 2) and two outgoing chiral modes (labeled 3 and 4) connecting to an extended scattering region. The dark shading represents a region of capacitive charging, the quantum dot. Near the entrance and exit to the quantum dot, where translational invariance is broken, backscattering of quasiparticles across the Hall bar is possible, denoted by vertical dashed lines. In a possible realization with a compressible quantum dot region, the edge current spreads into the bulk, and both elastic transmission along the edge (indicated by horizontal dashed line) and inelastic transmission through the dot is possible.

a chiral lead and a QD. In order to achieve this goal, we follow Matveev: we employ four electrostatically coupled infinite chirals. For each of these infinite chirals, one half is identified with the chiral wire outside the dot, whereas the other half is assumed to be positioned inside the dot. In this way, low-energy excitations of the chiral parts “inside the dot” can be identified with dissipative processes within the QD. Transport processes between a chiral entering the dot and another chiral leaving the dot are mediated by (i) the charging energy within the dot, which prevents charge accumulation inside the QD, and (ii) by coherent elastic tunneling between incoming and outgoing chirals. The setup corresponding to our model is depicted in Fig. 3.

We are now in a position to write the total Hamiltonian. It consists of three terms. The low-energy dynamics of the original semi-infinite leads, as well as the part of the chiral edge channels that constitutes part of the quantum dot, are delegated to a Hamiltonian that represents four infinite chirals, H_{ch} . Separately, the charging energy of the QD is described by H_{dot} . Finally, the coherent part of the lead-QD tunneling is represented by H_{tun} . We resort to bosonization, and define bosonic fields $\phi_i(x)$ with $i = 1, 2, 3, 4$ for each chiral, see Fig. 2. We decompose the bosonic fields into finite momentum and zero modes according to

$$\phi_i(x) = \phi_i(x) + \phi_i^0(x). \quad (1)$$

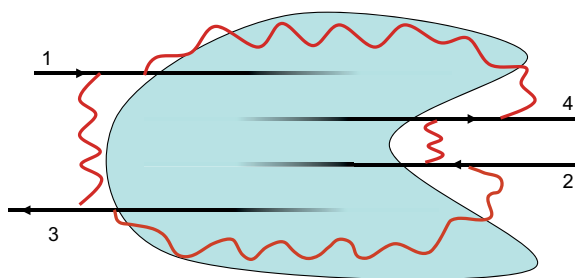


FIG. 3. (Color online) Effective modeling of the system (cf. Fig. 2) through four infinite chiral channels. Each channel consists of a half infinite wire (either incoming or outgoing), and a half infinite wire, which is part of the quantum dot (shaded area). Elastic scattering channels are represented by wiggly lines.

The finite momentum part has a Fourier representation according to

$$\phi_i(x) = \int \frac{dq}{2\pi} \theta(\pm q) \sqrt{\frac{2\pi v}{|q|}} (a_{i,q} e^{iqx} + a_{i,q}^\dagger e^{-iqx}), \quad (2)$$

and the structure of the zero modes will be discussed in the context of Klein factors in Sec. III. For the mode decomposition above, the plus sign should be used for the right-moving modes 1 and 4, and the minus sign for the left-moving modes 2 and 3. The $a_{i,q}$ and $a_{i,q}^\dagger$ are canonical boson operators with commutation relations $[a_{i,q}, a_{j,q'}^\dagger] = \delta_{i,j} 2\pi \delta(q - q')$, giving rise to $[\phi_i(x), \phi_j(y)] = (\pi g/2) \text{sgn}(x - y)$. The charge density is related to be boson fields via $\rho_i(x) = \frac{1}{\sqrt{2\pi}} \partial_x \phi_i(x)$, and the Hamiltonian for the chirals is given by

$$H_{\text{ch}} = \frac{\hbar v}{2\pi g} \sum_{i=1}^4 \int dx [\partial_x \phi_i(x)]^2. \quad (3)$$

In addition, there is a Coulomb charging energy for the dot:

$$H_{\text{dot}} = \frac{e^2}{2C} Q^2, \quad (4)$$

where

$$Q = \frac{1}{\sqrt{2\pi}} [-\phi_1(0) - \phi_2(0) + \phi_3(0) + \phi_4(0)] \quad (5)$$

denotes the charge on the dot. Coherent elastic tunneling between incoming and outgoing wires is described by the Hamiltonian

$$H_{\text{tun}} = t_{13} \cos(\phi_1 - \phi_3) + t_{24} \cos(\phi_2 - \phi_4) + t_{14} \cos(\phi_1 - \phi_4) + t_{23} \cos(\phi_2 - \phi_3). \quad (6)$$

Here we ignore retardation effects, assuming that tunneling is instantaneous. Thus the total Hamiltonian for the system is given by

$$H = H_{\text{ch}} + H_{\text{dot}} + H_{\text{tun}}. \quad (7)$$

As for some parts of our analysis, it will be useful to resort to a functional integral formulation of the problem, we also state the Matsubara finite temperature action of the system:

$$S = \frac{\hbar}{2\pi g} \sum_{j=1}^4 \int dx d\tau \partial_x \phi_j(x, \tau) [\pm i \partial_\tau \phi_j(x, \tau)] - \int d\tau H. \quad (8)$$

Here, the plus sign refers to the right moving branches 1 and 4, and the minus sign to the left moving branches 2 and 3. We note that the zero-mode part of the Hamiltonian H_0 and of the action S_0 will be discussed in detail in Sec. III.

A few comments are in order now. (i) Our analysis will be performed using the Matsubara technique. (ii) The tunneling is that of anyons (qps). We consider *only* this type of tunneling terms as they are (when allowed) potentially the most relevant terms. (iii) All channels are put on an equal footing. There is no distinction between forward scattering and backscattering any more. Particles on different chirals are different species, hence the finite momentum boson fields ϕ_1, ϕ_2, \dots commute. It is therefore vital (for the sake of preserving the appropriate commutation relations) to introduce Klein factors which, in

the case of qp tunneling, will be anyonic Klein factors. In the notation of Eq. (6), the Klein factors are implicit in the prefactors t_{ij} , their precise realization will be discussed below. We denote the c -number-valued scattering amplitudes without Klein factors by γ_{ij} , i.e., $t_{ij} = \gamma_{ij} \times$ (Klein factors). (iv) Each infinite chiral contributes half to the lead (either incoming or outgoing) and half to the degrees of freedom of a compressible puddle. The puddle (also known as quantum dot) accommodates gapless soft modes. (v) The charging (that is, electrostatic) interaction introduces coupling among the four channels. As we renormalize the model down in temperature (or applied voltage), flowing towards low-energy dynamics (long-time scales), the presence of a charging energy scale will force charge fluctuations on the QD to freeze-out. The QD then remains effectively neutral. (vi) The model is a hybrid of elastic (coherent) scattering between incoming and outgoing leads, and inelastic transport through the QD. The latter involves the excitation of low-lying modes in the QD, which can be described in terms of many particle-hole excitations. The former represents (to lowest order in the tunneling coupling constant) elastic tunneling through the QD. While one may imagine that backward tunneling, say between channels 1 and 3, is mediated by a resonant impurity level in a Hall bar constriction, one can motivate the forward tunneling, say between channels 1 and 4, as being due to coherent propagation along the edge of the QD. For any real QD, there is an energy scale defined by the mean single-particle level spacing in the QD, Δ . Once the maximum of temperature and bias voltage is driven below Δ , the soft modes, which constitute an essential part of our model are frozen out. At that point, our model ceases to be valid. Nevertheless, as long as the infrared cutoff of our RG flows is larger or comparable to Δ (which might be extremely small) the model and the predictions following from its analysis, are valid. (vii) The interesting regime of our RG flows stipulates running frequencies $\Delta \ll |\omega| \ll E_c$. Modes involving charge fluctuations are frozen out; for the model to be valid, soft modes in the QD still need to be available. Is such a regime possible? If we are focusing on a QD subject to a strong perpendicular magnetic field, then the relevant spectrum of the QD is 1D (on the edge). Ostensibly, both E_c and Δ will scale like $1/L$ (L = the linear size of the QD), and the difference between them may be only due to different coefficients. However, we may control the charging energy E_c , and actually render it larger than Δ , by considering a puddle realized by a sea of composite fermions, for instance at filling $\nu = 1/4$, whose motion does not follow 1D edges but instead explores the available 2D area of the QD.

Our analysis presented here is done with a view to the chiral edges of a FQHE. We stress here that it may apply to quantum wires described by Luttinger liquid models. Applying our analysis to Luttinger liquids implies certain assumptions. This is discussed in Appendix A.

III. GENERALIZED KLEIN FACTORS, AND THE EFFECTIVE ACTION

As was commented in Sec. II, we need to establish a convention as far as Klein factors, underlying the anyonic field operators, are concerned. We choose to establish such generalized Klein factors in relation to a geometry where all four infinite chirals are assumed to be segments of a single

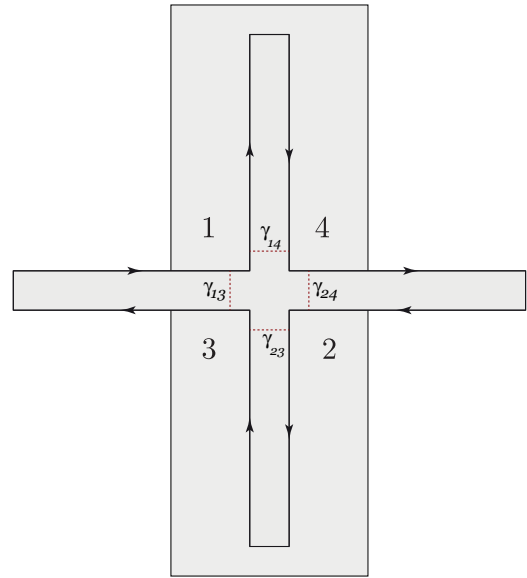


FIG. 4. (Color online) Putting all four chiral wires onto a single contour. Our previously disconnected chiral wires become segments denoted 1, 2, 3, and 4. They are connected by scattering as shown [cf. Eq. (7)]. The quantum dot is represented by the shaded area. All segments should be imagined “infinitely long,” e.g., a quasiparticle entering the dot along 1 will then propagate vertically upwards but will never make it to the “top of the contour” in finite time. It has to scatter to get to 4 (henceforth to 2). Note that the vertical scattering amplitude $\gamma_v = |\gamma_{13} + e^{-i\pi g} \gamma_{24}|$ and the horizontal scattering amplitude $\gamma_h = |\gamma_{23} + \gamma_{14}|$ will be the parameters of our RG flow as shown in Fig. 6. The relative phase factor in the definition of γ_v is due to the phases of scattering operators in Eq. (17).

chiral contour. In that convention, the exchange of two anyons positioned on channel i and channel j respectively has an unambiguous chirality. The geometry is shown in Fig. 4, and the related bosonic field configuration is depicted in Fig. 5.

A. Effective geometry

As we will consider qp scattering between the wires, we need to find a faithful representation of quasiparticle operators in terms of bosonic constituents. For the purpose of establishing unique commutation relations between qp

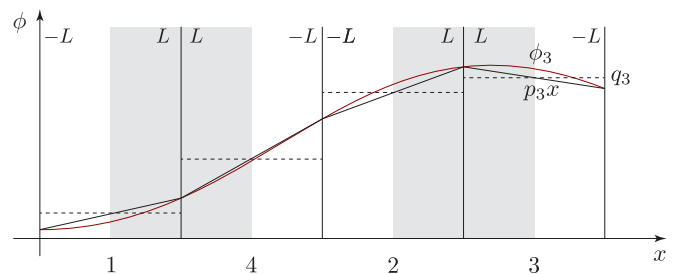


FIG. 5. (Color online) Field configuration on the closed contour. We introduce zero modes q_i , p_i to describe the increment of the field throughout any of the four segments. The “finite momentum components” ϕ_i obey periodic boundary conditions. Shaded regions indicate the dot regions.

operators acting on different chirals, we will put the four chiral edges defining our problem onto a single contour, as shown in Fig. 4. We now follow the recipe for representing quasiparticle operators as discussed by Oshikawa, Chamon, Afeck (OCA in the following) in Appendix E 5 of Ref. [26]. The starting point is a representation of chiral bosonic field, generalized for the presence of zero modes. In operator notation,

$$\hat{\phi}_i(x_i) = \frac{1}{\sqrt{2}} \left(\hat{q}_i - \frac{2\pi}{2L} \hat{p}_i x_i s_i \right) + \phi_i(x_i). \quad (9)$$

Here, x_i is confined to the i th segment of the contour, and L is an extension of that contour. The sign factor is $s_i = 1$ for $i = 1, 2$ and $s_i = -1$ for $i = 3, 4$. The Fourier components of the oscillator modes are such that these modes are algebraically independent on the different contours, and hence they commute. The role of zero modes and finite momentum modes in manufacturing a field continuously covering the whole contour is indicated in Fig. 5. We note that the zero modes satisfy the commutator relation

$$[\hat{q}_j, \hat{p}_k] = g i \delta_{j,k}. \quad (10)$$

At this point, a comment on normalization is due: we represent our chiral fields as $\exp(i\sqrt{2}\phi) \rightarrow \exp[i(q - \frac{2\pi}{L} p x + \sqrt{2}\phi)]$. This differs from the OCA convention by the factor $\sqrt{1/g}$. OCA have a convention $\exp[i(\sqrt{g}q + \frac{2\pi\sqrt{g}}{L} p + \phi)]$, and unit commutation relations $[q, p] = i$.

When substituted into a functional integral, $\hat{q}_i \rightarrow q_i(t)$ becomes a time-dependent field. In the literature, one often finds the additional dynamical substitution rule, $\hat{p}_i x_i \rightarrow \hat{p}_i(x_i - t)$, where, in order to simplify the notation, we have taken the velocity to be unity. This we should interpret in terms of the interaction picture dynamics generated by the zero-mode Hamiltonian

$$\hat{H}_0 \equiv \frac{\pi}{4gL} \sum_i \hat{p}_i^2, \quad (11)$$

and the corresponding zero-mode action

$$S_0[q, p] = \int d\tau \left(\frac{i}{g} p_i d_\tau q_i - H_0 \right). \quad (12)$$

The corresponding equations of motion

$$\begin{aligned} d_t q_i &= \frac{\pi}{L} p_i, \\ d_t p_i &= 0, \end{aligned}$$

are solved by $q_i(t) = q_i + p_i \frac{\pi}{L} t$, with constant (q_i, p_i) . Substitution into the field definition (9) leads to the dynamical interpretation

$$\phi_i(x_i, t) = \frac{1}{\sqrt{2}} \left(q_i - \frac{\pi}{L} p_i(x_i s_i - t) \right) + \phi_i(x_i, t)$$

with time-independent (q_i, p_i) . At this stage, the meaning of the sign factors s_i becomes transparent. They make sure that on contours 1, 2 causality is towards increasing coordinates (“incoming”), and on contours 3, 4 towards decreasing coordinates (“outgoing”).

B. Generalized Klein factors

Following OCA, we represent a quasiparticle operator obeying proper *intrawire* commutation relations as

$$\psi_i = e^{iq_i} e^{-i\frac{2\pi}{L} p_i x_i} e^{i\phi_i}, \quad (13)$$

as a product of zero-mode operators and the finite momentum vertex operators $\exp(i\phi_i)$. Quasiparticle interwire commutation relations are now generated by multiplication with a factor

$$\eta_i \equiv e^{i\frac{\pi}{2} \sum_j \alpha_{ij} p_j}, \quad (14)$$

where the matrix α is given by

$$\alpha = \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & 1 & 1 & 0 \end{pmatrix}. \quad (15)$$

The matrix structure encodes the sequential ordering of segments on the contour. For example, the first row indicates $1 < 2, 3, 4$, and the second $2 > 1, 4$ and $2 < 3$. We define new operators $\Psi_i = \eta_i \psi_i$, which are designed such that

$$\Psi_i \Psi_j = \Psi_j \Psi_i e^{i\pi g \alpha_{ij}}. \quad (16)$$

The derivation of these commutation relations from the current path integral formalism is detailed in Appendix A.

We aim to understand how the presence of these “Klein factors” will interfere with the scaling of the scattering operators. To achieve this, we define the latter in such a way that the tunneling points sit at $x_i = 0$. In order to avoid ambiguity, we actually remove the tunneling sites a little bit from zero as indicated in Fig. 4. This means that the $p_i x_i$ terms in the quasiparticle amplitudes drop out. We are then led to consider the following bilinears:

$$\begin{aligned} \bar{\Psi}_3 \Psi_1 &= e^{i(q_3 - q_1)} e^{i\frac{\pi}{2}(p_1 + p_3 + 2p_2 + 2p_4)} e^{i(\phi_1 - \phi_3)} e^{-i\frac{\pi}{2}g}, \\ \bar{\Psi}_4 \Psi_2 &= e^{i(q_4 - q_2)} e^{i\frac{\pi}{2}(-p_2 - p_4)} e^{i(\phi_2 - \phi_4)} e^{i\frac{\pi}{2}g}, \\ \bar{\Psi}_3 \Psi_2 &= e^{i(q_3 - q_2)} e^{i\frac{\pi}{2}(p_3 + p_2)} e^{i(\phi_2 - \phi_3)} e^{-i\frac{\pi}{2}g}, \\ \bar{\Psi}_4 \Psi_1 &= e^{i(q_4 - q_1)} e^{i\frac{\pi}{2}(p_1 + p_4)} e^{i(\phi_1 - \phi_4)} e^{-i\frac{\pi}{2}g}. \end{aligned} \quad (17)$$

In deriving these relations, we made use of the commutation relations

$$e^{is\frac{\pi}{2}p_i} e^{is'q_i} = e^{is'q_i} e^{is\frac{\pi}{2}p_i} e^{iss'\frac{\pi}{2}g}, \quad (18)$$

where $s, s' = \pm 1$. These bilinears enter the problem through the scattering action

$$\begin{aligned} S_s[\bar{\Psi}, \Psi] &\equiv S_v[\bar{\Psi}, \Psi] + S_h[\bar{\Psi}, \Psi], \\ S_v[\bar{\Psi}, \Psi] &= \int d\tau (\gamma_{13} \bar{\Psi}_3 \Psi_1 + \gamma_{24} \bar{\Psi}_4 \Psi_2 + \text{H.c.}), \\ S_h[\bar{\Psi}, \Psi] &= \int d\tau (\gamma_{23} \bar{\Psi}_3 \Psi_2 + \gamma_{14} \bar{\Psi}_4 \Psi_1 + \text{H.c.}) \end{aligned} \quad (19)$$

Another observable relevant to our discussion is the charge on the dot. Defining the charge density as $s_i \partial_x \phi_i = \sqrt{2}\pi \rho_i$, we get for the charge

$$Q_i = \frac{1}{\sqrt{2}\pi} \int_0^L dx s_i \partial_x \phi_i = -\frac{p_i}{2} - \frac{s_i}{\sqrt{2}\pi} \phi_i(0), \quad (20)$$

Abbreviating as $\phi_i(0) = \phi_i$, we thus obtain

$$Q = -\frac{1}{2}(p_1 + p_2 + p_3 + p_4) - \frac{1}{\sqrt{2\pi}}(\phi_1 + \phi_2 - \phi_3 - \phi_4) \quad (21)$$

for the charge on the dot. (The mnemonic behind the different sign configuration is that in $\varphi \sim px + \phi$, the zero-mode p comes with a spatial coordinate. That one changes sign when we effectively count $x \rightarrow -x$ on the outgoing wires).

We finally note that away from the scattering points x_i the action is quadratic, implying that the field amplitudes $\phi_i(x_i \neq 0)$ can be integrated out. As a result, we obtain a standard dissipative action

$$S_{\text{diss}}[\phi] = \frac{T}{\pi g} \sum_{i=1}^4 \sum_n |\omega_n| |\phi_{i,n}|^2. \quad (22)$$

C. Effective action

Equations (17), (21), and (B15) describe the basic constituents of the problem, the scattering operators, the charge operator, and the dissipative action in terms of the twelve fields ϕ_i, q_i , and p_i . Throughout, we will assume that the charging energy $E_c = e^2/C$ is the largest energy scale in the problem. Focusing on smaller scales, we may assume fluctuations of the charge Q to be strongly suppressed, which removes one degree of freedom from the problem. Assuming Q to be locked, eight more linear combinations of the above variables may be integrated out without further approximation, by a procedure detailed in Appendix B. To define the resulting effective action, we define the new set of variables,

$$\begin{aligned} \Phi_0 &= \frac{1}{2}(\phi_1 + \phi_2 + \phi_3 + \phi_4), \\ \Phi_1 &= \frac{1}{2}(\phi_1 - \phi_2 + \phi_3 - \phi_4), \\ \Phi_2 &= \frac{1}{2}(\phi_1 - \phi_2 - \phi_3 + \phi_4), \\ \Phi_3 &= \frac{1}{2}(\phi_1 + \phi_2 - \phi_3 - \phi_4). \end{aligned} \quad (23)$$

$\Phi_3 \propto Q$ couples to the Coulomb charging action and is gapped out at energy scales lower than the charging energy, while Φ_0 is a zero mode with free action

$$\tilde{S}_0[\Phi_0] \equiv \frac{T}{2\pi g} \sum_n |\omega_n| |\Phi_{0,n}|^2. \quad (24)$$

The remaining two fields $\Phi_{1,2}$ are the nontrivial degrees of freedom of the model and their effective action is given by

$$\begin{aligned} S[\Phi_1, \Phi_2] &= S_0[\Phi_1, \Phi_2] + S_v[\Phi_2] + S_h[\Phi_1], \\ S_0[\Phi_1, \Phi_2] &= \frac{T}{2\pi g} \sum_n \Phi_n^T \begin{pmatrix} |\omega_n| & -\omega_n \\ \omega_n & |\omega_n| \end{pmatrix} \Phi_{-n}, \\ S_v[\Phi_2] &= \gamma_v \int d\tau \cos(\sqrt{2}\Phi_2), \\ S_h[\Phi_1] &= \gamma_h \int d\tau \cos(\sqrt{2}\Phi_1). \end{aligned} \quad (25)$$

Here, $\gamma_v = |\gamma_{13} + e^{-i\pi g} \gamma_{24}|$ and $\gamma_h = |\gamma_{23} + \gamma_{14}|$ are effective scattering amplitudes characterizing the backward and forward strength, respectively, cf. Eq. (B18). To understand the structure of this result, first notice that our problem conserves global

charge. This is equivalent to the statement that it is invariant under a uniform static shift of all variables $\phi_i \rightarrow \phi_i + \theta$. Such shifts leave the fields $\Phi_{1,2,3}$ manifestly invariant, while the zero-mode action for Φ_0 does not change provided $\theta = \text{const}$.

Turning to the action of the remaining two fields $\Phi_{1,2}$ it is straightforward to verify that in the absence of Klein factors, i.e., for c -number valued scattering coefficients t_{ij} , the scattering Hamiltonian (6) becomes separable once $Q = \text{const}$ is frozen out. The action would then decouple into two independent actions of fields $\Phi_{1,2}$, describing forward (γ_{13}, γ_{24}) and backscattering (γ_{14}, γ_{23}), resp. These actions are obtained from S , Eq. (25), if one ignores the off-diagonal terms in the kernel of S_0 .

The Klein factors account for the noncommutativity of the scattering operators. In the reduced representation, after integration over all auxiliary fields, their heritage is the canonical contribution to the action (the off-diagonal terms in the matrix kernel), i.e., a term $\sim \int d\tau \Phi_1 \partial_\tau \Phi_2$ stating noncommutativity of the forward and the backward scattering field.

What are the commutation relations between the fields? Usually, the commutator between two conjugate variables $[q, p] = i\alpha$ is related to an action $\frac{1}{\alpha} \int dt p \dot{q}$. Combining the two off-diagonal elements in the action (25), this would lead to $[\Phi_1, \Phi_2] = i\pi g$. However, this result is incorrect. As usual in a functional integral based approach, operator commutation relations are obtained by analyzing the effects of time ordering in the integral, i.e., by subtracting the expressions $\langle \Phi_1(\pm\epsilon) \Phi_2 \rangle$ from each other. The evaluation of these terms must take the presence of the diagonal dissipative contributions to the action into account. We then obtain, with $\epsilon \rightarrow 0+$,

$$\begin{aligned} &\langle \Phi_1(\epsilon) \Phi_2(0) \rangle - \langle \Phi_1(-\epsilon) \Phi_2(0) \rangle \\ &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (e^{i\omega\epsilon} - e^{-i\omega\epsilon}) \frac{\pi g}{2\omega} \\ &= \frac{ig}{2} \int_{-\infty}^{\infty} d\omega \frac{\sin \omega}{\omega} = i \frac{\pi g}{2}. \end{aligned} \quad (26)$$

This implies the commutation relation

$$[\Phi_1, \Phi_2] = i \frac{\pi g}{2}, \quad (27)$$

which will play an important role throughout.

Before ending this section, let us comment on a point that will play a role in the interpretation of our results below: the effective action (25) describes the physics of the so-called dissipative Hofstadter model introduced by Callan and Freed [27]. Recall that the Hofstadter model describes the physics of a rectangular lattice subject to a perpendicular magnetic field. Describing nearest-neighbor hopping on the lattice in terms of two operators p_{12} as $\cos(p_{1,2})$, the action of the system contains a sum of two \cos operators, where different coupling constants may account for anisotropy. The presence of a magnetic field implies the lack of commutativity, $[p_x, p_y] \neq 0$, i.e., a term $\sim \int d\tau p_1 \partial_\tau p_2$ in the action, whose coupling constant is a measure of the field strength. Finally, dissipation may be introduced by including operators $\sim \sum_n |\omega_n| (|p_{1,n}|^2 + |p_{2,n}|^2)$. Adding everything up, we arrive at an action equivalent to (25) (an identification $p_i \leftrightarrow \Phi_i$ understood). More specifically, Eq. (25) describes the DHM fine tuned to a configuration

where field and dissipation strength balance each other. What motivated Callan and Freed to generalize the nondissipative Hofstadter to the presence of dissipation was the expectation that the fractal structure of its single particle spectrum might turn into a fractal pattern of *phase transitions*. Their analysis indeed confirmed that the phase plane spanned by field and dissipation strength is covered by a fractal network of phase transition lines separating phases of localization (cos-coupling constants scaling to zero) from phases of delocalization (diverging cos amplitudes.) They arrived at that conclusion within a nonperturbative analysis resting on an approximate and an exact self-duality symmetry of the model. The above-mentioned balancing of field and dissipation strength in our context means that we are cutting through this pattern along a line parameterized by the Luttinger interaction strength g . Later on, we will confirm within an instanton approach generalizing that of Ref. [27] that for repulsive interactions, $g \geq 1$, we are generically in a “delocalized” sector of the phase plane, which is another way of saying that the intermediate scattering fixed point is approached.

IV. THE WEAK-WEAK FIXED POINT: PERTURBATIVE ANALYSIS

In this section, we will analyze how initially small coherent tunneling terms S_h, S_v will grow under the RG, depending on the value of the Luttinger parameter g . To leading order in the couplings γ_h and γ_v , only one of the fields Φ_1 or Φ_2 enters the analysis, and hence we only need the correlation functions

$$\langle \Phi_{i,n} \Phi_{i,-n} \rangle = \frac{\pi g}{2} \frac{1}{|\omega_n|}. \quad (28)$$

This information allows us to work out the flow equations for γ_h, γ_v . Decomposing the fields Φ_i as usual into slow and fast modes and performing a RG transformation, one sees that the exponential of the field is renormalized by a factor

$$e^{i\sqrt{2}\Phi_i} \rightarrow b e^{i\sqrt{2}\Phi_i} e^{-T \sum_{n,f} (\Phi_{i,n} \Phi_{i,-n})}. \quad (29)$$

Here, the factor b in front of the exponential is due to the rescaling of the integration measure $d\tau$ in the action (25), and the \sum_f runs over fast frequencies $\Lambda/b < |\omega_n| < \Lambda$. Using the correlation function (28), we find

$$\begin{aligned} T \sum_{n,f} \langle \Phi_{i,n} \Phi_{i,-n} \rangle &= \frac{\pi T g}{2} \sum_{n,f} \frac{1}{|\omega_n|} \\ &= \frac{g}{2} \int_{\Lambda/b}^{\Lambda} \frac{d\omega}{\omega} = \frac{g}{2} \ln b. \end{aligned} \quad (30)$$

Taking everything together, we find that the scattering operators scale up with dimension $1 - g/2$, the result obtained earlier by us [11]. The reason for the deviation from KF scaling with dimension $1 - g$ is that fluctuations in the incoming and outgoing chiral are partially correlated with each other. This is due to the fact that for energies below the charging energy of the dot, every incoming charge fluctuation is split equally between the two outgoing chirals, such that the amount of independent fluctuations is effectively halved. Thus the presence of dissipation makes the quadratic action unstable with respect to coherent scattering even for the hypothetical case of chiral edges with an “attractive” interaction $1 < g < 2$.

The analysis described above relates to the point “1” in close vicinity of the Gaussian fixed point with $\gamma_h = 0, \gamma_v = 0$, denoted as (0,0) in Fig. 5. Due to the scaling dimension $1 - g/2$ of coherent tunneling terms, the Gaussian fixed point (0,0) (WWFP) is unstable in the physical regime of $g \leq 1$. We want to emphasize that the variables γ_h, γ_v are not conductances for tunneling in horizontal or vertical direction, but are instead the tunneling amplitudes, which act as mass terms for the fields Φ_1 and Φ_2 in the regime of strong coupling.

V. THE WEAK-STRONG FIXED POINT (WSFP): PHASE SLIP-INSTANTON MECHANISM

We now want to assume that one of the two tunnel couplings is much larger than the other one, say $\Gamma \equiv \gamma_v \gg \gamma \equiv \gamma_h$. To heuristically understand what is happening in this limit, let us temporarily go back to a Hamiltonian description in which we have two tunneling operators

$$\Gamma \cos(\sqrt{2}\Phi_2), \quad \gamma \cos(\sqrt{2}\Phi_1),$$

where the phase operators obey the commutation relations $[\Phi_1, \Phi_2] = ig\pi/2$. To leading order in γ , the interaction-picture time evolution of the exponential $\exp(-\Gamma \cos(\sqrt{2}\Phi_2))$ under the action of the perturbing operator $\propto \gamma$ is governed by the commutators of the type $[\gamma \cos(\sqrt{2}\Phi_1), \exp(-\Gamma \cos(\sqrt{2}\Phi_2))]$. The noncommutativity of Φ_1 and Φ_2 implies that the exponential of Φ_1 acts as a translation operator for Φ_2 by an amount $g\pi$. How does the system respond to such a translation? A naive and incorrect argument suggests that such a phase flip by π (in the case $g = 1$) of $\sqrt{2}\Phi_2$ will cost a large amount of energy due to the large magnitude of Γ , hence it should be accompanied by a second, almost immediate instanton that will add up $-\pi$ or $+\pi$ to the first one, taking the field configuration of Φ_2 back to a minimum. This argument suggests that the lowest terms effectively contributing to the perturbation expansion will be of order γ^2 . The less naive and correct heuristic argument takes into account that the argument of $\cos(\sqrt{2}\Phi_2)$ contains finite k modes besides the zero-modes generating the commutation relations of Φ_2 relative to Φ_1 . For this reason, the original phase change by $g\pi$ due to the Klein factors can be compensated by a kink in the finite momentum part of Φ_2 . This secondary kink enables the field Φ_2 to settle back into an energetically favorable configuration without the help of another tunneling event, but it *does* cost a finite amount of dissipative action. The more quantitative analysis to be detailed momentarily shows that the primary effect of the added action contribution is a change of the scaling dimension of γ to $1 - g$, which should be compared to the dimension $1 - g/2$ characterizing the weak-weak fixed point discussed in the previous section.

On a more formal level, we will apply an instanton analysis in Φ_2 . The complementary operator introduced within the duality transformation is assumed to be perturbatively weak. The instanton approach assumes that the relevant field configurations are rare phase slips $\phi_2 = \frac{1}{\sqrt{2}}(2l + 1)\pi \rightarrow \frac{1}{\sqrt{2}}(2(l \pm 1) + 1)\pi$ between 2π -consecutive minima of $\cos \sqrt{2}\Phi_2$. Assuming that these events occur at times τ_i , the

phase profile is best described by its time derivative

$$\begin{aligned}\partial_\tau \Phi_2 &= \sum_{i=1}^N \sqrt{2\pi} s_i \delta(\tau - \tau_i), \\ i\omega_n \Phi_{2,n} &= \sum_{i=1}^N \sqrt{2\pi} s_i e^{-i\omega_n \tau_i},\end{aligned}\quad (31)$$

where δ is a broadened δ function, and $s_i = \pm 1$ denote the direction (the charge) of the phase slip. To facilitate the evaluation of the instanton functional integral, we use a Hubbard-Stratonovich with a field Θ_n to partially decouple the quadratic term $|\omega_n| \Phi_{2,n} \Phi_{2,-n}$ according to

$$\begin{aligned}T \sum_n \left(\frac{1}{2\pi g} \Phi_{1,n} \Phi_{1,-n} |\omega_n| + \frac{1}{\pi g} \Phi_{1,n} \Phi_{2,-n} \omega_n \right) &\rightarrow \\ \rightarrow T \sum_n \left(\frac{1}{\pi g} \Phi_{1,n} \Phi_{1,-n} |\omega_n| + \frac{\pi g}{2} \Theta_n \Theta_{-n} |\omega_n| \right. \\ \left. + \Theta_n \Phi_{1,-n} |\omega_n| - \Theta_n \Phi_{2,-n} \omega_n \right).\end{aligned}\quad (32)$$

Doing the Gaussian integral over Θ , one gets from the second to the first line. We note that Θ and Φ_2 are a canonical pair. Denoting the instanton action by S_i , the functional integral now becomes

$$\begin{aligned}\mathcal{Z} &= \int D\Phi_1 D\Theta e^{-S_0[\Theta, \Phi_1] - S_\gamma[\Phi_1]} \\ &\times \sum_N \frac{1}{N!} \sum_{\{s_i\}} \int \prod d\tau_i e^{-NS_i} e^{i\sqrt{2\pi} \sum_{i=1}^N s_i T \sum_n \Theta_n e^{i\omega_n \tau_i}} \\ &= \int D\Phi_1 D\Theta e^{-S_0[\Theta, \Phi_1] + e^{-S_i} \int d\tau \cos(\sqrt{2\pi}\Theta) - S_\gamma[\Phi_1]}.\end{aligned}\quad (33)$$

Here, $S_\gamma[\Phi_1] = \gamma \int d\tau \cos(\sqrt{2}\Phi_1)$, and

$$\begin{aligned}S_0[\Theta, \Phi_1] &= T \sum_n \left(\frac{1}{\pi g} \Phi_{1,n} \Phi_{1,-n} |\omega_n| \right. \\ &\quad \left. + \frac{\pi g}{2} \Theta_n \Theta_{-n} |\omega_n| + \Theta_n \Phi_{1,-n} |\omega_n| \right) \\ &= \frac{T}{2} \sum_n (\Phi_{1,n}, \Theta_n) \begin{pmatrix} \frac{2}{\pi g} & 1 \\ 1 & \pi g \end{pmatrix} |\omega_n| \begin{pmatrix} \Phi_{1,-n} \\ \Theta_{-n} \end{pmatrix}.\end{aligned}\quad (34)$$

We are now in a position to explore scaling dimensions. Due to the assumed weakness of both $\exp(-S_i)$ and γ , we can treat the two operators independently of each other. The basic formula we use is that for a field ϕ with correlation $\langle \phi_n \phi_{-n} \rangle = \frac{\kappa}{T|\omega_n|}$, we have

$$e^{i c \phi} \rightarrow e^{i c \phi} b^{1 - \frac{\kappa c^2}{2\pi}}$$

in an RG step. In our particular case, $\langle X_n X_{-n} \rangle = \frac{1}{T|\omega_n|} M^{-1}$, where M is the matrix controlling the free action:

$$\begin{aligned}\langle \Phi_{1,n} \Phi_{1,-n} \rangle &= \frac{1}{T|\omega_n|} \pi g \Rightarrow \kappa = \pi g, c = \sqrt{2}, \\ \langle \Theta_n \Theta_{-n} \rangle &= \frac{1}{T|\omega_n|} \frac{2}{\pi g} \Rightarrow \kappa = \frac{2}{\pi g}, c = \sqrt{2\pi}.\end{aligned}\quad (35)$$

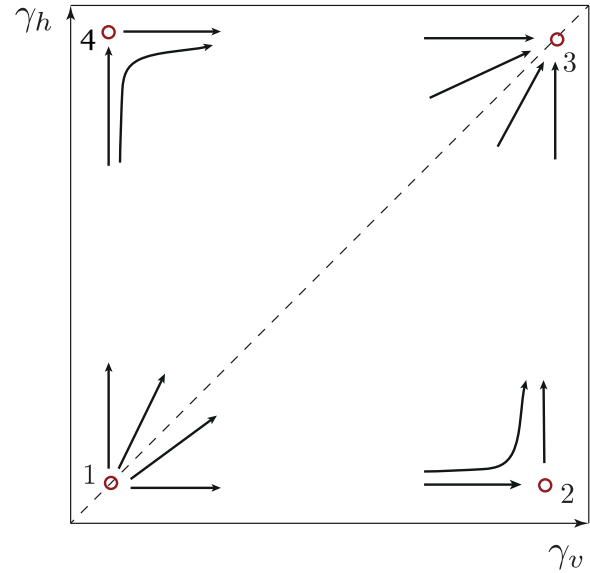


FIG. 6. (Color online) Description of the RG flow in terms of masses γ_h and γ_v , which denote the strength of elating tunneling in horizontal and vertical direction, respectively. Near the RG-unstable weak-weak fixed point 1, there are perturbative corrections to the Gaussian $(1/2, 1/2)$ conductance in both γ_h and γ_v , which grow under the RG. In the vicinity of the weak-strong fixed point 2, instantons in the strong scattering term γ_v are irrelevant, constituting a stable direction in parameter space, while the weak scattering amplitude γ_h grows under the RG and thus constitutes an unstable direction. The conductance is close to unity in the vertical direction, and perturbatively small in the horizontal direction. In the vicinity of the strong-strong fixed point 3, both vertical and horizontal tunnelings are strong, with RG irrelevant instanton corrections. The conductance is $(1/2, 1/2)$, and the fixed point is stable under RG scaling.

We then find that the two coupling constants scale as

$$e^{-S_i} \rightarrow e^{-S_i} b^{1 - \frac{\kappa c^2}{2\pi}}, \quad \gamma \rightarrow \gamma b^{1-g}. \quad (36)$$

Thus, in the regime $g < 1$, the weak coupling γ is a relevant perturbation, while the exponential of the instanton action scales to zero, indicating that instantons in the strongly coupled scattering channel are irrelevant. The weak-strong FP has an unstable direction corresponding to the growth of the “weak” scattering channel, while the irrelevance of instantons in the “strong” channel indicates that the channel stays at strong coupling. This situation corresponds to point “2” (and similarly to point “4”) in the phase diagram Fig. 6.

VI. THE STRONG-STRONG FIXED POINT (SSFP): INSTANTON ANALYSIS

We now want to discuss the situation when both the vertical and horizontal tunneling terms are strong, such that none can be discussed perturbatively anymore. In the ground state of the system both fields Φ_1 and Φ_2 then occupy one of the minima of the cos-terms, and elementary excitations above the ground state are jumps from one such minimum to the neighboring minimum, see Fig. 7. In this way, the field configurations move on a lattice of ground states, which are connected by instantons (jumps) in the fields. The scaling dimension for

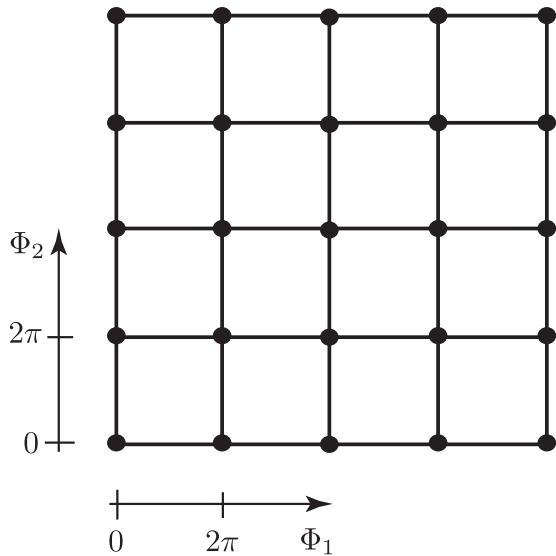


FIG. 7. Square lattice for instantons in the (Φ_1, Φ_2) fields. Jumps in the horizontal and vertical directions have equal length, and instantons in diagonal direction are less relevant than horizontal and vertical ones.

such a jump can be obtained by computing the action for the instanton due to the quadratic dissipative term in the fields, as described in the section above. In this way, we establish that the strong-strong fixed point (SSFP, denoted by the point 3 in Fig. 5) is indeed stable. In the following, we will present a more formal derivation of this result.

If both $\gamma_{v,h} > 0$ are strong, we may subject the problem to an instanton duality mapping. In this regime, the relevant excitations are rare phase slips $\Phi_i = (2l_i + 1)\pi \rightarrow (2(l_i \pm 1) + 1)\pi$ between consecutive minima of the cos potentials. Assuming these events to occur at times $\tau_{i,a}$, the phase profiles are best described by their time derivative,

$$\partial_\tau \Phi_i = \sum_{a=1}^{N_i} 2\pi s_{i,a} \delta(\tau - \tau_{i,a}), \quad (37)$$

$$i\omega_n \Phi_{i,n} = \sum_{a=1}^{N_i} 2\pi s_{i,a} e^{-i\omega_n \tau_{i,a}},$$

where δ is a broadened δ function, and $s_{i,a} = \pm 1$ denote the direction (the charge) of the phase slip. To facilitate the evaluation of the instanton functional integral we Hubbard-Stratonovich decouple the quadratic dependence in $\partial_\tau \Phi_i$ or $i\omega_n \Phi_{i,n}$. Writing the free Gaussian action as $S_0[\Phi] = \frac{T}{2} \sum_n \Phi_n^T \omega_n X_n^{-1} \Phi_{-n}$, where the matrix X is defined through Eq. (25) as $X_n = \pi g \begin{pmatrix} \text{sgn}(n) & \\ & 1 \\ & & \text{sgn}(n) \end{pmatrix}$. We complete the square in terms of a dual integration variable Θ as

$$\begin{aligned} S_0[\Phi] &\rightarrow S_0[\Theta, \Phi] \\ &\equiv \frac{T}{2} \sum_n \Theta_n \omega_n X_n \Theta_{-n} + T \sum_n \Theta_n^T \omega_n \Phi_{-n}. \end{aligned} \quad (38)$$

The second term tells us that Φ and Θ are canonical variables.

We denote the instanton actions as $S_{i,i}$ and, keeping in mind their exponential smallness in the coupling constants,

$\gamma_{v,h}$, write down a double expansion:

$$\begin{aligned} \mathcal{Z} &= \int D\Theta e^{-\frac{T}{2} \sum_n \Theta_n \omega_n X_n \Theta_{-n}} \sum_{N_i} \frac{1}{N_1! N_2!} \\ &\times \sum_{\{s_{i,a}\}} \int \prod d\tau_{i,a} e^{-\sum_i (N_i S_{i,i} + i2\pi \sum_a s_{i,a} T \sum_n e^{i\omega_n \tau_{i,a}} \Theta_{i,n})} \\ &= \int D\Theta e^{-\frac{T}{2(2\pi)^2} \sum_n \Theta_n \omega_n X_n \Theta_{-n} - \sum_i e^{-S_{i,i}} \int d\tau \cos(\Theta_i)}. \end{aligned} \quad (39)$$

In the last step, we scaled Θ by a factor π for convenience. Substituting the form of the X matrix, we arrive at an action

$$\begin{aligned} S[\Theta] &= \frac{T}{2} \frac{g}{4\pi} \sum_n \Theta_n^T \begin{pmatrix} |\omega_n| & \omega_n \\ -\omega_n & |\omega_n| \end{pmatrix} \Theta_{-n} \\ &+ \sum_i e^{-S_{i,i}} \int d\tau \cos(\Theta_i). \end{aligned} \quad (40)$$

dual to (25). The differences are in the coupling constants: $g \rightarrow 2/g$ and $\gamma_{v/h} \rightarrow e^{-S_{i,2/1}}$. This means that for very large and nearly equal coupling constants γ_i , the coupling constants of the dual theory can be renormalized by first-order RG analysis where the reasoning of the foregoing section obtains the scaling

$$e^{-S_{i,i}} \rightarrow e^{-S_{i,i}} b^{1-\frac{1}{g}}. \quad (41)$$

At first sight, this result looks surprising: for $g < 1$, the instanton insertions are irrelevant, and the system flows to strong coupling, while for $1 < g < 2$, the instanton operators are relevant, and the weak potential perturbation addressed in the previous section is also relevant. This suggests the existence of an intermediate fixed point. Finally, for $g = 1$, the instanton operators are marginal, a situation which seems inconclusive at first.

In order to clarify this situation in more depth, we follow Callan and Freed [27] and generalize the matrix X_n as

$$X_n^{-1} \rightarrow \frac{1}{2\pi} \begin{pmatrix} \alpha \text{sgn}(n) & \beta \\ -\beta & \alpha \text{sgn}(n) \end{pmatrix}, \quad (42)$$

i.e., we allow for independent coefficients of dissipation (α) and canonical term (β). Our current situation is recovered by setting $\alpha = \beta = \frac{1}{g}$. With this generalization, we have $\langle \Phi_{i,n} \Phi_{i,-n} \rangle = \frac{2\pi}{T} \frac{1}{|\omega_n|} \frac{\alpha}{\alpha^2 + \beta^2}$, and the perturbative scaling dimension becomes $1 - \frac{\alpha}{\alpha^2 + \beta^2}$. This tells us that the weak scattering operators are relevant if $\alpha < \alpha^2 + \beta^2$. Defining $z = \alpha + i\beta$, we realize that z must lie outside a circle in the complex plane centered around $(1/2, 0)$ and with radius $1/2$ (cf. Fig. 8). In our case, we are sitting on a line $z = \frac{1}{g}(1 + i)$, and for $g < 2$ we are outside that circle, i.e., we are dealing with relevant operators. We note that for $\beta = 1$, the condition collapses to $g < 1$, corresponding to the standard KF case.

In the dual case, we have the same situation, only that X_n^{-1} gets replaced by $X_n^{-1} \rightarrow \frac{2\pi}{\alpha^2 + \beta^2} \begin{pmatrix} \alpha \text{sgn}(n) & -\beta \\ \beta & \alpha \text{sgn}(n) \end{pmatrix}$. Defining $\tilde{z} = \frac{\alpha}{\alpha^2 + \beta^2} - i \frac{\beta}{\alpha^2 + \beta^2}$, we realize that the instanton duality amounts to a mapping $z \rightarrow \tilde{z} = z^{-1}$. The instanton operators are irrelevant, provided \tilde{z} lies inside the circle above. In our present setting, this requires that $1/(g^{-1}(1 + i)) = \frac{g}{2}(1 - i)$ lie

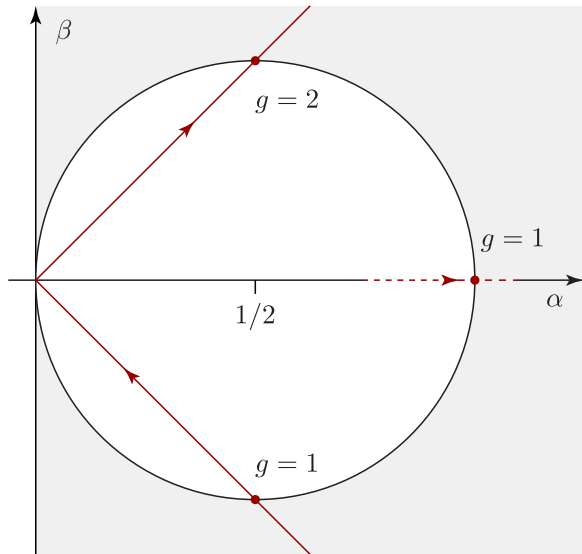


FIG. 8. (Color online) Relevancy of perturbations as a function of the two parameters α, β defined in the text. The white disk/shaded area is the region of the phase diagram where scattering terms are irrelevant/relevant. At the weak coupling fixed point, $\alpha = \beta = g^{-1}$ and the region of relevant perturbation (shaded) is reached as $g \searrow 2$ (line departing from origin corresponding to lowering g). At strong coupling, $\alpha = -\beta = g/2$ and the region of irrelevant perturbation (circle) is reached for $g \searrow 1$ (on the line $\beta = -\alpha$). The line $\beta = 0$ corresponds to the KF model, where perturbations around weak coupling become relevant at $g = 1$. This means that for repulsive lead interactions, $g < 1$, perturbations at weak (strong) coupling are relevant (irrelevant) implying the stability of the stability of the strong coupling fixed point.

inside the circle. For $g < 1$ this is the case. In the hypothetical case of attractive lead interactions, $g > 1$ we are outside and the instanton operators are relevant, while the weak scattering operators are also relevant.

For $g = 1$, we are borderline. We note that backscattering is a relevant perturbation in the weak-weak limit—with a scaling dimension of the backscattering probability equal one. This is in line with reference [28] (cf. Eq. (346) with $N_{ch} = 2$, see also Ref. [10]). In the strong-weak and in the strong-strong limits, the flow of the backscattering probability is marginal. This is indeed verified in the next section (cf. also Fig. 10). For this behavior to hold, we have utilized in our analysis the instantaneous character of backscattering. Reference [28] allows for more general scenarios which may lead to truncation of the scaling flows [cf. Eq. (348) there].

In the next section, we treat the model for the specific value $g = 1$ of the Luttinger parameter by using the method of fermionization, and find agreement with the scenario discussed here. Even if we consider noninteracting leads, yet, there is still interaction in the model: the electrostatic charging energy, which couples the four chirals among each other. It is clear from the scaling obtained above that the flows in the vicinity of both the WSFP/SWFP and the SSFP are marginal. This nicely reproduces the expectation that for Fermi liquid leads, scattering should be marginal in the low-energy limit, and that arbitrary ratios between forward and backward scattering strength can be realized.

VII. GENERAL COMMENTS ON THE ANALYSIS

We would like to discuss some subtleties of the analysis presented above: (i) We assume a cutoff (voltage or temperature), which may be small, yet larger than the level spacing Δ of the QD (which for a typical quantum Hall geometry may be exceedingly small). Had we pursued the RG all the way down to Δ , the character of the model would have changed: there are no inelastic excitations then, hence the inelastic channel is frozen. In that case, in the presence of only elastic scattering, we are driven to either full transmission or full reflection, in agreement with the Kane-Fisher result. In the present analysis, we formally take the limit of an infinite QD, before pursuing our RG procedure all the way down to zero temperature or zero voltage. (ii) In the absence of forward scattering channels ($\gamma_{14} = \gamma_{23} = 0$), our model and the results obtained for its scaling reduce to those considered by Furusaki and Matveev. (iii) Except for the highly degenerate points of perfect scattering amplitude cancellation $\gamma_{13} + e^{-i\pi g} \gamma_{24} = 0$ or $\gamma_{23} + \gamma_{14} = 0$, our analysis holds regardless of scattering phase shifts and strengths (at the degenerate point, the model collapses to a variant of the Kane-Fisher problem). However, these points require fine tuning and are nongeneric. The specific conditions for perfect scattering amplitude cancellations are related to the choice of the Aharonov-Bohm phase enclosed by the the corresponding tunneling path. Here we have followed special choice, that of a zero Aharonov-Bohm flux. We note that OCA Ref. [26] (in their three lead geometry) have allowed the freedom of choosing the Aharonov-Bohm flux. (iv) An extension of our analysis from FQHE to interacting Luttinger wires is presented in Appendix.

VIII. REFERMIONIZATION FOR THE CASE $g = 1$

We consider a special model, where there are only two nonzero scattering amplitudes $\gamma_{14} \equiv \gamma_h$ and $\gamma_{24} \equiv \gamma_v$, see Fig. 9, and where the leads are noninteracting. The model has an interesting dynamics all the same, since at low energies

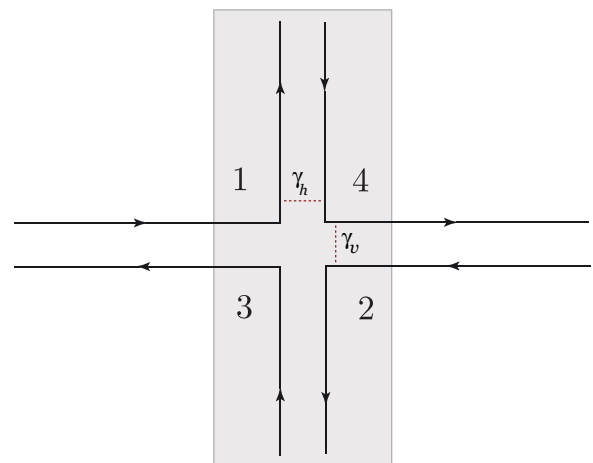


FIG. 9. (Color online) Setup with two competing scattering amplitudes $\gamma_{14} \equiv \gamma_h$ and $\gamma_{24} \equiv \gamma_v$. We consider the special case of noninteracting leads $g = 1$, and denote the probability for elastic scattering from segment 1 to segment 4 by P_{14} , and the probability for elastic scattering from segment 2 to segment 4 by P_{24} .

the mode Φ_3 describing charge fluctuations in the quantum dot is already frozen. In this limit, the tunneling amplitudes γ_h and γ_v are both relevant perturbations. Using a mapping onto noninteracting fermions, we derive an exact solution of the model, which describes the competition between the competing scattering processes γ_h and γ_v . We find that at high temperatures and small γ_h and γ_v , both γ_h and γ_v increase $\propto T^{-1/2}$, in agreement with a weak coupling perturbative RG in Sec. IV. The asymptotic value for $T \rightarrow 0$ is nonuniversal and depends on the ratio of initial values of γ_h and γ_v . This finding complements the results of Secs. V and VI, where it was found that noninteracting leads with $g = 1$ separate two different regimes of RG flow.

The starting point for our discussion are the tunneling terms and the dissipative term in Eq. (25). Due to the off-diagonal terms in S_0 , the fields Φ_1 and Φ_2 satisfy the commutation relation $[\Phi_1, \Phi_2] = i\pi \frac{g}{2}$. In this Appendix, we specialise to noninteracting leads with $g = 1$. Due to the presence of the charging energy on the dot, the diagonal entries of S_0 describing dissipative processes are given by $|\omega_n|/2\pi$. We compare this value with the dissipative action $|\omega_n|/4\pi\nu$ for the charge density field describing backscattering of quasiparticles (in the absence of a QD) between two counterpropagating fractional quantum Hall edges with Luttinger parameter ν . We find that noninteracting leads with a QD charging energy have a dissipative action equivalent to that of interacting $\nu = 1/2$ LL edges in the absence of a QD. The special case of backscattering between $\nu = 1/2$ LL edge states is can be solved exactly by the method of refermionization [29], which we will adapt to our problem of two competing scattering processes in the following. In order to achieve this goal, we undo the shift Eq. (B17), such that in the following the fields Φ_1 and Φ_2 contain finite frequency modes only. The noncommutativity between the fields is now described by Majorana operators η_1, η_3 , and η_4 with $\{\eta_i, \eta_j\} = 2\delta_{ij}$. Switching to a Hamiltonian formalism suitable for refermionization, the tunneling Hamiltonian is given by

$$H_{\text{tun}} = i\eta_1\eta_4\gamma_h[e^{i\sqrt{2}\Phi_1(0)} + e^{-i\sqrt{2}\Phi_1(0)}] + i\eta_2\eta_4\gamma_v[e^{i\sqrt{2}\Phi_2(0)} + e^{-i\sqrt{2}\Phi_2(0)}]. \quad (43)$$

Here, we have absorbed a factor 1/2 into the tunneling amplitudes γ_h and γ_v as compared to Eq. (25). Taking into account that the dissipative actions of Φ_1, Φ_2 are equal to that of an infinite *chiral* Luttinger liquid, the operators $e^{i\sqrt{2}\Phi_{1/2}}$ can be interpreted as the operator $\Psi_{1/2}$ of free chiral electrons. However, to make this equivalence more precise, we need to introduce Majorana fermions $f_{1/2}$ as a Klein factor for the new electrons [29], and define

$$e^{i\sqrt{2}\Phi_{1/2}(0)} = \sqrt{2\pi a} \Psi_{1/2}(0) f_{1/2}. \quad (44)$$

Here, a denotes the short distance cutoff of the theory, which is needed to make sure that the fermion fields $\Psi_{1/2}$ have the proper dimension of one over square root of length. In this way, we obtain the fermionized tunnel Hamiltonian

$$H_{\text{tun}} = i\eta_2\eta_4\gamma_v\sqrt{2\pi a}[\Psi_2(0) f_2 + f_2 \Psi_2^\dagger(0)] + i\eta_1\eta_4\gamma_h\sqrt{2\pi a}[\Psi_1(0) f_1 + f_1 \Psi_1^\dagger(0)]. \quad (45)$$

In addition, the fermions have a kinetic term (the velocity is taken to be unity)

$$H_0 = \int dx [\Psi_1^\dagger(x)(-i\partial_x)\Psi_1(x) + \Psi_2^\dagger(x)(-i\partial_x)\Psi_2(x)]. \quad (46)$$

Since the leads are noninteracting, we can discuss transport through the QD in terms of a scattering formalism. For an incoming particle on edge 1, we define the probability P_{14} to scatter onto the outgoing edge mode 4. Similarly, for an incoming particle on edge 2, we denote the probability to scatter onto the outgoing edge mode 4 by P_{24} . Then, in Appendix D, we derive the result

$$P_{14} = \frac{\Gamma_h}{\Gamma} F\left(\frac{T}{\Gamma}\right), \quad (47a)$$

$$P_{24} = \frac{\Gamma_v}{\Gamma} F\left(\frac{T}{\Gamma}\right), \quad (47b)$$

with

$$\Gamma_h = 4\pi a\gamma_h^2, \quad \Gamma_v = 4\pi a\gamma_v^2, \quad \Gamma = \Gamma_h + \Gamma_v, \quad (48)$$

and

$$F(x) = \frac{2}{x} \arctan \frac{x}{2}. \quad (49)$$

The scaling function has the limiting behaviors $F(x) = 1 - \frac{1}{3}(x/2)^2$ for small x , and $F(x) = \frac{2}{x}$ for large x . Using these asymptotics, we find

$$P_{14} = \begin{cases} \frac{2\Gamma_h}{T} & \text{for } T \gg \Gamma \\ \frac{\Gamma_h}{\Gamma} & \text{for } \frac{T}{\Gamma} \rightarrow 0 \end{cases}, \quad (50)$$

and similarly for P_{24} , with Γ_h and Γ_v interchanged. In the high-temperature limit, this result agrees with the perturbative analysis presented in Sec. IV. Since $P_{14} \propto \gamma_h^2$ in the high-temperature limit, the flow of P_{14} is representative for the flow of the tunneling amplitude γ_h , and similarly for P_{24} and γ_v . The flow of P_{14} and P_{24} is shown in Fig. 10. For weak bare values of γ_h, γ_v with $\Gamma_h/T \ll 1, \Gamma_v/T \ll 1$, the flow starts in the vicinity of the unstable weak-weak fixed point. In terms of the variables P_{14} and P_{24} , the flow occurs along straight line trajectories, which stop at the line of fixed points defined by $P_{14} + P_{24} = 1$. The end point of the flow on this fixed line is determined by the ratio of initial couplings γ_h/γ_v . In particular, in the weak-strong limit with, say, $\gamma_v \gg \gamma_h$, the flow is truncated at a temperature $T \approx \Gamma_v$, and the weak coupling γ_h is marginal. This situation is intermediate between the case $g > 1$, where the weak coupling is irrelevant, and the case $g < 1$, where the weak coupling is relevant. Thus the exact solution for the case of noninteracting leads with $g = 1$ is in full agreement with the perturbative analysis presented in Sec. V. We find that for $g = 1$ there is no strong-strong fixed point, which again is in agreement with the analysis in Sec. VI, in which it was found that $g = 1$ separates the case of a stable strong-strong fixed point for repulsive lead interactions with $g < 1$ from the case of an unstable strong-strong fixed point for $g > 1$.

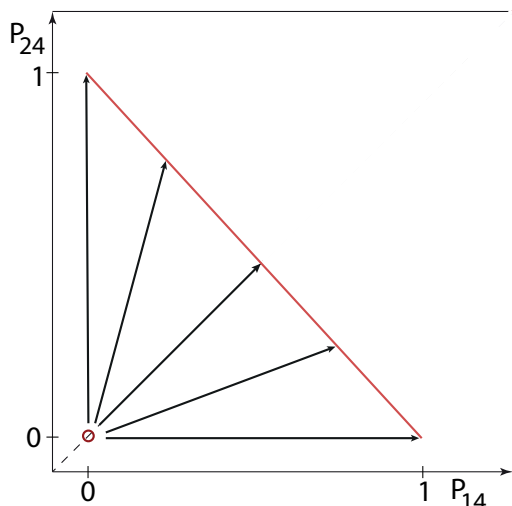


FIG. 10. (Color online) Description of the RG flow in terms of scattering probabilities P_{14} and P_{24} , which in the weak coupling limit are proportional to the square of tunneling amplitudes γ_h^2 and γ_v^2 , respectively. Near the RG-unstable weak-weak fixed point (red circle), there are perturbative corrections to the Gaussian $(1/2, 1/2)$ conductance in both γ_h and γ_v , which grow under the RG. The RG flow terminates on a line of fixed points (red line), which are characterized by scattering probabilities and a conductance which depend on the ratio γ_v/γ_h of bare scattering amplitudes.

IX. SUMMARY

In summary, we have considered a model of scattering between one-dimensional chiral leads in the presence of both inelastic and elastic scattering channels. We have found that this combination of scattering can stabilize a new $\frac{1}{2} - \frac{1}{2}$ fixed point, with probability $1/2$ for the transmission and probability $1/2$ for the reflection of an incoming particle. This $\frac{1}{2} - \frac{1}{2}$ fixed point is intermediate between the $1 - 0$ and $0 - 1$ fixed points in the presence of only elastic scattering channels. In order to establish the existence of this fixed point, we employed a nonperturbative instanton analysis in either one or both of the scattering channels. Our main result is that the intermediate fixed point is stable for Luttinger parameters $g < 1$ in the leads. This conclusion is backed up by a fermionization analysis for the special value $g = 1$. For noninteracting fermions with $g = 1$, we recover the well-known marginal relevance of scattering.

In general, even the cases with large γ_v (or large γ_h) that resemble the KF flow, are not identical to the KF model given the fact that there is an additional inelastic term in the Hamiltonian. This can be seen from the action (34) in Sec. V, where the prefactor of the dissipative term is twice that of KF due to the extra dissipation in our model. The fact that we recover the KF scaling is only due to the coupling to instantons in the field with the strong elastic term. We note that in those cases the stable fixed point one flows into, namely, point 3 in Fig. 6, differs from the KF stable fixed point by the presence of an extra dissipative channel. In short, our starting point is different from that of KF, even if certain parts of our RG flow diagram give rise to a scaling which is identical to that of KF.

While this $1/2 - 1/2$ fixed point is the main result of the present analysis, we recall that our previous paper [11] pointed out novel results concerning noise and current-current correlations. One possible realization of our inelastic puddle is to define some region of a quantum Hall strip at a compressible filling factor. That may be achieved with (inter alia) low density filling fractions. The latter have been shown to be achievable in the experiments of Rodarro *et al.* [30]. A conductance plateau at $1/2$, which is predicted for the stable fixed point discussed in this manuscript, has been observed by Nakaharai *et al.* [31]. Thus the experiment [31] on inhomogeneous n -type and p -type graphene may be a manifestation of our model, although further quantitative studies of that system are needed. The experimentally obtained transmission coefficients scale (with temperature) towards intermediate values, a phenomenon that has been interpreted as plateaux formation in the original paper. This is consistent with the simultaneous relevance of all scattering channels and the ensuing equilibration of transport coefficients found in our paper, although for $g = 1$ we do not predict the termination of the flow at a universal $(1/2, 1/2)$ beam splitter configuration. While the present paper focuses on dc conductance, we recall that our theory provides smoking gun predictions via a vis shot noise in such systems [11].

ACKNOWLEDGMENTS

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APPENDIX A: PROOF OF EQ. (16)

As usual, the path integral fixes operator ordering through time order. Consider, for example, two quasiparticle operators:

$$\begin{aligned} \Psi_i(\tau + \delta\tau)\Psi_j(\tau) &\rightarrow e^{i(\frac{\pi}{2} \sum_{i'} \alpha_{i'} p_{i'}(\tau + \delta\tau) - q_i(\tau + \delta\tau))} \\ &\times e^{i(\frac{\pi}{2} \sum_{j'} \alpha_{j'} p_{j'}(\tau) - q_j(\tau))} \\ &\times \tilde{\psi}_i(\tau + \delta\tau)\tilde{\psi}_j(\tau). \end{aligned} \quad (\text{A1})$$

The expression we want to compare with looks the same, except for an exchange of time arguments $(\tau - \delta\tau) \leftrightarrow \tau$. Now, imagine this expression inserted in the functional integral. We may always integrate over $\{q_i\}$, as these fields enter the action linearly (assuming a perturbative approach, where all Ψ_k are expanded out of the exponent). Integration over the q 's generates the step function profiles

$$p_k(\tau) = p_{k,0} + g \sum_a s_a \Theta(\tau - \tau_a),$$

where $p_{k,0}$ is the initial value, the sum runs over all appearances of q_k in the exponents, τ_a are the respective times, and $s_a = 1$ for a $-iq$ and -1 for a $+iq$ factor. Now, with these structures in place, we can explore the behavior of the exponent above. Denoting by $\tilde{\Phi}_{ij}$ the cumulative phase which will not respond to an exchange of the time arguments, we have

$$\begin{aligned} \Psi_i(\tau + \delta\tau)\Psi_j(\tau) &\rightarrow e^{i\tilde{\Phi}_{ij} + i\frac{\pi g}{2}\alpha_{ij}}, \\ \Psi_i(\tau)\Psi_j(\tau + \delta\tau) &\rightarrow e^{i\tilde{\Phi}_{ij} + i\frac{\pi g}{2}\alpha_{ji}}, \end{aligned}$$

where the phase in the first line reflects the fact that at time τ p_j jumped by $g/\sqrt{2}$ and this phase change is read out by the contribution $\alpha_{ij} p_j(\tau + \delta\tau)$ to the first phase. Similarly for the second line. The relative phase is given by

$$e^{i\frac{\pi g}{2}(\alpha_{ij}-\alpha_{ji})} = e^{i\pi g\alpha_{ij}},$$

as required.

APPENDIX B: DERIVATION OF THE EFFECTIVE ACTION (25)

In this section, we derive the effective three-variable action (25) from the original description in terms of twelve fields, ϕ_i, q_i, p_i . We start by casting the orthogonal transformation (23) into the more compact notation

$$\Phi = M\phi, \quad M = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}.$$

This transformation is orthogonal, $M^T = M^{-1}$, and thus the inverse transformation is given by $\phi = M^{-1}\Phi = M^T\Phi$. For the zero modes, this allows to define $p = M^{-1}P$ and $q = M^{-1}Q$ as above to obtain

$$Q = -2P_0 - \frac{\sqrt{2}}{\pi}\Phi_3. \quad (\text{B1})$$

Then, the zero-mode part of the scattering operators assumes the form

$$\begin{aligned} \bar{\Psi}_3\Psi_1 &= e^{i(-Q_2-Q_3)} e^{i\frac{\pi}{2}(-P_1+3P_0)} \bar{\psi}_3\psi_1 e^{-i\frac{\pi}{2}g}, \\ \bar{\Psi}_4\Psi_2 &= e^{i(+Q_2-Q_3)} e^{i\frac{\pi}{2}(+P_1-P_0)} \bar{\psi}_4\psi_2 e^{i\frac{\pi}{2}g}, \\ \bar{\Psi}_3\Psi_2 &= e^{i(+Q_1-Q_3)} e^{i\frac{\pi}{2}(-P_2+P_0)} \bar{\psi}_3\psi_2 e^{-i\frac{\pi}{2}g}, \\ \bar{\Psi}_4\Psi_1 &= e^{i(-Q_1-Q_3)} e^{i\frac{\pi}{2}(+P_2+P_0)} \bar{\psi}_4\psi_1 e^{-i\frac{\pi}{2}g}. \end{aligned} \quad (\text{B2})$$

The transformation $(q, p) \rightarrow (MQ, MP)$ is orthogonal, which means that the canonical piece of the action remains unaltered, as

$$\frac{i}{g} \int d\tau P_i d\tau Q_i. \quad (\text{B3})$$

In principle, there is also the zero-mode charging term $H \sim \sum_i P_i^2$ to consider, however, it does not play any role due to the prefactor of inverse contour length, which becomes zero in the thermodynamic limit.

We note that the scattering terms do not couple to Q_0 . This means that we may do the integral over Q_0 to conclude that $d_t P_0 = \text{const.}$: the total charge on the system is constant. Without loss of generality, we may call this constant 0. Our scattering terms thus simplify to

$$\begin{aligned} \bar{\Psi}_3\Psi_1 &= e^{i(-Q_2-Q_3)} e^{-i\frac{\pi}{2}P_1} \bar{\psi}_3\psi_1 e^{-i\frac{\pi}{2}g}, \\ \bar{\Psi}_4\Psi_2 &= e^{i(+Q_2-Q_3)} e^{+i\frac{\pi}{2}P_1} \bar{\psi}_4\psi_2 e^{i\frac{\pi}{2}g}, \\ \bar{\Psi}_3\Psi_2 &= e^{i(+Q_1-Q_3)} e^{-i\frac{\pi}{2}P_2} \bar{\psi}_3\psi_2 e^{-i\frac{\pi}{2}g}, \\ \bar{\Psi}_4\Psi_1 &= e^{i(-Q_1-Q_3)} e^{+i\frac{\pi}{2}P_2} \bar{\psi}_4\psi_1 e^{-i\frac{\pi}{2}g}. \end{aligned} \quad (\text{B4})$$

Next, ignoring the overall charging term, we observe that P_3 does not appear in the scattering operators, nor in the dot

charging term. This leads to the constraint $d_t Q_3 = 0$, and we may set $Q_3 = 0$ without loss of generality. As a result, we have the further simplification, down to

$$\bar{\Psi}_3\Psi_1 = e^{-iQ_2} e^{-i\frac{\pi}{2}P_1} \bar{\psi}_3\psi_1 e^{-i\frac{\pi}{2}g}, \quad (\text{B5a})$$

$$\bar{\Psi}_4\Psi_2 = e^{+iQ_2} e^{+i\frac{\pi}{2}P_1} \bar{\psi}_4\psi_2 e^{i\frac{\pi}{2}g}, \quad (\text{B5b})$$

$$\bar{\Psi}_3\Psi_2 = e^{+iQ_1} e^{-i\frac{\pi}{2}P_2} \bar{\psi}_3\psi_2 e^{-i\frac{\pi}{2}g}, \quad (\text{B5c})$$

$$\bar{\Psi}_4\Psi_1 = e^{-iQ_1} e^{+i\frac{\pi}{2}P_2} \bar{\psi}_4\psi_1 e^{-i\frac{\pi}{2}g}. \quad (\text{B5d})$$

We note that the zero-mode sectors of the scattering operators in Eqs. (B5a), (B5b) and Eqs. (B5c), (B5d) now pairwise commute among themselves. However, those in Eqs. (B5a), (B5b) and Eqs. (B5c), (B5d) do not commute. This structure suggests a further simplification: from the commutation relation

$$[Q_i, P_j] = gi\delta_{ij} \quad (\text{B6})$$

we compute

$$\left[Q_2 + \frac{\pi}{2}P_1, Q_1 - \frac{\pi}{2}P_2 \right] = -ig\pi. \quad (\text{B7})$$

Now this relation motivates the canonical transformation

$$\begin{aligned} X &= \frac{Q_2}{\sqrt{\pi}} + \frac{\sqrt{\pi}}{2}P_1, & Y &= -\frac{Q_1}{\sqrt{\pi}} + \frac{\sqrt{\pi}}{2}P_2, \\ X' &= \frac{Q_1}{\sqrt{\pi}} + \frac{\sqrt{\pi}}{2}P_2, & Y' &= -\frac{Q_2}{\sqrt{\pi}} + \frac{\sqrt{\pi}}{2}P_1. \end{aligned} \quad (\text{B8})$$

The new variables obey the relations $[X, Y] = ig$ and the same for X', Y' . The primed and unprimed variables are mutually commutative. All this means that the canonical piece of the action remains invariant. In particular, we have a contribution

$$S[X, Y] = \frac{i}{g} \int d\tau Y d_t X. \quad (\text{B9})$$

The scattering operators assume the form

$$\begin{aligned} \bar{\Psi}_3\Psi_1 &= e^{-i\sqrt{\pi}X} \bar{\psi}_3\psi_1 e^{-i\frac{\pi}{2}g}, \\ \bar{\Psi}_4\Psi_2 &= e^{i\sqrt{\pi}X} \bar{\psi}_4\psi_2 e^{i\frac{\pi}{2}g}, \\ \bar{\Psi}_3\Psi_2 &= e^{-i\sqrt{\pi}Y} \bar{\psi}_3\psi_2 e^{-i\frac{\pi}{2}g}, \\ \bar{\Psi}_4\Psi_1 &= e^{i\sqrt{\pi}Y} \bar{\psi}_4\psi_1 e^{-i\frac{\pi}{2}g}, \end{aligned} \quad (\text{B10})$$

which means that we do not need the variables X', Y' , and do not consider them in the following.

Using the canonical representation (23) and assuming locking of the charge mode we have

$$\begin{aligned} \bar{\Psi}_3\Psi_1 &\sim e^{i(-\sqrt{\pi}X + \sqrt{2}\Phi_2)}, \\ \bar{\Psi}_3\Psi_2 &\sim e^{i(-\sqrt{\pi}Y - \sqrt{2}\Phi_1)}, \end{aligned} \quad (\text{B11})$$

where we have absorbed the $\exp(\pm i\sqrt{\pi}/2g)$ factors in the scattering phases γ_{13} and γ_{23} .

The structure of the scattering operators suggests a shift

$$\begin{aligned}\Phi_2 &\rightarrow \Phi_2 + \sqrt{\frac{\pi}{2}}X, \\ \Phi_1 &\rightarrow \Phi_1 - \sqrt{\frac{\pi}{2}}Y,\end{aligned}\tag{B12}$$

whereupon the scattering operators simplify to

$$\begin{aligned}\bar{\Psi}_3\Psi_1 &\sim e^{i\sqrt{2}\Phi_2}, \\ \bar{\Psi}_3\Psi_2 &\sim e^{-i\sqrt{2}\Phi_1}.\end{aligned}\tag{B13}$$

As a result of this shift, the fields Φ_1 and Φ_2 no longer commute with each other, and we have

$$[\Phi_2, \Phi_1] = ig\frac{\pi}{2}.\tag{B14}$$

Now we are in a position to express the local effective action in terms of variables which allow an efficient analysis of the competing scattering processes. The dissipative part of the action, obtained by integrating out the finite wave vector modes of the bosonic fields, is given by

$$S_{\text{diss}}[\Phi] = \frac{T}{\pi g} \sum_{i=0}^2 \sum_n |\omega_n| |\Phi_{i,n}|^2,\tag{B15}$$

where $\Phi_{i,n}$ denotes the n th Matsubara component of the field Φ_i . The sum runs only up to $i = 2$, on account of the locking of the field Φ_3 . The $i = 0$ contribution to the sum defines the zero-mode action (24), and will be ignored throughout. Introducing the shorthand notation

$$\Phi = (\Phi_1, \Phi_2)^T, \quad \Xi = (X, Y)^T,\tag{B16}$$

the quadratic part of the action then assumes the form

$$\begin{aligned}S_0[X, \Phi] &= \frac{T}{\pi g} \sum_n \Phi_n^T \Phi_{-n} |\omega_n| - \frac{T}{g} \frac{2}{\pi} \sum_n \Phi_n^T \sigma_3 \Xi_{-n} |\omega_n| \\ &\quad + \frac{T}{2g} \sum_n \Xi_n^T \begin{pmatrix} |\omega_n| & \omega_n \\ -\omega_n & |\omega_n| \end{pmatrix} \Xi_{-n}.\end{aligned}\tag{B17}$$

The coupling between Φ and Ξ makes sure that the commutation relation Eq. (B14) is reproduced in correlation functions of the Φ_i . Since the vector variable Ξ no longer appears in the scattering part of the action, it can be integrated out. Doing the Gaussian integral, we obtain the action $S_0[\Phi_1, \Phi_2]$ as given in Eq. (25).

We finally note that the two ‘‘vertical’’ scattering operators $\sim \bar{\Psi}_3\Psi_1/\bar{\Psi}_4\Psi_2$ (and similarly for the horizontals) may couple at arbitrary strength/scattering phase. This leads us to $S[\Phi] = S_0[\Phi_1, \Phi_2] + S_v[\Phi_2] + S_h[\Phi_1]$, where the scattering action reads [cf. Eq. (19)]

$$\begin{aligned}S_v[\Phi_2] &= \sum_{x=13,24} \int d\tau |\gamma_x| \cos(\sqrt{2}\Phi_2 + \phi_x), \\ S_h[\Phi_1] &= \sum_{x=14,23} \int d\tau |\gamma_x| \cos(\sqrt{2}\Phi_1 + \phi_x).\end{aligned}\tag{B18}$$

Here, the phases ϕ_{ij} absorb the phase of the complex scattering amplitudes γ_{ij} , the phases $\exp(i\pi g/2)$ appearing in (B17), and

the relative sign change of the field variables in 13 versus 24 [cf. Eq. (B10)].

Without loss of generality, we can use an addition theorem for trigonometric functions followed by a constant shift of the fields Φ_1 and Φ_2 to transform the scattering actions into the form given in Eq. (25), with $\gamma_v = |\gamma_{13} + e^{-i\pi g} \gamma_{24}|$ and $\gamma_h = |\gamma_{23} + \gamma_{14}|$.

APPENDIX C: NONCHIRAL LUTTINGER LIQUIDS

Throughout the analysis presented in this paper, we have assumed a setup consisting of a quantum dot connected to four chiral wires, which are geometrically separated from each other. These chiral wires talk to each other through the tunneling to the QD (and the charging interaction thereon), and through elastic tunneling terms between the incoming and the outgoing channels. Such channels can be realized, for example, as the edges of a fractional quantum Hall strip; the most relevant tunneling operators are those of fractionally charged quasiparticles (the latter possess fractional statistics as well), and we have assumed that such tunneling terms are allowed. While intra(chiral)channel interaction is allowed, interchannels interaction is excluded.

When it comes to realizing our theory with nonchiral Luttinger liquid wires, the situation is trickier. There are two main issues that should be noted. First, throughout our analysis, we have assumed that forward and backward (elastic) scattering are treated on equal footing. There is no concrete significance associated with forward or backward. When it comes to nonchiral Luttinger wires, unless special conditions are specified, forward (elastic) scattering is irrelevant, and is clearly distinct from backward scattering. Second, as was noted above, in the case of chiral edges supported by an incompressible fractional quantum Hall electron gas, it is clear that tunneling of quasiparticles is allowed. The situation with nonchiral Luttinger wires is different, however. In the analysis depicted below we allow only for electron tunneling; in general the results of the ensuing analysis will be qualitatively different. However, for a specific choice of the interaction [compare Eq. (C10)], our beam splitter can be realized using nonchiral Luttinger liquid wires.

Let us briefly review these two types of processes. We briefly repeat the analysis of Ref. [32]. Consider the Luttinger liquid Hamiltonian

$$H = \frac{v}{2\pi} [K(\partial_x\phi)^2 + K^{-1}(\partial_x\theta)^2],\tag{C1}$$

where the bosonic fields satisfy a Kac-Moody algebra

$$[\phi(x), \theta(x')] = \frac{i\pi}{2} \text{sgn}(x - x').\tag{C2}$$

Here, K is the Luttinger liquid interaction parameter. One can define left and right moving chiral fields, $\Phi_{R/L} = \phi \pm \theta$, and respective electronic field operators $\psi_{R/L} = e^{\pm ik_F x} e^{i\Phi_{R/L}}$, with commutation relations $[\Phi_R(x), \Phi_R(x')] = -[\Phi_L(x), \Phi_L(x')] = i\pi \text{sgn}(x - x')$. In terms of these chiral modes, the Hamiltonian reads

$$H = \pi v_0 [\rho_R^2 + \rho_L^2 + 2\lambda_{\rho R} \rho_L],\tag{C3}$$

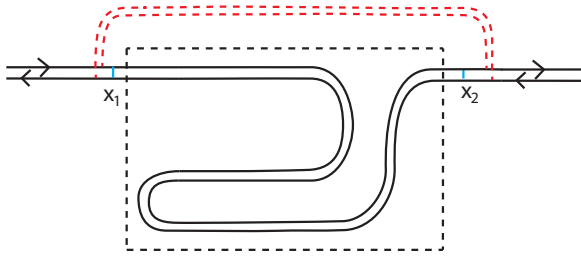


FIG. 11. (Color online) It is not possible to realize a beam splitter with two standard nonchiral LLs because backscattering between left- and right-moving electrons in the same LL (short blue lines) is relevant, whereas forward scattering across the dot (dashed red lines) is irrelevant. The part of the wires inside the dashed box constitutes the quantum dot with a nonlocal charging interaction.

with an inter-channel interaction (between the two chiral modes). Here,

$$\rho_{R/L}(x) = \pm \frac{1}{2\pi} \partial_x \Phi_{R/L}. \quad (\text{C4})$$

One can define other modes, $\varphi_{R/L} = K\phi \pm \theta$, in terms of which the Hamiltonian decouples into “left” and “right” sectors. These new modes have commutation relations $[\varphi_R(x), \varphi_R(x')] = -[\varphi_L(x), \varphi_L(x')] = i\pi K \text{sgn}(x - x')$. The operators $e^{i\varphi_{R/L}}$ are field operators of chiral quasiparticles that carry charge Ke . However, experimentally realizable tunneling processes can only involve electrons, and in the following, we will discuss the RG relevance of such processes.

1. Interactions within one LL only

A sketch of a possible realization is shown in Fig. 11. It is well known that a local potential (or, in the microscopic model, a weak bond) gives rise to backscattering and is a relevant perturbation in the RG sense. More problematic is the microscopic realization of an RG-relevant forward scattering term, which coherently transfers electrons across the quantum dot (dashed red lines in Fig. 1). In order to compute the scaling dimension of such a forward scattering term, we describe the LL by chiral bosonic eigenstates φ_{\pm} with imaginary time action

$$S_{L/R} = \frac{u}{4\pi K} \int dx d\tau \partial_x \varphi_{R/L} (\pm i \partial_\tau + \partial_x) \varphi_{R/L}. \quad (\text{C5})$$

Here, u denotes the renormalized velocity, K the LL parameter, and the smooth part of the total electron density is given by $\rho(x) = \frac{1}{2\pi} (\partial_x \varphi_R + \partial_x \varphi_L)$. In this basis, right and left moving electrons (i.e., electronic states at the left and right Fermi point of the noninteracting system) have the bosonized form

$$\psi_{R/L}(x) = \frac{1}{\sqrt{2\pi a}} e^{i[K_{\pm} \varphi_R(x) - K_{\mp} \varphi_L(x)]}, \quad (\text{C6})$$

with $K_{\pm} = (K^{-1} \pm 1)/2$. Thus a local backscattering operator is given by

$$\begin{aligned} \hat{O}(x_1) &= \psi_R^\dagger(x_1) \psi_L(x_1) \\ &= \frac{1}{2\pi a} e^{i(K_+ - K_-)[\varphi_R(x_1) + \varphi_L(x_1)]}. \end{aligned} \quad (\text{C7})$$

Using $K_+ - K_- = 1$, we reproduce the well known result that the correlation function $\langle \hat{O}_{\text{back}}(t) \hat{O}_{\text{back}}(0) \rangle \sim t^{-2K}$, which

makes backscattering a relevant perturbation for a repulsive interaction with $K < 1$. On the other hand, a forward scattering term involves electron operators at two different spatial positions:

$$\begin{aligned} \hat{O}_f(x_1, x_2) &= \psi_R^\dagger(x_2) \psi_R(x_1) \\ &= \frac{1}{2\pi a} e^{iK_+[\varphi_R(x_1) - \varphi_R(x_2)] - iK_-[\varphi_L(x_1) - \varphi_L(x_2)]}. \end{aligned} \quad (\text{C8})$$

Since positions x_1 and x_2 are separated by an “infinitely long” dot region, correlation functions between fields at positions x_1 and x_2 vanish. As a consequence, the correlation function in time of the forward scattering operator factorizes according to

$$\begin{aligned} &\langle \hat{O}_f(t) \hat{O}_f(0) \rangle \\ &\propto e^{-\frac{1}{2} K_+^2 ([\varphi_R(x_1, t) - \varphi_R(x_1, 0)]^2) - \frac{1}{2} K_+^2 ([\varphi_R(x_2, t) - \varphi_R(x_2, 0)]^2)} \\ &\quad \times e^{-\frac{1}{2} K_-^2 ([\varphi_L(x_1, t) - \varphi_L(x_1, 0)]^2) - \frac{1}{2} K_-^2 ([\varphi_L(x_2, t) - \varphi_L(x_2, 0)]^2)} \\ &\propto t^{-K(2K_+^2 + 2K_-^2)} \\ &\propto t^{-(K^{-1} + K)}. \end{aligned} \quad (\text{C9})$$

Since the function $f(K) = K^{-1} + K$ has its minimum value 2 at $K = 1$, the forward scattering operator Eq. (C8) seems to be irrelevant for all values of K . As a consequence, finding a microscopic description of an RG relevant forward scattering operator requires a modification of the interaction term.

2. Local interactions between both nonchiral Luttinger liquids

Here, we want to allow for the possibility that there is a local interaction term including both LL wires as shown in Fig. 12. Following the notation of Wen [33], we describe the system by a K matrix with $K = \text{diag}(1, -1, 1, -1)$, where we imagine that the first two branches belong to the first LL with fields $\varphi_{1R,0}$ and $\varphi_{1L,0}$, and the second two branches belonging to the second LL. These are eigenmodes in the absence of interactions. A local charging term acting on both LL wires, i.e., equally on all four modes is described by the velocity

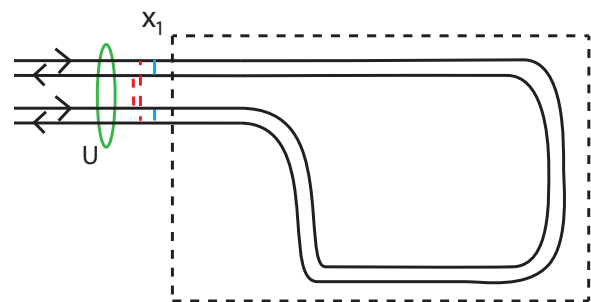


FIG. 12. (Color online) In order to realize a beam splitter with a nonchiral setup, two LLs with a local charging interaction which does not discriminate between the two wires is needed (indicated by green oval, strength $U > 0$). In this setup, both intrawire backscattering (short blue lines) and interwire backscattering (dashed red lines) are relevant, whereas interwire forward scattering (not shown in the figure) is irrelevant. The part of the wires inside the dashed boxed constitutes the quantum dot with a nonlocal charging interaction.

matrix

$$V = \begin{pmatrix} 1+U & U & U & U \\ U & 1+U & U & U \\ U & U & 1+U & U \\ U & U & U & 1+U \end{pmatrix}. \quad (\text{C10})$$

We now diagonalize the real symmetric velocity matrix by an orthogonal transformation according to $V \rightarrow V_1 = AVA^T$ with

$$A = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1/2 & 1/2 & -1/2 & -1/2 \end{pmatrix}. \quad (\text{C11})$$

We obtain $V_1 = \text{diag}(1+4U, 1, 1, 1)$. In the following, we introduce the charge mode velocity

$$u = 1 + 4U. \quad (\text{C12})$$

Next, we rescale fields such that the velocity matrix becomes equal to the unit matrix, i.e., $V_2 = \Lambda V_1 \Lambda$ with $\Lambda = \text{diag}(1/\sqrt{u}, 1, 1, 1)$. With the help of these transformations, we find $K_2 = \Lambda A K A^T \Lambda$ with

$$K_2 = \begin{pmatrix} 0 & 1/\sqrt{2u} & 1/\sqrt{2u} & 0 \\ 1/\sqrt{2u} & 0 & 0 & 1/\sqrt{2} \\ 1/\sqrt{2u} & 0 & 0 & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}. \quad (\text{C13})$$

Since K_2 is real symmetric, it can be diagonalized by an orthogonal transformation according to $K_d = BK_2B^T$ with

$$B = \begin{pmatrix} -1/\sqrt{2} & 1/2 & 1/2 & 0 \\ 1/\sqrt{2} & 1/2 & 1/2 & 0 \\ 0 & -1/2 & 1/2 & 1/\sqrt{2} \\ 0 & 1/2 & -1/2 & 1/\sqrt{2} \end{pmatrix}, \quad (\text{C14})$$

and we obtain $K_d = \text{diag}(-1/\sqrt{u}, 1/\sqrt{u}, -1, 1)$. The entries of K_d are the inverse of the scaling dimensions of the new fields. Through this series of transformations, the new fields $\phi_1, \phi_2, \phi_3, \phi_4$ are related to the original fields according to the transformation $\phi = B\Lambda^{-1}A\varphi$. In order to compute scaling dimensions of tunneling operators, we need the inverse transformation, which is given by

$$C \equiv (B\Lambda A)^{-1} = A^T \Lambda B^T = \begin{pmatrix} \frac{-1+\sqrt{u}}{2\sqrt{2u}} & \frac{1+\sqrt{u}}{2\sqrt{2u}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1+\sqrt{u}}{2\sqrt{2u}} & \frac{1-\sqrt{u}}{2\sqrt{2u}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1+\sqrt{u}}{2\sqrt{2u}} & \frac{1+\sqrt{u}}{2\sqrt{2u}} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1+\sqrt{u}}{2\sqrt{2u}} & \frac{1-\sqrt{u}}{2\sqrt{2u}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}. \quad (\text{C15})$$

Using the matrix C , we can express the original fields in terms of the new ones according to

$$\varphi = C\phi. \quad (\text{C16})$$

In order to compute the scaling dimension of the operator for backscattering a right mover in wire one into a left mover in

wire two, we use the expression

$$\varphi_{1R} + \varphi_{2L} = -\frac{1}{\sqrt{2u}}\phi_1 + \frac{1}{\sqrt{2u}}\phi_2 - \frac{1}{\sqrt{2}}\phi_3 + \phi_4, \quad (\text{C17})$$

and find for this tunneling operator the scaling dimension

$$g_{1R,2L} = \frac{1}{2} \left(\sqrt{u} \frac{1}{2u} + \sqrt{u} \frac{1}{2u} + \frac{1}{2} + \frac{1}{2} \right) = \frac{1}{2} \left(\frac{1}{\sqrt{u}} + 1 \right). \quad (\text{C18})$$

In the presence of a repulsive interaction $U > 0$, the charge mode velocity $u > 1$, and hence $g_{1R,2L} < 1$ makes interwire backscattering relevant. Similarly, we find that intra-wire backscattering is relevant with $g_{1R,1L} = g_{1R,2L}$. Thus this model allows to realize the intermediate fixed point discussed in the main text in the framework of nonchiral LL wires.

APPENDIX D: SOLUTION OF THE REFERMIONIZED MODEL FOR $g = 1$

In this Appendix, we discuss the derivation of the scattering probabilities Eqs. (47a) and (47b).

1. Solution for a single scattering amplitude

We first consider the simple situation where $\gamma_v = 0$ and only $\gamma_h \neq 0$. In order to construct scattering states as eigenstates of the Hamiltonian, we need to make assumptions about the commutators of $\eta_1\eta_4$ with f_1 and Ψ_1 . If $\eta_1\eta_4$ commutes with both f_1 and Ψ_1 , we will arrive at the standard solution of refermionization [29]. As there is no reason to expect that our result should differ from the standard one, we will make this assumption. Then, we define the new Majorana operator $\tilde{f}_1 = i\eta_1\eta_4 f_1$, and with $H = H_0 + H_{\text{tun}}$ find the following equations of motion:

$$-i\partial_t \Psi_1(x) = [H, \Psi_1(x)] = i\partial_x \Psi_1(x) + \sqrt{2\pi a} \gamma_h \tilde{f}_1 \delta(x) \quad (\text{D1a})$$

$$-i\partial_t \Psi_1^\dagger(x) = [H, \Psi_1^\dagger(x)] = i\partial_x \Psi_1^\dagger(x) - \sqrt{2\pi a} \gamma_h \tilde{f}_1 \delta(x) \quad (\text{D1b})$$

$$-i\partial_t \tilde{f}_1 = [H, \tilde{f}_1] = 2\sqrt{2\pi a} \gamma_h [\Psi_1(0) - \Psi_1^\dagger(0)]. \quad (\text{D1c})$$

Away from $x = 0$, $\Psi_1(x)$ satisfies the free fermion equation and has plane waves as a solution. At $x = 0$, the field is discontinuous and acquires a phase shift. As a consequence, in the above equations of motion, $\Psi(0)$ needs to be interpreted as $\frac{1}{2}[\Psi_1(0+) + \Psi_1(0-)]$. We solve the equations of motions by using the most general expression for scattering eigenstates:

$$A_{1,\epsilon}^\dagger = \int_x (\varphi_{11,\epsilon} \Psi_1^\dagger + \varphi_{14,\epsilon}^* \Psi_1 + a_{11,\epsilon} \tilde{f}_1). \quad (\text{D2})$$

For the wave functions, we make the ansatz

$$\varphi_{11,\epsilon}(x) = \theta(-x)e^{i\epsilon x} + \theta(x)e^{i\delta_{11}}e^{i\epsilon x}, \quad (\text{D3a})$$

$$\varphi_{14,\epsilon}^*(x) = \theta(x)e^{i\delta_{14}}e^{i\epsilon x}. \quad (\text{D3b})$$

Following Ref. [29], we interpret the scattering states in the following way: the region $x < 0$ corresponds to the incoming wave packet, the region $x > 0$ the scattered components of the wave packet. The wave function φ_{11} multiplying the creation operator ψ^\dagger describes the amplitude for forward scattering, i.e., is associated with the component of the scattering state which continues to propagate along segment 1 of the edge, in the interior of the QD. The wave function φ_{14}^* multiplying the annihilation operator ψ_1 on the other hand is interpreted as the amplitude for scattering onto the outgoing part of edge segment 4. Clearly, unitarity demands that the sum of the probabilities for forward and backwards scattering is equal to one, $|\varphi_{11}|^2 + |\varphi_{14}|^2 = 1$.

Using the ansatz Eq. (D3b) in the equations of motion Eqs. (D1a)–(D1c), we find

$$i[1 - e^{i\delta_{11}}] - 2\sqrt{2\pi a}\gamma_h a_{11} = 0, \quad (\text{D4a})$$

$$-ie^{i\delta_{14}} + 2\sqrt{2\pi a}\gamma_h a_{11} = 0, \quad (\text{D4b})$$

$$\epsilon a_{11} = \frac{\sqrt{2\pi a}}{2}\gamma_h[-(1 + e^{i\delta_{11}}) + e^{i\delta_{14}}], \quad (\text{D4c})$$

and finally the solutions

$$e^{i\delta_{11}} = \frac{i\epsilon}{i\epsilon - 4\pi a\gamma_h^2}, \quad (\text{D5a})$$

$$e^{i\delta_{14}} = \frac{-4\pi a\gamma_h^2}{i\epsilon - 4\pi a\gamma_h^2}. \quad (\text{D5b})$$

We note that, different from the case of truly noninteracting fermions, $|\varphi_{11,\epsilon}|^2$ and $|\varphi_{14,\epsilon}|^2$ cannot be interpreted as scattering probabilities for each value of energy separately. Instead, in order to obtain the scale dependent scattering probability, they need to be integrated over the derivative of the Fermi function to obtain

$$P_{14} = \int d\omega(-f'(\omega))|\varphi_{14,\omega}|^2 \approx \frac{2\Gamma_h}{T} \arctan \frac{T}{2\Gamma_h}. \quad (\text{D6})$$

In the last step, the derivative of the Fermi function was approximated as a box function of width temperature T and height $1/T$. Identifying the width of the resonance as $\Gamma_h = 4\pi a\gamma_h^2$, and introducing the scaling function $F(x) = \frac{2}{x} \arctan \frac{x}{2}$, this result can be written in a more compact form as $\tilde{P}_{14} = F(T/\Gamma)$. The scaling function has the limiting

behaviors $F(x) = 1 - \frac{1}{3}(x/2)^2$ for small x and $F(x) = \frac{2}{x}$ for large x . Using these asymptotics, we find

$$P_{14} = \begin{cases} \frac{2\Gamma_h}{T} & \text{for } T \gg \Gamma_h \\ 1 - \frac{T^2}{12\Gamma_h^2} & \text{for } T \ll \Gamma_h \end{cases}. \quad (\text{D7})$$

This result agrees with the standard perturbative analysis of a single scatterer in a LL.

We now discuss a complementary scenario, in which $\gamma_h = 0$ and $\gamma_v \neq 0$. By symmetry, it is clear that the solution can be found in an analogous way as discussed above. For future reference, the solution for the scattering eigenstate is given by

$$A_{2,\epsilon}^\dagger = \int_x (\varphi_{22,\epsilon}\Psi_2^\dagger + \varphi_{44,\epsilon}^*\Psi_2 + a_{22,\epsilon}\tilde{f}_1). \quad (\text{D8})$$

Now, for $x < 0$, the operator ψ_2^\dagger describes an incoming state along contour 2. For $x > 0$, the operator ψ_2^\dagger creates a state which is forward scattered into the interior of the QD along contour 2, whereas ψ_2 describes a partial wave which is scattered onto the outgoing part of edge 4. In what follows, it will be important that both $\psi_1(x)$ and $\psi_2(x)$ create an outgoing wave on edge 4.

2. Solution for two competing scattering amplitudes

We now consider the interesting case in which both γ_h and γ_v are nonzero. Due to the Klein factors represented by products of Majorana fermions $\eta_1\eta_4$ and $\eta_2\eta_4$, the two different scattering operators do not commute with each other, and the respective equations of motion are coupled. In addition to $\tilde{f}_1 = i\eta_1\eta_4 f_1$, we now introduce $\tilde{f}_2 = i\eta_2\eta_4 f_1$. Now we need to take into account the nontrivial anticommutator

$$\{\tilde{f}_1, \tilde{f}_2\} = -2\eta_1 f_1 \eta_2 f_2 \equiv -2\hat{C}, \quad (\text{D9})$$

where we have made the assumption that f_1 and f_2 commute with each other and with $\eta_1\eta_4$ and $\eta_2\eta_4$. Since \hat{C} is a product of an even number of Majoranas, it trivially commutes with Ψ_1 , Ψ_1^\dagger , Ψ_2 , and Ψ_2^\dagger . In addition,

$$[\tilde{f}_1, \hat{C}] = 0, \quad \text{and} \quad [\tilde{f}_2, \hat{C}] = 0 \quad \Rightarrow \quad [H_{\text{tun}}, \hat{C}] = 0. \quad (\text{D10})$$

As \hat{C} commutes with all operators, it commutes with the Hamiltonian and hence does not have any dynamics. In addition, it squares to one $\hat{C}^2 = 1$. With this in mind, we find the following equations of motion for the full system:

$$-i\partial_t \Psi_1(x) = [H, \Psi_1(x)] = i\partial_x \Psi_1(x) + \sqrt{2\pi a}\gamma_h \tilde{f}_1 \delta(x), \quad (\text{D11a})$$

$$-i\partial_t \Psi_1^\dagger(x) = [H, \Psi_1^\dagger(x)] = i\partial_x \Psi_1^\dagger(x) - \sqrt{2\pi a}\gamma_h \tilde{f}_1 \delta(x), \quad (\text{D11b})$$

$$-i\partial_t \tilde{f}_1 = [H, \tilde{f}_1] = 2\sqrt{2\pi a}[\gamma_h \Psi_1(0) - \gamma_h \Psi_1^\dagger(0) - \hat{C}\gamma_v \Psi_2(0) + \hat{C}\gamma_v \Psi_2^\dagger(0)], \quad (\text{D11c})$$

$$-i\partial_t \Psi_2(x) = [H, \Psi_2(x)] = i\partial_x \Psi_2(x) + \sqrt{2\pi a}\gamma_v \tilde{f}_2 \delta(x), \quad (\text{D11d})$$

$$-i\partial_t \Psi_2^\dagger(x) = [H, \Psi_2^\dagger(x)] = i\partial_x \Psi_2^\dagger(x) - \sqrt{2\pi a}\gamma_v \tilde{f}_2 \delta(x), \quad (\text{D11e})$$

$$-i\partial_t \tilde{f}_2 = [H, \tilde{f}_2] = 2\sqrt{2\pi a}[\gamma_v \Psi_2(0) - \gamma_v \Psi_2^\dagger(0) - \hat{C}\gamma_h \Psi_1(0) + \hat{C}\gamma_h^* \Psi_1^\dagger(0)]. \quad (\text{D11f})$$

In the following, we will derive scattering states as a solution of these equations by using the fact that $\hat{C}^2 = 1$. We make the most general ansatz for operators

$$A_{1,\epsilon}^\dagger = \int_x (\varphi_{11,\epsilon} \Psi_1^\dagger + \varphi_{14,\epsilon}^* \Psi_1 + \varphi_{12,\epsilon} \Psi_2^\dagger + \tilde{\varphi}_{14,\epsilon}^* \Psi_2 + a_{11,\epsilon} \tilde{f}_1 + a_{12,\epsilon} \tilde{f}_2), \quad (\text{D12a})$$

$$A_{2,\epsilon}^\dagger = \int_x (\varphi_{22,\epsilon} \Psi_2^\dagger + \varphi_{24,\epsilon}^* \Psi_2 + \varphi_{21,\epsilon} \Psi_1^\dagger + \tilde{\varphi}_{24,\epsilon}^* \Psi_1 + a_{22,\epsilon} \tilde{f}_2 + a_{21,\epsilon} \tilde{f}_1) \quad (\text{D12b})$$

creating eigenstates of the Hamiltonian H and satisfying

$$\epsilon A_{1,\epsilon}^\dagger = [H, A_{1,\epsilon}^\dagger], \quad \epsilon A_{2,\epsilon}^\dagger = [H, A_{2,\epsilon}^\dagger]. \quad (\text{D13})$$

In the following, we only discuss the expression for $A_{1,\epsilon}^\dagger$, as the corresponding expression for $A_{2,\epsilon}^\dagger$ is obtained by interchanging γ_h with γ_v . Specifically, we make the ansatz

$$\varphi_{11,\epsilon}(x) = \theta(-x)e^{i\epsilon x} + \theta(x)e^{i\delta_{11}}e^{i\epsilon x}, \quad (\text{D14a})$$

$$\varphi_{14,\epsilon}^*(x) = \theta(x)e^{i\delta_{14}}e^{i\epsilon x}, \quad (\text{D14b})$$

$$\varphi_{12,\epsilon}(x) = \theta(x)e^{i\delta_{12}}e^{i\epsilon x}, \quad (\text{D14c})$$

$$\tilde{\varphi}_{14,\epsilon}^*(x) = \theta(x)e^{i\delta_{14}}e^{i\epsilon x}. \quad (\text{D14d})$$

Imposing the condition (D13), the equations

$$i[1 - e^{i\delta_{11}}] - 2\sqrt{2\pi a} \gamma_h [a_{11,\epsilon} - \hat{C}a_{12,\epsilon}] = 0, \quad (\text{D15a})$$

$$-ie^{i\delta_{14}} + 2\sqrt{2\pi a} \gamma_h [a_{11,\epsilon} - \hat{C}a_{12,\epsilon}] = 0, \quad (\text{D15b})$$

$$-ie^{i\delta_{12}} - 2\sqrt{2\pi a} \gamma_v [a_{12,\epsilon} - \hat{C}a_{11,\epsilon}] = 0, \quad (\text{D15c})$$

$$-ie^{i\delta_{14}} + 2\sqrt{2\pi a} \gamma_v [a_{12,\epsilon} - \hat{C}a_{11,\epsilon}] = 0, \quad (\text{D15d})$$

$$\epsilon a_{11,\epsilon} = \frac{\sqrt{2\pi a}}{2} [-(1 + e^{i\delta_{11}})\gamma_h + e^{i\delta_{14}}\gamma_h], \quad (\text{D15e})$$

$$\epsilon a_{12,\epsilon} = \frac{\sqrt{2\pi a}}{2} [-e^{i\delta_{12}}\gamma_v + e^{i\delta_{14}}\gamma_v] \quad (\text{D15f})$$

are found. Using $\hat{C}^2 = 1$, we see that the first four equations only depend on the combination $a_{11,\epsilon} - \hat{C}a_{12,\epsilon}$, and if we take

an appropriate linear combination of Eqs. (D15e) and (D15f), we can solve for

$$e^{i\delta_{11}} = \frac{i\epsilon - 4\pi a \gamma_h^2}{i\epsilon - 4\pi a (\gamma_h^2 + \gamma_v^2)}, \quad (\text{D16a})$$

$$e^{i\delta_{14}} = \frac{-4\pi a \gamma_h^2}{i\epsilon - 4\pi a (\gamma_h^2 + \gamma_v^2)}, \quad (\text{D16b})$$

$$e^{i\delta_{12}} = \frac{-4\pi a \gamma_h \gamma_v}{i\epsilon - 4\pi a (\gamma_h^2 + \gamma_v^2)}, \quad (\text{D16c})$$

$$e^{i\delta_{14}} = \frac{4\pi a \gamma_h \gamma_v}{i\epsilon - 4\pi a (\gamma_h^2 + \gamma_v^2)}. \quad (\text{D16d})$$

The probability for forward scattering is given by $|e^{i\delta_{11}}|^2$, the probability for scattering onto lead 2 is given by $|e^{i\delta_{12}}|^2$, and the probability for scattering onto lead 4 is given by $|e^{i\delta_{14}}|^2 + |e^{i\delta_{14}}|^2$. There are two contributions for scattering onto lead 4, one due to the operator ψ_1 in the scattering state Eq. (D12a), and a second one due to the operator ψ_2 in the same scattering state. We will see in a moment that indeed both these contributions are needed in order to reproduce the perturbative result for the tunneling probability from lead 1 onto lead 4.

Integrating over the derivative of the Fermi function as in the case of a single scatterer, we obtain the probabilities for transmission:

$$P_{11} = \frac{\Gamma_v^2}{\Gamma^2} F\left(\frac{T}{\Gamma}\right), \quad (\text{D17a})$$

$$P_{14} = \frac{\Gamma_h}{\Gamma} F\left(\frac{T}{\Gamma}\right), \quad (\text{D17b})$$

$$P_{12} = \frac{\Gamma_h \Gamma_v}{\Gamma^2} F\left(\frac{T}{\Gamma}\right). \quad (\text{D17c})$$

We now argue that these results agree with a perturbative calculation in γ_h and γ_v . In the limit of large T/Γ , one finds $P_{14} = 2\Gamma_h/T$, exactly the same result as in the case $\gamma_v = 0$ discussed previously. This is to be expected, since corrections due to γ_v can enter only in an additive fashion, and have to be of order $O(\gamma_h^2 \gamma_v^2)$. On the other hand, in the same limit, one finds $P_{12} = 2\Gamma_h \Gamma_v / \Gamma T$, which is again to be expected since scattering from lead 1 into lead 2 has to be a two-step process and has to be proportional to $\gamma_h^2 \gamma_v^2$.

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