Transport in ultradilute solutions of ^{3}He in superfluid ^{4}He

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We calculate the effect of a heat current on transporting ^{3}He dissolved in superfluid ^{4}He at ultralow concentration, as will be utilized in a proposed experimental search for the electric dipole moment of the neutron (nEDM). In this experiment, a phonon wind will be generated to drive (partly depolarized) ^{3}He down a long pipe. In the regime of ^{3}He concentrations $\lesssim 10^{-9}$ and temperatures ~ 0.5 K, the phonons comprising the heat current are kept in a flowing local equilibrium by small angle phonon-phonon scattering, while they transfer momentum to the walls via the ^{4}He first viscosity. On the other hand, the phonon wind drives the ^{3}He out of local equilibrium via phonon-^{3}He scattering. For temperatures below 0.5 K, both the phonon and ^{3}He mean free paths can reach the centimeter scale, and we calculate the effects on the transport coefficients. We derive the relevant transport coefficients, the phonon thermal conductivity, and the ^{3}He diffusion constants from the Boltzmann equation. We calculate the effect of scattering from the walls of the pipe and show that it may be characterized by the average distance from points inside the pipe to the walls. The temporal evolution of the spatial distribution of the ^{3}He atoms is determined by the time dependent ^{3}He diffusion equation, which describes the competition between advection by the phonon wind and ^{3}He diffusion. As a consequence of the thermal diffusivity being small compared with the ^{3}He diffusivity, the scale height of the final ^{3}He distribution is much smaller than that of the temperature gradient. We present exact solutions of the time dependent temperature and ^{3}He distributions in terms of a complete set of normal modes.

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I. INTRODUCTION

The physics underlying the transport properties of mixtures of ^{3}He and superfluid ^{4}He changes markedly as the concentration of ^{3}He varies. We determine here the transport properties of these mixtures at very low concentrations, $x_3 = n_3/(n_3 + n_4) \lesssim 10^{-9}$, where n_3 and n_4 are the ^{3}He and ^{4}He densities, and low temperatures, $T \lesssim 0.6$ K, where phonons are the dominant superfluid excitation. In this case, the phonons are in local thermal equilibrium; their interactions with the ^{3}He distort the ^{3}He distribution and dominate the ^{3}He diffusion. For concentrations $x_3 \gtrsim 10^{-4}$, the reverse situation holds: the ^{3}He are in local equilibrium due to rapid ^{3}He-^{3}He scattering and the phonon distribution is distorted due to phonon-^{3}He interactions [1]. In the intermediate concentration regime, the phonon and ^{3}He distributions are both distorted and must be determined by solving the coupled evolution, or Boltzmann, equations [2]. At the highest concentrations, $x_3 \sim 1\%$, Fermi-Dirac statistics for the ^{3}He becomes important [3]. These transport properties are of interest as an example of a two-component fluid with excitations of comparable energy but very different momenta, and where the excitations of the two species obey different statistics.

The transport properties of ^{3}He in superfluid ^{4}He at low concentrations are also important for the proposed experiment [4] to measure the neutron electric dipole moment (nEDM) at the Oak Ridge National Laboratory Spallation Neutron Source. There, the neutron precession frequency will be determined using the absorption of polarized ultracold neutrons on polarized ^{3}He atoms in solution in superfluid ^{4}He via the reaction

$$\mathrm{n} + {}^3\mathrm{He} \rightarrow \mathrm{p} + \mathrm{t} + 764\ \mathrm{keV}, \quad (1)$$

which has a strong spin dependence, since capture proceeds primarily through the spin-singlet channel. Two key considerations accrue from this choice of detection technique. In order to maximize the precision with which the precession frequency can be measured, the optimal ^{3}He concentration, $x_3 \sim 10^{-10}$, corresponds to a capture rate comparable to the decay rate of the neutrons. However, primarily due to wall collisions, the ^{3}He will gradually become depolarized. In order to reduce the background from neutron capture on unpolarized ^{3}He, it is crucial to be able to periodically sweep out the ^{3}He by means of a heat current [5]. In this paper, we calculate both the heat and ^{3}He particle currents based on well-established microscopic theory of phonon-phonon [6] and phonon-^{3}He scatterings [7], as well as the evolution of both the temperature and ^{3}He concentration.

At the concentrations and temperatures of interest in the experiment, in addition to phonon-phonon, phonon-^{3}He, and ^{3}He-^{3}He scattering, the scattering of both phonons and ^{3}He from the walls of the containers can also be important. Here, we extend the solution of the Boltzmann equations in Ref. [2] to include these effects, in addition to providing some examples for $x_3 \lesssim 10^{-9}$. For illustration, we consider the effect of a heat current in an essentially one-dimensional geometry, with the ^{3}He superfluid ^{4}He mixture in a long pipe with a diameter of a few centimeters. The phonon-wall interactions affect the thermal conductivity as well as the phonon velocity distribution within the pipe; the ^{3}He-wall interactions affect the transport of the ^{3}He in the presence of a heat current.

This paper is arranged as follows. Section II describes the basic scattering mechanisms the calculation of transport coefficients from the Boltzmann equation is given in Sec. III. Subsequently, we calculate the temporal and spatial evolution of the temperature (Sec. IV) and the ^{3}He density (Sec. V). We summarize our results in Sec. VI. In Appendix A, we analyze the transport when the scattering of phonons is predominantly from the walls of the pipe, and in Appendix B, we solve analytically the equation for the temporal evolution of the ^{3}He concentration.

II. PHONON AND ^{3}He RELAXATION

We begin by considering the relevant microscopic relaxation mechanisms (detailed in Ref. [2]), first for the phonons. The momentum-dependent mean free path of a phonon of momentum q scattering against the ^{3}He,

$$\ell_{\rm ph3}(q) = \frac{s}{\gamma_q} = \frac{4\pi n_4}{x_3 J}\frac{1}{q^4}, \tag{2}$$

is typically greater than 1 km for $x_3 \lesssim 10^{-9}$ and $T \sim 0.5$ K [2]; here γ_q is the corresponding scattering rate, s is the phonon velocity, and J is an angle-integrated rate constant. Therefore phonon-wall and phonon-phonon scatterings determine the phonon distribution. As discussed in Refs. [2] and [8], rapid, small angle phonon-phonon scattering establishes thermal equilibrium along phonon "rays," i.e., given directions in momentum space, with the distribution

$$n^{le}_{\vec q} = \frac{1}{e^{(sq-\vec q\cdot\vec v_{\rm ph})/T(\vec r)} - 1}, \tag{3}$$

where $T(\vec r)$ is the local temperature and $v_{\rm ph}$ is the mean phonon drift velocity. Large angle phonon-phonon scattering, either in a single event or a succession of small-angle processes, is slower and gives rise to the phonon first viscosity,

$$\eta_{\rm ph} = \frac{1}{5}\frac{TS_{\rm ph}}{s}\ell_{\rm visc}, \tag{4}$$

where $S_{\rm ph}$ is the phonon entropy density and $\ell_{\rm visc}$ is the viscous mean free path. At a pressure of 0.1 bar,

$$\ell_{\rm visc} \simeq \frac{3.2\times10^{-3}}{T_K^5}\ \text{cm}, \tag{5}$$

to a good approximation [6,9], where T_K is the temperature in Kelvin; at $T = 0.45$ K, $\ell_{\rm visc} \simeq 0.17$ cm.

In the presence of a heat flux, $\vec Q = TS_{\rm ph}\vec v_{\rm ph}$, small-angle phonon-phonon scattering keeps the phonons in local thermal equilibrium, where the mean phonon drift velocity $\vec v_{\rm ph}$ is

$$TS_{\rm ph}\vec v_{\rm ph} = -K_{\rm ph}\nabla T. \tag{6}$$

The thermal conductivity of the phonons $K_{\rm ph}$ can, at low concentrations, be written as [2,9]

$$K_{\rm ph} = \frac{5}{8}\frac{sS_{\rm ph}R^2}{\ell_{\rm eff}}, \tag{7}$$

in a pipe of radius R and where $\ell_{\rm eff}$ is the effective mean free path:

$$\frac{1}{\ell_{\rm eff}} = \frac{1}{\ell_{\rm visc}} + \frac{16}{5R}. \tag{8}$$

The second term represents scattering of the phonons on the walls; the numerical coefficient 16/5 is chosen to give the correct Casimir limit. In this limit, $\ell_{\rm visc}$ large compared to the pipe diameter (see Appendix A), the phonon thermal conductivity assumes the Casimir form [10]

$$K_{\rm ph,Casimir} = 2RsS_{\rm ph}. \tag{9}$$

In the opposite limit, $\ell_{\rm visc} \ll R$, the thermal conductivity becomes

$$K_{\rm ph,visc} = \frac{5}{8}\frac{sS_{\rm ph}R^2}{\ell_{\rm visc}}. \tag{10}$$

The ^{3}He contribution to the overall heat flux is negligible at low x_3 [1,2].

The mean free path of a ^{3}He scattering on unpolarized ^{3}He is [2]

$$\ell_{33} = \frac{1}{(n_3/2)\sigma_{33}} = \frac{8.66\times10^{-8}}{x_3}\ \text{cm}, \tag{11}$$

where σ_{33} is the corresponding cross section. Thus, for $x_3 < 10^{-9}$, one has $\ell_{33} \gtrsim 1$ m; we therefore neglect ^{3}He-^{3}He scattering. On the other hand, the mean free path for ^{3}He scattering on phonons [2],

$$\ell_{\rm 3ph} = \frac{\sqrt3}{2J}\left(\frac{n_4}{S_{\rm ph}}\right)^2\frac{m^{*1/2}s^2}{T^{3/2}} = 0.077\left(\frac{0.45\ K}{T}\right)^{15/2}\ \text{cm}, \tag{12}$$

where m^* is the ^{3}He effective mass in superfluid ^{4}He, is small compared to the pipe diameter for $T \gtrsim 0.3$ K. Thus the dominant process for bringing the ^{3}He toward equilibrium for the temperatures of interest in the experiment is scattering against phonons. In the next section, we outline the calculation of the ^{3}He transport coefficients; more details can be found in Ref. [2].

III. ^{3}He BOLTZMANN EQUATION AND TRANSPORT COEFFICIENTS

The ^{3}He Boltzmann equation has the general form

$$\begin{aligned}\frac{\partial f_{\vec p}}{\partial t} &+ \frac{\vec p}{m^*}\cdot\nabla_r f_{\vec p}\\ &= \sum_{p',q,q'}\mathcal{T}\big[f_{\vec p'}n^{le}_{\vec q'}(\vec r)\big(1+n^{le}_{\vec q}(\vec r)\big)\\ &\quad - f_{\vec p}\,n^{le}_{\vec q}(\vec r)\big(1+n^{le}_{\vec q'}(\vec r)\big)\big] - \frac{\delta f_{\vec p} - \beta f^0_p v_3}{\tau_{33}}\\ &\quad - \frac{\delta f_{\vec p} - \beta f^0_p v_3}{\tau_{3ws}} - \frac{\delta f_{\vec p}}{\tau_{3wd}}.\end{aligned} \tag{13}$$

Here, $f_{\vec p}$ is the ^{3}He distribution function,

$$f^0_p = e^{-\beta(p^2/2m^*-\mu_3)} \tag{14}$$

is the equilibrium distribution function, and we write the deviations from local equilibrium as

$$\delta f_{\vec p} = f_{\vec p} - f^{le0}_p, \tag{15}$$

where

$$f^{le0}_{\vec p} = e^{-\left(p^2/2m^*-\vec p\cdot\vec v_{\rm ph}-\mu_3(\vec r)\right)/T(\vec r)} \tag{16}$$

is the local equilibrium distribution function, i.e., the distribution towards which collisions with phonons drive the ^{3}He. The first term on the right represents the scattering on the phonons, the second term the scattering from other ^{3}He (numerically insignificant for the concentrations of interest), and the last two terms isotropic diffuse (d) and specular (s) scattering of the ^{3}He from the walls. In the term describing collisions with phonons, $\vec{p}$ and $\vec{p}\,'$ are the initial and final ^{3}He momenta, respectively, and $\vec{q}$ and $\vec{q}\,'$ are the corresponding phonon momenta. The phonon-^{3}He scattering kernel is $\mathcal{T} \equiv |\langle p'q'|T|pq\rangle|^2 2\pi\delta(p^2/2m^* + sq - p'^2/2m^* - sq')$, and momentum conservation, $\vec{p}\,' + \vec{q}\,' = \vec{p} + \vec{q}$, is understood in the collision term.

In order to calculate the effects of a phonon wind on the ^{3}He, we solve Eq. (13) for $\delta f_{\vec{p}}$. On the left side of the Boltzmann equation, we approximate the distribution by its local equilibrium form, f_p^{le0}. We neglect the contribution from $\partial v_{\rm ph}/\partial z$ because of the relatively small temperature gradient [see Eq. (38) below], while the gradient of β in this term gives a second-order contribution, which we neglect. The left side of the Boltzmann equation is then

$$\frac{\partial f_p^{le0}}{\partial z} = \left(\frac{p^2}{2m^*} - \frac{3}{2}T\right) f_p^0 \frac{1}{T^2}\frac{\partial T}{\partial z} + f_p^0 \frac{1}{n_3}\frac{\partial n_3}{\partial z}. \tag{17}$$

On the right side, we write

$$\delta f_{\vec{p}} \equiv \beta f_p^0 p_z w_p; \tag{18}$$

as shown in Ref. [2], the ^{3}He-phonon collision term is diagonalized by expanding w_p in Sonine polynomials [2].

To solve for w_p, we multiply the Boltzmann equation by p_z and integrate over all $\vec{p}$. Noting that the distortion $\delta f_{\vec{p}}$ is proportional to p_z, we see first that on the right side of the Boltzmann equation the two terms $\propto \delta f_{\vec{p}} - \beta f_p^0 v_3$ do not contribute, since both the ^{3}He-^{3}He scattering and the specular ^{3}He scattering from the walls conserve momentum in the z direction. Following Eq. (81) of Ref. [2] for the phonon-^{3}He scattering, we see that the remaining terms on the right side comprise

$$-\frac{\beta\Gamma}{3m^*}\left(\delta f_{\vec{p}} - \beta f_p^0 p_z v_{\rm ph}\right) - \frac{\delta f_{\vec{p}}}{\tau_{3wd}}. \tag{19}$$

With the inclusion of the recoil effect in the phonon-^{3}He scattering to the lowest order (see the Appendix of Ref. [1] and Sec. VI of Ref. [2]), the solution of the Boltzmann equation is

$$\delta f_{\vec{p}} = \tau_3' \frac{p_z}{m^*}\left(\frac{\beta^2\Gamma_{\rm rec}}{3} f_p^0 v_{\rm ph} - \frac{\partial f_p^{le0}}{\partial z}\right), \tag{20}$$

where

$$\frac{1}{\tau_3'} \equiv \frac{\beta\Gamma_{\rm rec}}{m^*} + \frac{1}{\tau_{3wd}} \tag{21}$$

is the effective ^{3}He scattering rate, including both scattering from phonons, encoded in $\Gamma_{\rm rec}$, and diffuse scattering from the walls of the pipe.

Integrating the Boltzmann equation, Eq. (13), over $\vec{p}$ we recover the continuity equation

$$\frac{\partial n_3}{\partial t} + \nabla\cdot\vec{j}_3 = 0, \tag{22}$$

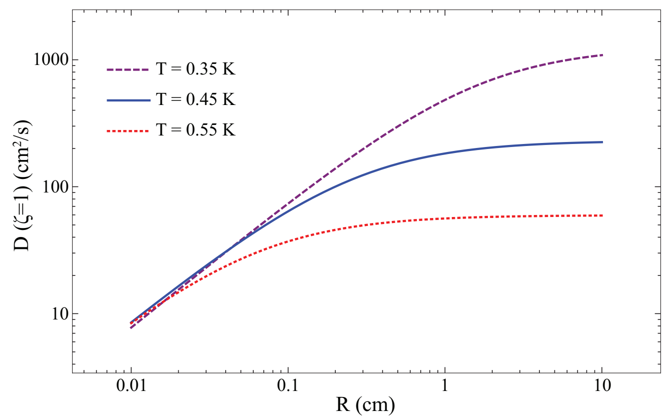

FIG. 1. (Color online) The diffusion constant, $D_{\rm rec}$, Eq. (25) as a function of the pipe radius, R, for $T = 0.35$ (purple, dashed), 0.45 blue, solid), and 0.55 K (red, dotted). We assume here that the ^{3}He wall scattering is diffuse, $\zeta = 1$.

with the ^{3}He particle current given by

$$\vec{j}_3 = \nu \int \frac{d^3p}{(2\pi)^3}\frac{\vec{p}}{m^*}\delta f_{\vec{p}}, \tag{23}$$

where ν is the number of spin degrees of freedom of the ^{3}He: 1 for a fully polarized sample and 2 in the unpolarized case. It is straightforward to evaluate the current (including the effect of recoil in the phonon-^{3}He scattering)

$$j_3 = n_3 v_{\rm ph} - D_{\rm rec}\frac{\partial n_3}{\partial z} - D_{T,\rm rec}\frac{\partial T}{\partial z}, \tag{24}$$

where the ^{3}He diffusion constant, including recoil corrections, is [2]

$$D_{\rm rec} = 3\xi(R,\zeta)T^2/\Gamma_{\rm rec}, \tag{25}$$

and the "thermoelectric" coefficient is

$$D_{T,\rm rec} = 3\xi(R,\zeta)Tn_3/\Gamma_{\rm rec}. \tag{26}$$

In these expressions the basic forms of D and D_T are modified by the wall scattering factor

$$\xi(R,\zeta) = \left(1 + \frac{3m^*}{\beta\Gamma_{\rm rec}}\frac{\zeta}{\tau_{3wd}(R)}\right)^{-1}, \tag{27}$$

where

$$\zeta = \frac{\tau_{3wd}^{-1}}{\tau_{3wd}^{-1} + \tau_{3ws}^{-1}} \tag{28}$$

is the fraction of the ^{3}He wall scattering rate that is diffuse.

To see the effect of scattering of ^{3}He from the walls, we take the total wall scattering rate to be simply

$$\frac{1}{\tau_{3wd}(R)} = \frac{\bar{v}_3}{\mathcal{D}_{\rm eff}}, \tag{29}$$

where $\bar{v}_3 = \sqrt{3T/m*}$ is the mean ^{3}He thermal velocity, and $\mathcal{D}_{\rm eff} = 2R/3$ [see Eq. (A8)] is the effective average distance from an interior point to the wall of an infinitely long pipe of radius R entering the transport [11]. The effect of wall scattering on the diffusion constant $D_{\rm rec}$, for example, is shown in Fig. 1 for $\zeta = 1$. As we see, the effect becomes more

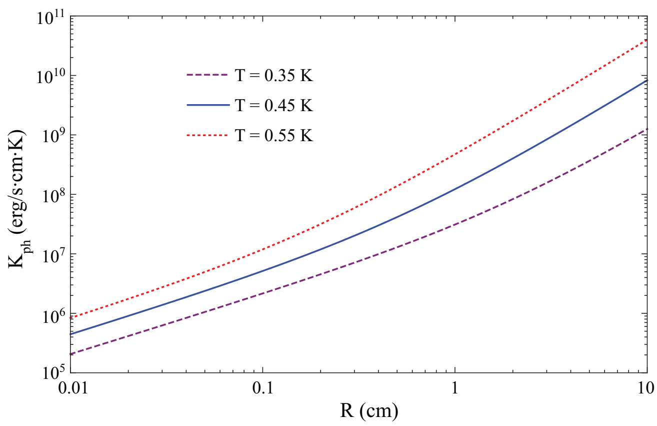

FIG. 2. (Color online) The phonon thermal conductivity, $K_{\rm ph}$, from Eq. (7) for $T = 0.35$ (dashed), 0.45 solid), and 0.55 K (dotted) as a function of pipe radius. For $R \ll \ell_{\rm visa}$, the conductivity rises as R [Eq. (9)], while for $R \gg \ell_{\rm visc}$, it rises as R^2 [Eq. (10)].

important for lower temperatures because of the decrease in the phonon density. As $R \to \infty$, $D_{\rm rec}$ approaches the result without wall scattering, which in the vicinity of the operating temperature regime of the nEDM experiment, $T \sim 0.45$ K, is

$$D_{\rm rec} \cong \frac{0.88}{T_K^7}\ {\rm cm}^2/{\rm s}. \tag{30}$$

We also show, in Fig. 2, the corresponding effect of phonon-wall scattering on the phonon thermal conductivity, Eq. (7).

At temperatures of approximately 0.3 K and below, the mean free path of a ^{3}He quasiparticle in the bulk medium is greater than 1 cm, which is comparable to the radii of the pipes considered for the nEDM experiment. In this regime, the ^{3}He quasiparticles can lose momentum not only by collisions with phonons but also directly to the walls of the pipe. This situation is analogous to the (Knudsen) flow of a low-density gas in response to a pressure gradient, except that in the dilute helium solutions the force driving the flow of ^{3}He has two components, one due to the ^{3}He pressure gradient and another due to the collisions with phonons. When the ^{3}He distribution function is stationary, the two contributions to the force are equal and opposite. The ^{3}He distribution function in principle depends not only on direction in momentum space but also on the radial coordinate in the pipe. However, the time scales for smoothing out the radial dependence via diffusion or ballistic transport are of order several milliseconds, and therefore short compared with the overall evolution time scales of the system. A detailed study of this regime lies outside the scope of the present article.

IV. TIME EVOLUTION OF THE TEMPERATURE

We now estimate, using the heat diffusion equation, the time scale for heating the fluid. We assume, as above, that the heat is carried by the phonons [see also Eq. (89) of Ref. [2] and discussion following] and that the relative temperature variation and, hence, the variation of $K_{\rm ph}$ is small, so that

$$\frac{\partial \epsilon}{\partial t} = 3S_{\rm ph}\frac{\partial T}{\partial t} = -\nabla\cdot\vec{Q} = K_{\rm ph}\nabla^2 T, \tag{31}$$

where ϵ is phonon energy density. Within a few scattering times after the application of heat at one end of the pipe ($z = 0$), the temperature there is approximately fixed at $T_0 + \Delta T$. We assume that the temperature at $z = L$, the other end of the pipe, is kept constant at temperature T_0 by a refrigerator.

To solve the heat diffusion equation it is sufficient to consider the average of the temperature, $T(z,t)$, over the cross-section of the pipe, thus avoiding having to take into account details of the counterflows within the pipe. The solution is given in terms of the modes in the pipe that vanish at $z = 0$ and L, $\sin k_\nu z$, where $k_\nu = \nu\pi/L$, with ν here a positive integer:

$$T(z,t) = T_0 + \Delta T(1 - z/L) + \sum_{\nu\neq 0} c_\nu e^{-D_{\rm th}k_\nu^2 t}\sin k_\nu z. \tag{32}$$

We denote the thermal diffusivity by

$$D_{\rm th} = K_{\rm ph}/3S_{\rm ph}, \tag{33}$$

and recognize $3S_{\rm ph} = (2\pi^2/15)(T/s)^3$ as the ^{4}He specific heat. The condition that $T(z,t=0) = T_0$ except immediately at $z = 0$, implies that the mode weights are given by $c_\nu = -2\Delta T/\nu\pi$. The characteristic time, $\tau_{\rm th}$, to set up a steady-state phonon wind is essentially that of the $\nu = 1$ mode,

$$\tau_{\rm th} = \frac{1}{D_{\rm th}k_1^2} = \frac{L^2 K_{\rm ph}}{3\pi^2 S_{\rm ph}}. \tag{34}$$

For typical conditions in the experiment, 5 mW of heat in a 3-cm-diameter, 100-cm long pipe at $T = 0.45$ K, the phonon thermal conductivity is 2.4×10^8 erg/s cm K, $\tau_{\rm th} \sim 11$ ms, and $\Delta T = 3$ mK.

V. TIME EVOLUTION OF THE ^{3}He CONCENTRATION

To begin examining the ^{3}He concentration, we consider its steady-state distribution in the presence of a heat current or phonon wind. Because, as we shall see below, the term involving D_T is relatively small for low concentrations, the condition that the ^{3}He particle current, Eq. (24), vanishes, is simply

$$D_{\rm rec}\frac{\partial n_3}{\partial z} = v_{\rm ph}n_3, \tag{35}$$

which has the solution

$$n_3(z) = \tilde{n}_3 e^{z/h} \equiv n_{3,\infty}(z), \tag{36}$$

where we define the scale height, $h = D_{\rm rec}/v_{\rm ph}$ (for the example parameters above, $D_{\rm rec} = 225$ cm^2/s, $v_{\rm ph} = 17$ cm/s, and $h = 13$ cm), and

$$\tilde{n}_3 = \frac{n_0}{e^{L/h} - 1}\frac{L}{h}, \tag{37}$$

with n_0 the initial uniform ^{3}He density. We note that the relative size of the term involving $D_{T,\rm rec}$ is simply the ratio of the scale heights of the concentration and the temperature,

$$\frac{D_{T,\mathrm{rec}}|\partial T/\partial z|}{D_{\mathrm{rec}}|\partial n_3/\partial z|} = \frac{|\partial \ln T/\partial z|}{|\partial \ln n_3/\partial z|} = \frac{D_{\mathrm{rec}} S_{\mathrm{ph}}}{K_{\mathrm{ph}}}, \tag{38}$$

about $1/1000$ for the example parameters given above.

In the nEDM experiment, the ^{3}He in the system depolarizes in time, primarily due to interactions with walls. The depolarized ^{3}He will be removed by a phonon wind before the system is recharged with more highly polarized ^{3}He. As above, we consider the simple situation of a long pipe with a heater at $z = 0$ and closed ends. The evolution of the ^{3}He is governed by a competition between two processes: the phonon wind, which were it to act alone would push all the ^{3}He to the downstream (large z) end of the pipe, and diffusion of the ^{3}He, limited by scattering with the phonons, which allows the ^{3}He to drift back towards smaller z.

This evolution of the ^{3}He concentration in the presence of a phonon wind is described by the diffusion equation resulting from Eq. (22),

$$\frac{\partial n_3(z,t)}{\partial t} + v_{\mathrm{ph}}\frac{\partial n_3}{\partial z} - D_{\mathrm{rec}}\frac{\partial^2 n_3}{\partial z^2} = 0, \tag{39}$$

where we have dropped the D_T term in Eq. (24). Once a steady phonon wind, with a small temperature gradient, is established, we may neglect the temperature dependence of D_{rec} and take v_{ph} and D_{rec} to be constant. For a pipe with a large length to diameter ratio, we may treat the problem as one dimensional, averaging over its cross section as we did above for the heat flow. The boundary conditions are that the ^{3}He current, j_3, Eq. (24), vanishes at the two ends of the pipe,

$$\frac{\partial n_3}{\partial z} = \frac{n_3}{h} \qquad (z = 0, L). \tag{40}$$

To solve Eq. (39) with constant v_{ph} and D, we write the ^{3}He density as $e^{z/2h}\hat{n}(z,t)$ and decompose $\hat{n}(z,t)$ as a sum of time dependent modes $\hat{n}_\nu(z,t)$ periodic in $2L$ (see Appendix B):

$$n_3(z,t) = e^{z/2h}\sum_{\nu=0}^{\infty}\hat{n}_\nu(z,t), \tag{41}$$

where $\hat{n}$ satisfies the boundary condition

$$\frac{\partial \hat{n}}{\partial z} = \frac{\hat{n}}{2h} \qquad (z = 0, L). \tag{42}$$

The spatial parts of the mode functions $\hat{n}_\nu(z,t)$ are the complete orthonormal set

$$\phi_\nu(z) = \begin{cases} \frac{e^{z/2h}}{[h(e^{L/h}-1)]^{1/2}}, & \nu = 0,\\ \alpha_\nu\left(\cos k_\nu z + \frac{1}{2hk_\nu}\sin k_\nu z\right), & \nu \geqslant 1 \end{cases} \tag{43}$$

with $k_\nu = \pi\nu/L$ and $\alpha_\nu = [(L/2)(1 + 1/(2hk_\nu)^2)]^{-1/2}$. The time dependence of the modes is e^{-t/τ_ν}, where

$$\frac{1}{\tau_\nu} = \begin{cases} 0, & \nu = 0,\\ k_\nu^2 D + v_{\mathrm{ph}}^2/4D = \left(k_\nu^2 + 1/4h^2\right)D, & \nu \geqslant 1. \end{cases} \tag{44}$$

The solution of Eq. (39) for an initially uniform density $n_{3,0}$ is then

$$n_3(z,t) = n_{3,0}\, e^{z/2h}\sum_{\nu=0}^{\infty} c_\nu \phi_\nu(z) e^{-t/\tau_\nu}, \tag{45}$$

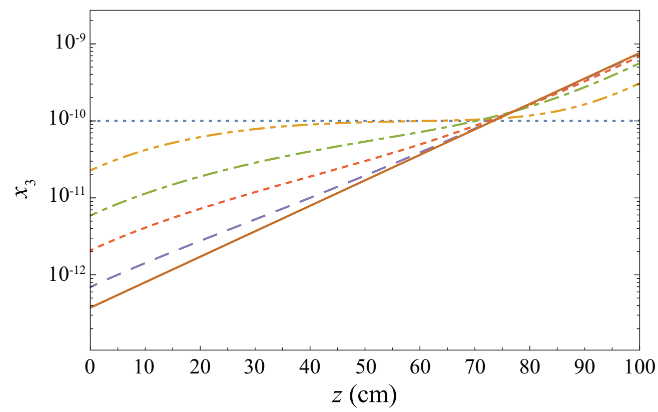

FIG. 3. (Color online) The ^{3}He concentration, x_3, from the solution of Eq. (39) as a function of z, the distance along the pipe, for various times: $t = 0$ (dotted), 1 (dash double dot), 3 (dash dot), 5 (short dash), 8 (long dash), and 20 s (solid). The result is shown for typical parameters in the nEDM experiment: $x_{3,0} = 10^{-10}$ and 5 mW of heat into a 3-cm-diameter, 100-cm-long pipe at a nominal temperature of 0.45 K.

where

$$\begin{aligned} c_\nu &= \int_0^L \left(n(z,0)/n_3^0\right)\phi_\nu(z)dz \\ &= \begin{cases} \frac{L}{[h(e^{L/h}-1)]^{1/2}}, & \nu = 0,\\ \frac{8h\alpha_\nu}{1+(2hk_\nu)^2}\left(1 + (-1)^{\nu+1}e^{-L/2h}\right), & \nu \geqslant 1. \end{cases} \end{aligned} \tag{46}$$

As we show in Appendix B, the general solution may be written in compact form in terms of a Green's function

$$\hat{n}(z,t) = \int_0^L \mathcal{G}(z,z',t)\hat{n}(z',0)dz', \tag{47}$$

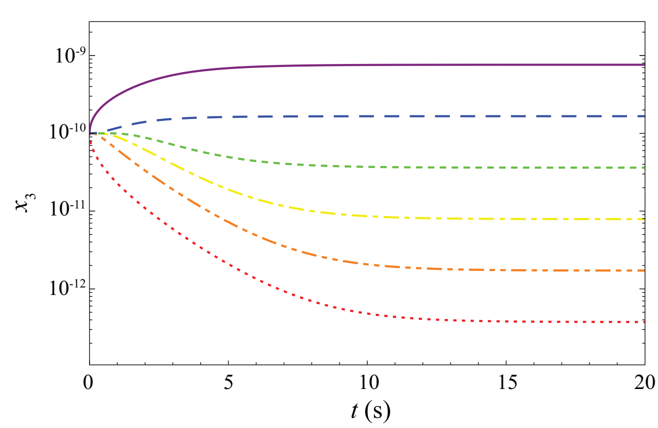

FIG. 4. (Color online) The ^{3}He concentration, x_3, from the solution of Eq. (39) as a function of time, t, for various positions along the pipe: the curves correspond to $z = 0$ (dotted), 20 (dash double dot), 40 (dash dot), 60 (short dash), 80 (long dash), and 100 cm (solid). Note that it takes about 5.5 time constants, τ_1, for the distribution at the hot end of the pipe ($z = 0$) to reach equilibrium. The result is shown for typical parameters in the nEDM experiment: $x_{3,0} = 10^{-10}$, and 5 mW of heat into a 3-cm-diameter, 100-cm-long pipe at a nominal temperature of 0.45 K, for which $\tau_1 = 1.8$ s.

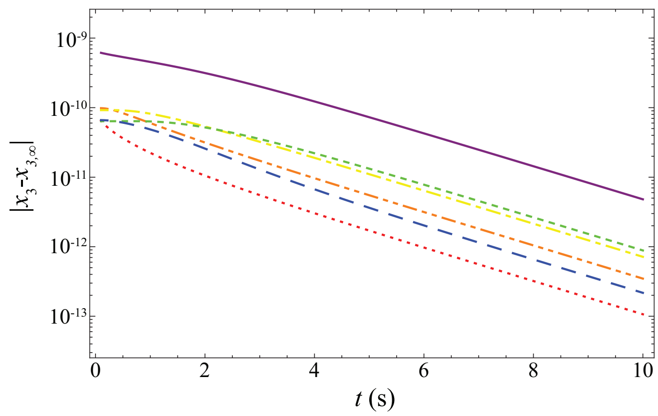

FIG. 5. (Color online) The absolute value of the ^{3}He concentration difference, $|x_3 - x_{3,\infty}|$, from the solution of Eq. (22) as a function of time, t, for various positions along the pipe: the curves correspond to $z = 0$ (dotted), 20 (dash double dot), 40 (dash dot), 60 (short dash), 80 (long dash), and 100 cm (solid). We note that the lowest mode, corresponding to the time constant τ_1, dominates the time evolution after a few seconds. The result is shown for typical parameters in the nEDM experiment: $x_{3,0} = 10^{-10}$, and 5 mW of heat into a 3-cm-diameter, 100-cm-long pipe at a nominal temperature of 0.45 K, for which $\tau_1 = 1.8$ s.

where

$$\mathcal{G}(z,z',t) = \sum_{\nu=0}^{\infty} \phi_\nu(z)\phi_\nu(z') e^{-t/\tau_\nu}\theta(t). \tag{48}$$

The z and t dependencies of x_3 are shown in Figs. 3 and 4, respectively, for the case of the uniform initial distribution and typical experimental values (5-mW heat into a 3-cm-diameter, 100-cm-long pipe at $T = 0.45$ K). As the figures illustrate, the concentration scale height, h, is substantially smaller than the pipe length. Figure 5 plots the difference between x_3 and its steady-state value for several points along the pipe, showing that, after a few seconds, the lowest mode, with $\tau_1 = 1.8$ s, dominates the time evolution throughout the pipe.

The results for the evolution of the ^{3}He concentration presented here are equally applicable to a heat flush experiment being carried out at Harvard at natural ^{3}He concentration [12]. There one must use the more general phonon thermal conductivity as derived in Ref. [2]; phonon-wall scattering in this regime plays a negligible role.

VI. SUMMARY

We have calculated the transport properties of dilute mixtures, $x_3 \lesssim 10^{-9}$, of ^{3}He in superfluid ^{4}He at temperatures around 0.5 K where phonons are the dominant excitations of the superfluid. In this regime, we considered a simple one-dimensional geometry (a pipe), a heat current generates a phonon wind with phonons in local equilibrium corresponding to the temperature at that point in the pipe. On the other hand, phonon scattering distorts the ^{3}He distribution from equilibrium. Starting from the known phonon-phonon and phonon-^{3}He scattering, we calculate the transport coefficients from the Boltzmann equation. We show that, in the presence of a heat current which generates a temperature scale height much larger than the length of the pipe (i.e., a small relative temperature gradient), the scale height for the ^{3}He concentration can be much less than the pipe length. This leads to a large decrease in concentration at the hot boundary and a corresponding increase at the cold end. For temperatures below 0.5 K, the mean free paths of both the phonons and the ^{3}He can reach the centimeter scale; in these cases, scattering from the walls of the container becomes important. Finally, we calculate the time scales associated with the evolution of both the temperature and concentration distributions; because of the large superfluid thermal conductivity, the thermal time scales are on the microsecond scale, whereas the corresponding scale for the evolution of the concentration is on the scale of seconds.

ACKNOWLEDGMENTS

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APPENDIX A: PHONON DRIFT VELOCITY IN THE BALLISTIC LIMIT

Here, we derive the spatial dependence of the phonon drift velocity when the phonon-phonon scattering mean free path for large angle scattering, Eq. (5), is much larger than the pipe radius, R. For the pipe geometry considered in the text, this condition holds at a temperature $T \lesssim 0.3$ K. We assume that the pipe axis is in the z direction, and that the transverse coordinates are x and y. We also assume that a phonon striking the cylinder wall is diffusively reflected, with a distribution of final momenta given by the local temperature, $T(z) = T_0 + T'z$, where $T' < 0$ is the temperature gradient. Then $n_{\vec{q}}(\vec{r})$, the number of phonons of momentum $\vec{q}$ at point $\vec{r}$, is given by the equilibrium distribution $n_q^0(z')$ at the point $\vec{r}\,' = (x',y',z')$ on the pipe wall where the phonon at $\vec{r}$ originated:

$$n_{\vec{q}}(\vec{r}) = \frac{1}{e^{sq/T(z'(\vec{q}))} - 1} \simeq n_q^0 - z' q \frac{T'}{T_0}\frac{\partial n_q^0}{\partial q}, \tag{A1}$$

to lowest order in T', where n_q^0 is the equilibrium distribution.

The point of origin is determined by simple geometry, namely, $\vec{r} - \vec{r}\,' = \hat{q}\,\mathcal{D}$, where $\mathcal{D} = |\vec{r} - \vec{r}\,'|$. We measure $\hat{q}$ in polar coordinates θ_q and ϕ_q. Then $z' = z - \mathcal{D}\cos\theta_q$, $x = x' - \mathcal{D}\sin\theta_q\cos\phi_q$, and $y' = y - \mathcal{D}\sin\theta_q\sin\phi_q$. Using $x'^2 + y'^2 = R^2$ on the cylinder wall, we have then

$$\sin^2\theta_q\,\mathcal{D}^2 - 2\rho\mathcal{D}\sin\theta_q\cos(\phi_q - \phi_r) - R^2 + \rho^2 = R^2, \tag{A2}$$

where ϕ_r is the azimuthal angle of $\vec{r}$ and $\rho = \sqrt{x^2+y^2}$. The solution is

$$\mathcal{D} = \frac{1}{\sin\theta_q}[\rho\cos(\phi_q - \phi_r) + \sqrt{R^2 - \rho^2\sin^2(\phi_q - \phi_r)}]; \tag{A3}$$

without loss of generality we take $\phi_r = 0$.

The local phonon flow velocity $v_{\mathrm{ph},z}(\rho)$ is given by the local total momentum flux density in the z direction divided by the normal mass density of the phonons, ρ_{ph},

$$v_{\mathrm{ph},z}(\rho) \equiv \frac{1}{\rho_{\mathrm{ph}}}\int \frac{d^3q}{(2\pi)^3} q\cos\theta_q n_{\vec{q}}\,(\vec{r}). \tag{A4}$$

To first order in T', only the $\mathcal{D}$ term in the distribution function survives the angular average in the numerator, so that

$$v_{\mathrm{ph},z}(\rho) = s\frac{T'}{T_0}\frac{\int d^3q\cos^2\theta_q q^2\mathcal{D}\,\partial n_q^0/\partial q}{\int d^3q q^2\,\partial n_q^0/\partial q}, \tag{A5}$$

independent of z. Since $\mathcal{D}$ is independent of q, the integrals over q in numerator and denominator cancel, and

$$v_{\mathrm{ph},z}(\rho) = -3s\frac{T'}{T_0}\langle\mathcal{D}\cos^2\theta_q\rangle, \tag{A6}$$

where the angular brackets denote the average over angles of $\vec{q}$.

The flow velocity averaged over the cross section of the pipe (denoted by an overline) is simply

$$\overline{v}_{\mathrm{ph},z} = \frac{1}{\pi R^2}\int_0^R 2\pi\rho\,d\rho\,v_{\mathrm{ph},z}(\rho) = -3s\frac{T'}{T_0}\mathcal{D}_{\mathrm{eff}}, \tag{A7}$$

where

$$\mathcal{D}_{\mathrm{eff}} = \frac{1}{\pi R^2}\int_0^R 2\pi\rho\,d\rho\,\langle\mathcal{D}\cos^2\theta_q\rangle. \tag{A8}$$

In evaluating $\mathcal{D}_{\mathrm{eff}}$, the integral over θ_q decouples from those over ρ and ϕ_q, and the latter are easily performed if one integrates over ρ before integrating over ϕ_q. One finds

$$\mathcal{D}_{\mathrm{eff}} = \frac{2R}{3} = \frac{1}{2}\overline{\langle\mathcal{D}\rangle}, \tag{A9}$$

which expresses the fact that the length important for thermal conduction is *one-half* of $\overline{\langle\mathcal{D}\rangle}$, the average distance to the wall of the pipe, averaged over the cross section of the pipe. Since $\overline{v}_{\mathrm{ph},z} = -3(K_{\mathrm{ph}}/TC_{\mathrm{ph}})T'$, where C_{ph} is the phonon heat capacity, we find the phonon thermal conductivity,

$$K_{\mathrm{ph}} = sC_{\mathrm{ph}}\mathcal{D}_{\mathrm{eff}} = \frac{2}{3}sC_{\mathrm{ph}}R, \tag{A10}$$

which is the Casimir result, Ref. [10].

Locally,

$$\int\frac{d\Omega_q}{4\pi}\mathcal{D}\cos^2\theta_q = \frac{\pi R}{4}I(\rho^2/R^2), \tag{A11}$$

where the elliptic integral

$$I(t^2) = \frac{2}{\pi}\int_0^{\pi/2} d\phi\sqrt{1-t^2\sin^2\phi} \tag{A12}$$

must be done numerically. As we see in Fig. 6, the velocity profile

$$v_{\mathrm{ph},z}(\rho) = -\frac{3\pi sRT'}{4T_0}I(\rho^2/R^2) \tag{A13}$$

is independent of z and nearly quadratic almost to the edge of the pipe, where it falls more rapidly, but unlike when viscosity dominates, it does not fall to zero at the pipe wall.

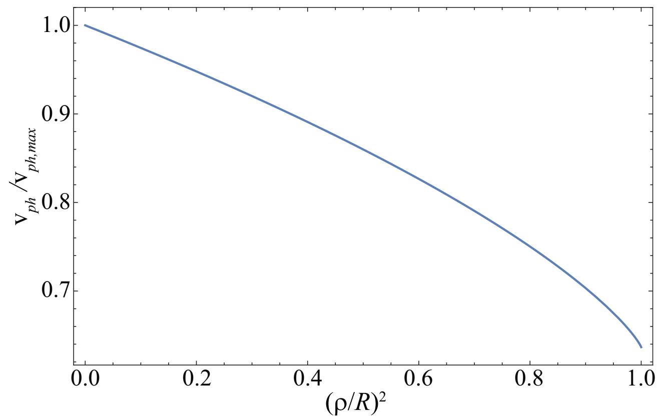

FIG. 6. (Color online) Normalized velocity distribution across the cylinder as a function of $(\rho/R)^2$ in the Casimir limit.

APPENDIX B: EXACT SOLUTION OF THE TIME DEPENDENT DIFFUSION EQUATION WITH ADVECTION

Here, we construct the general solutions of the time-dependent diffusion equation (39) by first transforming the equation into self-adjoint form by writing $n_3(z,t) = e^{z/2h}\hat{n}(z,t)$. As a result, $\hat{n}(z,t)$ obeys

$$\left(\frac{\partial}{\partial t} + \frac{v_{\mathrm{ph}}^2}{4D} - D\frac{\partial^2}{\partial z^2}\right)\hat{n}(z,t) = 0, \tag{B1}$$

in the interval $0 \leqslant z \leqslant L$ (we simply write a generic diffusion constant D to simplify the notation). The boundary condition of vanishing current at the two ends of the pipe then becomes

$$\frac{\partial\hat{n}}{\partial z} = \frac{\hat{n}}{2h} \qquad (z = 0,L), \tag{B2}$$

where $h = D/v_{\mathrm{ph}}$. To realize the boundary conditions, we expand $\hat{n}$ in a complete set of normalized solutions of Eq. (B1) that are periodic in the interval 0 to $2L$,

$$\phi_\nu(z) = \alpha_\nu\left(\cos kz + \frac{1}{2hk}\sin kz\right), \tag{B3}$$

with $k = \pi\nu/L$ ($\nu \geqslant 1$), and

$$\alpha_\nu = [(L/2)(1 + 1/(2hk_\nu)^2)]^{-1/2}, \tag{B4}$$

together with the stationary solution

$$\phi_0(z) = \frac{e^{z/2h}}{[h(e^{L/h}-1)]^{1/2}}. \tag{B5}$$

The modes $\phi_\nu(z)$ form an orthonormal set obeying

$$\int_0^L \phi_\nu(z)\phi_{\nu'}(z)dz = \delta_{\nu,\nu'} \tag{B6}$$

as well as the completeness relation in the interval $0 < z,z' < L$,

$$\sum_{\nu=0}^{\infty}\phi_\nu(z)\phi_\nu(z') = \delta(z-z'). \tag{B7}$$

The time dependence of the modes $\phi_\nu(z,t)$ is e^{-t/τ_ν}, with

$$\frac{1}{\tau_\nu} = \left(\frac{1}{4h^2} + k_\nu^2\right)D, \quad \nu \geqslant 1, \tag{B8}$$

and $1/\tau_0 = 0$. Then

$$\hat{n}(z,t) = \sum_{\nu=0}^{\infty} b_\nu \phi_\nu(z) e^{-t/\tau_\nu} \tag{B9}$$

with

$$b_\nu = \int_0^L \hat{n}(z,0)\phi_\nu(z)dz. \tag{B10}$$

We can thus write the solution $\hat{n}(z,t)$ in terms of the initial density distribution as

$$\hat{n}(z,t) = \int_0^L \mathcal{G}(z,z',t)\hat{n}(z',0)dz', \tag{B11}$$

where

$$\mathcal{G}(z,z',t) = \sum_{\nu=0}^{\infty} \phi_\nu(z)\phi_\nu(z') e^{-t/\tau_\nu}\theta(t) \tag{B12}$$

is the Green's function for the diffusion equation in the form (B1), with θ the Helmholtz unit step function. At $t = 0$, the completeness relation implies $\mathcal{G}(z,z',t) = \delta(z - z')$, and thus

$$\left(\frac{\partial}{\partial t} + \frac{v_{\rm ph}^2}{4D} - D\frac{\partial^2}{\partial z^2}\right)\mathcal{G}(z,z',t) = \delta(z - z')\delta(t), \tag{B13}$$

in the interval $0 \leqslant z \leqslant L$.

We now convert back to the original form of the diffusion equation, (39), and have

$$n_3(z,t) = \int_0^L G(z,z',t)\, n_3(z',0)dz', \tag{B14}$$

where

$$G(z,z',t) = e^{(z+z')/2h}\mathcal{G}(z,z',t) \tag{B15}$$

is the Green's function for the original diffusion equation (39), i.e.,

$$\left(\frac{\partial}{\partial t} + v_{\rm ph}\frac{\partial}{\partial z} - D\frac{\partial^2}{\partial z^2}\right)G(z,z',t) = e^{(z+z')/2h}\delta(z - z')\delta(t). \tag{B16}$$

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