

Abelian and non-Abelian states in $\nu = 2/3$ bilayer fractional quantum Hall systemsMichael R. Peterson,¹ Yang-Le Wu,² Meng Cheng,³ Maissam Barkeshli,³ Zhenghan Wang,^{3,4} and Sankar Das Sarma²¹*Department of Physics & Astronomy, California State University Long Beach, Long Beach, California 90840, USA*²*Joint Quantum Institute and Condensed Matter Theory Center, Department of Physics, University of Maryland, College Park, Maryland 20742, USA*³*Station Q, Microsoft Research, Santa Barbara, California 93106-6105, USA*⁴*Department of Mathematics, University of California, Santa Barbara, California 93106, USA*

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There are several possible theoretically allowed non-Abelian fractional quantum Hall (FQH) states that could potentially be realized in one- and two-component FQH systems at total filling fraction $\nu = n + 2/3$, for integer n . Some of these states even possess quasiparticles with non-Abelian statistics that are powerful enough for universal topological quantum computation, and are thus of particular interest. Here we initiate a systematic numerical study, using both exact diagonalization and variational Monte Carlo, to investigate the phase diagram of FQH systems at total filling fraction $\nu = n + 2/3$, including in particular the possibility of the non-Abelian Z_4 parafermion state. In $\nu = 2/3$ bilayers we determine the phase diagram as a function of interlayer tunneling and repulsion, finding only three competing Abelian states, without the Z_4 state. On the other hand, in single-component systems at $\nu = 8/3$, we find that the Z_4 parafermion state has significantly higher overlap with the exact ground state than the Laughlin state, together with a larger gap, suggesting that the experimentally observed $\nu = 8/3$ state may be non-Abelian. Our results from the two complementary numerical techniques agree well with each other qualitatively.

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I. INTRODUCTION

Multicomponent fractional quantum Hall (FQH) states appear in a wide variety of two-dimensional electron systems (2DES) [1], such as multilayer or multisubband quantum wells [2], systems with small Zeeman energy where the electron spin plays an active role, and systems with multiple valley degrees of freedom, such as graphene [3–7], silicon [8], and AIAs [9,10]. These systems offer several tunable parameters, which allow the observation of rich zero temperature phase diagrams involving topologically distinct FQH states even at a fixed total filling fraction, and indeed novel FQH phases of multicomponent systems have been experimentally observed. Most notable perhaps is the observation of the so-called 331 Abelian even-denominator FQH state in half-filled bilayer systems [11–13]. However, in many cases, little is known about the myriad possible FQH phases and phase transitions that can be experimentally realized in multicomponent 2DES.

Recently, motivated by the possibility of a non-Abelian state at $\nu = 5/2$ in GaAs quantum wells [14,15], there have been detailed numerical studies at total filling fraction $\nu = n + 1/2$ (n integer) in two-component systems [16]. While $\nu = n + 1/2$ has been studied in great detail, the problem at $\nu = n + 2/3$ has received comparatively less attention from numerical studies [17–24]. Such systems were first studied experimentally over 20 years ago, where a two-component to single-component phase transition was observed in monolayer (presumably due to spin) and bilayer systems [25–28]. There are three Abelian FQH states that can be realized at $\nu = 2/3$: the 330 state,

$$\Psi_{330} = \prod_{i<j} (z_i - z_j)^3 (w_i - w_j)^3 \prod_{i,j} (z_i - w_j)^0, \quad (1)$$

consisting of two decoupled $1/3$ Laughlin states in each layer where z_i and w_i , for $i = 1, \dots, \frac{1}{2}N$, are the complex

coordinates of the electrons in the two layers, and here and hereafter we have omitted the Gaussian factor $\exp(-\sum_i |z_i|^2/4l^2)$ for all wave functions, a pseudospin singlet Abelian state, here called the 112 state,

$$\Psi_{\text{singlet}} = \mathcal{P}_{\text{LLL}} \prod_{i<j} |z_i - z_j|^2 |w_i - w_j|^2 \Psi_{112}^*, \quad (2)$$

where \mathcal{P}_{LLL} is the LLL projection operator, and the particle-hole conjugate of the $1/3$ Laughlin state, referred to here as the $2/3$ Laughlin state,

$$\Psi_{\text{P-H}} = \mathcal{P}_{\text{LLL}} \prod_{i<j} (z_i - z_j)^2 \Phi_{\nu=-2}, \quad (3)$$

where $\Phi_{\nu=-2}$ is the wave function for the $\nu = -2$ integer quantum Hall state. The pseudospin singlet 112 state can be easily understood within composite fermion theory as composite fermions filling the lowest spin-up and spin-down levels in a reversed effective magnetic field [23,29,30]. Early numerical work on $\nu = 2/3$ bilayers considered the overlap of model wave functions with the exact ground state of the system for $N = 6$ electrons on a torus [31], finding these three phases in the two-component 2D system (for the monolayer spinful system, the 330 state is unlikely).

Different theoretical studies have suggested five possible exotic non-Abelian FQH states can occur at $\nu = 2/3$, yet have not been numerically investigated (see Table I). These include: the Z_4 parafermion FQH state [32–34],

$$\Psi_{Z_4} = \mathcal{A}[\Psi_{330}], \quad (4)$$

where \mathcal{A} is an antisymmetrization over all electron coordinates, a Fibonacci state based on $SU(3)_2$ Chern-Simons theory [35,36], interlayer and intralayer Pfaffian states [37,38],

$$\Psi_{\text{Inter Pf}} = \text{Pf}\left(\frac{1}{z_i - z_j}\right) \text{Pf}\left(\frac{1}{w_i - w_j}\right) \Psi_{221} \quad (5)$$

TABLE I. Candidate Abelian and non-Abelian FQH states at total filling fraction $\nu = 2/3$. On the sphere these states occur at different shifts $\mathcal{S} \equiv \frac{3}{2}N - N_\Phi$, where N_Φ is the number of flux quanta. The Fibonacci state, as a single-component system, has a shift of 6; as a two-component system, it has a shift of 3 per layer.

Possible states at $\nu = 2/3$	Type	Shift, \mathcal{S}
330 state	Abelian	3
Pseudospin singlet 112	–	1
Particle-hole conjugate of 1/3 Laughlin	–	0
Z_4 parafermion	Non-Abelian	3
Bilayer Fibonacci	–	3
Interlayer Pfaffian	–	3
Intralayer Pfaffian	–	3
Bonderson-Slingerland hierarchy	–	4

and

$$\Psi_{\text{Inter Pf}} = \text{Pf}\left(\frac{1}{x_i - x_j}\right) \Psi_{221}, \quad (6)$$

respectively, with x_i running over the coordinates of the N electrons in both layers with

$$\Psi_{221} = \prod_{i < j} (z_i - z_j)^2 (w_i - w_j)^2 \prod_{i,j} (z_i - w_j)^1, \quad (7)$$

and a Bonderson-Slingerland hierarchy state [39]. It should be noted that the Z_4 parafermion and the intralayer Pfaffian states are defined only when the number of electrons N is divisible by 4.

As a result, the $N = 6$ overlap study [31] did not actually rule out the possibility of having stable non-Abelian phases even in the lowest Landau level. The Z_4 and Fibonacci states have been shown theoretically to exhibit continuous phase transitions from the 330 state [36,40], suggesting these states might be stabilized nearby more conventional ones if appropriate microscopic parameters are found and tuned experimentally. The goal of our work is to investigate numerically the possible existence of exotic non-Abelian $2/3$ (or generally, $n + 2/3$) FQH states in realistic 2DES.

The Fibonacci FQH state contains the non-Abelian Fibonacci quasiparticle, whose braiding statistics is known to be powerful enough to be utilized for universal topological quantum computation (TQC) [41]. The Z_4 parafermion FQH state is based on the $SU(2)_4$ topological quantum field theory, which has been discovered to allow for universal TQC [42,43]. The Bonderson-Slingerland hierarchy state at $\nu = n + 2/3$, and the interlayer Pfaffian state, can also be used for universal TQC if realized on topologically nontrivial spaces with topological operations known as Dehn twists [44,45] that can be realized in a physically realistic experimental setup [46,47]. It is thus timely to revisit the $\nu = 2/3$ bilayer phase diagram numerically and investigate the possibility of realizing these non-Abelian states.

In this work we carry out a study of two-component FQH systems at total filling fraction $\nu = n + 2/3$. We analyze the relative stability of the three Abelian states and the non-Abelian Z_4 state through exact diagonalization and variational Monte Carlo studies that also consider the inter/intralayer Pfaffian states. In the lowest Landau level (LLL), our results

are consistent with the phase diagram proposed previously [31] and we importantly find that the Z_4 state is not competitive relative to the other Abelian states. However, in the limit of large interlayer tunneling in the second Landau level (SLL), at $\nu = 8/3$, our results suggest that the Z_4 state is preferable relative to the possible Abelian states. This unexpected new finding suggests the already experimentally observed $8/3$ FQH state may be the exotic Z_4 non-Abelian state, rather than the usual Abelian Laughlin state. Given the existence of the $5/2$ FQH state in the SLL, thought to be the non-Abelian Moore-Read state, the possibility that the SLL $8/3$ FQH state might also be a (different) non-Abelian state is plausible and consistent with the fact that the experimental $8/3$ state typically is considerably weaker than the $5/2$ state as manifested in the measured energy gaps [48–59].

We consider the Hamiltonian describing two quantum Hall layers with N total spin-polarized electrons, separated by a distance d , with interlayer electron tunneling strength Δ :

$$H = \sum_{i < j}^N \left[\sum_{\sigma}^{\uparrow, \downarrow} V_{\text{intra}}(|\mathbf{r}_i^{\sigma} - \mathbf{r}_j^{\sigma}|) + V_{\text{inter}}(|\mathbf{r}_i^{\uparrow} - \mathbf{r}_j^{\downarrow}|) \right] - \frac{e^2}{\epsilon l} \Delta S_x, \quad (8)$$

where \mathbf{r}_i^{σ} is the position of the i th electron in layer σ , and l is the magnetic length. The intralayer Coulomb interaction is given by $V_{\text{intra}}(r) = \frac{e^2}{\epsilon r}$, while the interlayer interaction is given by $V_{\text{inter}}(r) = \frac{e^2}{\epsilon \sqrt{r^2 + d^2}}$ (ϵ is the dielectric of the host semiconductor). The interlayer tunneling term is written as the total pseudospin S_x operator, with Δ the interlayer tunneling strength in units of $\frac{e^2}{\epsilon l}$.

II. PHASE DIAGRAM IN THE LOWEST LANDAU LEVEL

We first consider the $\nu = 2/3$ bilayer quantum phase diagram in the LLL. The Hamiltonian Eq. (8) has two dimensionless parameters: d/l , the ratio of the inter and intralayer Coulomb interactions, and Δ , the ratio of the interlayer tunneling to the intralayer Coulomb interaction. The relative stability of the three Abelian states was studied through wave function overlap calculations for $N = 6$ electrons (3 per layer) on the torus [31]. We revisit this for larger systems using exact diagonalization (ED) for up to $N = 12$ electrons in the spherical geometry. In this setup, states with different topological orders may appear at different shifts $\mathcal{S} \equiv \frac{3}{2}N - N_\Phi$, where N_Φ is the number of flux quanta.

Figures 1(a)–1(f) displays our numerical results for the overlaps of the model wave functions for the 330, singlet 112, and $2/3$ Laughlin states, with the exact Coulomb ground state at shifts $\mathcal{S} = 3, 1$, and 0 , respectively, together with the energy gaps for $N = 8$. The energy gap is taken as the difference between the angular momentum $L = 0$ ground state and the first excited state (if the ground state has $L \neq 0$, the gap is set to zero). We can combine the energy gaps at different shifts into a single function $\delta(d/l, \Delta)$ by choosing the maximal gap among the different shifts. We note this gap is not necessarily the transport gap measured experimentally but the gap connected to the robustness of the phase—in many cases they are known to be qualitatively similar. Similar results are obtained for $N = 6$ and 10 (see Figs. 2 and 3). We do not compute the overlap with the Z_4 parafermion state for $\mathcal{S} = 3$ and $N = 8$

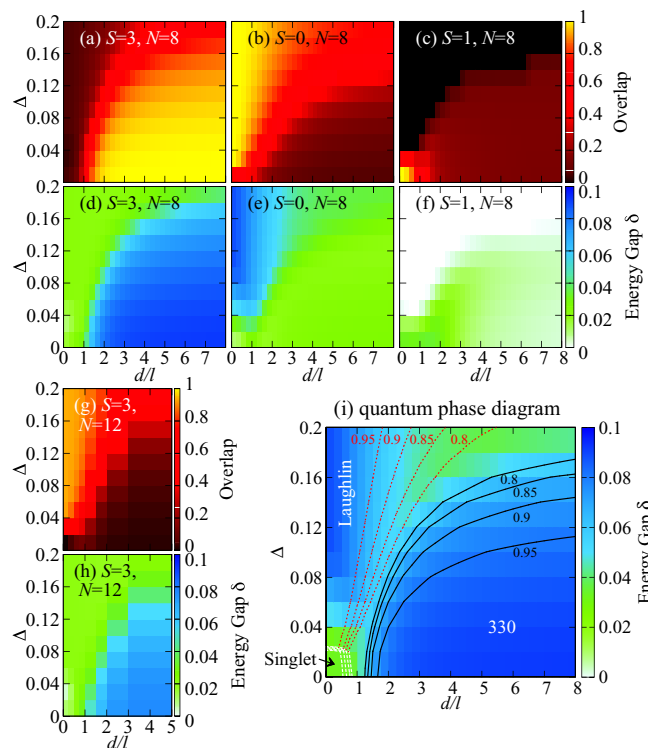


FIG. 1. (Color online) (a)–(f) Shifts of $\mathcal{S} = 3, 0$, and 1 from left to right for $N = 8$ electrons. (a)–(c) The overlap with Ψ_{330} , $2/3$ Laughlin, and the pseudospin singlet (112), respectively. (d)–(f) The energy gap. (g) The overlap with Z_4 state for $N = 12$ at $\mathcal{S} = 3$. (h) The gap for $N = 12$ and shift $\mathcal{S} = 3$. (i) Displays the resulting quantum phase diagram.

for three reasons. One is the Z_4 state is a single-component state and for $N = 8$ electrons there is only one possible $L = 0$ state. The second reason is the Z_4 state has four-electron clustering properties that cause it to vanish exactly unless $N \bmod 4 = 0$. Hence, one must consider at least $N = 12$ electrons, see Figs. 1(g) and 1(h). The third reason is the gap at $\mathcal{S} = 3$ in the single-component limit is significantly below the gap at $\mathcal{S} = 0$ corresponding to the $2/3$ Laughlin state.

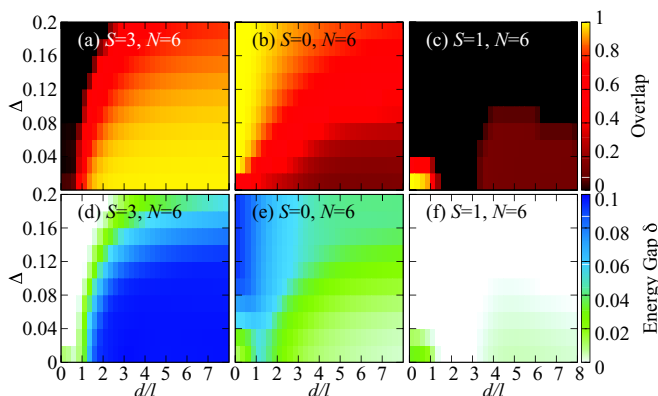


FIG. 2. (Color online) (a)–(f) Shifts of $\mathcal{S} = 3, 0$, and 1 from left to right for $N = 6$ electrons in the lowest Landau level. (a)–(c) The overlap with Ψ_{330} , $2/3$ Laughlin, and the pseudospin singlet (112), respectively. (d)–(f) The energy gap.

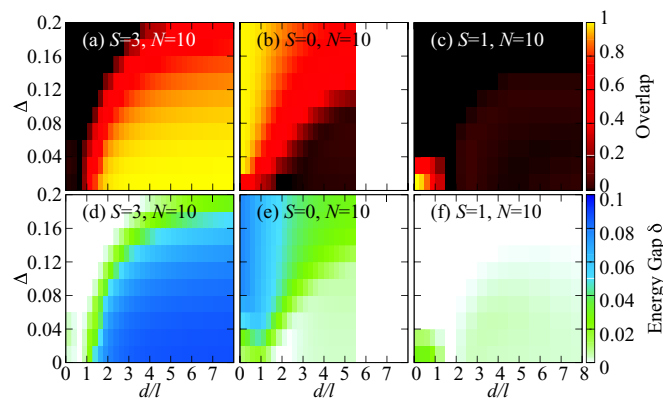


FIG. 3. (Color online) (a)–(f) Shifts of $\mathcal{S} = 3, 0$, and 1 from left to right for $N = 10$ electrons in the lowest Landau level. (a)–(c) The overlap with Ψ_{330} , $2/3$ Laughlin, and the pseudospin singlet (112), respectively. (d)–(f) The energy gap. The white space beyond $d/l > 5.5$ for $\mathcal{S} = 0$ was not calculated and therefore left blank.

We produce a phase diagram for the bilayer system taking into account both the overlaps and the energy gaps at different shifts. The topological order is identified from the model wave function with the highest overlap with the ground state, and its stability is characterized by the energy gap. Figure 1(i) shows a plot of the gap function $\delta(d/l, \Delta)$ for $N = 8$ and contour lines showing the wave function overlaps. Our results for $N = 6, 10$ electrons (not shown) are consistent with Fig. 1(i). We emphasize this approximate phase diagram matches remarkably well with Ref. [31], determined by wave function overlap and topological degeneracy on the torus.

To investigate the relative stability of the Z_4 parafermion state, we consider the two-component system for $N = 12$ particles. In Figs. 1(g) and 1(h) we show the overlap of the exact ground state with the Z_4 parafermion state, together with the value of the energy gap at shift $\mathcal{S} = 3$. While the Z_4 state has a maximum overlap of ≈ 0.93 in the single-component limit, the Laughlin state has a much higher overlap of ≈ 0.99 . Furthermore, the system possesses a much larger energy gap at the $2/3$ -Laughlin shift relative to the shift of the Z_4 parafermion state.

To further assess the stability of the Z_4 state compared to the $2/3$ Laughlin state, we can consider the single-component limit of Eq. (8), obtained for strong tunneling $\Delta \gg 1$ and small $d/l \ll 1$, i.e., a single quantum well. The smaller dimension of the Fock space in this limit allows us to consider $N = 16$ electrons. Here we study a particularly realistic model that includes Landau level (LL) mixing [parametrized by the ratio of the cyclotron energy to the Coulomb energy $\kappa = (\hbar\omega_c)/(e^2/\epsilon l)$] and finite width of the single quantum well (parametrized by well width w/l) [60]. Specifically, our realistic Hamiltonian is

$$H_{\text{realistic}} = \sum_m V_m^{(2)}(w/l, \kappa) \sum_{i < j} P_{ij}(m) + \sum_m V_m^{(3)}(w/l, \kappa) \sum_{i < j < k} P_{ijk}(m), \quad (9)$$

where $P_{ij}(m)$ and $P_{ijk}(m)$ are operators that project onto pairs (i, j) or triplets (i, j, k) of electrons with relative angular

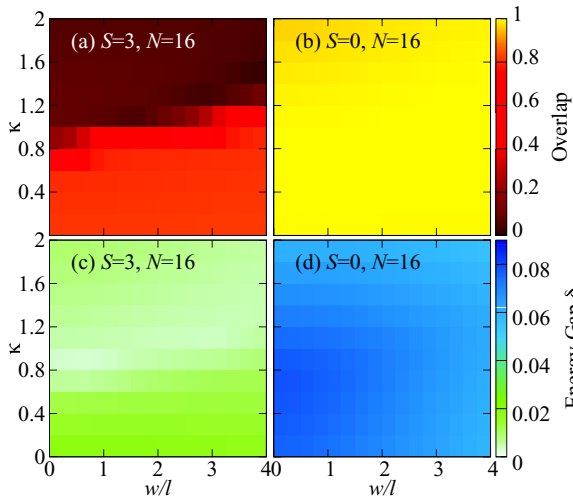


FIG. 4. (Color online) Overlap and gap calculations for $N = 16$ electrons in the LLL in the single-component limit. Top left (a) shows overlaps with the Z_4 state, at $S = 3$; top right (b) shows overlaps with the Laughlin state, at $S = 0$. Lower panels (c) and (d) show the gaps at $S = 3$ and $S = 0$.

momentum m . The Hamiltonian is parametrized by two- and three-body pseudopotentials $V_m^{(2)}(w/l, \kappa)$ and $V_m^{(3)}(w/l, \kappa)$, respectively. The two-body pseudopotentials are renormalized by Landau level mixing corrections which include the effects of virtual transitions of electrons (holes) to unoccupied (occupied) Landau and subband levels to lowest order in the Landau level mixing parameter κ . Landau level mixing also produces an emergent three-body term that explicitly breaks particle-hole symmetry. This Hamiltonian is described in great detail in Ref. [60] and was recently implemented in a numerical study of the FQHE at $\nu = 5/2$ [61]. In this work we restrict our attention to $w/l < 4$; wide quantum well systems are often better described as bilayers. Our results for the LLL are displayed in Fig. 4 where the Laughlin state is clearly shown to be preferable. The Z_4 overlap at $S = 3$ is large (≈ 0.82) and essentially decreases monotonically with κ and is robust to width w/l . At $S = 0$ the Laughlin state has an overlap of nearly unity (≈ 0.99) and is robust to κ and w/l . Both $S = 0$ and 3 have nonzero gaps, but the gap at $S = 0$ is nearly three times larger than $S = 3$. Both overlaps and gaps are robust to varying κ and w/l . Based on these results, we do not expect the Z_4 state in the bilayer system at $\nu = 2/3$ in the LLL. Our conclusions for the LLL, based on ED, are further corroborated with variational Monte Carlo (see below).

III. SECOND LANDAU LEVEL

Now that we have confidently ruled out the Z_4 state in the LLL, we turn our attention to the second Landau level (SLL). Repeating the overlap and gap calculations with the SLL pseudopotentials, we obtain results that are quite different from the LLL. In particular, we find in the single-component limit ($d/l \ll 1$ and $\Delta \gg 1$) the gap at shift $S = 3$ is significantly larger than at $S = 0$ suggesting the ground state in this limit might not be the $2/3$ Laughlin state, but rather an alternative state with $S = 3$.

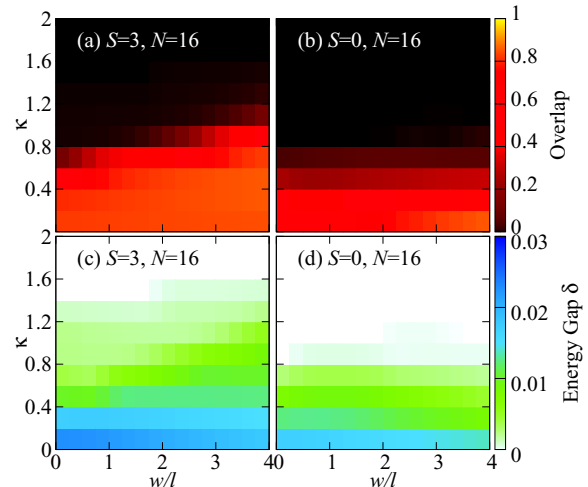


FIG. 5. (Color online) Overlaps (a) and (b) and energy gaps (c) and (d) for $N = 16$ electrons in the SLL in the single-component limit. See caption of Fig. 4.

Leaving the exploration of the full two-component phase diagram for the next section, we now take a closer look at the SLL in the single-component limit. We use the same realistic model introduced earlier (but at filling $8/3$), and focus on the competition between the Z_4 and the Laughlin states. Surprisingly, we find that the Z_4 state appears favored over the Laughlin state according to both overlap and gap calculations, as shown in Fig. 5. In the SLL, the overlap with the Z_4 state is qualitatively similar to the LLL, i.e., it is nearly 0.83 – 0.84 for small κ and decreases to zero as κ is increased. The Laughlin state at $S = 0$ has a smaller overlap of 0.64 – 0.8 , increases with w/l , and monotonically decreases with κ . Importantly, the gap is approximately 1.5 times larger at $S = 3$ than it is at $S = 0$. Our results for $N = 12$ electrons are qualitatively similar, but with quantitatively higher overlaps for both the Z_4 and Laughlin states. Last, we note that in the limit of zero LL mixing ($\kappa = 0$) our Hamiltonian is particle-hole symmetric and our results for the spin-polarized $\nu = 8/3$ state should translate to $\nu = 7/3$ where some recent theoretical studies have suggested that the FQH state at $7/3$ is likely in the Laughlin universality class [62,63].

IV. VARIATIONAL ENERGIES

The discussion so far has focused on the fate of the Z_4 state in comparison with the three Abelian phases, without exploring other non-Abelian possibilities. One may also worry about the finite-size effect in the ED results. To address these concerns, we have performed variational energy calculations that also include the interlayer and intralayer Pfaffian states at shift $S = 3$, for much larger system sizes. The energy expectation value of Eq. (8) of the three Abelian states and the Pfaffians are computed using Monte Carlo for up to $N = 60$ electrons, with sample size 10^7 . We refer to Ref. [64] for the details of the Monte Carlo energy calculation and Ref. [23] for the efficient evaluation of composite fermion wave functions. It turns out that the antisymmetrization used to construct Ψ_{Z_4} [Eq. (4)] is prohibitively expensive for numerical calculations,

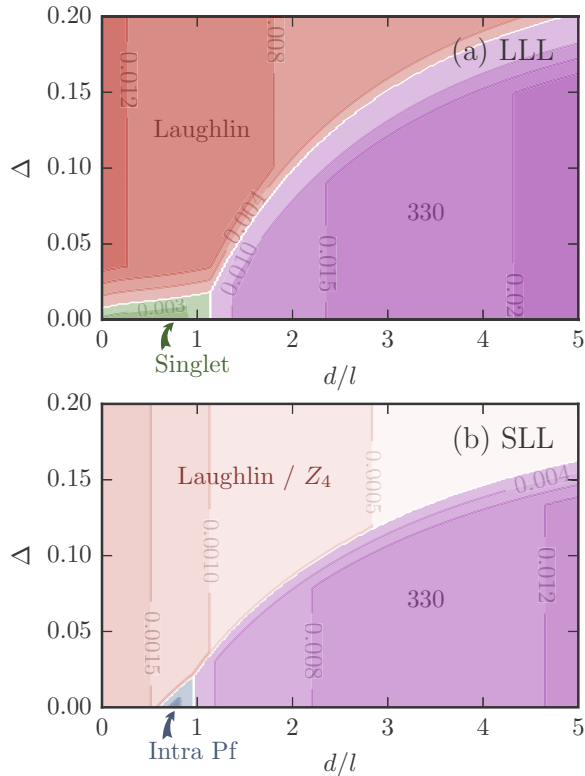


FIG. 6. (Color online) The quantum phase diagram determined by variational energies at thickness $w = 0$. The contour lines depict the energy advantage δE of each dominant phase. In the SLL, the Laughlin state does not have a clear energy advantage over the Z_4 state.

hence, we leverage the Jack polynomial representation of Ψ_{Z_4} to directly obtain its second-quantized amplitudes [65]. This technique allows us to obtain the state and compute its variational energy, i.e., the expectation value of Hamiltonian for Ψ_{Z_4} , for up to $N = 28$ electrons, well beyond the scope of exact diagonalization. We assess the relative stability of different phases of the two-component system by comparing their variational energies. To estimate the energy per particle in the thermodynamic limit, we use quadratic extrapolation in $1/N$, weighted by the statistical error on each data point [23]. The extrapolated energy has an error between 10^{-3} and 10^{-4} .

The phase diagram is determined according to the wave function with the lowest energy, and we characterize the phase stability using the energy advantage δE of the dominant wave function over its closest competitor. Figure 6 shows the contour plots of $\delta E(d/l, \Delta)$. In the LLL, the phase diagram determined by variational energies is in qualitative agreement with the ED results. We find that the non-Abelian Z_4 and the interlayer/intralayer Pfaffian states remain energetically

unfavorable throughout the phase diagram, and the singlet 112 occupies a very small corner of the parameter space. In the SLL, while the Halperin 330 state still dominates at large layer separation d/l , the Laughlin state at large Δ is now much less stable compared with the LLL. The main competition comes from the non-Abelian Z_4 state. In fact, for much of the phase diagram, the energy difference between the two is on the same order as the estimated extrapolation error ($\lesssim 10^{-3}$) for the Z_4 state. This is in strong agreement with our gap and overlap calculations using ED, namely, that the non-Abelian Z_4 state is highly competitive with the Laughlin state. Incidentally, we also find a small region in the parameter space that favors the intralayer Pfaffian state, but we do not find any parameter set that stabilizes the interlayer Pfaffian state.

V. CONCLUSION

Based on our exact diagonalization and Monte Carlo studies, we find that $\nu = 2/3$ bilayers in the LLL, in the limit of weak LL mixing, most likely do not realize the non-Abelian Z_4 parafermion state. Most remarkably, in the single-component limit of the SLL, the non-Abelian Z_4 phase may be favorable for the $8/3$ FQH state relative to the Laughlin state. Indeed, previous studies of the experimentally obtained energy gaps of FQH states in the SLL have already indicated the possibility that the electron correlations are sufficiently different from those of the LLL and that novel exotic states might be realized [48,51–53,55,56,59]. While weak quasiparticle tunneling experiments through a quantum point contact [58] suggest that $\nu = 8/3$ is the $2/3$ Laughlin state, it cannot be considered to be definitive yet and more experiments are necessary. The implication of our finding that the observed $8/3$ SLL FQH state may be the parafermionic Z_4 non-Abelian phase is enormous since this state can be utilized for universal topological quantum computation.

As this work was being completed, we became aware of a related manuscript [66] by Geraedts *et al.* By utilizing primarily the density matrix renormalization group technique, they reported that the interlayer Pfaffian is stabilized for a modified LLL interaction with a hollow core, which is very different from the LLL and SLL realistic Coulomb interactions considered in our work. It is an interesting open question whether the disparity between the present work and that of Ref. [66] is due to differences in techniques, differences in models, or both.

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