Correlation effects on topological crystalline insulators

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We study interaction effects on the topological crystalline insulators protected by time-reversal (T) and reflection symmetry (R) in two and three spatial dimensions. From the stability analysis of the edge states with bosonization, we find that the classification of the two-dimensional symmetry-protected topological (SPT) phases protected by $Z_2 \times [\mathrm{U}(1) \rtimes T]$ symmetry is reduced from $\mathbb{Z}$ to $\mathbb{Z}_4$ by interactions where the Z_2 symmetry denotes the reflection whose mirror plane is the two-dimensional plane itself. By extending the approach recently proposed by Isobe and Fu, we show that the classification of the three-dimensional SPT phases (i.e., topological crystalline insulators) protected by $R \times [\mathrm{U}(1) \rtimes T]$ symmetry is reduced from $\mathbb{Z}$ to $\mathbb{Z}_8$ by interactions.

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I. INTRODUCTION

Recently, topological structures of gapped quantum states have attracted much attention. The topologically nontrivial phases are many-body states with gapped excitation spectra in the bulk and characterized by nontrivial topological structure of wave functions. A remarkable property of topological phases is the existence of stable gapless boundary modes, which is a source of various exotic properties; the boundary modes are the origin of quantization of the Hall conductivity in integer quantum Hall systems [1] and topological magnetoelectric effects in three-dimensional topological insulators [2]. One of the important questions in this field is classification of topological phases, i.e., to count how many topologically distinct phases exist under given symmetry. The first answer to this question is obtained for free-fermion systems, for which classification is summarized in the so-called periodic table of topological insulators and superconductors [3–5]. Free-fermion systems are categorized into 10 Altland-Zirnbauer symmetry classes [6] in terms of time-reversal, particle-hole, and sublattice symmetry. The periodic table tells us that, in every spatial dimension, 3 and 2 of the 10 Altland-Zirnbauer symmetry classes have topological insulators/superconductors characterized by topological indices from $\mathbb{Z}$ and $\mathbb{Z}_2$, respectively. The periodic table of topological insulators provides helpful information for searching for topological materials.

The notion of topological insulators and superconductors is extended to systems of interacting fermions or bosons. The symmetry-protected topological (SPT) phases [7] are many-body ground states with short-range entanglement which have gapped excitations in the bulk and gapless excitations on the boundary which are stable against any symmetry-preserving perturbation. The Haldane gap phase of integer-spin chains [8,9] and bosonic integer quantum Hall states [10,11] are examples of bosonic SPT phases in one and two dimensions. As for fermionic SPT phases, topologically nontrivial phases are also expected in several $4f$- or $5d$-electron systems, and the Kondo insulator $\mathrm{SmB_6}$ is a candidate of a topological insulator of correlated electrons [12–15]. However, it was shown by Fidkowski and Kitaev that interactions can change the topological classification (periodic table) of free-fermion systems [16]. They demonstrated that interchain interactions in eight Kitaev chains gap out all Majorana zeromodes without symmetry breaking. This implies that topological classification of one-dimensional topological insulators/superconductors in class BDI (or chiral orthogonal class) is changed from $\mathbb{Z}$ to $\mathbb{Z}_8$ by interactions. Such a collapse of topological classification is also observed in two dimensions [17–20] and three dimensions [21–24]. Several theoretical frameworks for classification of SPT phases in interacting systems have been developed. For example, theory based on group cohomology [7,8,25] or cobordisms [26–28] classifies possible topological actions for the bulk states, while a theoretical approach using the Chern-Simons theory [29,30] or nonlinear sigma models [21–24,31] examines the stability of gapless boundary modes.

In many cases SPT phases are protected by local symmetry such as time-reversal and other internal symmetries. However, spatial symmetry (e.g., reflection, rotation, etc.) can also protect topological phases, as pointed out by Fu [32] and realized in the topological crystalline insulator SnTe [33,34]. This compound respects the time-reversal and reflection symmetries and has four Dirac cones on the (0,0,1) surface [34]. Having a trivial strong Z_2 index, SnTe is a trivial band insulator as the four gapless Dirac cones are not protected from opening of a gap by time-reversal symmetry. However, the reflection symmetry about the (1,1,0) plane prohibits any Dirac mass term that could gap out Dirac cones, and the stability of surface Dirac cones is guaranteed by mirror Chern numbers defined on mirror planes in the Brillouin zone; the topological classification is thus $\mathbb{Z}$ [34–37]. Besides SnTe, there are other candidate materials theoretically proposed as topological crystalline insulators in strongly correlated electron systems: a heavy-fermion compound $\mathrm{YbB_{12}}$ [38] and a d-electron system $\mathrm{Sr_3PbO}$ [39].

Recently, effects of interactions on topological crystalline insulators have been addressed by Isobe and Fu [40]. Motivated by first-principles calculations [41,42] predicting that thin films of SnTe become two-dimensional topological insulators (i.e., quantum spin Hall insulators), they studied stability of gapless edge states with internal Z_2 symmetry that comes from reflection symmetry about the two-dimensional plane; they obtained $\mathbb{Z}_4$ classification for thin films. Furthermore, they have developed a theoretical approach to classify three-dimensional topological crystalline insulators which utilizes the stability analysis of the gapless edge states. This approach led to

$\mathbb{Z}_8$ classification for three-dimensional topological crystalline insulators. Strictly speaking, however, the symmetry of the models studied in Ref. [40] is U(1)$\times Z_2$, where the U(1) symmetry denotes charge conservation [43]. In general, the gapless edge states are chiral under the U(1)$\times Z_2$ symmetry. The time-reversal symmetry of topological crystalline insulators is taken into account in the assumption of nonchiral gapless edge structure in Ref. [40].

In this paper we study interaction effects on topological crystalline insulators protected by time-reversal and reflection symmetries. That is, we classify fermionic SPT phases under $Z_2 \times [\mathrm{U}(1) \rtimes T]$ symmetry in two dimensions and $R \times [\mathrm{U}(1) \rtimes T]$ symmetry in three dimensions, where T and R denote time-reversal and reflection. Using the models studied by Isobe and Fu, we carefully derive transformation laws of fields under symmetry transformations and examine the stability of gapless edge modes under symmetry-preserving perturbations and interactions. We obtain $\mathbb{Z}_4$ classification for two dimensions and $\mathbb{Z}_8$ classification for three dimensions, in agreement with Isobe and Fu.

The rest of this paper is organized as follows. In Sec. II, we consider a two-dimensional model respecting the reflection symmetry whose reflection plane is parallel to the two-dimensional plane. We elucidate the collapse of topological classification from $\mathbb{Z}$ to $\mathbb{Z}_4$ due to interactions. In Sec. III, by extending the argument of Ref. [40] to time-reversal-invariant systems, we show the collapse of topological classification from $\mathbb{Z}$ to $\mathbb{Z}_8$ in three-dimensional topological crystalline insulators. This means that eight Dirac cones on the surface of a topological crystalline insulator can be gapped out. In addition, in Sec. IV we point out that two Dirac cones can be gapped out without symmetry breaking by attaching a fractionalized quantum spin Hall insulator with topological order. In Sec. V we summarize our results.

II. TWO-DIMENSIONAL TOPOLOGICAL CRYSTALLINE INSULATOR

Recent first-principles calculations predict that quantum spin Hall states can be realized in the (111) thin films of the SnTe class of three-dimensional topological crystalline insulators [41,42]. The surface Dirac fermions on the top and bottom surfaces of the (111) thin films are gapped by intersurface coupling and turn into a topological state. Following Ref. [41], we take the effective Hamiltonian for the Dirac fermions on the top and bottom surfaces of a thin film with an odd number of layers,

$$H_{2\mathrm{D}} = \int d^2\boldsymbol{x}\,\boldsymbol{\psi}^\dagger(\boldsymbol{x})[(i\partial_x\sigma^y - i\partial_y\sigma^x)\otimes\tau^z + m\tau^x]\boldsymbol{\psi}(\boldsymbol{x}), \tag{1}$$

where $\boldsymbol{x} = (x,y)$, $\boldsymbol{\psi}^\dagger = (\psi^\dagger_{1\uparrow},\psi^\dagger_{1\downarrow},\psi^\dagger_{2\uparrow},\psi^\dagger_{2\downarrow})$, and the velocity is set equal to unity. Here $\psi^\dagger_{\alpha\sigma}$ is the creation operator of Dirac fermions with spin $\sigma = \uparrow,\downarrow$ on the top ($\alpha = 1$) or the bottom ($\alpha = 2$) surface. The Pauli matrices σ^i and τ^i ($i = x,y,z$) act on the spin and surface indices, respectively. The coupling of the top and bottom surface Dirac fermions gives the mass term $m\tau^x$.

Notice that the Hamiltonian (1) respects the time-reversal (T) symmetry and the local Z_2 symmetry (g) arising from the reflection symmetry with respect to the two-dimensional plane. Transformations of $\boldsymbol{\psi}$ under the symmetry operations are described by

$$T = -i\sigma^y K, \quad g = i\sigma^z\tau^x, \tag{2}$$

where K is the operator for complex conjugation.

The presence of the Z_2 symmetry promotes the topological index characterizing the two-dimensional Hamiltonian from $\mathbb{Z}_2$ ($\mathbb{Z}_2$ index of time-reversal invariant insulators) to $\mathbb{Z}$ (a mirror Chern number). To understand this, we first note that the Chern number is defined for each eigenspace of g. The operators T and g commute with each other, $[-i\sigma^y K, i\sigma^z\tau^x] = 0$, and the time-reversal symmetry interchanges the eigenspaces of g. For example, if $|+\rangle$ and $|-\rangle$ are eigenstates of g with the eigenvalues $+i$ and $-i$, respectively, then $T|\pm\rangle$ are eigenstates with the eigenvalues $\mp i$, respectively. Thus, the Hamiltonian is block diagonalized into eigenspaces of g which are related by the time-reversal symmetry and characterized by the Chern numbers with opposite signs [44]. Having the integer topological number, N_0 copies of $H_{2\mathrm{D}}$ can have N_0 pairs of helical edge modes protected by time-reversal and reflection symmetries.

To describe a Kramers pair of helical edge modes of the two-dimensional topological insulator, we introduce smooth spatial modulation in the mass term in Eq. (1). This generates gapless states moving along a line where the sign of the mass changes. Let us replace $m\tau^x$ by $m(x)\tau^x$, where $m(x)$ takes a positive (negative) value for $x > 0$ ($x < 0$). This yields helical edge states localized at the kink ($x = 0$); solving the Dirac equation for the zero mode,

$$[i\partial_x\sigma^y\otimes\tau^z + m(x)\tau^x]\,|\pm y\rangle = 0, \tag{3}$$

we obtain a Kramers pair of gapless modes $|+y\rangle$ and $|-y\rangle$ propagating to the $+y$ and $-y$ directions, respectively:

$$\langle\boldsymbol{x}|\pm y\rangle = \exp\left[\pm ik_y y - \int_0^x dx' m(x')\right]|\pm y\rangle_0, \tag{4a}$$

with

$$|\pm y\rangle_0 = \begin{pmatrix}1\\ i\end{pmatrix}_\sigma \otimes \begin{pmatrix}1\\ i\end{pmatrix}_\tau \pm i\begin{pmatrix}1\\ -i\end{pmatrix}_\sigma \otimes \begin{pmatrix}1\\ -i\end{pmatrix}_\tau, \tag{4b}$$

where $|\boldsymbol{x}\rangle$ is the eigenstate of $\boldsymbol{x}$ and k_y denotes the momentum along the y direction. These states are transformed as

$$T\begin{pmatrix}|+y\rangle\\ |-y\rangle\end{pmatrix} = \begin{pmatrix}|-y\rangle\\ -|+y\rangle\end{pmatrix}, \quad g\begin{pmatrix}|+y\rangle\\ |-y\rangle\end{pmatrix} = \begin{pmatrix}i\,|+y\rangle\\ -i\,|-y\rangle\end{pmatrix}. \tag{5}$$

In order to examine the stability of the gapless helical edge modes in the presence of interactions, we bosonize the fermionic gapless edge modes [29,30,45,46]. The Lagrangian for the bosonic fields representing the helical edge modes is given by

$$\mathcal{L} = \int \frac{dx}{4\pi}[K_{I,J}\partial_t\phi_I(x)\partial_x\phi_J(x) - \partial_x\phi_I(x)\partial_x\phi_I(x)], \tag{6a}$$

with $K = \rho^z$, where ρ^i ($i = x,y,z$) are the Pauli matrices and summation over the repeated indices I,J is assumed (this is assumed throughout this paper). The bosonic fields, $\boldsymbol{\phi} =$

$(\phi_1,\phi_2) = (\phi^+,\phi^-)$, describe the edge modes propagating to the $+y$ and $-y$ directions, respectively. More specifically, the vertex operators $:e^{i\phi^+}:$ and $:e^{i\phi^-}:$ create fermions in the $|+y\rangle$ and $|-y\rangle$ states, respectively. The fields ϕ_I are defined modulo 2π. The commutation relations of these fields are given by

$$[\phi_I(x),\phi_J(y)] = i\pi\rho^z_{I,J}\mathrm{sgn}(x-y) + i\pi\,\mathrm{sgn}(I-J). \quad (6b)$$

Here, $\mathrm{sgn}(x)$ equals $+1$, 0, and -1 for $x>0$, $x=0$, and $x<0$, respectively. The second term on the right-hand side of Eq. (6b) accounts for the anticommutation relation of fermions from different edge modes. Symmetry transformations of the bosonic fields ϕ_I can be deduced from Eq. (5), which should be compared with $\mathcal{G}:e^{i\phi_{1,2}}:\mathcal{G}^{-1}$ $(\mathcal{G} = \hat{T},\hat{g},\hat{u}_\theta)$. We thus obtain the following transformation rules for the bosonic fields:

$$\hat{T}\boldsymbol{\phi}(x)\hat{T}^{-1} = -\sigma^x\boldsymbol{\phi}(x) + \pi\boldsymbol{e}_2, \quad (7a)$$

$$\hat{g}\boldsymbol{\phi}(x)\hat{g}^{-1} = \boldsymbol{\phi}(x) + \frac{\pi}{2}(\boldsymbol{e}_1 - \boldsymbol{e}_2), \quad (7b)$$

$$\hat{u}_\theta\boldsymbol{\phi}(x)\hat{u}_\theta^{-1} = \boldsymbol{\phi}(x) + \theta(\boldsymbol{e}_1 + \boldsymbol{e}_2), \quad (7c)$$

where $\boldsymbol{e}_I$ is the unit vector whose Ith entry is one and the other entries are zero. The operators $\hat{T}$, $\hat{g}$, and $\hat{u}_\theta$ denote the time-reversal, the local reflection, and the charge U(1) rotation, respectively. The minus sign in Eq. (7a) is due to the antiunitary nature of $\hat{T}$. The symmetry group of the system is denoted as $Z_2 \times [\mathrm{U}(1) \rtimes T]$ in the notation of Ref. [29].

We will examine whether interactions can gap out N_0 copies of the gapless helical modes defined above without symmetry breaking for $N_0 = 1,\ldots,4$. The Lagrangian for the $2N_0$-component bosonic field $\boldsymbol{\phi} = (\phi_1,\ldots,\phi_{2N_0})^T$, which describes the N_0 copies of helical edge modes, is given by Eq. (6a) with the K matrix

$$K = \rho^z \otimes \mathbb{1}_{N_0}, \quad (8a)$$

where $\mathbb{1}_{N_0}$ is the $N_0 \times N_0$ unit matrix. The Ith component of the bosonic field ϕ_I corresponds to the edge mode propagating to the $+y$ $(-y)$ direction for odd I (even I), respectively. The commutation relations of the bosonic fields ϕ_I $(I = 1,\ldots,2N_0)$ are given by

$$[\phi_I(x),\phi_J(y)] = i\pi(K^{-1})_{I,J}\mathrm{sgn}(x-y) + i\pi\,\mathrm{sgn}(I-J). \quad (8b)$$

The transformation rules in Eq. (7) hold if we replace $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ with the following $2N_0$-dimensional vectors: $\boldsymbol{e}_1 = (1,0,1,0,\ldots,1,0)$ and $\boldsymbol{e}_2 = (0,1,0,1,\ldots,0,1)$.

The N_0 pairs of helical edge modes are gapped out if there exist potential

$$\mathcal{L}_{\mathrm{int}} = \sum_\alpha C_\alpha \int dx :\cos(\boldsymbol{l}_\alpha\cdot\boldsymbol{\phi} + a_\alpha): \quad (9)$$

with N_0 linearly independent vectors $\boldsymbol{l}_\alpha$ satisfying the Haldane's null vector condition [47]

$$\boldsymbol{l}^T_\alpha K^{-1}\boldsymbol{l}_\beta = 0, \qquad (\alpha,\beta = 1,\ldots,N_0) \quad (10)$$

so the fields satisfy $[\boldsymbol{l}_\alpha\cdot\boldsymbol{\phi}(x),\boldsymbol{l}_\beta\cdot\boldsymbol{\phi}(y)] = 0$ up to $2\pi in$ $(n \in \mathbb{Z})$ for $\alpha,\beta = 1,\ldots,N_0$. The coupling constants C_α and the phases a_α are real numbers. As indicated by the colons in Eq. (9), the vertex operators are normal ordered,

$$:e^{i\boldsymbol{l}\cdot\boldsymbol{\phi}}: = e^{i\frac{\pi}{2}\sum_{I<J}l_Il_J}:e^{il_1\phi_1}::e^{il_2\phi_2}:\cdots:e^{il_{2N_0}\phi_{2N_0}}:, \quad (11)$$

where the phase factor $\exp(\frac{i\pi}{2}\sum_{I<J}l_Il_J)$ is in accordance with the second term in the commutator in Eq. (8b). It makes $:e^{-i\boldsymbol{l}\cdot\boldsymbol{\phi}}:$ the Hermitian conjugate of $:e^{i\boldsymbol{l}\cdot\boldsymbol{\phi}}:$.

There are cases when the symmetry is broken spontaneously, even when $\mathcal{L}_{\mathrm{int}}$ in Eq. (9) is invariant under the symmetry transformations. The occurrence of spontaneous symmetry breaking can be judged by finding elementary bosonic variables as follows. The N_0 linearly independent vectors $\{\boldsymbol{l}_1,\ldots,\boldsymbol{l}_{N_0}\}$ form a N_0-dimensional lattice as $\boldsymbol{R} = j_1\boldsymbol{l}_1 + j_2\boldsymbol{l}_2 + \cdots + j_{N_0}\boldsymbol{l}_{N_0}$, where j_α are arbitrary integers $(\alpha = 1,\ldots,N_0)$. From the integer vectors in the lattice, we find a set of linearly independent vectors $\{\tilde{\boldsymbol{l}}_1,\tilde{\boldsymbol{l}}_2,\ldots,\tilde{\boldsymbol{l}}_{N_0}\}$, from which we define a primitive lattice vectors $\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{N_0}\}$,

$$\boldsymbol{v}_\alpha = \frac{1}{\gcd(\tilde{l}_{\alpha,1},\ldots,\tilde{l}_{\alpha,2N_0})}\tilde{\boldsymbol{l}}_\alpha, \quad (12)$$

where gcd denotes the greatest common divisor of the integers in the parentheses. The vectors $\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{N_0}\}$ form a primitive cell of the smallest volume (see examples below). Finally, elementary bosonic variables [29] are given by $\boldsymbol{v}_\alpha\cdot\boldsymbol{\phi}$ $(\alpha = 1,\ldots,N_0)$. The set of elementary bosonic variables $\{\boldsymbol{v}_1\cdot\boldsymbol{\phi},\ldots,\boldsymbol{v}_{N_0}\cdot\boldsymbol{\phi}\}$ are pinned to constant values, when the fields $\{\boldsymbol{l}_1\cdot\boldsymbol{\phi},\ldots,\boldsymbol{l}_{N_0}\cdot\boldsymbol{\phi}\}$ are pinned by the potentail in $\mathcal{L}_{\mathrm{int}}$. We see that the gapless edge modes can be gapped out without symmetry breaking if and only if the set $\{\boldsymbol{v}_1\cdot\boldsymbol{\phi},\ldots,\boldsymbol{v}_{N_0}\cdot\boldsymbol{\phi}\}$ is invariant,

$$\mathcal{G}\{\boldsymbol{v}_1\cdot\boldsymbol{\phi},\ldots,\boldsymbol{v}_{N_0}\cdot\boldsymbol{\phi}\}\mathcal{G}^{-1} = \{\boldsymbol{v}_1\cdot\boldsymbol{\phi},\ldots,\boldsymbol{v}_{N_0}\cdot\boldsymbol{\phi}\}, \quad (13)$$

modulo 2π under the symmetry transformations in Eq. (7), where $\mathcal{G} = \hat{T},\hat{g},\hat{u}_\theta$.

Here two comments on the symmetry transformations of the cosine terms are in order. First, the vector $\boldsymbol{l}_0 = K^{-1}(1,1,\ldots,1)^T = (1,-1,\ldots,1,-1)$ is in the lattice generated by the elementary vectors $\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{N_0}\}$ when the cosine terms with $\{\boldsymbol{l}_1,\ldots,\boldsymbol{l}_{N_0}\}$ in $\mathcal{L}_{\mathrm{int}}$ respect the symmetry and gap out all the edge modes. In other words, the field $\boldsymbol{l}_0\cdot\boldsymbol{\phi}$ is pinned when all the edge modes are gapped out. This is understood by noting that $:\cos(2\boldsymbol{l}_0\cdot\boldsymbol{\phi}):$ is invariant under the symmetry transformations. Second, it is important to take into account additional phase shifts coming from the Klein factors [the second term in Eq. (8b)] when we examine the invariance of the cosine terms in $\mathcal{L}_{\mathrm{int}}$, in particular, under the time-reversal transformation, which changes the direction of propagation of edge modes.

We show below that four copies of the helical edge modes (6a) can be gapped out without symmetry breaking. We begin with the case of $N_0 = 1$. A gapping potential respecting the symmetry is given by

$$\mathcal{L}_{\mathrm{int}} = C\int dx{:}\cos[2\boldsymbol{v}\cdot\boldsymbol{\phi}(x)]{:} \quad (14a)$$

with

$$\boldsymbol{v} = (1,-1)^T. \quad (14b)$$

This potential is invariant under the operations of $\hat{T}$, $\hat{g}$, and $\hat{u}_\theta$. Indeed, the invariance under $\hat{g}$ and $\hat{u}_\theta$ can be seen as

$$\hat{g}(2\boldsymbol{v}\cdot\boldsymbol{\phi})\hat{g}^{-1} = 2\boldsymbol{v}\cdot\boldsymbol{\phi} + 2\pi, \tag{15a}$$

$$\hat{u}_\theta(2\boldsymbol{v}\cdot\boldsymbol{\phi})\hat{u}_\theta^{-1} = 2\boldsymbol{v}\cdot\boldsymbol{\phi}. \tag{15b}$$

To verify the invariance under $\hat{T}$, we note from Eqs. (7a) and (11) that

$$\begin{aligned}\hat{T}{:}e^{2i\boldsymbol{v}\cdot\boldsymbol{\phi}}{:}\hat{T}^{-1} &= \hat{T}{:}e^{2i\phi_1}{:}{:}e^{-2i\phi_2}{:}\hat{T}^{-1}\\ &= {:}e^{2i\phi_2}{:}{:}e^{-2i\phi_1}{:}\\ &= {:}e^{-2i\boldsymbol{v}\cdot\boldsymbol{\phi}}{:}\end{aligned} \tag{16a}$$

and vice versa and hence

$$\hat{T}{:}\cos(2\boldsymbol{v}\cdot\boldsymbol{\phi}){:}\hat{T}^{-1} = {:}\cos(2\boldsymbol{v}\cdot\boldsymbol{\phi}){:}. \tag{16b}$$

It turns out, however, that the ground state breaks the symmetry as the elementary bosonic variable $\boldsymbol{v}\cdot\boldsymbol{\phi}$ is transformed as

$$\hat{g}\,\boldsymbol{v}\cdot\boldsymbol{\phi}\,\hat{g}^{-1} = \boldsymbol{v}\cdot\boldsymbol{\phi} + \pi. \tag{17}$$

The vector $\boldsymbol{v}$ in Eq. (14b) is the elementary bosonic variable respecting the U(1) symmetry. Hence, it is impossible to gap out edge modes by any symmetry-preserving potential without spontaneous symmetry breaking when $N_0 = 1$.

For $N_0 = 2$, an example of pinning potentials allowed by the symmetry is given by

$$\mathcal{L}_{\text{int}} = C\sum_{\alpha=1,2}\int dx{:}\cos[2\boldsymbol{v}_\alpha\cdot\boldsymbol{\phi}(x)]{:} \tag{18a}$$

with

$$\boldsymbol{v}_1 = (1,0\,|\,0,-1),\quad \boldsymbol{v}_2 = (0,1\,|-1,0), \tag{18b}$$

where C is an arbitrary real number and the vertical lines in $\boldsymbol{v}_{1,2}$ are inserted between the copies of helical edge modes. The invariance under $\hat{g}$ and $\hat{u}_\theta$ can be checked directly as in the $N_0 = 1$ case. The time-reversal transformation of $\boldsymbol{\phi} = (\phi_1,\phi_2,\phi_3,\phi_4)^T$,

$$\hat{T}\boldsymbol{\phi}\hat{T}^{-1} = -\sigma^x \oplus \sigma^x\boldsymbol{\phi} + \pi(\boldsymbol{e}_2 + \boldsymbol{e}_4), \tag{19}$$

yields

$$\hat{T}{:}\cos(2\boldsymbol{v}_1\cdot\boldsymbol{\phi}){:}\hat{T}^{-1} = {:}\cos(2\boldsymbol{v}_2\cdot\boldsymbol{\phi}){:}, \tag{20a}$$

$$\hat{T}{:}\cos(2\boldsymbol{v}_2\cdot\boldsymbol{\phi}){:}\hat{T}^{-1} = {:}\cos(2\boldsymbol{v}_1\cdot\boldsymbol{\phi}){:}, \tag{20b}$$

where we have used Eq. (11). However, the spontaneous symmetry breaking occurs once the elementary bosonic variables $\{\boldsymbol{v}_1\cdot\boldsymbol{\phi},\boldsymbol{v}_2\cdot\boldsymbol{\phi}\}$ are pinned by $\mathcal{L}_{\text{int}}$, as these variables are not invariant under $\hat{g}$ [in a similar way to Eq. (17)]. We note that this observation holds for arbitrary $\boldsymbol{l}$'s of the pinning potentials respecting the U(1) symmetry. For example, if the cosine terms with

$$\boldsymbol{l}_1 = (1,-1\,|\,1,-1)^T, \tag{21a}$$

$$\boldsymbol{l}_2 = (1,1\,|-1,-1)^T, \tag{21b}$$

are used to gap out the edge modes, then the elementary bosonic fields $\{\boldsymbol{v}_1\cdot\boldsymbol{\phi},\boldsymbol{v}_2\cdot\boldsymbol{\phi}\}$ are also pinned (because $\boldsymbol{l}_1 + \boldsymbol{l}_2 = 2\boldsymbol{v}_1$ and $\boldsymbol{l}_1 - \boldsymbol{l}_2 = -2\boldsymbol{v}_2$), which gives rise to spontaneous symmetry breaking. Thus, we cannot gap out edge modes without symmetry breaking.

Let us move on to the case of $N_0 = 3$. When all the helical edge modes are gapped out, the field

$$\boldsymbol{l}_0\cdot\boldsymbol{\phi} = \phi_1 - \phi_2 + \phi_3 - \phi_4 + \phi_5 - \phi_6 \tag{22}$$

necessarily takes a classical value (i.e., $\langle\boldsymbol{l}_0\cdot\boldsymbol{\phi}\rangle = \text{const}$). This implies that the ground state spontaneously breaks the symmetry, as $\boldsymbol{l}_0\cdot\boldsymbol{\phi}$ is not invariant under $\hat{g}$,

$$\hat{g}\,\boldsymbol{l}_0\cdot\boldsymbol{\phi}\,\hat{g}^{-1} = \boldsymbol{l}_0\cdot\boldsymbol{\phi} + 3\pi. \tag{23}$$

For $N_0 = 4$, we can gap out all the edge modes while respecting the symmetry by taking the following cosine terms:

$$\mathcal{L}^{(4)}_{\text{int}} = C\sum_{\alpha=1}^{4}\int dx{:}\cos[\boldsymbol{v}_\alpha\cdot\boldsymbol{\phi}(x)]{:} \tag{24a}$$

with

$$\boldsymbol{v}_1 = (1,0\,|\,1,0\,|\,0,-1\,|\,0,-1)^T, \tag{24b}$$

$$\boldsymbol{v}_2 = (0,1\,|\,0,1\,|-1,0\,|-1,0)^T, \tag{24c}$$

$$\boldsymbol{v}_3 = (1,-1\,|-1,1\,|\,0,0\,|\,0,0)^T, \tag{24d}$$

$$\boldsymbol{v}_4 = (0,0\,|\,0,0\,|\,1,-1\,|-1,1)^T. \tag{24e}$$

The time-reversal invariance can be seen as follows:

$$\hat{T}{:}\cos(\boldsymbol{v}_1\cdot\boldsymbol{\phi}){:}\hat{T}^{-1} = {:}\cos(\boldsymbol{v}_2\cdot\boldsymbol{\phi}){:}, \tag{25a}$$

$$\hat{T}{:}\cos(\boldsymbol{v}_2\cdot\boldsymbol{\phi}){:}\hat{T}^{-1} = {:}\cos(\boldsymbol{v}_1\cdot\boldsymbol{\phi}){:}, \tag{25b}$$

$$\hat{T}{:}\cos(\boldsymbol{v}_3\cdot\boldsymbol{\phi}){:}\hat{T}^{-1} = {:}\cos(\boldsymbol{v}_3\cdot\boldsymbol{\phi}){:}, \tag{25c}$$

$$\hat{T}{:}\cos(\boldsymbol{v}_4\cdot\boldsymbol{\phi}){:}\hat{T}^{-1} = {:}\cos(\boldsymbol{v}_4\cdot\boldsymbol{\phi}){:}. \tag{25d}$$

Invariance under the transformations of $\hat{g}$ and $\hat{u}_\theta$ can be seen by straightforward calculations:

$$\hat{g}\,\boldsymbol{v}_\alpha\cdot\boldsymbol{\phi}\,\hat{g}^{-1} = \boldsymbol{v}_\alpha\cdot\boldsymbol{\phi}\pmod{2\pi}, \tag{26a}$$

$$\hat{u}_\theta\boldsymbol{v}_\alpha\cdot\boldsymbol{\phi}\,\hat{u}_\theta^{-1} = \boldsymbol{v}_\alpha\cdot\boldsymbol{\phi}, \tag{26b}$$

for $\alpha = 1,\ldots,4$. When elementary bosonic variables $\boldsymbol{v}_\alpha\cdot\boldsymbol{\phi}$ ($\alpha = 1,\ldots,4$) are pinned at some constant values such that $\langle\boldsymbol{v}_1\cdot\boldsymbol{\phi}\rangle = \langle\boldsymbol{v}_2\cdot\boldsymbol{\phi}\rangle$, the edge modes are completely gapped out without symmetry breaking. Hence, we conclude that two-dimensional SPT phases form a $\mathbb{Z}_4$ group, which is reduced from $\mathbb{Z}$ of the noninteracting fermions.

III. THREE-DIMENSIONAL TOPOLOGICAL INSULATOR

In this section, we discuss three-dimensional topological insulators with the time-reversal and reflection symmetries. In Ref. [40], Isobe and Fu studied interaction effects on topological crystalline insulators characterized by mirror Chern numbers. They introduced a spatially varying mass term to the surface Dirac Hamiltonian and examined the stability of one-dimensional gapless modes propagating along domain walls where the sign of the Dirac mass changes. We note that the topological crystalline insulating phase in SnTe is an SPT phase protected by both reflection and time-reversal symmetries. In Ref. [40], however, the time-reversal symmetry is not explicitly considered and is used only in the assumption of the nonchiral structure of boundary (domain wall) states [48]. Here we improve on their approach by keeping time-reversal symmetry intact at every step of calculations and show that topological classification collapse from $\mathbb{Z}$ to $\mathbb{Z}_8$.

Let us start with discussion on the topological invariant in the noninteracting case. As is well known, topological phases of class AII in three dimensions form a $\mathbb{Z}_2$ group. In the presence of additional reflection symmetry, the group structure of the topological phases is promoted to $\mathbb{Z}$ [34]. When the reflection plane is the yz plane, the reflection operator is given by

$$R = i\sigma^x P, \tag{27}$$

where σ^x acts on the electron spin and P changes the sign of x coordinate, $P{:}(x,y,z) \to (-x,y,z)$. The relevant symmetry group is denoted as $Z_2 \times [\mathrm{U}(1) \rtimes T]$ in the notation of Ref. [29]. The topological invariant for insulating phases under this symmetry group is the mirror Chern number $n_M = (n^+ - n^-)/2 \in \mathbb{Z}$, where $n^\pm$ are the Chern numbers defined in the eigenspaces of R on the two-dimensional fixed plane under the reflection in the momentum space. The time-reversal symmetry $T = -i\sigma^y K$ is not closed in each eigenspace of R on the fixed plane.

Following Ref. [40], we classify the correlated topological crystalline insulators by introducing a spatially varying Dirac mass term to the surface Hamiltonian. We will show below that the gapless modes from eight copies of Dirac cones can be gapped out without symmetry breaking. The classification of three-dimensional topological crystalline insulators is reduced from $\mathbb{Z}$ to $\mathbb{Z}_8$ by interactions.

The Hamiltonian of a single Dirac cone on the surface of a topological crystalline insulator is given by

$$H_{\mathrm{surf}} = \int d^2\boldsymbol{x}\, \boldsymbol{\psi}^\dagger(\boldsymbol{x})(i\partial_x\sigma^y - i\partial_y\sigma^x)\boldsymbol{\psi}(\boldsymbol{x}), \tag{28}$$

where $\boldsymbol{\psi}^\dagger = (\psi_\uparrow^\dagger, \psi_\downarrow^\dagger)$. Without reflection symmetry, two copies of the surface Dirac cones are gapped out without breaking the time-reversal symmetry by introducing a mass term in the Hamiltonian,

$$H_{\mathrm{surf}}^{(2)} = \int d^2\boldsymbol{x}\, \boldsymbol{\psi}^\dagger(\boldsymbol{x})[(i\partial_x\sigma^y - i\partial_y\sigma^x)\otimes\tau_0 + m_0\sigma^z\otimes\tau^y]\boldsymbol{\psi}(\boldsymbol{x}). \tag{29}$$

Here τ acts on the pseudospin space in which the eigenvalues ±1 of τ^z distingsuish the two Dirac cones and $\boldsymbol{\psi}^\dagger$ is a four-component vector, $\boldsymbol{\psi}^\dagger = (\psi_{1\uparrow}^\dagger, \psi_{1\downarrow}^\dagger, \psi_{2\uparrow}^\dagger, \psi_{2\downarrow}^\dagger)$, where $\psi_{\alpha,\sigma}^\dagger$ is the creation operator of a Dirac fermion with spin $\sigma = \uparrow,\downarrow$ and pseudospin $\alpha = 1,2$. The instability of two Dirac cones is consistent with the $\mathbb{Z}_2$ classification of symmetry class AII. However, when the reflection symmetry is imposed, the mass term is not allowed by the symmetry ($R\, m_0\sigma^z\otimes\tau^y R^{-1} = -m_0\sigma^z\otimes\tau^y$), and the two Dirac cones remain gapless; the $\mathbb{Z}$ classification follows.

One possible way to respect both time-reversal and reflection symmetries while introducing a mass to Dirac fermions is to modulate the sign of the mass in space. This gives rise to one-dimensional gapless modes at the domain wall where the mass changes its sign. To see this, let us consider the following simple model:

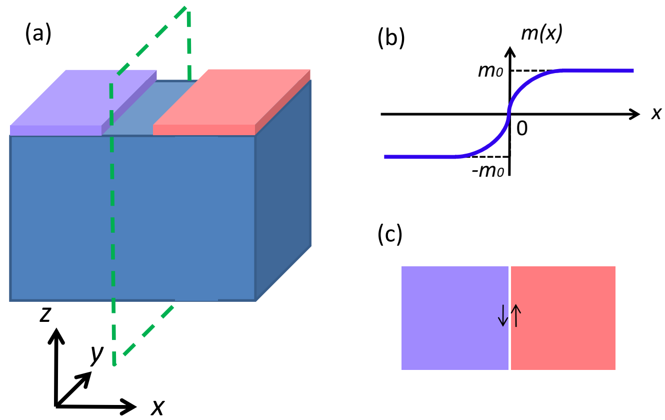

FIG. 1. (Color online) (a) Sketch of a spatially varying Dirac mass introduced to surface Dirac fermions of a three-dimensional topological crystalline insulator. The green dashed square denotes the reflection plane. We introduce a positive (negative) mass in the region where the red (violet) thin rectangle is attached. (b) Spatial dependence of the mass in Eq. (30). (c) Top view of panel (a). Helical edge modes are localized at the kink $x = 0$.

$$\widetilde{H}_{\mathrm{surf}}^{(2)} = \int d^2\boldsymbol{x}\, \boldsymbol{\psi}^\dagger(\boldsymbol{x})[(i\partial_x\sigma^y - i\partial_y\sigma^x)\otimes\tau_0 + m(x)\sigma^z\otimes\tau^y]\boldsymbol{\psi}(\boldsymbol{x}), \tag{30}$$

where $m(x)$ takes a positive (negative) value for $x > 0$ ($x < 0$), respectively. A sketch of the model is shown in Fig. 1. When the mass changes its sign as drawn in Fig. 1(b), a Kramers pair of gapless states propagating along the y direction are present at the domain wall at $x = 0$ [see Fig. 1(c)]. Their wave functions are written as

$$\langle\boldsymbol{x}|\pm y\rangle = \exp\left[\pm ik_y y - \int_0^x dx' m(x')\right]|\pm y\rangle_0, \tag{31a}$$

with

$$|+y\rangle_0 = i\begin{pmatrix}1\\1\end{pmatrix}_\sigma \otimes \begin{pmatrix}1\\i\end{pmatrix}_\tau, \tag{31b}$$

$$|-y\rangle_0 = i\begin{pmatrix}1\\-1\end{pmatrix}_\sigma \otimes \begin{pmatrix}1\\-i\end{pmatrix}_\tau, \tag{31c}$$

where $|+y\rangle$ and $|-y\rangle$ propagate to the $+y$ and $-y$ directions, respectively. Again, $|\boldsymbol{x}\rangle$ denotes the eigenstate of $\boldsymbol{x}$.

Under the operations of the time-reversal and reflection symmetries, these states are transformed as follows:

$$T\,|+y\rangle = |-y\rangle, \quad T\,|-y\rangle = -|+y\rangle, \tag{32a}$$

and

$$R\,|+y\rangle = i|+y\rangle, \quad R\,|-y\rangle = -i|-y\rangle. \tag{32b}$$

Hence, introducing the bosonic fields $\phi^\pm$ for the gapless modes propagating along the $\pm y$ directions, we obtain the transformation laws of $\phi^\pm$ which are identical to Eqs. (7) with $\boldsymbol{\phi} = (\phi^+,\phi^-)^T$. Thus, proceeding in the same way as in the previous section, we see that four copies of gapless modes in Eq. (31) can be gapped out without symmetry breaking by introducing cosine potentials similar to Eq. (24) that pin the gapless bosonic fields. Since the helical edge modes in Eq. (31) are obtained by introducing a domain wall in the Dirac mass that couples two Dirac cones, we conclude that the SPT phases of three-dimensional topological crystalline insulators protected by time-reversal and reflection form a $\mathbb{Z}_8$ group, which is reduced from the $\mathbb{Z}$ group of the noninteracting fermions.

IV. INSTABILITY FROM COUPLING TO TOPOLOGICAL ORDERED PHASES

So far, we have discussed the possibility of gapping out boundary modes of two- and three-dimensonal topological crystalline insulators without symmetry breaking by interactions among themselves. In this section, we consider a different situation in which gapless edge modes of a two-dimensional topological crystalline insulator are interacting with edge modes of a fractionalized quantum spin Hall insulator (QSHI*) [49]. In this case we find that even a single Kramers pair of edge modes in Eq. (4) can be gapped out, without spontaneous breaking of time-reversal and reflection symmetry. This result implies that two copies of Dirac cones on the surface of a three-dimensional topological crystalline insulator can also be gapped out without symmetry breaking by the coupling to edge modes of a QSHI*. In fact, gapped edge states between a QSHI and a QSHI* are discussed by Lu and Lee in Ref. [49]. Here we show that the same symmetric gapped states are obtained in the presence of additional reflection symmetry.

The QSHI* is obtained by gauging the symmetry of the fermion number parity from a QSHI; see the Appendix. The Lagrangian of edge modes of a QSHI* is given by

$$\mathcal{L} = \int \frac{dx}{4\pi}\left(-2\rho^x_{I,J}\partial_t\phi_I\partial_x\phi_J - V_{I,J}\partial_x\phi_I\partial_x\phi_J\right), \tag{33a}$$

with

$$\boldsymbol{\phi} = (\phi^s,\phi^c)^T, \tag{33b}$$

where V is a symmetric and positive definite matrix. The bosonic fields ϕ^c and ϕ^s describe chargeon and spinon, respectively. These bosonic fields are transformed as follows:

$$\hat{T}\begin{pmatrix}\phi^s\\ \phi^c\end{pmatrix}\hat{T}^{-1} = \begin{pmatrix}\phi^s\\ -\phi^c\end{pmatrix} + \frac{\pi}{2}\begin{pmatrix}-1\\ 1\end{pmatrix}, \tag{34a}$$

$$\hat{g}\begin{pmatrix}\phi^s\\ \phi^c\end{pmatrix}\hat{g}^{-1} = \begin{pmatrix}\phi^s\\ \phi^c\end{pmatrix} + \frac{\pi}{2}\begin{pmatrix}1\\ 0\end{pmatrix}, \tag{34b}$$

$$\hat{u}_\theta\begin{pmatrix}\phi^s\\ \phi^c\end{pmatrix}\hat{u}_\theta^{-1} = \begin{pmatrix}\phi^s\\ \phi^c\end{pmatrix} + \theta\begin{pmatrix}0\\ 1\end{pmatrix}. \tag{34c}$$

Thus, the total Lagrangian of the helical edge modes at the interface of a two-dimensional topological insulator (with a mirror Chern number) and a QSHI* is found from Eqs. (6a) and (33) to be given by

$$\mathcal{L} = \int \frac{dx}{4\pi}[K_{I,J}\partial_t\phi_I\partial_x\phi_J - V'_{I,J}\partial_x\phi_I\partial_x\phi_J], \tag{35a}$$

with

$$K = (-2\rho^x)\oplus\rho^z, \quad V' = V\oplus(v\mathbb{1}_2), \tag{35b}$$

and

$$\boldsymbol{\phi} = (\phi^s,\phi^c,\phi^+,\phi^-)^T, \tag{35c}$$

where $\mathbb{1}_2$ is the two-dimensional identity matrix and v is the velocity. We seek cosine potentials of the bosonic fields $:\cos(\boldsymbol{l}\cdot\boldsymbol{\phi}+\alpha_l):$ that can gap out the edge modes. The cosine potentials should satisfy the Haldane's null vector condition in Eq. (10). In addition, their l vectors should be of the form $K\boldsymbol{l}'$ with $\boldsymbol{l}'\in\mathbb{Z}^4$, since the cosine potentials are composed of bosonic or fermionic vertex operators. The following potential terms satisfy these conditions:

$$\mathcal{L}_{\rm int} = C\int dx[:\cos(2\phi^c-\phi^+-\phi^-): \\ -:\cos(2\phi^s-\phi^++\phi^-):], \tag{36}$$

where C is a negative constant. These terms respect all the symmetry we impose: time-reversal, reflection, and U(1) charge. Invariance under $\hat{g}$ or $\hat{u}_\theta$ operation can be seen by straightforward calculations, and the time-reversal invariance can be seen as follows:

$$\hat{T}:e^{i(2\phi^c-\phi^+-\phi^-)}:\hat{T}^{-1} = :e^{i(2\phi^c-\phi^+-\phi^-)}:, \tag{37a}$$

$$\hat{T}:e^{i(2\phi^s-\phi^++\phi^-)}:\hat{T}^{-1} = :e^{-i(2\phi^s-\phi^++\phi^-)}:. \tag{37b}$$

In contrast to the $N_0=2$ case discussed in Sec. II, spontaneous symmetry breaking does not occur in the present case. The elementary vectors associated with the $\boldsymbol{l}$ vectors of the cosine potentials in Eq. (36) are given by

$$\boldsymbol{v}_1 = (-1,-1,1,0)^T, \tag{38a}$$

$$\boldsymbol{v}_2 = (1,-1,0,1)^T. \tag{38b}$$

Applying the operator $\hat{T}$ transforms the vertex operators as

$$\hat{T}:e^{i\boldsymbol{v}_1\cdot\boldsymbol{\phi}}:\hat{T}^{-1} = -:e^{i\boldsymbol{v}_2\cdot\boldsymbol{\phi}}:, \tag{39a}$$

$$\hat{T}:e^{i\boldsymbol{v}_2\cdot\boldsymbol{\phi}}:\hat{T}^{-1} = -:e^{i\boldsymbol{v}_1\cdot\boldsymbol{\phi}}:, \tag{39b}$$

which implies that time-reversal symmetry is preserved as long as the pinned fields satisfy the relation

$$\langle\boldsymbol{v}_1\cdot\boldsymbol{\phi}\rangle - \langle\boldsymbol{v}_2\cdot\boldsymbol{\phi}\rangle = \pi. \tag{40}$$

This is indeed the case with the cosine potentials with $C<0$ in Eq. (36). The invariance of the elementary bosonic variables $\boldsymbol{v}_i\cdot\boldsymbol{\phi}$ under $\hat{g}$ or $\hat{u}_\theta$ can also be verified easily. Thus, we conclude that the helical edge modes between a QSHI and a QSHI* are gapped without symmetry breaking even in the presence of reflection symmetry.

In Sec. III we have considered a Kramers pair of gapless mode induced at a domain wall of a spatially varying Dirac mass that couples two Dirac cones on the surface of a three-dimensional topological crystalline insulator. As a corollary of the above result, we conclude that a gapless helical domain-wall mode is gapped out without symmetry breaking when coupled to a helical edge mode of a QSHI*.

V. CONCLUSION

Motivated by theoretical proposals of topological crystalline insulators in correlated electron systems, we have studied fermionc SPT phases which respect the time-reversal and reflection symmetry by employing the Chern-Simons approach. For two-dimensional systems respecting the reflection symmetry whose reflection plane is parallel to the two-dimensional plane, our analysis elucidates that the SPT phases form $\mathbb{Z}_4$ group, while topological classification in the noninteracting case is $\mathbb{Z}$. Furthermore, we have addressed classification of three-dimensional topological crystalline insulators by extending the argument proposed in Ref. [40] with keeping time-reversal symmetry intact. Our analysis revealed that eight Dirac cones on the surface are completely gapped out without symmetry breaking by two-particle backscattering of gapless modes on the domain wall of varying Dirac mass. This leads to collapse of topological classification from $\mathbb{Z}$ to $\mathbb{Z}_8$. Finally, we have pointed out the instability of gapless boundary modes through coupling to topological ordered phases.

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APPENDIX: GAUGING FERMION PARITY SYMMETRY TO FERMIONIC SPT PHASES

In this Appendix we review the QSHI*, which is obtained by gauging the symmetry of the fermion number parity P_f of a QSHI [49,50]. The effective action of the QSHI is written as

$$\mathcal{L}_{\mathrm{CS}} = \int d^2\boldsymbol{x}\left(\frac{\epsilon^{\mu\nu\rho}}{4\pi}K_{I,J}a^I_\mu\partial_\nu a^J_\rho - j^\mu l_I a^I_\mu\right), \tag{A1}$$

with $K = -\rho^z$. Here $\epsilon^{\mu\nu\rho}$ is the antisymmetric tensor ($\epsilon^{012} = 1$, $\mu,\nu,\rho = 0,1,2$), $\boldsymbol{\partial} := (\partial_t,\partial_x,\partial_y)$, $\boldsymbol{x} := (x,y)$. The internal Chern-Simons gauge fields $a_{I\mu}$ ($I = 1,2$) describe the bulk low-energy excitations in the QSHI. Summation over repeating indices is assumed. The integer vector $\boldsymbol{l} \in \mathbb{Z}^2$ characterizes quasiparticles, whose equation of motion is given by

$$l_I j^\mu = \frac{\epsilon^{\mu\nu\rho}}{2\pi}K_{I,J}\partial_\nu a^J_\rho. \tag{A2}$$

Equation (A1) predicts gapless modes at the boundary, which is set to be at $y = 0$. The Lagrangian of the boundary modes is given by

$$\mathcal{L}_{\mathrm{edge}} = \int \frac{dx}{4\pi}(K_{I,J}\partial_t\phi_I\partial_x\phi_J - v_F\partial_x\phi_I\partial_x\phi_I), \tag{A3}$$

with $\boldsymbol{\phi} := (\phi^\uparrow,\phi^\downarrow)^T$ [$I,J = 1,2$ and $(\uparrow,\downarrow) = (1,2)$]. Here $\phi^\uparrow$ and $\phi^\downarrow$ are the bosonic fields describing gapless modes of up-spin and down-spin quasiparticles and v_F denotes the Fermi velocity. The commutation relations of these fields are given by

$$[\phi_I(x),\phi_J(y)] = -i\pi\rho^z_{I,J}\mathrm{sgn}(x-y) + i\pi\,\mathrm{sgn}(I-J). \tag{A4}$$

The transformation laws of these fields under $\hat{T},\hat{g},\hat{u}_\theta$ are given by Eq. (7) with (ϕ^+,ϕ^-) replaced by $(\phi^\uparrow,\phi^\downarrow)$. Let us gauge the symmetry of fermion number parity. The corresponding operator $\hat{P}_f$ transforms the bosonic fields as

$$\hat{P}_f\boldsymbol{\phi}\hat{P}_f^{-1} = \boldsymbol{\phi} + \delta\boldsymbol{\phi}_{P_f}, \tag{A5a}$$

where

$$\delta\boldsymbol{\phi}_{P_f} = \begin{pmatrix}-\pi\\ -\pi\end{pmatrix}. \tag{A5b}$$

Gauging this symmetry is carried out by extending the action as follows:

(i) A vortex (symmetry flux) is attached to quasiparticles such that a quasiparticle with $\boldsymbol{l} \in \mathbb{Z}^2$ going around a vortex acquires a phase shift $\boldsymbol{l}\cdot\delta\boldsymbol{\phi}_{P_f}$ which is equal to the phase shift from the symmetry transformation generated by $\hat{P}_f$. This makes the $\boldsymbol{l}$ vectors of quasiparticles noninteger valued.

(ii) The action (A1) is extended so the quasiparticles with noninteger $\boldsymbol{l}$ vectors become elementary excitations.

The symmetry flux introduced in step (i) is described by the vector

$$\boldsymbol{l}_{P_f} = \begin{pmatrix}\frac{1}{2}\\ -\frac{1}{2}\end{pmatrix}. \tag{A6}$$

We can see this by calculating from Eq. (A1) the phase acquired by a quasiparticle with an $\boldsymbol{l}$ vector going around the symmetry flux $\boldsymbol{l}_{P_f}$. The phase is $2\pi\boldsymbol{l}^TK^{-1}\boldsymbol{l}_{P_f}$, which should be compared with $\boldsymbol{l}\cdot\delta\boldsymbol{\phi}_{P_f}$. With the flux contribution $\boldsymbol{l}_{P_f}$, quasiparticles are effectively described by the following vectors:

$$\boldsymbol{l}' := M\boldsymbol{n}, \tag{A7a}$$

with

$$M := \begin{pmatrix}\frac{1}{2} & -1\\ -\frac{1}{2} & 0\end{pmatrix}, \quad \boldsymbol{n} \in \mathbb{Z}^2. \tag{A7b}$$

In step (ii), we modify the action (A1) so all the quasiparticles are described by integer vectors, which we identify with $\boldsymbol{n}$ in Eq. (A7). The modified action then reads

$$\mathcal{L}_g = \int d^2\boldsymbol{x}\frac{\epsilon^{\mu\nu\rho}}{4\pi}K_{g\,I,J}\tilde{a}^I_\mu\partial_\nu\tilde{a}^J_\rho - n_I j^\mu\tilde{a}^I_\mu, \tag{A8a}$$

with

$$K_g = \begin{pmatrix}4 & 2\\ 2 & 0\end{pmatrix}, \quad I = 1,2, \tag{A8b}$$

where fields $\tilde{a}^I_\mu = M_{J,I}a^J_\mu$ are the internal Chern-Simons gauge fields of this gauged system. The K matrix for the gauged system gives the mutual statistics of quasiparticles with integer vectors ($\boldsymbol{n},\boldsymbol{n}' \in \mathbb{Z}^2$) as

$$2\pi\boldsymbol{n}^TK_g^{-1}\boldsymbol{n}' = 2\pi\boldsymbol{n}^TM^TK^{-1}M\boldsymbol{n}'. \tag{A9}$$

Without loss of generality, we can redefine the fields as $\tilde{a}^I_\mu = X_{I,J}\tilde{a}'^J_\mu$ with $X \in GL(2,\mathbb{Z})$. Choosing

$$X = \begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}, \tag{A10}$$

we obtain the following action of the QSHI* phase:

$$\mathcal{L}_g = \int d^2\boldsymbol{x}\left(\frac{\epsilon^{\mu\nu\rho}}{4\pi}2\rho^x\tilde{a}'^I_\mu\partial_\nu\tilde{a}'^J_\rho - n'_I j^\mu\tilde{a}'^I_\mu\right). \tag{A11}$$

This action predicts gapless edge modes at the boundary $y = 0$, which are described by the Lagrangian

$$\mathcal{L}_{g,\mathrm{edge}} = \int \frac{dx}{4\pi}\left(2\rho^x_{I,J}\partial_t\phi_I\partial_x\phi_J - V_{g\,I,J}\partial_x\phi_I\partial_x\phi_J\right), \tag{A12a}$$

with

$$\boldsymbol{\phi}^T = (\phi^s, -\phi^c). \tag{A12b}$$

Here V_g is a positive-definite symmetric matrix. This edge Lagrangian is equivalent to the Lagrangian in Eq. (33). The bosonic fields (ϕ^s,ϕ^c) are related to $(\phi^\uparrow,\phi^\downarrow)$ as follows:

$$(\phi^s, -\phi^c) \leftrightarrow (\phi^\uparrow,\phi^\downarrow)M(X^T)^{-1} \tag{A13}$$

or, equivalently,

$$\begin{pmatrix} 2\phi^s \\ 2\phi^c \end{pmatrix} \leftrightarrow \begin{pmatrix} \phi^\uparrow - \phi^\downarrow \\ \phi^\uparrow + \phi^\downarrow \end{pmatrix}. \tag{A14}$$

From this correspondence, we obtain the transformation law of fields ϕ^s and ϕ^c in Eq. (34). It follows from the correspondence that

$$\hat{T}{:}e^{2i\phi^s}{:}\hat{T}^{-1} \leftrightarrow -{:}e^{-i(\phi^\uparrow-\phi^\downarrow)}{:}, \tag{A15a}$$

$$\hat{T}{:}e^{2i\phi^c}{:}\hat{T}^{-1} \leftrightarrow -{:}e^{i(\phi^\uparrow+\phi^\downarrow)}{:}. \tag{A15b}$$

One can see how the time-reversal operator transforms the vertex operators in the pinning potentials in Eq. (36) in the following way. With the correspondence

$${:}e^{i(\phi^+-\phi^--2\phi^s)}{:} \leftrightarrow {:}e^{i(\phi^+-\phi^--\phi^\uparrow+\phi^\downarrow)}{:}, \tag{A16a}$$

$${:}e^{i(\phi^++\phi^--2\phi^c)}{:} \leftrightarrow {:}e^{i(\phi^++\phi^--\phi^\uparrow-\phi^\downarrow)}{:}, \tag{A16b}$$

applying the time-reversal operator yields

$$\hat{T}{:}e^{i(\phi^+-\phi^--2\phi^s)}{:}\hat{T}^{-1} \leftrightarrow {:}e^{i(\phi^+-\phi^--\phi^\uparrow+\phi^\downarrow)}{:}, \tag{A17a}$$

$$\hat{T}{:}e^{i(\phi^++\phi^--2\phi^c)}{:}\hat{T}^{-1} \leftrightarrow {:}e^{-i(\phi^++\phi^--\phi^\uparrow-\phi^\downarrow)}{:}, \tag{A17b}$$

from which Eq. (36) follows. Here the bosonic fields ϕ^+ and ϕ^- describe the helical modes of a QSHI. In a similar way, we obtain Eq. (39) by noting that the vertex operators have the following correspondence:

$${:}e^{i\boldsymbol{v}_1\cdot\boldsymbol{\phi}}{:} \leftrightarrow {:}e^{i(\phi^+-\phi^\uparrow)}{:} = -i{:}e^{i\phi^+}{:}{:}e^{-i\phi^\uparrow}{:}, \tag{A18a}$$

$${:}e^{i\boldsymbol{v}_2\cdot\boldsymbol{\phi}}{:} \leftrightarrow {:}e^{i(\phi^--\phi^\downarrow)}{:} = -i{:}e^{i\phi^-}{:}{:}e^{-i\phi^\downarrow}{:}, \tag{A18b}$$

which implies the following relations:

$$\hat{T}{:}e^{i\boldsymbol{v}_1\cdot\boldsymbol{\phi}}{:}\hat{T}^{-1} \leftrightarrow -{:}e^{i(\phi^--\phi^\downarrow)}{:}, \tag{A19a}$$

$$\hat{T}{:}e^{i\boldsymbol{v}_2\cdot\boldsymbol{\phi}}{:}\hat{T}^{-1} \leftrightarrow -{:}e^{i(\phi^+-\phi^\uparrow)}{:}. \tag{A19b}$$

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