Topological symmetry breaking: Domain walls and partial instability of chiral edges

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In two-dimensional topological systems chiral edges may exhibit a spectral change due to the formation of a Bose condensate and partial confinement in the bulk according to the topological symmetry breaking (TSB) mechanism. We analyze in detail what happens in the bulk as well as on the edge for a set of simple $c = 1$ chiral fractional quantum Hall systems. What this paper achieves is an explicit matching of the detailed field theoretic treatments of the bulk and boundary chiral boson and Chern-Simons theories, which allows for a precise interpretation of the TSB mechanism. TSB corresponds to the spontaneous breaking of a global discrete symmetry both in the bulk and on the edge and therefore to the appearance of domain walls in the bulk that may terminate in kinks on the edge. The walls, however, are locally metastable and break if a confined particle-hole pair is created. We list the $c = 1$ chiral models for which this type of instability may occur.

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I. INTRODUCTION

Topologically ordered phases of matter exhibit many fascinating properties, such as fractionalization of spin and charge and the possibility of non-Abelian braiding statistics [1]. One phenomenon that plays a pivotal role in almost all topological phases—be they topological insulators, topological superconductors, fractional quantum Hall (FQH) states *et cetera*—is the appearance of protected edge modes. In fact, for phases of which the bulk can be described by a Chern-Simons (CS) theory there is a bulk-boundary correspondence stating that the (1+1)-dimensional gapless edge can be described by a conformal field theory (CFT) and the bulk wave functions by the conformal blocks of the same CFT [2,3]. This strongly suggests that as long as the bulk is topologically ordered, no perturbations can destroy the chiral gapless edge theory. For nonchiral edges there is the possibility of counter-propagating edge modes gapping out and a criterium for stable edges is given in terms of the Lagrange subgroup [4,5], which has also been formulated for so-called symmetry-enriched phases [6–9] and classifications of gapped boundaries and domain walls for Abelian as well as non-Abelian models are given in Ref. [10–12].

In this paper, we point out a particular incompleteness of this picture. We show that a careful treatment of the problem necessarily has to take into account the possibility of Bose condensation in the bulk, corresponding to TSB. This formalism describes phase transitions between different topologically ordered phases due to the condensation of bosonic quasiparticles breaking the quantum group symmetry [13–15]. It has been successfully applied to many transitions between topological phases [16–19] and a more mathematical treatment has been formulated since [20–22]. Here we will show how simple chiral models corresponding to specific Laughlin states may be unstable due to TSB and decay into a different topological phase. The simplest of these are states describing quantum Hall fluids at filling fractions $\nu = 1/8$ and $\nu = 1/9$.

The second part of this paper will be devoted to showing how a careful treatment of TSB gives rise to a degeneracy in vacuum states in the broken phase. This eventually results in a spontaneous breaking of the symmetry when the system chooses one of these vacua as its ground state. This picture allows for the possibility of different domains within one broken phase and we give a description of the domain walls between them. It turns out that such a wall is metastable and can break through the creation of a confined particle-antiparticle pair. This pair may either annihilate in the bulk, or both particle and antiparticle will be expelled to the boundary of the sample. Furthermore, we find that the particles that are confined in the bulk are not confined at the boundary, but as they correspond to solitons in the edge theory they are gapped. This mechanism is interesting because it shows that gapping out by creating a conventional mass term and therefore breaking the chiral symmetry, is not the only way to create massive excitations.

The structure of this paper is as follows. In Sec. II we present all the ingredients of TSB that we use in the rest of this paper. The reader who is familiar with this formalism may skip this section. In Sec. III we apply TSB to chiral $U(1)$ states. First we show how the bosonic and fermionic Laughlin states can be expressed in terms of the $U(1)$ CFT and which of these states have a bosonic sector in their spectrum. Then we give the resulting phase structure after the condensation of such a boson. Section IV is devoted to domain walls and confinement. We show that a ground-state degeneracy is present and how it results in the formation of different domains. The confined particles in the bulk are associated to these walls and we find that they are no longer confined on the edge of our system. We end the paper in Sec. V by showing how to generalize the discussion to all unstable Laughlin states and we list the conclusions of the present paper.

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II. TOPOLOGICAL SYMMETRY BREAKING

We will recall the main ingredients of TSB here. The formalism involves three steps.

(i) We start from a (2+1)-dimensional topologically ordered phase with topological excitations that fall into irreps of some quantum group $\mathcal{A}$. The different sectors are labeled by their topological charge $a,b,\ldots \in \mathcal{A}$ and carry quantum numbers which are associated with their interaction under fusion and braiding.

The *fusion* of two particle types labeled by a and b is denoted by

$$a \times b = \sum_c N_{ab}^c\, c, \tag{1}$$

where $N_{ab}^c \in \mathbb{Z}_{\geqslant 0}$ gives the number of independent ways that a and b can fuse to c. These fusion rules are symmetric under interchange of a and b, i.e., $N_{ab}^c = N_{ba}^c$. Any physical system must have a unique sector representing the vacuum, $\mathbb{I}$, which means no particle at all. Also, we need to demand that for every particle a there is an antiparticle $\bar{a}$, in the sense that when these sectors fuse, they must have the vacuum sector in one of their fusion channels, i.e., $N_{a\bar{a}}^{\mathbb{I}} = 1$ for all sectors in the theory.

Quite naturally, when there is fusion the reverse can also be defined, which is called *splitting*. The fusion of two or more particles $\{a_1,\ldots,a_n\}$ to a sector b, spans a Hilbert space with dimension equal to the number of ways they can fuse to b. When a sector labeled by a is fused N times with itself the asymptotic growth of available fusion channels is given by $(d_a)^N$, where the positive real number d_a is called the *quantum dimension* of a. This quantum number is preserved under fusion, which means that for sectors that have fusion rules Eq. (1), their quantum dimensions obey

$$d_a d_b = \sum_c N_{ab}^c d_c. \tag{2}$$

Another way the topological excitation may interact is through *braiding*. When we have only two spatial dimensions at our disposal, the world lines of particles cannot cross and they form braids that encode the precise phase evolution of a multiparticle state. This is the origin of the existence of so-called *anyons* exhibiting fractional spin and statistics properties.

An important quantum number that we will use extensively is associated to a rotation of a particle of type a over a 2π angle, which we call the *topological spin* h_a. Under such a rotation, the wave function picks up a phase $\theta_a = e^{2\pi i h_a}$.

Interchanging two particles twice is equivalent to moving one particle around the other. Such an operation is called a *monodromy* and it may introduce a nontrivial phase factor in the two-particle wave function. The monodromy of anyons can also be expressed in terms of the spin. Assume that two particles a and b fuse to a specific channel c, then the monodromy is given by

$$M_{ab}^c = h_c - h_a - h_b. \tag{3}$$

(ii) A transition to a new phase can be driven whenever there is a bosonic sector present in the initial phase $\mathcal{A}$. A bosonic sector must have trivial spin, $h_b \in \mathbb{Z}$. A second condition is partial trivial self-monodromy, which means that upon fusion with itself it must have the trivial sector as one of it fusion channels, so $N_{bb}^{\mathbb{I}} = 1$. If such a sector is present, it may condense and form a new ground state, which breaks the initial symmetry $\mathcal{A}$ down to a residual one denoted by $\mathcal{T}$. Topological charge is now defined up to the charge of the condensate, which means that sectors of $\mathcal{A}$ may become identified with each other in the new phase $\mathcal{T}$. When there are non-Abelian excitations in the spectrum, certain sectors need to split up in order to obtain consistent fusion rules, but we will not encounter this situation in the present paper.

The splitting and identification of sectors can be summarized in the *branching rules*

$$a \rightarrow \sum_t n_a^t t, \tag{4}$$

where $a \in \mathcal{A}$, $t \in \mathcal{T}$, and n_a^t is a positive integer. For future reference, we call the sectors $a \in \mathcal{A}$ that branch to the same sector $t \in \mathcal{T}$ the *lifts* of t. Branching and fusion should commute, which severely restricts the branching rules and implies that quantum dimensions are preserved under branching, i.e., $d_a = \sum_t n_a^t d_t$.

(iii) Once the sectors of $\mathcal{T}$ and their fusion rules have been determined, there remains one last step in the process. When different sectors of the initial phase $\mathcal{A}$ become identified with each other in the intermediate phase, it does not imply that they have well-defined braiding interactions in $\mathcal{T}$. To illustrate this, consider two sectors $a_1, a_2 \in \mathcal{A}$, which become identified with each other, i.e., $a_1 \times b = a_2$, where b is a bosonic sector that condenses and drives the transition to $\mathcal{T}$. Now imagine we bring a_1 around b in the condensed phase $\mathcal{T}$. Since b represents the vacuum in $\mathcal{T}$ there should not be any nontrivial interaction between a_1 and b, and to ensure this we have to demand that the monodromy of any sector in the new phase with the condensate is trivial, i.e., $\theta_{a_2}\theta_{a_1}^{-1}\theta_b^{-1} = 1$. As b is a bosonic sector this boils down to $h_{a_2} - h_{a_1} \in \mathbb{Z}$. The sectors of $\mathcal{T}$ that do not have trivial braiding with the new vacuum are expelled from the bulk, because they cause a domain wall of finite energy in the condensate. In Sec. IV we will go into more detail regarding the interpretation of these confined particles.

After following all the steps presented above, we are left with a broken unconfined phase $\mathcal{U}$, which carries topological excitations with well-defined fusion and braiding relations. The boundary of $\mathcal{U}$ with a trivial phase, for instance the vacuum, is not simply described by $\mathcal{U}$, as would have been the case had we started from a phase $\mathcal{U}$ without applying TSB to an initial phase $\mathcal{A}$. The boundary must also contain the confined sectors that were expelled from the bulk. The correct description of the boundary is therefore given by the intermediate phase $\mathcal{T}$.

In the next section we recall the operator content of the compactified chiral boson theory and how it is related to the Laughlin states. We then proceed by showing how some of these states have a nontrivial boson in their particle spectrum that may drive a transition to a different Laughlin state.

III. CHIRAL $U(1)$ STATES AND TSB

We will derive for which Laughlin states at filling fraction $\nu = 1/M$ the particle spectrum contains a nontrivial boson, which could drive a phase transition to a state at different filling fractions.

The edge as well as the bulk can be described by a chiral boson with compactification radius $R = \sqrt{M}$. It is well known that for rational radius $R = \sqrt{\frac{2p'}{p}}$, with p,p' coprime there are only a finite number of sectors [23]. The primary fields of the extended theory are of the form

$$V_n = e^{in\phi/\sqrt{2pp'}}, \tag{5}$$

and their weights are

$$h_n = \frac{n^2}{4pp'}, \quad n = 0,\ldots,2pp' - 1. \tag{6}$$

The Hilbert space falls into irreps of the extended algebra, which will be denoted by $U(1)_{pp'}$. Note that all of the above is invariant under the interchange $p \leftrightarrow p'$, which corresponds to the invariance under modular transformations.

A. Laughlin states

We first turn to the bosonic Laughlin states at filling fraction $\nu = 1/M$, with M even. The theory can be described by a compactified chiral boson at radius $R = \sqrt{M}$ where we add the bosonic operator $V_M = e^{i\sqrt{M}\phi}$ to the chiral algebra, resulting in a finite number of sectors that form an algebra denoted by $U(1)_{M/2}$. The theory we start from has M sectors $V_n = e^{in\phi/\sqrt{M}}$, with conformal weights (spins)

$$h_n = \frac{n^2}{2M}, \quad n = 0,\ldots,M - 1. \tag{7}$$

The vertex operator that describes the physical boson of charge e is precisely the operator that is added to the chiral algebra $V_e = V_M$, with weight $h_e = M/2$, which is an integer for even M.

Moore and Read treated the fermionic case [2]. They choose the electron operator $V_e = e^{i\sqrt{M}\phi}$ as extended generator even though it has half-integer spin. The other operators are chosen such that they are mutually local with the electron operator. The weights of these operators are $h_n = \frac{n^2}{2M}$, with $n = 0,\ldots,M - 1$, and they indeed carry the right quantum numbers associated with the quasiholes of the FQH state. The reason why we do not adopt their formulation is that they do not distinguish between the electron operator and the trivial operator; therefore, we would never be able to distinguish between a fully gapped (edge) system and a $\nu = 1$ state.

We will follow a different strategy. For $\nu = 1/M$, with M odd, we start from a compactified boson at $R = \sqrt{M}$. Since M is odd we can choose $p' = M$ and $p = 2$ as coprime integers, resulting in a $U(1)_{2M}$ theory. The weights are

$$h_n = \frac{n^2}{8M}, \quad n = 0,...,4M - 1. \tag{8}$$

The sector with $n = 2M$ corresponds to the electron and it has spin $h_{2M} = M/2$. We want all the sectors to be local with respect to the electron operator. The monodromy is given by

$$\mathrm{M}_{n,e} = \frac{4Mn}{8M} = -\frac{n}{2}, \tag{9}$$

which means that only the even sectors are local and are good operators in this theory. Rewriting $2m = n$, we are left with $2M$ sectors $V_m = e^{im\phi/\sqrt{M}}$, labeled by $m = 0,...,2M - 1$, with weights $h_m = \frac{m^2}{2M}$. Let us call this theory $U^+(1)_{2M}$, where the $+$ denotes the even sectors. Effectively, the only difference with the literature is that we count up until twice the electron.

B. Unstable Laughlin states

We now turn to investigating which states have a nontrivial boson in their spectrum. Starting from a phase $\mathcal{A} = U(1)_{M/2}$, which has M sectors with spins given in Eq. (7), we will show that phases with filling fraction

$$\nu = \frac{1}{M} = \frac{1}{2l^2k}, \quad l = 2,3,\ldots, \quad k = 1,2,\ldots, \tag{10}$$

have at least one nontrivial boson that can drive a transition to a broken phase carrying less sectors.

The initial phase is $\mathcal{A} = U(1)_{l^2k}$, corresponding to a chiral boson compactified at $R = l\sqrt{2k}$, which has $2l^2k$ sectors with spins

$$h_n = \frac{n^2}{4l^2k}, \quad n = 0,1,\ldots,2l^2k - 1. \tag{11}$$

The smallest nontrivial bosonic sector is $b = 2lk$, which has spin $h_b = k$. When this boson forms a condensate, the other sectors arrange in orbits of length l under fusion with the boson

$$n \sim n + 2lk \sim \cdots \sim n + 2lk(l - 1), \tag{12}$$

which means that the sectors belonging to the same orbit get identified with each other, resulting in an intermediate phase $\mathcal{T}$, which has fusion rules Z_{2lk}.

To check which sectors become confined we consider the monodromy of a sector n with the boson

$$M_{n,b} = h_{n+b} - h_n - h_b = \frac{n}{l}. \tag{13}$$

The unconfined sectors can be expressed as $n = ml$, where $m = 0,1,\ldots,2k - 1$ and we see that they have spin $h_m = \frac{m^2}{4k}$, which we recognize as a $\mathcal{U} = U(1)_k$ theory. Of course, if this k is again of the form $k = l^2k'$ there will be other bosons left in the theory, which can also condense. The highest filling fractions that are not stable are $\nu = 1/8,1/16,1/18,\ldots$.

The same analysis can be performed for the fermionic Laughlin states and we will show that for filling fraction

$$\nu = \frac{1}{M} = \frac{1}{l^2k}, \quad l = 3,5,\ldots, \quad k = 1,3,\ldots, \tag{14}$$

a phase transition can occur.

Starting from $\mathcal{A} = U^+(1)_{2l^2k}$ corresponding to a chiral boson compactified at radius $R = l\sqrt{k}$, there are $2l^2k$ sectors with spins

$$h_n = \frac{n^2}{2l^2k}, \quad n = 0,1,\ldots,2l^2k - 1. \tag{15}$$

The charge e fermion is given by $n_e = l^2k$ and has spin $h_e = l^2k/2$. There is a nontrivial boson in this theory $b = 2lk$, which has spin $h_b = 2k$. When these particles condense, the original $\mathcal{A}$ sectors rearrange into orbits of length l, similar to Eq. (12).

The broken intermediate phase $\mathcal{T}$ has fusion rules Z_{2lk} and the monodromy of these sectors with the bosonic particle is

$$\mathrm{M}_{n,b} = \frac{2n}{l}. \tag{16}$$

This demonstrates that the unconfined particles are those for which $n = lm$ with $m = 0,1,\ldots,2k-1$ and their spins are given by $h_m = \frac{m^2}{2k}$.

They form a broken unconfined phase $\mathcal{U} = U^+(1)_{2k}$ at $R = \sqrt{k}$, corresponding to a fermionic Laughlin state at filling fraction $\nu = 1/k$. The highest fractions are $\nu = 1/9, 1/25, 1/27, \ldots$. For instance, the $\nu = 1/9$ breaks to a Laughlin state at $\nu = 1$, which is an IQH phase.

In this section we showed that simple chiral models are not completely stable. For those models that have a nontrivial bosonic sector in their spectrum a condensate may form. Topological charge is no longer conserved and certain sectors may disappear into the new vacuum or become confined. We would like to stress that we are dealing with chiral theories in this paper and we indeed find that the edge to the vacuum cannot be fully gapped even after TSB took place.

IV. DOMAIN WALL AND CONFINEMENT

In the previous section we applied TSB to drive a phase transition from a Laughlin state, which describes an Abelian FQH liquid at filling fraction $\nu = 1/M$, to another Laughlin state at larger filling fraction. In this section we take a closer look at the nature of the confined particles. We know that they must be expelled from the bulk of a broken phase, but what happens at the boundary? Moreover, we will demonstrate that different domains can appear in the broken $\mathcal{U}$ phase and show how the confined particles play a prominent role in the stability of the domain walls separating different domains.

A. Vertex operators and Wilson loops

To be definite and explicit we will consider the specific example of a bosonic Laughlin state at $\nu = 1/8$. At the end of this paper we will comment on how to generalize this to the other unstable Laughlin states discussed in the previous section.

Instead of simply considering the distinct topological sectors and their quantum numbers, we will cast them in a more familiar CFT form. The bulk is described by a $U(1)$ CS field at level $k = 8$ and the edge has gapless edge modes corresponding to a chiral boson theory compactified at radius $R = \sqrt{8}$. In the bulk the CS field can be written as a pure gauge $a_i(z) = \partial_i \phi(z)$, but on the edge there are dynamical degrees of freedom corresponding to $U(1)_4$. The Lagrangian describing the edge is given by

$$\mathcal{L} = \frac{1}{4\pi} \int dx (\partial_x \phi \partial_t \phi - \partial_x \phi \partial_x \phi), \tag{17}$$

where x is the coordinate along the edge [3]. It is a chiral boson with a global $U(1)$ invariance corresponding to the transformation $\phi(x) \to \phi(x) + f$.

The mode expansion of $\phi(x)$ compactified on a radius $R = \sqrt{8}$ on a cylinder of circumference L is

$$\phi(x) = \frac{2\pi \hat{N}}{\sqrt{8}L}\, x + \sqrt{8}\, \hat{\chi} + \text{oscillator modes}, \tag{18}$$

where we will discard the oscillator modes as we are only interested in the distinct topological sectors. The charge operator $\hat{N}$ and the zero mode $\hat{\chi}$ have commutation relations $[\hat{\chi}, \hat{N}] = i$.

Let us define several operators that play a crucial role in our subsequent analysis. The operators that create a localized topological charge are the normal ordered vertex operators,

$$V_n(x) =: e^{i \frac{n}{\sqrt{8}} \phi(x)} : , \tag{19}$$

where we will omit writing the normal ordering symbol from now on. The vertex operators are invariant under $\phi(x) \to \phi(x) + 2\pi R$ and have conformal weights $h_n = \frac{n^2}{16}$. They transform as irreps under the global symmetry $\phi(x) \to \phi(x) + f$: $V_n \to e^{inf/R} V_n$. The operators that measure charge correspond to nonlocal Wilson loops, defined as

$$W_q = \exp\left[\frac{iq}{\sqrt{8}} \int_{-L/2}^{L/2} a_x dx\right] = e^{2\pi i q \hat{N}/8}. \tag{20}$$

It is the exponentiated conserved global charge operator that is invariant under the global $U(1)$ symmetry. Note that this operator can be extended to the bulk where it becomes locally gauge invariant under the transformation $\phi(z) \to \phi(z) + f(z)$ for any value of q and we will use it at various values of q to probe the phase structure of the theory later on.

We will also employ open Wilson line operators,

$$W_q(x_1,x_2) = \exp\left[\frac{iq}{\sqrt{8}} \int_{x_1}^{x_2} \partial_x \phi(x) dx\right], \tag{21}$$

which are still invariant under the global $U(1)$ symmetry, but if extended to the bulk are not invariant under the local $U(1)$, which takes $\phi(z) \to \phi(z) + f(z)$. An interesting gauge invariant operator is obtained by attaching charged quasiparticles represented by vertex operators to the integer charged Wilson lines,

$$V_n(z_1) W_n(z_1,z_2) V_n^\dagger(z_2). \tag{22}$$

This operator is well defined in the bulk as well as on the boundary. From this expression it follows naturally that a quasiparticle of charge n cannot exist alone. There is always an antiparticle present and they are connected by a Wilson line that in the present case is just a Dirac string, a gauge artifact that can be moved around without changing the physics. This reflects the bulk-boundary correspondence: if we want to insert a single quasiparticle in the bulk there has to be an antiparticle somewhere on the boundary too.

Let us return to the edge theory. We label the distinct topological charge sectors by $n = 0, \ldots, 7$, which are the eigenvalue of $\hat{N}$, and they can be created by acting with the nonlocal operator

$$\bar{V}_n = \exp\left[i \frac{n}{\sqrt{8}L} \int_{-L/2}^{L/2} \phi(x) dx\right]. \tag{23}$$

These are nonlocal operators that act on the vacuum state as $\bar{V}_n|0\rangle = |n\rangle$ and commute with the charge operator as

$[\hat{N},\bar{V}_n] = n\bar{V}_n$, so they act as ladder operators on the charge eigenstates:

$$\bar{V}_n|n'\rangle = |n+n'\rangle, \quad \text{mod } 8. \tag{24}$$

The eight topological sectors form irreps of a global Z_8 group generated by $W_1 = e^{\frac{2\pi i}{8}\hat{N}}$ ($W_1^8 = 1$) and the ground state $|0\rangle$ is unique. The algebra of the operators that create a unit of charge and measure charge is

$$W_m V_n = V_n W_m e^{2\pi nmi/8}, \tag{25}$$

which holds for both $\bar{V}_n$ and $V_n(x)$.

Now that we have presented the relevant operator content of the chiral compactified boson of the $\mathcal{A}$ theory, we will move to a description of the formation of a condensate in this context.

B. Ground state degeneracy and vacuum expectation value

We are interested in which part of the topological structure of the bulk and boundary theory is preserved and what novel structures we may encounter if we assume that some nontrivial operator condenses. Phase transitions are usually accompanied by some order parameter obtaining a finite expectation value in the new phase, and the operators we have at our disposal are the vertex operators. Clearly, in the unbroken phase $\mathcal{A}$, they all have vanishing vacuum expectation value, but in the broken phase we will assume that for some V_b we have

$$|\langle V_b\rangle|^2 \neq 0. \tag{26}$$

This means that the ground state should be of the form

$$|n\rangle_b = \sum_t c_t|n+tb\rangle, \tag{27}$$

where t is an integer and c_t some coefficient, which should be chosen such that the states form an orthonormal set. In addition we require this new ground state to be invariant under rotations of 2π generated by $\hat{R} = e^{i2\pi\hat{N}^2/16}$, which acts on the states as

$$\hat{R}\ |n\rangle_b = \sum_t c_t e^{i2\pi(n+tb)^2/16}|n+tb\rangle. \tag{28}$$

This will further restrict the state to $n=0$ or $n=4$, and for both of these values of the charge we need to set $b=4$. So in this new phase the two possible ground states are

$$|0\rangle_\pm = \frac{1}{\sqrt{2}}(|0\rangle \pm |4\rangle). \tag{29}$$

The condition that the ground state is invariant under $\hat{R}$ is equivalent to demanding integer spin, which is a property a condensable sector should have. The difference with older work on TSB is that we recognize two different ground states instead of only $|0\rangle_+$, which may result in different domains due to spontaneous symmetry breaking as we will see below.

The operator V_4 has nonzero expectation value, which is very similar to creating Cooper pairs in a superconductor. In our case, we can freely create and annihilate particles of topological charge $n=4$. The states rearrange themselves as eigenstates of V_4,

$$|n\rangle_\pm = \frac{1}{\sqrt{2}}(|n\rangle \pm |n+4\rangle), \quad n = 0,\dots,3. \tag{30}$$

They form an orthonormal and complete set of representations of the group $G = Z_4 \otimes Z_2$, generated by W_2 and V_4. These two operators commute, $[W_2,V_4]=0$, and the action on the states is $W_2|n\rangle_\pm = e^{2\pi in/4}|n\rangle_\pm$ and $V_4|n\rangle_\pm = \pm|n\rangle_\pm$. In the broken phase the ground state is twofold degenerate and W_1 maps the two states onto each other, $W_1|0\rangle_\pm = |0\rangle_\mp$. These states carry a Z_2 charge generated by V_4, and the trivial Z_4 charge under W_2.

C. Domain walls

Since there is a twofold degenerate ground state in the broken phase, the system eventually chooses one of these states resulting in a spontaneous breaking of the Z_2 symmetry. However, it could happen that part of the system is in the $+$ ground state and the other part in the $-$ state. Intuitively this would result in different domains separated by domain walls carrying energy.

Let us first focus on the edge and start from a state where the entire edge is in the ground state $|0\rangle_+$. We apply the open Wilson line operator Eq. (21), which creates a domain in the $|0\rangle_-$ phase in between the points x_1 and x_2. At the edge we only have the global symmetry of shifting $\phi(x)$ by a constant. Clearly the Wilson line on the edge is invariant under this transformation and creating these different domains does not require the introduction of the V_1 sectors at the endpoints. The kinks located around x_1 and x_2, however, have finite energy, so restricting our consideration to the edge theory, we may conclude that the kinks are massive solitons, which are not confined as there is only vacuum in between them.

We may extend the operators $W_q(\mathcal{C})$ and $V_n(z)$ to well-defined operators referring to closed loops $\mathcal{C}$ and points (punctures) z in the bulk. In discussing the phase structure of the broken phase $\mathcal{U}$, it is important to make a clear distinction between whether we permit insertions in the bulk of the confined sector $V_1(z)$ or not. This distinction can be made because there are two scales in the problem: the gap or mass of the V_n excitations and the presumably smaller energy scale associated with the condensate. It is most natural to start with a situation where we do not include them, but we may still consider the Wilson loop operators with arbitrary q and in particular also with $q=1$. The interpretation of closed loops is similar to that on boundary, the Wilson loop operator now creates a domain of $-$ vacuum in the bulk, and a domain wall along the contour $\mathcal{C}$. It is interesting to deform this configuration as indicated in Fig. 1. In the figures on the left we have sketched the situation just discussed. However, the loop can be moved around at will, and in particular we may put it partially along the edge as in the third figure. How do we interpret this physically? When looked at from the perspective of the boundary we see that at the points A and B where the closed loop leaves the boundary, the vacuum on the boundary flips and therefore there should be a kink in the field at these points. The other part of the contour, going from B to A through the bulk, is a massive domain wall ending at the kink-antikink pair. The situation is comparable to the states created by $V_n(z)$, which represent massive localized anyons in the bulk and massless modes on the edge.

When there are different domains in the bulk we have to probe the system locally to measure which domain we are

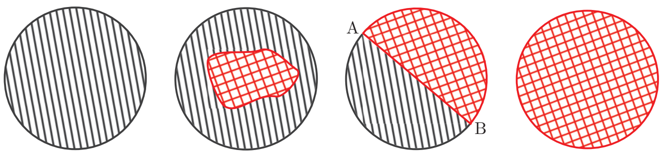

FIG. 1. (Color online) A disk in the broken phase $\mathcal{U}$ with two possible $\pm$ vacua indicated by gray stripes and a red mesh, respectively. On the left, the disk is entirely in the $+$ vacuum. Next, a Wilson loop W_1 is created, which has a $-$ vacuum inside and a physical domain wall along the loop. It can be deformed to lie partially on the boundary as indicated in the third figure. The boundary has two different domains and the kinks located around point A and B carry energy. The wall going across the disk through the bulk also carries energy. In the last figure the entire Wilson loop lies on the boundary.

in. We cannot simply use $V_4(z)$, as it is not gauge invariant. Instead, we need to use the gauge invariant object in Eq. (22) with $n = 4$, which gives $+1$ if the end points are in the same vacuum and -1 if they are in different vacua, i.e., crossing the wall either an even or odd number of times. The coloring of domains in Fig. 1 is well defined at this stage; therefore, in this restricted setting without V_1 quasiparticles the vacuum states can be unambiguously carried over to the bulk.

D. Confined particles

So far we have established a detailed, consistent picture of the physics of TSB except that we have to consider one more ingredient, and that is the role of the confined vertex operator $V_1(z)$ in the bulk of $\mathcal{U}$. We will see below that it plays a major role in the stability of domain walls.

We can distinguish two types of instability. One is a *global* instability, meaning that a closed loop in the bulk can shrink to zero. Since the domain wall has a fixed energy per unit length, shrinking lowers the total energy of the configuration and there is no topological obstruction to fully contract. More interesting is a *local* topological instability of the wall, where it can in principle break upon the creation of a V_1-$V_1^\dagger$ particle-hole pair attached to the new endpoints of the broken wall. This process is depicted in Fig. 2. Whereas the V_1 quasiparticles were not confined on the boundary as we argued before, they are linearly confined in the bulk exactly because they have to be attached to a domain wall of finite energy. The walls are metastable because the creation of a massive pair requires an energy of at least twice the particle gap. Another important consequence of the metastability of the wall in the bulk is that the domain structure of the vacuum is no longer protected.

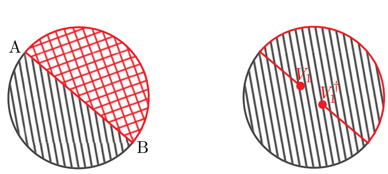

FIG. 2. (Color online) At the left a closed W_1 loop is depicted and on the right the loop is broken through the creation of a V_1-$V_1^\dagger$ pair. Probing with a W_4 Wilson line from a point in the bulk to the boundary is still consistently defined.

A topological argument explaining this goes as follows [24]. In the unbroken phase V_1 is present and we have a full $U(1)/Z_8$ gauge group in the bulk with topological flux and particle sectors $\pi_1(U(1)/Z_8) = \pi_0(Z_8) = Z_8$, corresponding to representations of the Z_8 group generated by W_1. After breaking, the gauge group is formally changed to $U(1)/Z_4$, corresponding to the Z_4 subgroup of Z_8 consisting of the even elements, and $\pi_0(Z_4)$ refers to the even sectors V_{2n}. The homotopy sequence of interest here is

$$\pi_0(Z_4) \to \pi_0(Z_8) \to \pi_0(Z_8/Z_4), \tag{31}$$

implying that the image of the first mapping is the kernel of the second. In physical terms this means that the even sectors of $\pi_0(Z_8)$ get mapped onto the trivial sector of $\pi_0(Z_8/Z_4) = Z_2$, where the latter group labels per definition the new types of domain wall that arise in the broken phase. In other words, the odd charges of the $\pi_0(Z_8)$ are mapped onto the nontrivial domain walls and are therefore confined, exactly as advertised.

The overall picture remains completely consistent if one takes into account that now there are two ways to go from a $+$ state created by $(V_n(x) + V_{n+4}(x))|0\rangle$ at a position x on the edge to the left of point A in Fig. 2, to the corresponding $-$ state with x to the right. The first option is to "cross" the wall by transforming a vertex operator with W_1 using Eq. (25). The other is by moving the vertex operators involved through the bulk and around the endpoint at the opening in the wall, which in fact means acting with the monodromy operator.

Let us demonstrate this explicitly by considering this question in the original unbroken $\mathcal{A}$ theory, and see what can be carried over to the broken phase. Given that the monodromy, i.e., encircling an anyon V_n with V_m (in the original $\mathcal{A}$ theory with $n,m = 0,\dots,7$), yields a phase factor

$$\exp[2\pi i(h_{m+n} - h_n - h_m)] = e^{2\pi i nm/8}, \tag{32}$$

we can make some important observations. Since the $\mathcal{A}$ sectors become combined (identified) as in Eq. (30) in the broken phase, encircling them around another $\mathcal{T}$ sector gives different monodromy phases. For $n,m = 0,\dots 3$ and $k,k' = 0,1$, the different phases of the monodromy can be expressed as

$$\frac{2\pi}{16}[(n+m+4k+4k')^2 - (n+4k)^2 - (m+4k')^2]$$
$$= \frac{2\pi}{8}(nm + 4nk' + 4mk) \mod 2\pi. \tag{33}$$

Two $\mathcal{T}$ sectors have consistent braiding if their monodromy is independent of k and k', which leaves us with only the sectors $n = 0$ and $n = 2$ as expected. One also may verify that only these sectors are mutually local with respect to the new vacuum and therefore survive as unconfined particles.

Note that from the monodromy phase we learn that if we have the fundamental quasiparticle corresponding to V_1 in the bulk and bring the new vacuum $V_0 \pm V_4$ around it, that would map the two vacua onto each other; i.e., $(V_0 \pm V_4) \to (V_0 \mp V_4)$. This means that the net effect of moving around the confined particle is the same as crossing the wall as we have described above, where the $\pm$ state transforms under W_1 and gets mapped to the $\mp$ state at the other side of the wall.

This shows once more that the wall is not locally stable, and it can break under the creation of a fundamental quasiparticle-hole pair, each of them remaining attached to the newly created end points. Alternatively one may consider starting from a disk entirely in one of the vacua and creating the V_1-$V_1^\dagger$ pair somewhere in the bulk. When the particles are moved apart they stay connected by a wall which explicitly follows from Eq. (22).

So their appearance will be exponentially suppressed not only because of their mass but also because of their interaction energy that rises linearly with distance, and they indeed are confined.

V. GENERALIZATIONS AND CONCLUSIONS

In this paper we mainly focused on the specific Laughlin state at $\nu = 1/8$ in order to clearly present our results, but the construction is easily generalized to the other unstable Laughlin states.

Even though the notation for fermionic and bosonic states is a bit different,

$$\text{Bosonic:} \quad M = 2l^2k \quad \mathcal{A} = U(1)_{l^2k} \quad l \geqslant 2, k \geqslant 1, \tag{34}$$

$$\text{Fermionic:} \quad M = l^2k \quad \mathcal{A} = U^+(1)_{2l^2k} \quad l \geqslant 3, k \geqslant 1, \tag{35}$$

what they have in common is that V_{2lk} acquires a vacuum expectation value, resulting in l different vacua. There are $l-1$ different Wilson loops, Eq. (20), that create the different domains, and there are $l-1$ distinct confined particles corresponding to V_n, with $n = 1,2,\ldots,l-1$, that can be attached at the end of a Wilson line as in Eq. (21).

The generalization to other Laughlin states is straightforward, but this work also suggests many other interesting generalizations, such as analyzing the field theory of TSB in general non-Abelian theories with a WZW theory on the edge.

Let us end this chapter by summarizing our results. Even though there can be no backscattering in a chiral system, we have shown that certain chiral edges labeled by two integers l and k are not entirely protected because TSB may occur. When a condensate of bosonic particles forms, certain topological sectors can disappear in this condensate and others become confined. After breaking the topological symmetry we are left with a phase that is still chiral but has less sectors in its spectrum.

Furthermore, we have extended our understanding of the original TSB picture proposed in 2002, by finding an explicit expression of an order parameter that obtains a finite expectation value in the broken phase. This leads to degenerate ground states and different domains separated by domain walls. Moreover this gives us a good understanding of the confined particles in the bulk, which turn out to be unconfined on the edge of the sample. We give simple criteria for the stability of the domain walls. This work also clearly shows the essential observable differences between an exact $\mathcal{U}$ theory and the $\mathcal{U}$ phase obtained after applying TSB to an $\mathcal{A}$ theory.

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