



# Topological invariants for gauge theories and symmetry-protected topological phases

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We study the braiding statistics of particlelike and looplike excitations in two- (2D) and three-dimensional (3D) gauge theories with finite, Abelian gauge group. The gauge theories that we consider are obtained by gauging the symmetry of gapped, short-range entangled, lattice boson models. We define a set of quantities, called *topological invariants*, that summarize some of the most important parts of the braiding statistics data for these systems. Conveniently, these invariants are always Abelian phases, even if the gauge theory supports excitations with non-Abelian statistics. We compute these invariants for gauge theories obtained from the exactly soluble group cohomology models of Chen, Gu, Liu, and Wen, and we derive two results. First, we find that the invariants take different values for every group cohomology model with finite, Abelian symmetry group. Second, we find that these models exhaust all possible values for the invariants in the 2D case, and we give some evidence for this in the 3D case. The first result implies that every one of these models belongs to a distinct symmetry-protected topological (SPT) phase, while the second result suggests that these models may realize all SPT phases. These results support the group cohomology classification conjecture for SPT phases in the case where the symmetry group is finite, Abelian, and unitary.

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## I. INTRODUCTION

Topological insulators [1,2] are a special case of symmetry-protected topological (SPT) phases [3–10]. These phases can occur in quantum many-body systems of arbitrary dimension and arbitrary symmetry. By definition, a gapped quantum many-body system belongs to a (nontrivial) SPT phase if it satisfies three properties. First, the Hamiltonian is invariant under some set of internal symmetries, none of which are broken spontaneously. Second, the ground state is short-range entangled: that is, it can be transformed into a product state or atomic insulator using a local unitary transformation [11–13]. Third, the ground state cannot be continuously connected with a product state, by varying some parameter in the Hamiltonian, without breaking one of the symmetries or closing the energy gap. In addition, nontrivial SPT phases typically have robust boundary modes [1,2,14–17] analogous to that of topological insulators, but this property is not part of the formal definition.

Chen, Gu, Liu, and Wen [10] have proposed a general classification scheme for SPT phases built out of bosons. Their classification scheme is based on their construction of a collection of exactly soluble lattice boson models of arbitrary symmetry and spatial dimension. The authors conjecture that these models, called group cohomology models, have two basic properties: (i) every group cohomology model belongs to a distinct SPT phase and (ii) every SPT phase can be realized by a group cohomology model. If both properties hold, then it follows logically that there is a one-to-one correspondence between the group cohomology models and SPT phases. In Ref. [10], the authors assumed this to be the case, and thereby derived a classification scheme for SPT phases based on group cohomology.

While the results and arguments of Ref. [10] represent a major advance in our understanding of SPT phases, they leave several questions unanswered. First, it is not obvious that properties (i) and (ii) hold in general [in fact, property (ii) is known to fail for SPT phases with antiunitary symmetries [17–19]. Second, even if these properties do hold at some level, the resulting classification scheme is not completely satisfying

since it does not tell us how to determine to which SPT phase a microscopic Hamiltonian belongs.

Motivated by these problems, several proposals have been made for how to physically characterize and distinguish SPT phases [15–17,20–32]. Here, we will focus on the suggestion of Refs. [15,24] which applies to two- (2D) and three-dimensional (3D) SPT models with *unitary* symmetries. Reference [15] showed, via a simple example, that one can probe 2D SPT models by gauging their symmetries and studying the braiding statistics of the excitations in the resulting gauge theory. These braiding statistics data are useful because they are invariant under arbitrary symmetry-preserving deformations of the Hamiltonian, as long as the energy gap remains open. Therefore, if two SPT models give rise to different braiding statistics, then they must belong to distinct SPT phases [33]. The braiding statistics approach can also be applied to 3D SPT phases [24,25]. In that case, after gauging the symmetry, one studies the braiding statistics of the vortex loop excitations in the resulting gauge theory. More specifically, different SPT phases can be distinguished by examining their *three-loop* braiding statistics, the statistics associated with braiding a loop  $\alpha$  around another loop  $\beta$ , while they are both linked to a third loop  $\gamma$ .

When considered together, the braiding statistics approach and the group cohomology construction raise several questions:

- (1) Does every group cohomology model lead to distinct braiding statistics?<sup>1</sup>
- (2) Do the group cohomology models exhaust all possible types of braiding statistics that can occur in SPT systems?
- (3) If two SPT models give rise to the same braiding statistics, do they always belong to the same phase?

<sup>1</sup>Here, when we say braiding statistics, we mean the complete set of algebraic data for anyon systems, including quantum dimensions and fusion rules. For more details, see e.g. Appendix E of Ref. [34].

The answers to these questions have powerful implications, especially if we can answer them affirmatively. For example, if we can answer the first question in the affirmative, we can immediately conclude that every group cohomology model belongs to a distinct phase. Likewise, if we can answer the second and third questions in the affirmative, then we can conclude that the group cohomology models realize all possible SPT phases. If we can answer all three questions affirmatively, then it follows that (1) the group cohomology classification is correct and (2) the braiding statistics data provide a universal probe for characterizing and distinguishing different SPT phases with unitary symmetries.

In this work, we consider the first and second questions for the case of 2D and 3D SPT phases with finite Abelian unitary symmetry group  $G = \prod_{i=1}^K \mathbb{Z}_{N_i}$ . We answer the first question in the affirmative and we find evidence that the same is true for the second question, as we explain in the following.

We obtain our results by focusing on a subset of the braiding statistics data that summarizes some of its most important features (in fact, for systems with Abelian statistics, this subset is equivalent to the full set of braiding data, see Sec. VIII). In the 2D case, this subset consists of three tensors  $\{\Theta_i, \Theta_{ij}, \Theta_{ijk}\}$  that take values in  $[0, 2\pi]$ , where  $1 \leq i, j, k \leq K$ . In the 3D case, it consists of three tensors  $\{\Theta_{i,l}, \Theta_{ij,l}, \Theta_{ijk,l}\}$  with  $1 \leq i, j, k, l \leq K$ . These tensors, which we call *topological invariants*, are defined by considering the Berry phase associated with certain composite braiding processes of vortices or vortex loops. Conveniently, these Berry phases are always Abelian phases regardless of whether the full set of braiding statistics is Abelian or non-Abelian.

We report two main results. First, we show that the topological invariants take different values in every group cohomology model. Second, we show that the group cohomology models exhaust all possible values for the topological invariants in the 2D case and we give some evidence for this in the 3D case. Our first result implies that the group cohomology models all belong to distinct phases. Our second result can be interpreted as evidence that the group cohomology models realize all possible SPT phases with finite Abelian unitary symmetry group.

Some of our results have appeared previously in the literature, although in a slightly different form. In particular, in the 2D case, Ref. [32] introduced invariants similar to ours and showed that the invariants can distinguish all the 2D group cohomology models. Also, much of our analysis of 3D gauge theories is similar to that of our previous work [24]. However, this paper goes further than Ref. [24] in three key ways. First, we study both Abelian and non-Abelian loop braiding statistics, while Ref. [24] only studied Abelian statistics. Second, we consider a general finite Abelian symmetry group  $\prod_{i=1}^K \mathbb{Z}_{N_i}$  while Ref. [24] only considered groups of the form  $(\mathbb{Z}_N)^K$ . Finally, we make a systematic comparison between the topological invariants and the group cohomology classification, while Ref. [24] only made this comparison in a few examples.

A note on our terminology: throughout the paper, we will refer to gauged SPT models as simply *gauge theories*. Also, we will refer to the gauged group cohomology models as *Dijkgraaf-Witten models* [35]. The Dijkgraaf-Witten models were studied long before the discovery of SPT phases, however, it can be shown that they are equivalent to the gauged

group cohomology models (the equivalence is discussed in Appendix C).

The rest of the paper is organized as follows. In Sec. II, we introduce the models that we will study, both the general gauged SPT models and the more specific Dijkgraaf-Witten models. In Sec. III, we discuss the general structure of braiding statistics in gauged SPT models and we define the topological invariants. Next, we compute the topological invariants in 2D and 3D Dijkgraaf-Witten models in Sec. IV. In Sec. V, we show that the topological invariants take different values in every Dijkgraaf-Witten model. In Sec. VI, we derive general constraints that the topological invariants must satisfy in any gauged SPT model. In Sec. VII, we discuss whether the Dijkgraaf-Witten models exhaust all possible values for the invariants. The relation between the topological invariants and the full set of braiding statistics in the case of Abelian statistics is discussed in Sec. VIII. Finally, in Sec. IX, we conclude and discuss the implications of our results for SPT phases. The Appendices contain several technical details.

## II. MODELS

### A. Gauge theories

The main systems we will study in this paper are 2D and 3D lattice gauge theories with finite Abelian gauge group  $G = \prod_{i=1}^K \mathbb{Z}_{N_i}$ . More specifically, we will study a particular class of gauge theories that are obtained from a two-step construction. The first step of the construction is to pick a 2D or 3D lattice boson or spin model with a global  $\prod_{i=1}^K \mathbb{Z}_{N_i}$  symmetry. This boson model can be quite general, with the only restrictions being that (1) it has local interactions, (2) the symmetry is an internal (onsite) symmetry rather than a spatial symmetry, and (3) its ground state is gapped and short-range entangled, that is, the ground state can be transformed into a product state by a local unitary transformation. Here, by a local unitary transformation, we mean a unitary transformation  $U$  of the form  $U = \exp(iHs)$ , where  $H$  is a local Hermitian operator and  $s$  is a finite constant that does not scale with the system size [11–13]. (Note that the transformation  $U$  need not commute with the symmetry.)

The second step of the construction is to gauge the global symmetry of the lattice boson model and couple it to a dynamical lattice gauge field with group  $G$ . In Appendix A, we give a precise prescription for how to implement this gauging procedure. This prescription mostly follows the usual minimal coupling scheme [36]. However, there is one nonstandard element that is worth mentioning: our procedure is defined so that the gauge coupling constant is *exactly zero*. More precisely, what we mean by this is that the Hamiltonian for the gauged model commutes with the flux operators that measure the gauge flux through each plaquette in the lattice. This property is convenient because it makes the low-energy physics of our models well controlled. In particular, using this property it can be shown that the gauge theories constructed via our gauging procedure are guaranteed to be gapped and deconfined as long as the original boson models are gapped and do not break the symmetry spontaneously (see Appendix A).

The above two-step construction defines the class of models that we will study in this paper. From now on, when we use the

term *gauge theory* we will be referring exclusively to models of this type, unless we state otherwise.

Before concluding this section, we would like to mention that although we find it convenient to use the particular gauging prescription in Appendix A, we do not expect that our results actually depend on the details of the gauging procedure, or on the fact that the resulting gauge theories have zero gauge coupling. Indeed, our results are guaranteed to hold for any model that can be continuously connected to one of the above gauge theories without closing the energy gap. We expect that the latter category includes models obtained from generic gauging procedures, as long as the gauge coupling constant is sufficiently small.

### B. Dijkgraaf-Witten models

In part of this work we will study a particular set of exactly soluble gauge theories, known as Dijkgraaf-Witten models [35] which are obtained by gauging the group cohomology models of Ref. [10]. We now briefly review the properties of the group cohomology models and the corresponding Dijkgraaf-Witten models. For the explicit definition of these models, see Appendix C.

The group cohomology models are exactly soluble lattice boson models that can be defined in any spatial dimension  $d$ . The basic input needed to construct a  $d$ -dimensional group cohomology model is a (finite) group  $G$  and a  $(d + 1)$ -cocycle  $\omega$ . Here, an  $n$ -cocycle  $\omega$  is a function  $\omega : G^n \rightarrow U(1)$  that satisfies certain conditions. One may define an equivalence relation on cocycles and the equivalence classes are labeled by the elements of the cohomology group  $H^n[G, U(1)]$  (a brief introduction to group cohomology [37] is given in Appendix B). It can be shown that the models constructed from equivalent cocycles are identical so we will say that the  $d$ -dimensional group cohomology models are labeled by elements of  $H^{d+1}[G, U(1)]$ .

Like the group cohomology models, the basic input needed to construct a Dijkgraaf-Witten model in spatial dimension  $d$  is a group  $G$  and a  $(d + 1)$ -cocycle  $\omega$ . Also, like the group cohomology models, the Dijkgraaf-Witten models constructed from equivalent cocycles are the same, so we will say that they are labeled by different elements of  $H^{d+1}[G, U(1)]$ . Here, we will focus on the case  $G = \prod_{i=1}^K \mathbb{Z}_{N_i}$  and  $d = 2, 3$ .

### C. Braiding statistics and phases of SPT models and gauge theories

Before proceeding further, we briefly review some results on the relationship between SPT models, gauge theories, and braiding statistics. We begin by defining *phases* of SPT models and phases of gauge theories. The former definition is relatively simple: we say that two lattice boson models with the same symmetry group belong to the same SPT phase if they can be continuously connected to one another by varying some parameter in the (symmetry-preserving) Hamiltonian, without closing the energy gap.

Defining phases of gauge theories is more subtle. In fact, there are two inequivalent ways to define this concept, both of which have their merits. In the first definition, two

gauge theories belong to the same phase if they can be continuously connected by varying some parameter in the (gauge-invariant) Hamiltonian without closing the energy gap. In the second definition, not only do we require the existence of an interpolating Hamiltonian with an energy gap, but we also demand that the interpolating Hamiltonian has *vanishing gauge coupling*, that is, the Hamiltonian must commute with the flux operators that measure the gauge flux through each plaquette in the lattice. While the first definition is very natural if one is interested in gauge theories for their own sake, the second definition is more relevant to the study of SPT phases. In this paper, our primary interest is in SPT phases, so we will use the second definition.

In parallel to the two ways of defining phases of gauge theories, there are also two ways to define what it means for two gauge theories to have the “same” braiding statistics data. In the first definition, two gauge theories have the same braiding statistics data if one can map the excitations of one gauge theory onto the excitations of the other gauge theory such that the corresponding excitations have identical braiding statistics. In the second definition, the corresponding excitations are required both to have the same braiding statistics *and* the same gauge flux. In this paper, we will use the second definition since it fits more naturally with our definition of phases of gauge theories.

With these definitions in mind, we can now discuss some results. An important observation is that if two lattice boson models belong to the same SPT phase, then the corresponding gauged models must also belong to the same phase. To see this, note that our gauging prescription (Appendix A) maps gapped lattice boson models onto gapped zero-coupling gauge theories; hence, any continuous interpolation between two SPT models can be gauged to give an interpolation between the two corresponding gauge theories.

Another important observation is that if two gauge theories belong to the same phase, then they must have the same braiding statistics. One way to see this is to note that braiding statistics data can only take on discrete values and cannot change continuously. (This discreteness property is known as *Oceanu rigidity* [34].) Combining the above two observations, we derive a useful corollary: if two lattice boson models belong to the same SPT phase, then they must give rise to the same braiding statistics after gauging their symmetries. The converse of this statement may also be true, but it is not obvious.

## III. DEFINING THE TOPOLOGICAL INVARIANTS

In this section, we construct a set of topological invariants for gauge theories with gauge group  $G = \prod_{i=1}^K \mathbb{Z}_{N_i}$ . (Here, when we say “gauge theory,” we mean a gauge theory of the type discussed in Sec. II.) These invariants are defined in terms of the braiding statistics [34,38] of the excitations of the gauge theory. They are denoted by  $\Theta_i, \Theta_{ij}, \Theta_{ijk}$  in the 2D case and  $\Theta_{i,l}, \Theta_{ij,l}, \Theta_{ijk,l}$  in the 3D case, where the indices  $i, j, k, l$  range over  $1, \dots, K$ . For pedagogical purposes, we first define the invariants in the case where the braiding statistics are Abelian, and then discuss the general case (where the statistics may be Abelian or non-Abelian).

### A. 2D Abelian case

We start with the simplest case: we consider 2D gauge theories with group  $G = \prod_{i=1}^K \mathbb{Z}_{N_i}$  and with Abelian braiding statistics.

#### 1. Excitations and braiding statistics

We first discuss the excitation spectrum of these gauge theories. In general, every excitation  $\alpha$  in a discrete gauge theory [36] can be labeled by the amount of gauge flux  $\phi_\alpha$  that it carries. In our case, the gauge flux  $\phi_\alpha$  can be described by a  $K$ -component vector  $\phi_\alpha = (\phi_{1\alpha}, \phi_{2\alpha}, \dots, \phi_{K\alpha})$  where each component  $\phi_{i\alpha}$  is a multiple of  $\frac{2\pi}{N_i}$ , and is defined modulo  $2\pi$ . Excitations can be divided into two groups: *charge* excitations that carry vanishing gauge flux and *vortex* excitations that carry nonzero gauge flux.

As far as their topological properties go, charge excitations are uniquely characterized by their gauge charge  $q = (q_1, q_2, \dots, q_K)$  where each component  $q_i$  is defined modulo  $N_i$ . In contrast, vortex excitations are *not* uniquely characterized by the amount of gauge flux that they carry: in fact, there are  $|G| = \prod_{i=1}^K N_i$  different types of vortices carrying the same flux  $\phi$ . All of these vortices can be obtained by attaching charge excitations to a fixed reference vortex with flux  $\phi$ . Throughout this paper, we will use Greek letters  $\alpha, \beta, \gamma, \dots$  to denote vortices as well as general excitations, and we will use the letter  $q$  to denote both a charge excitation and its gauge charge.

Before proceeding further, we would like to point out a possible source of confusion: Given what we have said about the different types of vortices, it is tempting to try to label vortex excitations by *both* their gauge flux and their gauge charge. The problem with this approach is that we do not know any physically meaningful way to define the absolute charge carried by a vortex excitation in a discrete gauge theory. Therefore, we will avoid using this notion in this paper. Instead, we will only use the concept of *relative* charge: we will say that two vortices  $\alpha, \alpha'$  *differ* by charge  $q$  if  $\alpha'$  can be obtained by attaching a charge excitation  $q$  to  $\alpha$ .

Let us now consider the braiding statistics of the different excitations. There are three different braiding processes to consider: braiding of two charges, braiding of a charge around a vortex, and braiding of two vortices. The first process is easy to analyze: indeed, it is clear that the charges correspond to local excitations in the ungauged short-range-entangled bosonic state. Therefore, the charges are all bosons and have trivial (bosonic) mutual statistics. The braiding between a charge and a vortex is also easy to understand, as it follows from the Aharonov-Bohm law. More specifically, the statistical Berry phase  $\theta$  associated with braiding a charge  $q$  around a vortex with flux  $\phi$  is given by

$$\theta = q \cdot \phi, \quad (1)$$

where “ $\cdot$ ” is the vector inner product. Note that attaching a charge to the vortex does not change the Aharonov-Bohm law since the charges have trivial mutual statistics with respect to one another.

From the above arguments, we see that the charge-charge and charge-vortex statistics are completely fixed by the gauge group, leaving no room for variation. Therefore, the only

braiding process that has potential for distinguishing gauge theories with the same gauge group is vortex-vortex braiding. Motivated by this observation, we will define the topological invariants  $\Theta_i, \Theta_{ij}$  in terms of the vortex-vortex braiding statistics.

#### 2. Topological invariants

Let  $\alpha$  be a vortex carrying a unit flux  $\frac{2\pi}{N_i} e_i$ , where  $e_i = (0, \dots, 1, \dots, 0)$  with a 1 the  $i$ th entry and 0 everywhere else. Let  $\beta$  be a vortex carrying a unit flux  $\frac{2\pi}{N_j} e_j$ . Here,  $i$  and  $j$  can take any value in  $1, \dots, K$ . We define

$$\Theta_{ij} = N^{ij} \theta_{\alpha\beta}, \quad \Theta_i = N_i \theta_\alpha, \quad (2)$$

where  $\theta_{\alpha\beta}$  is the mutual statistics between  $\alpha$  and  $\beta$ ,  $\theta_\alpha$  is the exchange statistics of  $\alpha$ , and  $N^{ij}$  is the least common multiple of  $N_i$  and  $N_j$  (More generally, throughout the paper, we use  $N^{ij\dots k}$  to denote the least common multiple of  $N_i, N_j, \dots, N_k$  and use  $N_{ij\dots k}$  to denote the greatest common divisor of  $N_i, N_j, \dots, N_k$ .)

For the quantities  $\Theta_{ij}$  and  $\Theta_i$  to be well defined, we need to check that  $N^{ij} \theta_{\alpha\beta}$  and  $N_i \theta_\alpha$  only depend on  $i$  and  $j$ , and not on the choice of the vortices  $\alpha, \beta$ . To see that this is the case, imagine that we replace  $\alpha, \beta$  with some other vortices  $\alpha', \beta'$  carrying flux  $\frac{2\pi}{N_i} e_i$  and  $\frac{2\pi}{N_j} e_j$ . Then, clearly the vortices  $\alpha$  and  $\alpha'$  differ only by the attachment of charge, as do  $\beta$  and  $\beta'$ . Therefore, according to the Aharonov-Bohm law, the change in  $\Theta_{ij}$  that occurs when we replace  $\alpha \rightarrow \alpha', \beta \rightarrow \beta'$  is

$$\Theta_{ij} \rightarrow \Theta_{ij} + 2\pi N^{ij} \left( \frac{x}{N_i} + \frac{y}{N_j} \right), \quad (3)$$

where  $x, y$  are integers that describe the amount of type- $i$  and type- $j$  charge that is attached to  $\beta$  and  $\alpha$ , respectively. But,  $N^{ij}$  is divisible by both  $N_i$  and  $N_j$  so we see that this replacement does not change  $\Theta_{ij}$  modulo  $2\pi$ . Similarly, the Aharonov-Bohm law tells us that the change in  $\Theta_i$  that occurs when we replace  $\alpha \rightarrow \alpha'$  is

$$\Theta_i \rightarrow \Theta_i + 2\pi N_i \frac{z}{N_i}, \quad (4)$$

where  $z$  is the type- $i$  charge that is attached to  $\alpha$ . Thus,  $\Theta_i$  is also unchanged modulo  $2\pi$ . We conclude that the quantities  $\Theta_{ij}$  and  $\Theta_i$  are both well defined.

In addition to being well defined, it is possible to show that  $\Theta_{ij}$  and  $\Theta_i$  have another nice property: they contain the same information as the full set of braiding statistics. We will derive this result in Sec. VIII.

### B. 2D general case

In this section, we move on to general 2D gauge theories with gauge group  $G = \prod_{i=1}^K \mathbb{Z}_{N_i}$ . Unlike the previous section, we do not assume that the braiding statistics of the excitations is Abelian. This additional generality is important because, contrary to naive expectations, gauge theories with Abelian gauge groups can sometimes have excitations with non-Abelian statistics. For example, this phenomenon occurs in 2D  $\prod_{i=1}^K \mathbb{Z}_{N_i}$  Dijkgraaf-Witten models when  $K \geq 3$  (cf. Ref. [39]).

In the general case, we will define three topological invariants  $\Theta_i, \Theta_{ij}, \Theta_{ijk}$ . The first two  $\Theta_i, \Theta_{ij}$  reduce to those defined in Eq. (2) when restricted to Abelian statistics. The third invariant  $\Theta_{ijk}$  is new to the non-Abelian case, and vanishes in the Abelian case.

### 1. General aspects: Excitations, fusion rules, and braiding statistics

Many features of the Abelian case carry over to the general case without change. First, we can still label every excitation  $\alpha$  by the amount of gauge flux  $\phi_\alpha = (\phi_{1\alpha}, \dots, \phi_{K\alpha})$  that it carries, where  $\phi_{i\alpha}$  is a multiple of  $\frac{2\pi}{N_i}$  and is defined modulo  $2\pi$ . Also, we can still divide excitations into two groups: *charges*, that carry vanishing flux, and *vortices*, that carry nonzero flux. Charge excitations are still characterized uniquely by their gauge charge  $q = (q_1, \dots, q_K)$  with  $q_i$  defined modulo  $N_i$ , while vortices are still characterized *nonuniquely* by their gauge flux. Finally, charges are still Abelian particles with trivial charge-charge statistics, and with charge-vortex statistics given by the Aharonov-Bohm law:  $\theta = q \cdot \phi$  where  $\phi$  is the gauge flux carried by the vortex. The main new element in the general case is that vortices can be non-Abelian, i.e., they can have non-Abelian fusion rules and non-Abelian braiding statistics with one another [34,38].

While the possibility of non-Abelian vortices complicates our analysis, we can still make some general statements about the fusion rules and braiding statistics in these systems. In what follows, we focus on the fusion rules, and we list some properties which will be useful in our later arguments (see Appendix D for proofs and details). To begin, imagine we fuse together two excitations  $\alpha$  and  $\beta$ . In general, there may be a number of possible fusion outcomes corresponding to other excitations  $\gamma$ :

$$\alpha \times \beta = \sum_{\gamma} N_{\alpha\beta}^{\gamma} \gamma, \quad (5)$$

where  $N_{\alpha\beta}^{\gamma}$  is the dimension of the fusion space  $\mathbb{V}_{\alpha\beta}^{\gamma}$ . One property of these fusion rules is that

$$\phi_{\gamma} = \phi_{\alpha} + \phi_{\beta} \quad (6)$$

for any fusion product  $\gamma$ . In particular, if  $\phi_{\alpha} + \phi_{\beta} = 0$ , then all the  $\gamma$ 's that appear on the right-hand side of (5) are pure charges.

A second property is that the fusion of a charge  $q$  and an excitation  $\alpha$  always results in a single excitation

$$\alpha \times q = \alpha', \quad (7)$$

where  $\alpha'$  is not necessarily distinct from  $\alpha$  and  $\phi_{\alpha'} = \phi_{\alpha}$ . A third property is that if two excitations  $\alpha, \alpha'$  have the same flux,  $\phi_{\alpha'} = \phi_{\alpha}$ , then there exists at least one charge  $q$  with  $\alpha' = \alpha \times q$ .

To describe the final property, let  $\alpha$  and  $\beta$  be two excitations, and let  $\gamma$  be one of their fusion channels. Let  $\alpha'$  and  $\beta'$  be two other excitations with  $\phi_{\alpha'} = \phi_{\alpha}$  and  $\phi_{\beta'} = \phi_{\beta}$ , and let  $\gamma'$  be one of their fusion channels. The final property states that there exist charges  $q_1$  and  $q_2$  such that  $\alpha' = \alpha \times q_1$ ,  $\beta' = \beta \times q_2$ , and  $\gamma' = \gamma \times q_1 \times q_2$ .

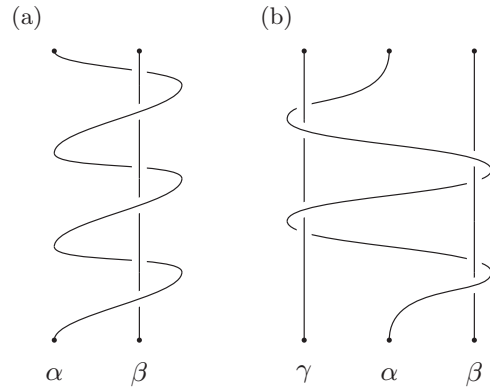


FIG. 1. Space-time trajectories of the vortices in the braiding processes associated with  $\Theta_{ij}$  [(a);  $N^{ij} = 3$ ] and  $\Theta_{ijk}$  [(b)]. The arrow of time is upward.

### 2. Topological invariants

Similarly to the Abelian case, we define the topological invariants  $\Theta_i, \Theta_{ij}, \Theta_{ijk}$  in terms of the braiding statistics of vortices. Let  $\alpha, \beta, \gamma$  be three vortices carrying unit fluxes  $\frac{2\pi}{N_i} e_i, \frac{2\pi}{N_j} e_j, \frac{2\pi}{N_k} e_k$ , respectively. The topological invariants  $\Theta_i, \Theta_{ij}, \Theta_{ijk}$  are defined as follows.

*Definitions:*

(1)  $\Theta_i = 2\pi N_i s_\alpha$ , where  $s_\alpha$  is the topological spin of  $\alpha$ .

(2)  $\Theta_{ij}$  is the Berry phase associated with braiding  $\alpha$  around  $\beta$  for  $N^{ij}$  times.

(3)  $\Theta_{ijk}$  is the Berry phase associated with the following braiding process:  $\alpha$  is first braided around  $\beta$ , then around  $\gamma$ , then around  $\beta$  in the opposite direction, and finally around  $\gamma$  in the opposite direction.

Figure 1 shows the space-time trajectories of the vortices in the braiding processes of  $\Theta_{ij}$  and  $\Theta_{ijk}$ . We note that the definitions of  $\Theta_i$  and  $\Theta_{ij}$  reduce to our previous definitions (2) in the Abelian case since  $2\pi s_\alpha = \theta_\alpha$  for Abelian quasiparticles. We can also see that  $\Theta_{ijk} = 0$  in the Abelian case.

Before we show that these quantities are well defined, we comment on the definition of  $\Theta_i$ . In defining  $\Theta_i$ , we have used the notion of topological spin. The topological spin  $s_\alpha$ ,  $0 \leq s_\alpha < 1$ , of an anyon  $\alpha$  is defined [34] to be

$$e^{i2\pi s_\alpha} = \frac{1}{d_\alpha} \sum_{\gamma} d_\gamma \text{tr}(R_{\alpha\alpha}^{\gamma}), \quad (8)$$

where  $d_\alpha$  and  $d_\gamma$  are the quantum dimensions of  $\alpha$  and  $\gamma$ , respectively,  $R_{\alpha\alpha}^{\gamma}$  is the braiding matrix associated with a half-braiding (exchange) of two  $\alpha$ 's, and the summation is over the  $\gamma$ 's appearing in the fusion product of  $\alpha \times \alpha$ .

We see that  $\Theta_i$  is rather abstract since  $s_\alpha$  does not have a direct physical interpretation. In contrast,  $\Theta_{ij}, \Theta_{ijk}$  are defined in terms of concrete physical braiding processes. One might wonder if there is a more concrete definition of  $\Theta_i$ . Indeed, when  $N_i$  is even, we find an alternative definition of  $\Theta_i$ :

$\Theta_i$  is the phase associated with exchanging two  $\alpha$ 's for  $N_i$  times, where  $\alpha$  is any vortex carrying unit flux  $\frac{2\pi}{N_i} e_i$ .

This alternative definition provides a direct way to “measure”  $\Theta_i$  when  $N_i$  is even. The equivalence between this alternative definition and the original topological spin definition of  $\Theta_i$  follows from two facts: First, exchanging two identical  $\alpha$

vortices  $N$  times is equivalent to braiding one around the other  $\frac{N}{2}$  times. Second, braiding two identical  $\alpha$  vortices around one another gives a pure phase  $e^{i4\pi s_\alpha}$ . (The latter claim, which is less obvious, is proved in Appendix E.)

One problem with the above definition is that it does not make sense when  $N_i$  is odd since in this case the unitary matrix associated with the exchange process is not necessarily a pure phase. Fortunately, we will see later that when  $N_i$  is odd,  $\Theta_i$  is uniquely determined by  $\Theta_{ii}$ , and the latter can be directly “measured” using a concrete braiding process. This point can be obtained from the constraints on the invariants, which we will study in Sec. VIB.

### 3. Proving the invariants are well defined

For the invariants to be well defined, we need to prove two points: (i) We need to show that the unitary transformations associated with the above braiding processes are always Abelian phases regardless of the fact that the vortices may be non-Abelian. (ii) We need to show that these Abelian phases are functions of  $i, j, k$  only and do not depend on the choice of vortices  $\alpha, \beta, \gamma$  as long as they carry fluxes  $\frac{2\pi}{N_i} e_i, \frac{2\pi}{N_j} e_j, \frac{2\pi}{N_k} e_k$ , respectively. Only point (ii) is needed for showing  $\Theta_i$  is well defined.

Let us start with proving point (ii) for  $\Theta_i$ . We first review some key properties of topological spin (a detailed discussion of topological spin can be found in Ref. [34]). If  $\alpha$  is an Abelian anyon,  $2\pi s_\alpha$  is just the exchange statistics of  $\alpha$ . In general,  $s_\alpha = s_{\bar{\alpha}}$ , where  $\bar{\alpha}$  is the antiparticle of  $\alpha$ . An important property of the topological spin is

$$R_{\beta\alpha}^\gamma R_{\alpha\beta}^\gamma = e^{i2\pi(s_\gamma - s_\alpha - s_\beta)} \text{id}_{\mathbb{V}_{\alpha\beta}^\gamma}. \quad (9)$$

Here,  $\mathbb{V}_{\alpha\beta}^\gamma$  is the fusion space of  $\alpha, \beta$  in the fusion channel  $\gamma$  and  $R_{\alpha\beta}^\gamma$  is the braiding matrix associated with a half-braiding of  $\alpha$  and  $\beta$  in the fusion channel  $\gamma$ . The notation  $\text{id}_{\mathbb{V}_{\alpha\beta}^\gamma}$  denotes the identity matrix in the fusion space  $\mathbb{V}_{\alpha\beta}^\gamma$ .

With these properties in mind, we now show that  $\Theta_i$  is well defined, i.e., we show that

$$2\pi N_i s_{\alpha'} = 2\pi N_i s_\alpha \quad (10)$$

for any two vortices  $\alpha, \alpha'$  carrying unit flux  $\frac{2\pi}{N_i} e_i$ . In the first step, we note that we can assume without loss of generality that  $\alpha' = q \times \alpha$  for some charge  $q$  since according to the properties discussed in Sec. III B1, any vortex with unit flux  $\frac{2\pi}{N_i} e_i$  can be constructed from a fixed vortex by fusing charges with it. To prove the result for this case, we substitute  $\beta = q$  and  $\gamma = \alpha' = q \times \alpha$  into Eq. (9), obtaining

$$R_{q\alpha}^{\alpha'} R_{\alpha q}^{\alpha'} = e^{i2\pi(s_{\alpha'} - s_\alpha - s_q)} = e^{i2\pi(s_{\alpha'} - s_\alpha)}, \quad (11)$$

where in the second equality we used the fact that  $q$  is a boson so  $s_q = 0$ . At the same time, we know

$$R_{q\alpha}^{\alpha'} R_{\alpha q}^{\alpha'} = e^{\frac{2\pi}{N_i} \times \text{integer}} \quad (12)$$

since the braiding of a charge around a vortex can be computed from the usual Aharonov-Bohm law. Combining these two relations, we see that  $2\pi N_i (s_{\alpha'} - s_\alpha)$  vanishes modulo  $2\pi$ , proving (10).

Next, we prove points (i) and (ii) in the case of  $\Theta_{ij}$ . Let  $\alpha$  and  $\beta$  be two vortices carrying unit flux  $\frac{2\pi}{N_i} e_i$  and  $\frac{2\pi}{N_j} e_j$ ,

respectively. Imagine we perform a full braiding of  $\alpha$  around  $\beta$  when they are in some fusion channel  $\delta$ . From the general theory of non-Abelian anyons [34], we know that the unitary matrix associated with a full braiding of  $\alpha$  around  $\beta$  in a fixed fusion channel  $\delta$  is a pure phase factor [this result is a corollary of Eq. (9)]. Denoting this phase factor by  $e^{i\theta_{\alpha\beta}^\delta}$ , the quantity  $\Theta_{ij}$  can be computed as

$$\Theta_{ij} = N^{ij} \theta_{\alpha\beta}^\delta. \quad (13)$$

In order to establish properties (i) and (ii) above, it suffices to show that

$$N^{ij} \theta_{\alpha\beta}^\delta = N^{ij} \theta_{\alpha'\beta'}^{\delta'} \quad (14)$$

for any other vortices  $\alpha', \beta'$  carrying unit flux  $\frac{2\pi}{N_i} e_i, \frac{2\pi}{N_j} e_j$ , and for any other fusion channel  $\delta'$ . Indeed, the independence of  $N^{ij} \theta_{\alpha\beta}^\delta$  with respect to the fusion channel  $\delta$  implies point (i), while the independence of  $N^{ij} \theta_{\alpha\beta}^\delta$  with respect to  $\alpha, \beta$  implies point (ii).

In fact, it is enough to prove (14) for the case where  $\alpha' = \alpha \times q_1$ ,  $\beta' = \beta \times q_2$ , and  $\delta' = \delta \times q_1 \times q_2$  for some charges  $q_1, q_2$  since according to the general properties discussed in Sec. III B1, any  $\alpha', \beta', \delta'$  can be obtained in this way. But, it is easy to prove (14) in this case. Indeed, from the Aharonov-Bohm law we can deduce the relation

$$N^{ij} (\theta_{\alpha'\beta'}^{\delta'} - \theta_{\alpha\beta}^\delta) = 2\pi N^{ij} \left( \frac{q_{2i}}{N_i} + \frac{q_{1j}}{N_j} \right), \quad (15)$$

where  $q_{2i}$  and  $q_{1j}$  are integers that describe the amount of type- $i$  and type- $j$  charge carried by  $q_2$  and  $q_1$ , respectively. We then observe that the expression on the right-hand side vanishes modulo  $2\pi$  since  $N^{ij}$  is divisible by both  $N_i$  and  $N_j$ . This establishes (14) and proves properties (i) and (ii) for  $\Theta_{ij}$ .

The proof of points (i) and (ii) for  $\Theta_{ijk}$  is more technical and is given in Appendix F.

### C. 3D Abelian case

Having warmed up with the 2D gauge theories, we now consider 3D gauge theories with gauge group  $G = \prod_i \mathbb{Z}_{N_i}$  and with Abelian loop statistics. The discussion that follows is a generalization of the case  $G = (\mathbb{Z}_N)^K$ , studied in Ref. [24].

#### 1. Excitations and three-loop braiding statistics

Discrete gauge theories in three dimensions support two types of excitations: charges and vortices. Charge excitations are particlelike and are characterized by the amount of gauge charge  $q$  that they carry, where  $q = (q_1, \dots, q_K)$  with  $q_i$  defined modulo  $N_i$ . Vortex excitations are stringlike and are characterized by the amount of gauge flux  $\phi$  that they carry where  $\phi = (\phi_1, \dots, \phi_K)$  with the component  $\phi_i$  being a multiple of  $\frac{2\pi}{N_i}$ , and defined modulo  $2\pi$ . We will refer to vortex excitations as *vortex loops* or simply *loops* since we will generally assume that the system is defined on a closed manifold with no boundary so that vortex excitations necessarily form closed loops [40]. We will use Greek letters  $\alpha, \beta, \gamma$  to denote vortex loop excitations, and will use  $\phi_\alpha$  to denote the gauge flux carried by the loop excitation  $\alpha$ .

As in the 2D case, it is important to keep in mind that while charge excitations are uniquely characterized by their gauge

charge, vortex loop excitations are *not* uniquely characterized by their gauge flux: in fact, there are  $|G| = \prod_{i=1}^K N_i$  different types of vortex loop excitations carrying the same gauge flux  $\phi$ . All of these excitations can be obtained by attaching charges to a fixed reference loop with flux  $\phi$ .

Also, just as in 2D, there is some subtlety in defining the absolute charge carried by a vortex loop excitation. Therefore, throughout this paper we will only use the concept of relative charge: we will say that two vortex loops  $\alpha$  and  $\alpha'$  differ by charge  $q$  if  $\alpha'$  can be obtained by attaching a charge excitation  $q$  to  $\alpha$ . The only exception to this rule involves *unlinked* vortex loops: when a vortex loop is not linked to any other loops, then there is a natural way to define how much charge it carries: we will say that such a vortex loop is *neutral* if it can be shrunk to a point and annihilated by local operators. Similarly, we will say that an unlinked vortex loop carries charge  $q$  if it can be obtained by attaching charge  $q$  to a neutral loop.

Let us now consider the braiding statistics of these excitations. There are several types of processes we can consider: braiding of two charges, braiding of a charge around a vortex loop, and braiding involving several vortex loops. As in the 2D case, it is easy to see that the charge-charge statistics are all bosonic and the statistics between a charge  $q$  and a vortex loop carrying a flux  $\phi$  follow the Aharonov-Bohm law

$$\theta = q \cdot \phi. \quad (16)$$

We note that the above Aharonov-Bohm law holds quite generally: it does not depend on the amount of charge attached to the loop, nor on whether the loop is unlinked or linked with other loops.

What is left are braiding processes involving several vortex loops. In general, there are many kinds of loop braiding processes we can consider, including processes involving two loops [41–43], three loops [24–26,44,45], or even more complicated configurations. Here, we will follow Ref. [24] and focus on the three-loop braiding process depicted in Fig. 2, in which a loop  $\alpha$  is braided around a loop  $\beta$  while both are linked to a third “base” loop  $\gamma$ . Reference [24] argued that this three-loop braiding process is a useful probe for characterizing and distinguishing 3D topological phases. As in Ref. [24], we denote the statistical phase associated with this braiding process by  $\theta_{\alpha\beta,c}$ , where  $c$  is an integer vector that characterizes the amount of flux carried by  $\gamma$ . More specifically,  $c$  is defined by  $\phi_\gamma = (\frac{2\pi}{N_1}c_1, \dots, \frac{2\pi}{N_K}c_K)$ . We use the notation  $\theta_{\alpha\beta,c}$  rather than  $\theta_{\alpha\beta,\gamma}$  because the statistical phase is insensitive to the amount of charge attached to  $\gamma$  and depends only on its flux  $\phi_\gamma$  which is parametrized by  $c$ . We will also consider an exchange or half-braiding process, in which two identical loops  $\alpha$ , both

linked to the base loop  $\gamma$ , exchange their positions. We denote the associated three-loop exchange statistics by  $\theta_{\alpha,c}$ .

## 2. Topological invariants

We define our topological invariants  $\Theta_{i,l}$  and  $\Theta_{ij,l}$  in terms of the the three-loop braiding statistics of vortex loops. Let  $\alpha$  and  $\beta$  be two vortex loops carrying unit flux  $\frac{2\pi}{N_i}e_i$  and  $\frac{2\pi}{N_j}e_j$ , where  $e_i = (0, \dots, 1, \dots, 0)$  with the  $i$ th entry being 1 and all others being 0. Suppose both  $\alpha$  and  $\beta$  are linked to a third vortex loop  $\gamma$  carrying unit flux  $\frac{2\pi}{N_l}e_l$ . We define

$$\Theta_{ij,l} = N^{ij}\theta_{\alpha\beta,e_l}, \quad \Theta_{i,l} = N_i\theta_{\alpha,e_l}. \quad (17)$$

Using arguments similar to those for  $\Theta_i, \Theta_{ij}$  from Sec. III A2, one can show that the quantities  $\Theta_{i,l}, \Theta_{ij,l}$  depend only on  $i, j, l$  and not on the choice of vortices  $\alpha, \beta, \gamma$ . Thus,  $\Theta_{ij,l}$  and  $\Theta_{i,l}$  are well-defined quantities.

In addition to being well defined, it is possible to show that  $\Theta_{ij,l}$  and  $\Theta_{i,l}$  contain all the information about the three-loop braiding statistics in the gauge theory. We will derive this result in Sec. VIII.

## D. 3D general case

In this section, we move on to general 3D gauge theories with gauge group  $G = \prod_{i=1}^K \mathbb{Z}_{N_i}$ . Unlike the last section, we do not assume that the three-loop braiding statistics is Abelian. We will define three topological invariants  $\Theta_{i,l}$ ,  $\Theta_{ij,l}$ , and  $\Theta_{ijk,l}$ . The first two,  $\Theta_{i,l}$ ,  $\Theta_{ij,l}$ , reduce to those defined in Eq. (17) when restricted to Abelian statistics. The third invariant  $\Theta_{ijk,l}$  is new to the non-Abelian case, and vanishes in the Abelian case.

### 1. General aspects of non-Abelian loop braiding

In order to analyze the general case, it is important to recognize the analogy between 3D loop braiding and 2D particle braiding. This analogy can be seen most easily by examining a 2D cross section of a loop braiding process, as shown in Fig. 2(b). Here, we see that a braiding process involving two loops  $\alpha, \beta$  that are linked to a base loop  $\gamma$  can be mapped onto a braiding process involving two pointlike particles in two dimensions. More generally, any braiding process involving loops  $\alpha_1, \dots, \alpha_N$  that are linked to a base loop  $\gamma$  can be mapped onto a braiding process involving  $N$  pointlike particles in two dimensions. It can be shown that this mapping between 3D loop braiding and 2D particle braiding is one-to-one, so that the 3D braid group for loops (when linked to a base loop) is identical to the 2D braid group for particles [46]. In addition to braiding, there is also a close analogy between fusion processes in two and three dimensions. Just as two particles can be fused together to form another particle, two loops  $\alpha, \beta$  that are linked to the same loop  $\gamma$  can be fused to form a new loop that is also linked to  $\gamma$  (Fig. 3).

This correspondence between the 2D and 3D cases implies that the algebraic structure of fusion and braiding in 2D anyon theories [34] can be carried over without change to the theory of 3D loop excitations. In particular, for any loops  $\alpha$  and  $\beta$  that are both linked with  $\gamma$ , we can define an associated fusion space  $\mathbb{V}_{\alpha\beta,c}^\delta$ , where  $\delta$  denotes their fusion channel. Also, we can define an  $F$  symbol  $F_{\alpha\beta\mu,c}^\delta$  that describes a unitary mapping

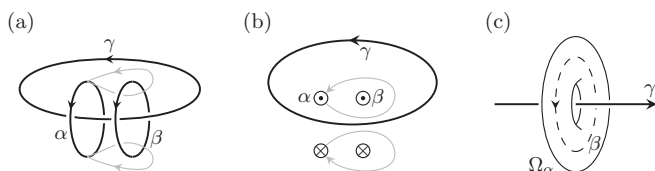


FIG. 2. Three-loop braiding process. (a) The gray curves show the paths of two points on the moving loop  $\alpha$ . (b) Cross section of the braiding process in the plane that  $\gamma$  lies in. (c) A torus  $\Omega_\alpha$  is swept out by  $\alpha$  during the braiding, which encloses the loop  $\beta$  (dashed circle).

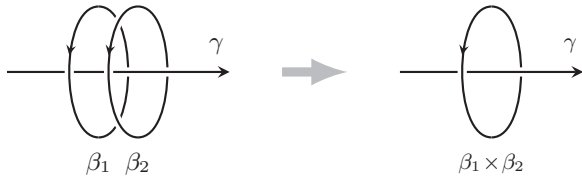


FIG. 3. Fusion of two loops  $\beta_1$  and  $\beta_2$ , both linked to  $\gamma$ . We denote this type of fusion by  $\beta_1 \times \beta_2$ . (This is different notation from Ref. [24], where this type of fusion was denoted by  $\beta_1 + \beta_2$ .)

between two different ways of parametrizing the fusion of three loops

$$F_{\alpha\beta\mu,c}^\delta : \bigoplus_{\xi} V_{\alpha\beta,c}^\xi \otimes V_{\xi\mu,c}^\delta \rightarrow \bigoplus_{\eta} V_{\alpha\eta,c}^\delta \otimes V_{\beta\mu,c}^\eta, \quad (18)$$

and that satisfies the pentagon equation [34]. Likewise, we can define an  $R$  symbol  $R_{\alpha\beta,c}^\delta$  which is a unitary transformation

$$R_{\alpha\beta,c}^\delta : V_{\alpha\beta,c}^\delta \rightarrow V_{\beta\alpha,c}^\delta \quad (19)$$

and that satisfies the hexagon equation [34]. As in the 2D case, the  $R$  symbol describes a half-braiding of loops: a full braiding of two loops  $\alpha, \beta$  that are in a fusion channel  $\delta$  is given by  $R_{\beta\alpha,c}^\delta R_{\alpha\beta,c}^\delta$ . Finally, we can define quantum dimensions and topological spins of loop excitations. The topological spin of a loop  $\alpha$  that is linked to  $\gamma$  is given by

$$e^{i2\pi s_{\alpha,c}} = \frac{1}{d_{\alpha,c}} \sum_{\delta} d_{\delta,c} \text{tr}(R_{\alpha\alpha,c}^\delta), \quad (20)$$

where  $d_{\alpha,c}$  and  $d_{\delta,c}$  are quantum dimensions.

For all of the above quantities,  $V_{\alpha\beta,c}^\delta, F_{\alpha\beta\mu,c}^\delta$ , etc., the dependence on the base loop  $\gamma$  enters through the index “ $c$ ” where  $c$  is an integer vector defined by  $\phi_\gamma = (\frac{2\pi}{N_1}c_1, \dots, \frac{2\pi}{N_K}c_K)$ . The reason we use the notation  $V_{\alpha\beta,c}^\delta$ , etc., rather than  $V_{\alpha\beta,\gamma}^\delta$ , etc., is because it is clear that these quantities depend only on the flux carried by  $\gamma$  which is parametrized by  $c$ .

### 2. Topological invariants

Having established the analogy between 3D loop braiding and 2D particle braiding, we now define the 3D topological invariants using the same approach as in the 2D case. Let  $\alpha, \beta, \gamma$  be three vortex loops that are linked with another loop  $\sigma$ . Suppose that  $\alpha, \beta, \gamma, \sigma$  carry unit flux  $\frac{2\pi}{N_i}e_i, \frac{2\pi}{N_j}e_j, \frac{2\pi}{N_k}e_k, \frac{2\pi}{N_l}e_l$ , respectively. The topological invariants  $\Theta_{i,l}, \Theta_{ij,l}, \Theta_{ijk,l}$  are defined as follows.

*Definitions:*

- (1)  $\Theta_{i,l} = 2\pi N_i s_{\alpha,e_l}$ , where  $s_{\alpha,e_l}$  is the topological spin of  $\alpha$  when it is linked to  $\sigma$ .
- (2)  $\Theta_{ij,l}$  is the Berry phase associated with braiding the loop  $\alpha$  around  $\beta$  for  $N^{ij}$  times, while both are linked to  $\sigma$ .
- (3)  $\Theta_{ijk,l}$  is the phase associated with the following braiding process:  $\alpha$  is first braided around  $\beta$ , then around  $\gamma$ , then around  $\beta$  in an opposite direction, and finally around  $\gamma$  in an opposite direction. Here,  $\alpha, \beta, \gamma$  are all linked with  $\sigma$ .

Similarly to the 2D case, there is an alternative and more concrete definition of  $\Theta_{i,l}$  when  $N_i$  is even:  $\Theta_{i,l}$  can be defined as the phase associated with exchanging two  $\alpha$  loops for  $N_i$  times.

We need to prove two points to show these quantities are well defined: (i) We need to show that the unitary transformations associated with the above braiding processes are always Abelian phases regardless of the fact that the vortex loops may be non-Abelian. (ii) We need to show that these Abelian phases are functions of  $i, j, k, l$  only and do not depend on the choice of vortex loops  $\alpha, \beta, \gamma, \sigma$  as long as they carry fluxes  $\frac{2\pi}{N_i}e_i, \frac{2\pi}{N_j}e_j, \frac{2\pi}{N_k}e_k, \frac{2\pi}{N_l}e_l$ . These two properties can be established using similar arguments to those given in the 2D case in Sec. III B.

### E. Examples

To see some examples of these invariants, we consider the gauged group cohomology models of Ref. [10] or, equivalently, the Dijkgraaf-Witten models of Ref. [35]. We will compute the invariants for these models in the next two sections. All the results listed below follow from two formulas which we will derive later, namely, (42a)–(42c) and (49a)–(49c).

The simplest nontrivial example is given by the 2D Dijkgraaf-Witten models with symmetry group  $G = \mathbb{Z}_2$ . In this case,  $H^3[\mathbb{Z}_2, U(1)] = \mathbb{Z}_2$  so we can construct two Dijkgraaf-Witten models [15]. The only independent invariant in this case is  $\Theta_1$ , which describes the phase associated with exchanging two identical  $\pi$  vortices *twice*. The values of  $\Theta_1$  in the two Dijkgraaf-Witten models are

$$\begin{aligned} \text{Trivial model:} & \quad \Theta_1 = 0, \\ \text{Nontrivial model:} & \quad \Theta_1 = \pi. \end{aligned}$$

Importantly, we can see that  $\Theta_1$  takes different values in the two models, which proves that they belong to distinct phases.

More generally, for  $G = \mathbb{Z}_N$ , we have  $H^3[\mathbb{Z}_N, U(1)] = \mathbb{Z}_N$ , so we can construct  $N$  Dijkgraaf-Witten models in this case. Similarly to the  $\mathbb{Z}_2$  case, these models can be distinguished from one another by the topological invariant  $\Theta_1$ , which takes a different value in  $0, \frac{2\pi}{N}, \dots, \frac{2\pi}{N}(N-1)$  for each of the  $N$  models.

Another interesting example is given by the 2D Dijkgraaf-Witten models with symmetry group  $G = \mathbb{Z}_N \times \mathbb{Z}_N \times \mathbb{Z}_N$ . In this case,  $H^3[\mathbb{Z}_N \times \mathbb{Z}_N \times \mathbb{Z}_N, U(1)] = \mathbb{Z}_N^7$  so we can construct  $N^7$  models. Interestingly, there are also seven independent topological invariants in this case:

$$\Theta_1, \Theta_2, \Theta_3, \Theta_{12}, \Theta_{13}, \Theta_{23}, \Theta_{123}.$$

The invariant  $\Theta_1$  is the topological spin of a vortex that carries  $\frac{2\pi}{N}(1,0,0)$  flux, multiplied by  $N$ . The invariant  $\Theta_{12}$  is the phase associated with braiding a vortex carrying  $\frac{2\pi}{N}(1,0,0)$  flux around a vortex carrying  $\frac{2\pi}{N}(0,1,0)$  flux for  $N$  times. The invariant  $\Theta_{123}$  is the phase associated with braiding  $\frac{2\pi}{N}(1,0,0)$  flux around  $\frac{2\pi}{N}(0,1,0)$  flux in the counterclockwise direction, then around  $\frac{2\pi}{N}(0,0,1)$  flux in the counterclockwise direction, then around the same  $\frac{2\pi}{N}(0,1,0)$  flux and  $\frac{2\pi}{N}(0,0,1)$  flux in the clockwise direction. The meanings of the other invariants are similar. The invariant  $\Theta_{123}$  is an indicator of non-Abelian statistics: if  $\Theta_{123} = 0$ , the corresponding statistics is Abelian; otherwise, the statistics is non-Abelian. We find that all seven



invariants take values in

$$0, \frac{2\pi}{N}, \frac{4\pi}{N}, \dots, \frac{2\pi(N-1)}{N},$$

and that these values distinguish all of the  $N^7$  2D  $\mathbb{Z}_N \times \mathbb{Z}_N \times \mathbb{Z}_N$  models. Again, this result proves that the  $N^7$  models each belong to a different phase according to the definition given in Sec. II C.

Finally, we consider 3D Dijkgraaf-Witten models with symmetry group  $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ . In this case,  $H^3[\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)] = \mathbb{Z}_2^2$  so we can construct four models [47]. We find there are two independent topological invariants in this case, namely,  $\Theta_{1,2}$  and  $\Theta_{2,1}$ . While there exist other invariants such as  $\Theta_{12,1}$ ,  $\Theta_{12,2}$ , they are not independent, as we show in Sec. VI. The invariant  $\Theta_{1,2}$  is the phase associated with exchanging two identical loops that carry a  $(\pi, 0)$  flux while both loops are linked to a third loop that carries a  $(0, \pi)$  flux. The meaning of  $\Theta_{2,1}$  is similar. We find that

$$\Theta_{1,2} = 0 \text{ or } \pi, \quad \Theta_{2,1} = 0 \text{ or } \pi.$$

Each of the four combinations of the values occurs in a different model. Hence, once again, the invariants distinguish all of the 3D  $\mathbb{Z}_2 \times \mathbb{Z}_2$  Dijkgraaf-Witten models and prove that they belong to distinct phases according to the definition given in Sec. II C.

#### IV. INVARIANTS IN DIJKGRAAF-WITTEN MODELS

In this section, we compute the topological invariants for all 2D and 3D Dijkgraaf-Witten models with Abelian gauge group  $G = \prod_{i=1}^K \mathbb{Z}_{N_i}$ . We obtain explicit expressions of the invariants in terms of the cocycle  $\omega$  that is used to define the Dijkgraaf-Witten model.

##### A. 2D topological invariants

###### 1. Review of braiding statistics in 2D Dijkgraaf-Witten models

In this section, we summarize some previously known results on the braiding statistics in 2D Dijkgraaf-Witten models. Although these results do not provide an *explicit* formula for the braiding statistics in Dijkgraaf-Witten models, they do the next best thing: they give a well-defined mathematical procedure for how to compute these statistics in terms of the 3-cocycle  $\omega$  that defines the model. This procedure involves a mathematical structure known as the *twisted quantum double algebra* [48]. The twisted quantum double formalism is quite general and can be applied to any finite group  $G$ , including non-Abelian groups. However, in the following discussion, we will specialize to the case of Abelian  $G$ , and we will only give a minimal review of the ingredients that are necessary for the computation of the invariants  $\Theta_i, \Theta_{ij}, \Theta_{ijk}$ . For more details, readers may consult Refs. [39,48].

The first component of the twisted quantum double formalism is a scheme for labeling quasiparticle excitations. Let us consider a Dijkgraaf-Witten model corresponding to an Abelian group  $G = \prod_{i=1}^K \mathbb{Z}_{N_i}$  and 3-cocycle  $\omega$ . According to the formalism, each excitation in this model can be uniquely labeled by a doublet

$$\alpha = (a, \rho), \quad (21)$$

where  $a = (a_1, \dots, a_K)$  is a group element of  $G$  with  $0 \leq a_i \leq N_i - 1$ , and  $\rho$  is an irreducible projective representation of  $G$  satisfying

$$\rho(b)\rho(c) = \chi_a(b, c)\rho(b+c), \quad (22)$$

for all  $b, c \in G$ . Here,  $\chi_a$  is a phase factor defined by

$$\chi_a(b, c) = \frac{\omega(a, b, c)\omega(b, c, a)}{\omega(b, a, c)}, \quad (23)$$

and is called the *slant product* of  $\omega$ . The two labels  $(a, \rho)$  have a simple physical meaning: the first component  $a$  describes the amount of flux  $\phi_\alpha = (\frac{2\pi}{N_1}a_1, \dots, \frac{2\pi}{N_K}a_K)$  carried by the excitation  $\alpha$ , while the second component  $\rho$  is related to the amount of charge attached to  $\alpha$ . For a more precise correspondence between the mathematical labels  $(a, \rho)$  and the physical notions of gauge flux and gauge charge, we refer the reader to Appendix G.

The second component of the formalism is a formula for the fusion rules of the excitations. Specifically,

$$(a, \rho) \times (b, \mu) = \sum_{\sigma} N_{\rho\mu}^{\sigma} (a+b, \sigma), \quad (24)$$

where the fusion multiplicities  $N_{\rho\mu}^{\sigma}$  are computed as follows. First, we define a projective representation  $\rho * \mu$  by

$$(\rho * \mu)(g) = \rho(g) \otimes \mu(g) \cdot \chi_g(a, b) \quad (25)$$

for any  $g \in G$ . The  $N_{\rho\mu}^{\sigma}$  are then defined in terms of the decomposition of  $\rho * \mu$  into irreducible projective representations  $\sigma$ :

$$\rho * \mu = \bigoplus_{\sigma} N_{\rho\mu}^{\sigma} \sigma. \quad (26)$$

In addition to fusion rules, this formalism provides a convenient way to parametrize the degenerate ground states associated with a collection of (non-Abelian) excitations. Consider a system of  $n$  excitations  $\alpha_i = (a_i, \rho_i)$ ,  $i = 1, \dots, n$ , and suppose that these excitations fuse to the vacuum. The ground-state manifold associated with these excitations can be obtained in two steps. First, we construct the tensor product  $V = V_1 \otimes \dots \otimes V_n$  where  $V_i$  is the vector space on which  $\rho_i$  is defined. Then, we project onto the subspace of  $V$  that corresponds to the vacuum fusion channel. This projection is implemented by the operator

$$P = \frac{1}{|G|} \sum_{g \in G} \rho_1(g) * \dots * \rho_n(g). \quad (27)$$

The degenerate ground states associated with  $\alpha_1, \dots, \alpha_n$  can be parametrized by vectors that lie in the image of  $P : V \rightarrow V$ .

We are now ready to present a formula for the braiding statistics of the excitations. Suppose that the above system of  $n$  excitations are arranged in a line in the order  $\alpha_1, \dots, \alpha_n$ . Then, the unitary transformation associated with braiding  $\alpha_i$  around its neighbor  $\alpha_{i+1}$ , is given by [39]

$$B_{\alpha_i \alpha_{i+1}} = P \cdot \text{id}_{V_1} \otimes \dots \otimes \text{id}_{V_{i-1}} \otimes \rho_i(a_{i+1}) \otimes \rho_{i+1}(a_i) \otimes \text{id}_{V_{i+2}} \otimes \dots \otimes \text{id}_{V_n} \cdot P, \quad (28)$$

where  $\text{id}_{V_i}$  denotes the identity matrix on the vector space  $V_i$  of  $\rho_i$ . (The exchange statistics for the excitations can be computed in a similar fashion, but we will not discuss them here as they are not necessary for our purposes.)

The final result we will need is an expression for the topological spin. According to the twisted quantum double formalism, the topological spin of an excitation  $\alpha = (a, \rho)$  is given by [39]

$$e^{i2\pi s_\alpha} = \frac{1}{\dim(\rho)} \text{tr} \rho(a), \quad (29)$$

where  $\dim(\rho)$  is the dimension of the representation  $\rho$ . This formula can equivalently be written as

$$e^{i2\pi s_\alpha} \text{id}_V = \rho(a), \quad (30)$$

where  $\text{id}_V$  is the identity matrix in the vector space  $V$  of  $\rho$ . The reason that (29) is equivalent to (30) is that  $\rho(a)$  is always a pure phase. Indeed, this property follows from Schur's lemma and the observation that  $\chi_a(a, b) = \chi_a(b, a)$  so that  $\rho(a)$  commutes with any other matrix  $\rho(b)$ . [As an aside, we note that the fact that  $\rho(a)$  is a pure phase is consistent with the results in Appendix E.]

## 2. Explicit formulas for the invariants

We now compute the invariants  $\Theta_i$ ,  $\Theta_{ij}$ , and  $\Theta_{ijk}$  for a 2D Dijkgraaf-Witten model with group  $G = \prod_{i=1}^K \mathbb{Z}_{N_i}$  and 3-cocycle  $\omega$ . Let  $\alpha, \beta, \gamma$  be three vortices carrying unit flux  $2\pi e_i/N_i$ ,  $2\pi e_j/N_j$ , and  $2\pi e_k/N_k$ , respectively. Using the notation from the previous section, we can label these vortices as  $\alpha = (e_i, \rho)$ ,  $\beta = (e_j, \mu)$ , and  $\gamma = (e_k, \nu)$  for some projective representations  $\rho, \mu, \nu$ . We will denote the vector spaces associated with  $\rho, \mu, \nu$  by  $V, W, X$ . To compute  $\Theta_i$ ,  $\Theta_{ij}$ , and  $\Theta_{ijk}$ , we need to find the topological spin of these vortices and to analyze various braiding processes involving them.

We begin with  $\Theta_i$ . From (30) we derive

$$e^{i\Theta_i} \text{id}_V = e^{i2\pi N_i s_\alpha} \text{id}_V = \rho(e_i)^{N_i}. \quad (31)$$

We then rewrite the right-hand side as

$$\begin{aligned} \rho(e_i)^{N_i} &= \prod_{n=0}^{N_i-1} \{\rho[(n+1)e_i]^{-1} \rho(e_i) \rho(ne_i)\} \\ &= \prod_{n=0}^{N_i-1} \chi_{e_i}(e_i, ne_i) \text{id}_V, \end{aligned} \quad (32)$$

where the second line follows from Eq. (22). We conclude that the invariant  $\Theta_i$  is given by

$$\exp(i\Theta_i) = \prod_{n=0}^{N_i-1} \chi_{e_i}(e_i, ne_i). \quad (33)$$

Similarly, we can obtain expressions for  $\Theta_{ij}, \Theta_{ijk}$  using (28):

$$\begin{aligned} e^{i\Theta_{ij}} \text{id}_V \otimes \text{id}_W &= \rho(e_j)^{N^{ij}} \otimes \mu(e_i)^{N^{ij}}, \\ e^{i\Theta_{ijk}} \text{id}_V &= \rho(e_k)^{-1} \rho(e_j)^{-1} \rho(e_k) \rho(e_j), \end{aligned}$$

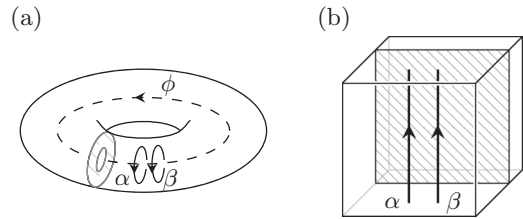


FIG. 4. (a) A thickened torus  $\mathbb{T}^2 \times [0,1]$  with a flux  $\phi$  threading the inner hole. (b) The thickened torus drawn as a cube with the top and bottom faces as well as the front and back faces identified.

which can then be related to  $\chi$  using (22):

$$\begin{aligned} \exp(i\Theta_{ij}) &= \prod_{n=1}^{N^{ij}} \chi_{e_i}(e_j, ne_j) \chi_{e_j}(e_i, ne_i), \\ \exp(i\Theta_{ijk}) &= \frac{\chi_{e_i}(e_k, e_j)}{\chi_{e_i}(e_j, e_k)}. \end{aligned} \quad (34)$$

Equations (33) and (34) are the formulas we seek, where  $\chi$  is defined by (23).

The properties of  $\Theta_i, \Theta_{ij}, \Theta_{ijk}$  that were derived in Sec. III B from more general considerations are manifest in the above expressions. We can see that  $\Theta_i, \Theta_{ij}, \Theta_{ijk}$  only depend on  $i, j, k$  and not on the choice of vortices  $\alpha, \beta, \gamma$  since the representations  $\rho, \mu, \nu$  do not appear in the final expressions (33) and (34). In addition, it is easy to verify that these formulas are invariant under the change  $\omega \rightarrow \omega \cdot \nu$  if  $\nu$  is a coboundary (see Appendix B for the definition of a coboundary). This is to be expected since two cocycles that differ by a coboundary are known to define the same Dijkgraaf-Witten model, and therefore must give the same values for the invariants.

## B. 3D topological invariants

### 1. Dimensional reduction

Our approach for computing the 3D invariants is based on dimensional reduction: we derive a relationship between vortex loop statistics in 3D Dijkgraaf-Witten models and vortex statistics in 2D Dijkgraaf-Witten models, and then we analyze the latter using previously known 2D results. A similar dimensional reduction approach was used in Ref. [24]. The derivation we discuss here is more general than that of Ref. [24] in some ways and less general in other ways. It is more general because it applies even if the vortex loop statistics are non-Abelian, but it is also less general because it is only valid for Dijkgraaf-Witten models, while Ref. [24] derived a dimensional reduction formula without restricting to these exactly soluble systems.

To begin, consider a Dijkgraaf-Witten model associated with a group  $G = \prod_{i=1}^K \mathbb{Z}_{N_i}$  and a 4-cocycle  $\omega$ . Let us define this model on a thickened 2D torus, i.e., the manifold  $\mathbb{T}^2 \times [0,1]$  (Fig. 4). Consider a state consisting of two loop excitations  $\alpha, \beta$  which wind around the inner hole of the torus, and suppose that there is a flux

$$\phi = \left( \frac{2\pi}{N_1} a_1, \dots, \frac{2\pi}{N_k} a_k \right) \quad (35)$$

threading the inner hole. This geometry is equivalent to a standard three-loop setup in which  $\alpha, \beta$  are linked with another loop carrying flux  $\phi$ . Our task is to find the unitary transformation associated with braiding  $\alpha$  around  $\beta$  in the presence of the flux  $\phi$ .

To this end, we redraw the thickened torus  $\mathbb{T}^2 \times [0, 1]$  as a cube whose top and bottom faces are identified as well as its front and back faces. In this representation, the shaded square in Fig. 4(b) corresponds to the shaded annulus in Fig. 4(a) and the left and right faces of the cube correspond to the inner and outer surfaces of the thickened torus. The flux loops  $\alpha, \beta$  can be drawn as lines connecting the top and bottom faces of the cube.

To proceed further, we make use of a special property of Dijkgraaf-Witten models: these models can be defined for any triangulation of space-time, and their properties do not depend on the choice of triangulation. Therefore, we are free to triangulate our cube however we like and it will not affect the braiding statistics between  $\alpha$  and  $\beta$ . Making use of this freedom, we consider a triangulation with translational symmetry in the  $x$ ,  $y$ , and  $z$  directions. More specifically, we consider triangulation in which there is only one unit cell in the  $z$  direction, but many unit cells in the  $x$  and  $y$  directions. With this choice of triangulation, the cube can be viewed as a 2D system. Furthermore, the braiding process involving the vortex loops  $\alpha$  and  $\beta$  can be viewed as a process involving two vortices in this effective 2D system.

By the same argument as in the Supplemental Material of Ref. [24], one can show that this effective 2D system is identical to a 2D Dijkgraaf-Witten model with group  $G$  and a 3-cocycle

$$\chi_a(b, c, d) = \frac{\omega(b, a, c, d)\omega(b, c, d, a)}{\omega(a, b, c, d)\omega(b, c, a, d)}, \quad (36)$$

where  $a = (a_1, \dots, a_K)$  is defined in terms of  $\phi$  as in Eq. (35). Here,  $\chi_a$  is known as the *slant product* of  $\omega$ . One can check that  $\chi_a$  is indeed a 3-cocycle for any choice of  $a$ .

Putting this all together, we conclude that the vortex loop statistics for a 3D Dijkgraaf-Witten model with 4-cocycle  $\omega$  and with a base loop with flux  $\phi$ , are identical to the vortex statistics in a 2D Dijkgraaf-Witten model with 3-cocycle  $\chi_a$ .

## 2. Explicit formulas for the invariants

We are now ready to compute the invariants  $\Theta_{i,l}, \Theta_{ij,l}, \Theta_{ijk,l}$  for a 3D Dijkgraaf-Witten model with group  $G = \prod_{i=1}^K \mathbb{Z}_{N_i}$  and 4-cocycle  $\omega$ .

Let  $\alpha, \beta, \gamma$  be three vortex loops carrying unit flux  $2\pi e_i/N_i$ ,  $2\pi e_j/N_j$ , and  $2\pi e_k/N_k$ , respectively, and suppose that all three are linked with another loop  $\sigma$  carrying unit flux  $2\pi e_l/N_l$ . To compute  $\Theta_{i,l}, \Theta_{ij,l}, \Theta_{ijk,l}$ , we need to analyze various braiding processes involving  $\alpha, \beta, \gamma$ . We can accomplish this task with the help of the dimensional reduction results of the previous section: according to those results, the braiding statistics of the vortex loops  $\alpha, \beta, \gamma$  in a 3D Dijkgraaf-Witten model with cocycle  $\omega$  are identical to the braiding statistics of vortices in a 2D Dijkgraaf-Witten model with 3-cocycle  $\chi_{e_l}$  [Eq. (36)]. Therefore, the 3D invariants  $\Theta_{i,l}, \Theta_{ij,l}, \Theta_{ijk,l}$  can be obtained from the corresponding 2D invariants [Eqs. (33) and (34)] simply by substituting  $\chi_{e_l}(a, b, c)$  for  $\omega(a, b, c)$ . In

this way, we obtain

$$\begin{aligned} \exp(i\Theta_{i,l}) &= \prod_{n=1}^{N_i} \chi_{e_l, e_i}(e_i, n e_i), \\ \exp(i\Theta_{ij,l}) &= \prod_{n=1}^{N^{ij}} \chi_{e_l, e_i}(e_j, n e_j) \chi_{e_l, e_j}(e_i, n e_i), \\ \exp(i\Theta_{ijk,l}) &= \frac{\chi_{e_l, e_i}(e_k, e_j)}{\chi_{e_l, e_i}(e_j, e_k)}, \end{aligned} \quad (37)$$

where  $\chi_{e_l, e_i}$  is defined as

$$\chi_{e_l, e_i}(b, c) = \frac{\chi_{e_l}(e_i, b, c) \chi_{e_l}(b, c, e_i)}{\chi_{e_l}(b, e_i, c)} \quad (38)$$

and where  $\chi_{e_l}$  is defined in Eq. (36).

## V. SHOWING THE INVARIANTS DISTINGUISH ALL DIJKGRAAF-WITTEN MODELS

In this section, we show that the invariants take different values for each of the Dijkgraaf-Witten models with group  $G$ . This result has two implications: (i) each of the Dijkgraaf-Witten models belongs to a distinct phase and (ii) the invariants can distinguish these phases.

### A. 2D case

We now show that the invariants  $\Theta_i, \Theta_{ij}, \Theta_{ijk}$  can distinguish all the 2D Dijkgraaf-Witten models corresponding to the group  $G = \prod_{i=1}^K \mathbb{Z}_{N_i}$ . (A similar result was obtained previously in Ref. [32].) Our proof is based on a counting argument: let  $\mathcal{N}_{2D}^{DW}$  be the number of 2D Dijkgraaf-Witten models with group  $G$ , let  $\mathcal{N}_{2D}^{\text{phase}}$  be the number of phases of 2D Dijkgraaf-Witten models with group  $G$ , and let  $\mathcal{N}_{2D}^{\Theta}$  be the number of distinct values that the invariants take over all 2D Dijkgraaf-Witten models with group  $G$ . Clearly, we must have

$$\mathcal{N}_{2D}^{\Theta} \leq \mathcal{N}_{2D}^{\text{phase}} \leq \mathcal{N}_{2D}^{DW}. \quad (39)$$

What we will prove is the opposite inequality:

$$\mathcal{N}_{2D}^{\Theta} \geq \mathcal{N}_{2D}^{DW}. \quad (40)$$

It will then follow that  $\mathcal{N}_{2D}^{\Theta} = \mathcal{N}_{2D}^{\text{phase}} = \mathcal{N}_{2D}^{DW}$ , which shows that each of the Dijkgraaf-Witten models belongs to a distinct phase and the invariants can distinguish all the phases (according to the definition of phases given in Sec. II C).

To prove (40) we consider the set of 3-cocycles of the form

$$\begin{aligned} \omega(a, b, c) &= \exp \left\{ i2\pi \sum_{ij} \frac{P_{ij}}{N_i N_j} a_i (b_j + c_j - [b_j + c_j]) \right\} \\ &\times \exp \left\{ i2\pi \sum_{ijk} \frac{Q_{ijk}}{N_{ijk}} a_i b_j c_k \right\}, \end{aligned} \quad (41)$$

where  $P_{ij}$  is an integer matrix and  $Q_{ijk}$  is an integer tensor with  $Q_{ijk} = 0$  if  $i, j, k$  are not all distinct. (The latter restriction on  $Q_{ijk}$  is not essential, and we include it only to simplify some of the formulas that follow.) Here, the symbol  $[b_j + c_j]$  is defined as the residue of  $b_j + c_j$  modulo  $N_j$  with values

taken in the range  $0, \dots, N_j - 1$ . [One can verify that  $\omega$  obeys the 3-cocycle condition (B2) with straightforward algebra.] Inserting the cocycle  $\omega$  into the expressions in Eqs. (33) and (34), one immediately obtains

$$\Theta_i = \frac{2\pi}{N_i} P_{ii}, \quad (42a)$$

$$\Theta_{ij} = \frac{2\pi}{N_{ij}} (P_{ij} + P_{ji}), \quad (42b)$$

$$\Theta_{ijk} = -\frac{2\pi}{N_{ijk}} (Q_{ijk} + Q_{jki} + Q_{kij} - Q_{jik} - Q_{ikj} - Q_{kji}). \quad (42c)$$

From the above formulas, we see that the invariants  $\Theta_{ij}$  can take on  $N_{ij}$  different values as  $P_{ij}$  ranges over all integer matrices. Similarly,  $\Theta_i$  can take on  $N_i$  different values while  $\Theta_{ijk}$  can take on  $N_{ijk}$  different values. Furthermore, the values of  $\Theta_i$ , and  $\Theta_{ij}$  with  $i < j$ , and  $\Theta_{ijk}$  with  $i < j < k$ , can be varied independently from one another. Therefore, we have the lower bound

$$\mathcal{N}_{2D}^{\Theta} \geq \prod_i N_i \prod_{i < j} N_{ij} \prod_{i < j < k} N_{ijk}. \quad (43)$$

At the same time, we know that the Dijkgraaf-Witten models are parametrized by elements of the group

$$H^3[G, U(1)] = \prod_i \mathbb{Z}_{N_i} \prod_{i < j} \mathbb{Z}_{N_{ij}} \prod_{i < j < k} \mathbb{Z}_{N_{ijk}}, \quad (44)$$

so that

$$\mathcal{N}_{2D}^{\text{DW}} = |H^3[G, U(1)]| = \prod_i N_i \prod_{i < j} N_{ij} \prod_{i < j < k} N_{ijk}. \quad (45)$$

Combining (43) and (45) gives the desired inequality (40) and proves the result.

### B. 3D case

We now show that the invariants  $\Theta_{i,l}, \Theta_{ij,l}, \Theta_{ijk,l}$  can distinguish all the 3D Dijkgraaf-Witten models corresponding to the group  $G = \prod_{i=1}^K \mathbb{Z}_{N_i}$ . The argument closely follows the 2D case: let  $\mathcal{N}_{3D}^{\text{DW}}$  be the number of 3D Dijkgraaf-Witten models with group  $G$ , let  $\mathcal{N}_{3D}^{\text{phase}}$  be the number of phases of 3D Dijkgraaf-Witten models with group  $G$ , and let  $\mathcal{N}_{3D}^{\Theta}$  be the number of distinct values that the invariants take over all 3D Dijkgraaf-Witten models with group  $G$ . Clearly, we have

$$\mathcal{N}_{3D}^{\Theta} \leq \mathcal{N}_{3D}^{\text{phase}} \leq \mathcal{N}_{3D}^{\text{DW}}. \quad (46)$$

What we will show is that

$$\mathcal{N}_{3D}^{\Theta} \geq \mathcal{N}_{3D}^{\text{DW}}. \quad (47)$$

It will then follow that  $\mathcal{N}_{3D}^{\Theta} = \mathcal{N}_{3D}^{\text{phase}} = \mathcal{N}_{3D}^{\text{DW}}$ , which shows that each of the Dijkgraaf-Witten models belongs to a distinct phase and the invariants distinguish all the phases (according to the definition of phases given in Sec. II C).

To prove (47), we consider the set of 4-cocycles of the form

$$\omega(a, b, c, d) = \exp \left\{ i 2\pi \sum_{ijk} \frac{M_{ijk}}{N_{ij} N_k} a_i b_j (c_k + d_k - [c_k + d_k]) \right\} \\ \times \exp \left\{ i 2\pi \sum_{ijkl} \frac{L_{ijkl}}{N_{ij} N_{kl}} a_i b_j c_k d_l \right\}, \quad (48)$$

where  $M_{ijk}$  is an arbitrary integer tensor and  $L_{ijkl}$  is an integer tensor with  $L_{ijkl} = 0$  if  $i, j, k$  are not all distinct. [As in the 2D case, it is simple to verify that  $\omega$  obeys the 4-cocycle condition (B3).] Inserting the 4-cocycle  $\omega$  into the expressions in Eq. (37), we obtain

$$\Theta_{i,l} = \frac{2\pi}{N_{il}} (M_{ili} - M_{lii}), \quad (49a)$$

$$\Theta_{ij,l} = \frac{2\pi N^{ij}}{N_{il} N_j} (M_{ilj} - M_{lij}) + \frac{2\pi N^{ij}}{N_{ji} N_i} (M_{jli} - M_{lji}), \quad (49b)$$

$$\Theta_{ijk,l} = -\frac{2\pi}{N_{ijkl}} \sum_{\hat{p}} \text{sgn}(\hat{p}) L_{\hat{p}(i)\hat{p}(j)\hat{p}(k)\hat{p}(l)}, \quad (49c)$$

where  $\hat{p}$  is a permutation of  $i, j, k, l$  and  $\text{sgn}(\hat{p}) = \pm 1$  is the parity of  $\hat{p}$ . From the above formulas, we see that different choices of  $M$  and  $L$  give different values of  $\Theta_{i,l}, \Theta_{ij,l}$ , and  $\Theta_{ijk,l}$ . More precisely, it can be shown that as  $M$  and  $L$  range over the set of allowed integer tensors, the invariants  $\Theta_{i,l}, \Theta_{ij,l}$ , and  $\Theta_{ijk,l}$  take on at least  $\prod_{i < l} (N_{il})^2 \prod_{i < j < l} (N_{ijl})^2 \prod_{i < j < k < l} N_{ijkl}$  different values (this counting is done in Appendix I). Therefore, we have the lower bound

$$\mathcal{N}_{3D}^{\Theta} \geq \prod_{i < l} (N_{il})^2 \prod_{i < j < l} (N_{ijl})^2 \prod_{i < j < k < l} N_{ijkl}. \quad (50)$$

On the other hand, we know that the Dijkgraaf-Witten models are parametrized by elements of the group

$$H^4[G, U(1)] = \prod_{i < j} (\mathbb{Z}_{N_{ij}})^2 \prod_{i < j < k} (\mathbb{Z}_{N_{ijk}})^2 \prod_{i < j < k < l} \mathbb{Z}_{N_{ijkl}}, \quad (51)$$

so that

$$\mathcal{N}_{3D}^{\text{DW}} = \prod_{i < l} (N_{il})^2 \prod_{i < j < l} (N_{ijl})^2 \prod_{i < j < k < l} N_{ijkl}. \quad (52)$$

Combining (50) and (52) gives the desired inequality (47) and proves the result.

## VI. GENERAL CONSTRAINTS ON THE INVARIANTS

In this section, we derive general constraints on the invariants that hold for any gauge theory with group  $G = \prod_{i=1}^K \mathbb{Z}_{N_i}$ . In the next section, we will discuss whether these constraints are complete, i.e., whether any solution to these constraints can be realized by an appropriate gauge theory.

### A. 2D Abelian case

Let us start with the case of 2D gauge theories with gauge group  $G = \prod_{i=1}^K \mathbb{Z}_{N_i}$  and Abelian braiding statistics. According to Sec. III A, there are two invariants  $\Theta_i$  and  $\Theta_{ij}$  in

this case. We will now argue that  $\Theta_i$  and  $\Theta_{ij}$  must satisfy the following general constraints:

$$\Theta_{ii} = 2\Theta_i, \quad (53a)$$

$$\Theta_{ij} = \Theta_{ji}, \quad (53b)$$

$$N_{ij}\Theta_{ij} = 0, \quad (53c)$$

$$N_i\Theta_i = 0, \quad (53d)$$

where all equations are defined modulo  $2\pi$ . To prove this statement, we first recall the following general properties of Abelian braiding statistics:

$$\theta_{\alpha\beta} = \theta_{\beta\alpha}, \quad (54a)$$

$$\theta_{\alpha\alpha} = 2\theta_\alpha, \quad (54b)$$

$$\theta_{\alpha(\beta_1 \times \beta_2)} = \theta_{\alpha\beta_1} + \theta_{\alpha\beta_2}, \quad (54c)$$

$$\theta_{(\alpha \times \beta)} = \theta_\alpha + \theta_\beta + \theta_{\alpha\beta}. \quad (54d)$$

Here,  $\theta_{\alpha\beta}$  denotes the mutual statistics of  $\alpha, \beta$ , while  $\theta_\alpha$  denotes the exchange statistics of  $\alpha$  and  $\alpha \times \beta$  denotes the excitation created by fusing together  $\alpha$  and  $\beta$ . Each of these identities follows from simple physical arguments. The symmetry relation (54a) comes from the fact that braiding  $\alpha$  around  $\beta$  is topologically equivalent to braiding  $\beta$  around  $\alpha$ . The relation (54b) follows immediately from the definition of exchange statistics. The linearity relation (54c) comes from the fact that fusing  $\beta_1$  and  $\beta_2$  must commute with braiding  $\alpha$  around them. The other linearity relation (54d) has a similar flavor.

We can see that Eqs. (53a) and (53b) follow immediately from these general constraints. To prove (53c), consider braiding a vortex  $\alpha$  that carries a unit flux  $\frac{2\pi}{N_i}e_i$  around  $N_j$  identical vortices  $\beta$ , with each  $\beta$  carrying a unit flux  $\frac{2\pi}{N_j}e_j$ . Clearly, the associated statistical phase is  $N_j\theta_{\alpha\beta}$ . At the same time, according to the linearity relation (54c), this statistical phase is equal to the phase associated with braiding  $\alpha$  around the fusion product of all the  $\beta$  vortices. Since the total flux of  $N_j$   $\beta$  vortices is zero, they fuse to a charge. It then follows from the Aharonov-Bohm law that the latter quantity is a multiple of  $\frac{2\pi}{N_i}$ . We conclude that

$$N_j\theta_{\alpha\beta} = \frac{2\pi}{N_i} \times \text{integer}.$$

Equation (53c) follows immediately using  $N_{ij}N^{ij} = N_iN_j$  together with the definition (2) of  $\Theta_{ij}$ .

To prove (53d), imagine exchanging a set of  $N_i$   $\alpha$ 's with another set of  $N_i$   $\alpha$ 's. According to the linearity relation (54d) and the relation (54b), the associated exchange statistical phase is  $N_i^2\theta_\alpha = N_i\Theta_i$ . However, this phase must be a multiple of  $2\pi$  since  $N_i$   $\alpha$ 's fuse to a charge which is a boson. Thus, Eq. (53d) must hold.

## B. 2D general case

We now consider the case of general 2D gauge theories with gauge group  $G = \prod_{i=1}^K \mathbb{Z}_{N_i}$ . There are three invariants,  $\Theta_i, \Theta_{ij}, \Theta_{ijk}$  in this case. We will now argue that  $\Theta_i, \Theta_{ij}$  again satisfy the constraints (53a)–(53d), as in the Abelian case. In

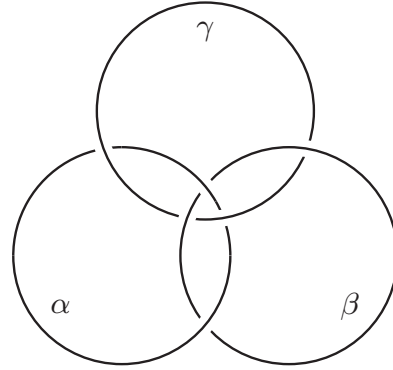


FIG. 5. A Borromean ring obtained by closing up the trajectories in Fig. 1(b).

addition, the third invariant  $\Theta_{ijk}$  satisfies

$$\Theta_{ijk} = \text{sgn}(\hat{p})\Theta_{\hat{p}(i)\hat{p}(j)\hat{p}(k)}, \quad (55a)$$

$$\Theta_{iij} = 0, \quad (55b)$$

$$N_{ijk}\Theta_{ijk} = 0, \quad (55c)$$

where  $\hat{p}$  is a permutation of the indices  $i, j, k$  and  $\text{sgn}(\hat{p}) = \pm 1$  is its parity and again all the equations are defined modulo  $2\pi$ . We note that the constraint (55a) tells us that  $\Theta_{ijk}$  is fully antisymmetric modulo  $2\pi$ , while (55b) is a stronger constraint on  $\Theta_{iij}$  than its antisymmetry since the antisymmetry only requires that  $2\Theta_{iij} = 0 \pmod{2\pi}$ .

We first prove the above constraints for  $\Theta_{ijk}$ ; afterwards we will prove (53a)–(53d). To prove that  $\Theta_{ijk}$  is fully antisymmetric, i.e., (55a), we consider the space-time trajectories [Fig. 1(b)] of the vortices in the braiding process associated with  $\Theta_{ijk}$ . Since the unitary transformation associated with the braiding process is Abelian, we can close up the space-time trajectories in Fig. 1(b), so that they form the Borromean ring in Fig. 5. The closed-up trajectories are associated with the following process: we first create three particle-antiparticle pairs  $\alpha, \bar{\alpha}, \beta, \bar{\beta}$ , and  $\gamma, \bar{\gamma}$  out of the vacuum, where  $\alpha, \beta, \gamma$  carry unit flux  $\frac{2\pi}{N_i}e_i, \frac{2\pi}{N_j}e_j, \frac{2\pi}{N_k}e_k$ , respectively, then braid  $\alpha, \beta, \gamma$  in the way that leads to the phase  $\Theta_{ijk}$ , and finally annihilate the three pairs to return to the vacuum. The fact that we can annihilate the particles at the end of the process is guaranteed by the fact that the braiding results in a pure phase, otherwise the particle-antiparticle need not be in the vacuum fusion channel after the braiding.

With this picture in mind, we can see that  $\Theta_{ijk}$  is equal to the phase associated with the Borromean ring space-time trajectories. Since the Borromean ring is cyclically symmetric, we deduce that  $\Theta_{ijk} = \Theta_{jki} = \Theta_{kij}$ . It is also not hard to see that reversing the braiding process associated with  $\Theta_{ijk}$  gives rise to the phase  $\Theta_{ikj}$ . So,  $\Theta_{ijk} = -\Theta_{ikj}$ . Putting these relations together, we see  $\Theta_{ijk}$  is fully antisymmetric.

To prove (55b), we make use of the result in Appendix E which shows that braiding a vortex  $\alpha$  around another  $\alpha$  gives only a pure phase. Consider three vortices  $\alpha, \alpha, \beta$  with  $\phi_\alpha = \frac{2\pi}{N_i}e_i$  and  $\phi_\beta = \frac{2\pi}{N_j}e_j$  and imagine the braiding process associated with  $\Theta_{iij}$ :  $\alpha$  is braided around the other  $\alpha$ , then around  $\beta$ , then around the other  $\alpha$  in the opposite direction, and finally around  $\beta$  in the opposite directions. In this four-step

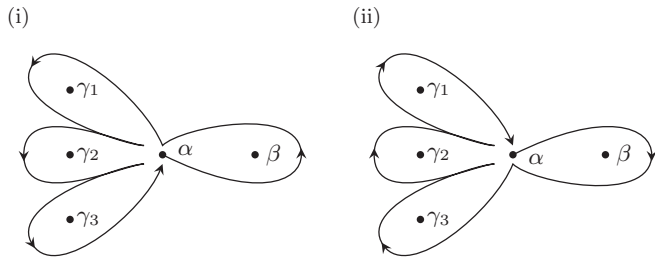


FIG. 6. Thought experiment to prove the constraint (55c) ( $N_k = 3$  is taken for illustration). For clarity, we split the composite braiding into two steps (i) and (ii). In the thought experiment, (ii) follows immediately after (i).

process, we can switch the order of the second and third steps since the third step gives only a pure phase. Thus, it is obvious that this four-step process neither changes the state of the system nor leads to any Abelian phase. Hence, Eq. (55b) holds.

To prove (55c), we consider a collection of excitations  $\alpha, \beta, \gamma_1, \dots, \gamma_{N_k}$ , where  $\phi_\alpha = \frac{2\pi}{N_i} e_i$ ,  $\phi_\beta = \frac{2\pi}{N_j} e_j$ , and  $\phi_{\gamma_t} = \frac{2\pi}{N_k} e_k$  for any  $t = 1, \dots, N_k$ . We consider a composite braiding process shown in Fig. 6: first we braid  $\alpha$  around  $\beta$  in the counterclockwise direction, then sequentially round  $\gamma_1, \dots, \gamma_{N_k}$  in the counterclockwise direction, then around  $\beta$  in the clockwise direction, and sequentially around  $\gamma_{N_k}, \dots, \gamma_1$  in the clockwise direction. This braiding process can be described by a product of operators

$$B \equiv B_{\alpha\gamma_1}^{-1} \dots B_{\alpha\gamma_{N_k}}^{-1} B_{\alpha\beta}^{-1} B_{\alpha\gamma_{N_k}} \dots B_{\alpha\gamma_1} B_{\alpha\beta},$$

where  $B_{\alpha\beta}, B_{\alpha\gamma_1}, \dots, B_{\alpha\gamma_{N_k}}$  are the operators associated with braiding  $\alpha$  around  $\beta, \gamma_1, \dots, \gamma_{N_k}$  respectively. Now, according to the definition of  $\Theta_{ijk}$ , we have  $B_{\alpha\gamma_t}^{-1} B_{\alpha\beta}^{-1} B_{\alpha\gamma_t} B_{\alpha\beta} = e^{i\Theta_{ijk}} \hat{I}$  for any  $t = 1, \dots, N_k$ . It then follows that

$$B = e^{iN_k \Theta_{ijk}} \hat{I}, \quad (56)$$

where  $\hat{I}$  is the identity operator. On the other hand, braiding  $\alpha$  around  $\gamma_1, \dots, \gamma_{N_k}$  in sequence is equivalent to braiding  $\alpha$  around the fusion product of  $\gamma_1, \dots, \gamma_{N_k}$ . Since  $\gamma_1, \dots, \gamma_{N_k}$  fuse to some charge  $q$ , we derive

$$B = B_{\alpha q}^{-1} B_{\alpha\beta}^{-1} B_{\alpha q} B_{\alpha\beta}, \quad (57)$$

where  $B_{\alpha q}$  denotes the operator associated with the braiding of  $\alpha$  around  $q$ . Now, by the Aharonov-Bohm formula, we know that  $B_{\alpha q}$  is a pure phase, which implies that  $B$  is just the identity operator  $B = \hat{I}$ . Therefore,  $e^{iN_k \Theta_{ijk}} = 1$ , i.e.,  $N_k \Theta_{ijk} = 0$ . Similarly, we can show  $N_i \Theta_{ijk} = N_j \Theta_{ijk} = 0$ . Putting this together, we derive the constraint (55c).

Next, we prove the constraints (53a)–(53d) in the non-Abelian case. The constraint (53a) follows immediately from Appendix E while (53b) is obvious. To prove (53c), we consider a vortex  $\alpha$  carrying a flux  $\frac{2\pi}{N_i} e_i$ , together with  $N_j$  vortices  $\beta_1, \dots, \beta_{N_j}$  all carrying a flux  $\frac{2\pi}{N_j} e_j$ . Imagine  $\alpha$  is braided around  $\beta_1$  for  $N^{ij}$  times, then around  $\beta_2$  for  $N^{ij}$  times, and so on. The result is a total phase  $N_j \Theta_{ij}$ . This sequence of braiding processes can be described by a product of operators

$$B' = B_{\alpha\beta_{N_j}}^{N^{ij}} \dots B_{\alpha\beta_2}^{N^{ij}} B_{\alpha\beta_1}^{N^{ij}}, \quad (58)$$

where  $B_{\alpha\beta_t}$  represents the operator associated with braiding  $\alpha$  around  $\beta_t$  once. Any two operators  $B_{\alpha\beta_t}$  and  $B_{\alpha\beta_s}$  commute because the commutator  $B_{\alpha\beta_t}^{-1} B_{\alpha\beta_s}^{-1} B_{\alpha\beta_t} B_{\alpha\beta_s} = e^{i\Theta_{ijj}}$  and  $\Theta_{ijj} = 0$  according to (55b). Therefore, the operator  $B'$  can be rewritten as

$$B' = (B_{\alpha\beta_{N_j}} \dots B_{\alpha\beta_2} B_{\alpha\beta_1})^{N^{ij}}, \quad (59)$$

which means  $B'$  is equivalent to braiding  $\alpha$  around  $\beta_1, \dots, \beta_{N_j}$  as a whole for  $N^{ij}$  times. However, the vortices  $\beta_1, \dots, \beta_{N_j}$  fuse to a pure charge, and when  $\alpha$  is braided around any charge for  $N^{ij}$  times, the result is no phase at all. Therefore, we obtain  $N_j \Theta_{ij} = 0$ . Similarly, one can show that  $N_i \Theta_{ij} = 0$ . Putting this together, we derive the constraint (53c).

Finally, to prove the constraint (53d), we use the diagrammatical representation of the topological spin [34]

$$\text{loop} = e^{i2\pi s_\alpha} \text{line} \quad (60)$$

Let us imagine  $N_i$  identical  $\alpha$ 's, which should fuse to some charge  $q$ . Consider the following diagram:

$$\text{braiding} = \text{fusion} \quad (61)$$

where the case  $N_i = 3$  is shown for simplicity. The left-hand side equals the topological spin  $e^{i2\pi s_q} = 1$ , while the right-hand side equals

$$e^{i2\pi N_i s_\alpha + \sum_{n=0}^{N_i-1} i4\pi n s_\alpha} = e^{i2\pi N_i^2 s_\alpha} = e^{iN_i \Theta_i}, \quad (62)$$

where the result of Appendix E is used. We conclude that  $N_i \Theta_i = 0$  modulo  $2\pi$ , which proves the constraint (53d).

### C. 3D Abelian case

We now consider the case of 3D gauge theories with gauge group  $G = \prod_{i=1}^K \mathbb{Z}_{N_i}$  and Abelian braiding statistics. In this case, there are two invariants  $\Theta_{i,l}$  and  $\Theta_{ij,l}$ . We will argue that these invariants must satisfy the following general constraints:

$$\Theta_{ii,l} = 2\Theta_{i,l}, \quad (63a)$$

$$\Theta_{ij,l} = \Theta_{ji,l}, \quad (63b)$$

$$N_{ijl} \Theta_{ij,l} = 0, \quad (63c)$$

$$N_{il} \Theta_{i,l} = 0, \quad (63d)$$

$$\frac{N^{ijl}}{N^{ij}} \Theta_{ij,l} + \frac{N^{ijl}}{N^{jl}} \Theta_{jl,i} + \frac{N^{ijl}}{N^{li}} \Theta_{li,j} = 0, \quad (63e)$$

$$\Theta_{i,l} \frac{N^{il}}{N_i} + \Theta_{il,i} = 0, \quad (63f)$$

$$\Theta_{i,i} = 0 \quad (\text{conjectured}). \quad (63g)$$

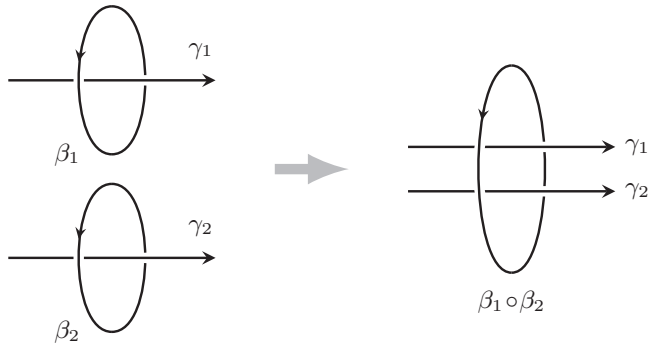


FIG. 7. Fusion of two loops  $\beta_1$  and  $\beta_2$  that carry the same amount of flux  $\phi_{\beta_1} = \phi_{\beta_2}$  and that are linked to different base loops. We denote this type of fusion by  $\beta_1 \circ \beta_2$ . This is different notation from Ref. [24], where this type of fusion was denoted by  $\beta_1 \oplus \beta_2$ .

We note that the above constraints are a generalization of those derived in Ref. [24].

To prove these constraints, we make use of the following general properties of Abelian three-loop statistics:

$$\theta_{\alpha\alpha,c} = 2\theta_{\alpha,c}, \quad (64a)$$

$$\theta_{\alpha\beta,c} = \theta_{\beta\alpha,c}, \quad (64b)$$

$$\theta_{\alpha(\beta_1 \times \beta_2),c} = \theta_{\alpha\beta_1,c} + \theta_{\alpha\beta_2,c}, \quad (64c)$$

$$\theta_{(\alpha \times \beta),c} = \theta_{\alpha,c} + \theta_{\beta,c} + \theta_{\alpha\beta,c}, \quad (64d)$$

$$\theta_{(\alpha_1 \circ \alpha_2)(\beta_1 \circ \beta_2), (c_1 + c_2)} = \theta_{\alpha_1\beta_1, c_1} + \theta_{\alpha_2\beta_2, c_2}, \quad (64e)$$

$$\theta_{\alpha \circ \beta, c_1 + c_2} = \theta_{\alpha, c_1} + \theta_{\beta, c_2}. \quad (64f)$$

Note that these equations involve two types of fusions of loops, i.e., the “ $\times$ ” and “ $\circ$ ” fusions, shown in Figs. 3 and 7 respectively. The “ $\times$ ” fusion involves two loops that are linked to the same base loop, while the “ $\circ$ ” fusion involves two loops that carry the same amount of fluxes but are linked to different base loops (note that these types of fusion were denoted by “+” and “ $\oplus$ ” in Ref. [24]). One can see that the first four equations resemble Eqs. (54a)–(54d) in 2D systems, while the last two are new to 3D systems. We call (64b) the *symmetry relation* and call (64c)–(64f) the *linearity relations*. Like the 2D relations, the linearity relations encode the fact that braiding and exchanging of loops commute with fusion of loops. The linearity relations (64c) and (64d) only involve a single base loop and are similar to the 2D linearity relations (54c) and (54d). The linearity relations (64e) and (64f) involve the “ $\circ$ ” fusion with different base loops, and have no analog in 2D systems. A graphical proof of (64e) can be found in Ref. [24] and the proof for (64f) is similar.

With the help of these general properties, we will now prove the constraints (63a)–(63f). The constraints (63a) and (63b) are the simplest to prove, as they follow immediately from (64a) and (64b). To prove (63c), we first imagine braiding a vortex loop  $\alpha$  around  $N_j$  identical vortex loops  $\beta$ , where both  $\alpha$  and  $\beta$  are linked to a base loop  $\gamma$  and  $\phi_\alpha = \frac{2\pi}{N_i} e_i$ ,  $\phi_\beta = \frac{2\pi}{N_j} e_j$ ,  $\phi_\gamma = \frac{2\pi}{N_l} e_l$ . Clearly, the total statistical phase is  $N_j \theta_{\alpha\beta, e_l}$ . On the other hand, from the linearity property (64c), we know that this phase is equal to that of braiding  $\alpha$  around the fusion product of the  $\beta$ ’s. Since the  $N_j$   $\beta$ ’s fuse to a pure charge, it

follows that the latter quantity is a multiple of  $2\pi/N_i$ . Hence, we derive

$$N_j \theta_{\alpha\beta, e_l} = \frac{2\pi}{N_i} \times \text{integer}. \quad (65)$$

Comparing with the definition of  $\Theta_{ij,l}$ , we deduce that

$$N_{ij} \Theta_{ij,l} = 0, \quad (66)$$

modulo  $2\pi$ . Next, we imagine a collection of  $N_l$  identical three-loop linked structures. In each structure, loops  $\alpha$  and  $\beta$  are linked with  $\gamma$ , and  $\alpha$  is braided around  $\beta$ . The total phase in the  $N_l$  braiding processes is  $N_l \theta_{\alpha\beta, e_l}$ . We then fuse together all the linked structures and make use of the linearity relation (64e), and obtain

$$N_l \theta_{\alpha\beta, e_l} = \theta_{AB, N_l e_l}, \quad (67)$$

where  $A, B$  denote the two loops  $A = \alpha \circ \dots \circ \alpha$  and  $B = \beta \circ \dots \circ \beta$ . Now, since the  $N_l e_l = 0$ , the two loops  $A$  and  $B$  are not linked to any base loop so the statistical phase on the right-hand side can be computed using the conventional Aharonov-Bohm law (see Ref. [24] for a more detailed argument):

$$\theta_{AB, N_l e_l} = \frac{2\pi}{N_j} q_A \cdot e_j + \frac{2\pi}{N_i} q_B \cdot e_i, \quad (68)$$

where  $q_A, q_B$  denote the amount of charge carried by the (unlinked) loops  $A, B$ . We conclude that

$$N_l \theta_{\alpha\beta, e_l} = \frac{2\pi}{N_i} \times \text{integer} + \frac{2\pi}{N_j} \times \text{integer}, \quad (69)$$

which implies that

$$N_l \Theta_{ij,l} = 0 \quad (70)$$

modulo  $2\pi$ . Combining (66) and (70), we immediately derive the constraint (63c). The proof of (63d) is similar, the only difference being that one needs to consider exchange statistics rather than mutual statistics, and the linearity relations (64d) and (64f) rather than (64c) and (64e).

The proof of the “cyclic relations” (63e) and (63f) follows the same philosophy as above, and involves considering certain thought experiments. These thought experiments are described in Ref. [24] in the case of  $G = (\mathbb{Z}_N)^K$ . It is not hard to extend these thought experiments to  $G = \prod_i \mathbb{Z}_{N_i}$ , so we do not repeat them here and instead refer the reader to Ref. [24].

It is unfortunate that we are not able to prove the last constraint (63g). Therefore, this relation is just a conjecture. However, from (63d) and (63f), we can prove the weaker constraint  $3\Theta_{i,i} = 0$ .

#### D. 3D general case

To complete our discussion, we now consider the case of general 3D gauge theories with gauge group  $G = \prod_{i=1}^K \mathbb{Z}_{N_i}$ . In this case, there are three invariants  $\Theta_{i,l}$ ,  $\Theta_{ij,l}$ , and  $\Theta_{ijk,l}$ . We will now argue that  $\Theta_{i,l}, \Theta_{ij,l}$  satisfy the constraints (63a)–(63d), as in the Abelian case. In addition, the third invariant  $\Theta_{ijk,l}$  satisfies

$$\Theta_{ijk,l} = \text{sgn}(\hat{p}) \Theta_{\hat{p}(i)\hat{p}(j)\hat{p}(k),l}, \quad (71a)$$

$$\Theta_{iij,l} = 0, \quad (71b)$$

$$N_{ijk} \Theta_{ijk,l} = 0. \quad (71c)$$

Unlike the other cases that we have discussed, we expect that the above list of constraints is *incomplete*: that is, there are likely further constraints on the invariants beyond the ones listed here. One reason for this belief is that in the case of the Dijkgraaf-Witten models, the invariants obey the cyclic constraints (63e)–(63g), and  $\Theta_{ijk,l}$  satisfies the stricter constraint

$$\begin{aligned}\Theta_{ijk,l} &= \text{sgn}(\hat{p})\Theta_{\hat{p}(i)\hat{p}(j)\hat{p}(k),\hat{p}(l)}, \\ \Theta_{ii,j,l} &= 0, \\ N_{ijkl}\Theta_{ijk,l} &= 0,\end{aligned}$$

where  $\hat{p}$  is a permutation of the indices  $i, j, k, l$  and  $\text{sgn}(\hat{p}) = \pm 1$  is its parity. We find it plausible that these additional constraints may apply more generally than to the Dijkgraaf-Witten models, but we have not been able to prove this fact.

The easiest constraints to establish are (71a)–(71c). These constraints follow identical arguments to the 2D results (55a)–(55c). Likewise, the two constraints (63a) and (63b) follow from the same logic as the 2D results (53a) and (53b), which were proved in the general case in Sec. VIB.

To prove (63c), we first notice that  $N_{ij}\Theta_{ij,l} = 0$  can be established in the same way as its 2D analog (53c). Next, we consider a thought experiment with  $N_l$  identical three-loop linked structures  $\{\alpha, \beta, \gamma\}$  with  $\phi_\alpha = \frac{2\pi}{N_i}e_i$ ,  $\phi_\beta = \frac{2\pi}{N_j}e_j$ ,  $\phi_\gamma = \frac{2\pi}{N_k}e_k$ . We imagine braiding each  $\alpha$  around the corresponding  $\beta$  for  $N^{ij}$  times. The result of each braiding process is an Abelian phase  $\Theta_{ij,l}$ . Therefore, the result of braiding all the  $\alpha$ 's simultaneously is  $N_l\Theta_{ij,l}$ . Now, since all the phases are Abelian, a linearity relation such as (64e) applies in this case. More specifically, one can argue that the phase associated with this braiding process is equal to that of braiding the loop  $A = \alpha \circ \dots \circ \alpha$  around  $B = \beta \circ \dots \circ \beta$  for  $N^{ij}$  times, while both  $A, B$  are linked to a base loop which carries a flux  $N_l\phi_l$ . Now, since  $N_l\phi_l = 0$  modulo  $2\pi$ ,  $A, B$  are not linked to any base loop so this phase can be computed from the Aharonov-Bohm law, as in Eq. (68). In this way, we deduce that

$$N_l\Theta_{ij,l} = N^{ij} \left( \frac{2\pi}{N_i} \times \text{integer} + \frac{2\pi}{N_j} \times \text{integer} \right) = 0. \quad (72)$$

Combining this with  $N_{ij}\Theta_{ij,l} = 0$ , we derive the constraint (63c). The proof of (63d) is similar and involves the use of a linearity relation for  $\Theta_{i,l}$  analogous to (64f).

### VII. DO THE DIJKGRAAF-WITTEN MODELS EXHAUST ALL POSSIBLE VALUES FOR THE INVARIANTS?

In this section, we ask and partially answer the following question:

*Do there exist Abelian gauge theories for which the invariants acquire values beyond those given by the Dijkgraaf-Witten models?*

If the answer to this question is “yes,” then it follows that there exist gauge theories that do not belong to the same phase as any of the Dijkgraaf-Witten models. On the other hand, if the answer is “no,” we cannot make any rigorous statements about the existence or nonexistence of gauge theories beyond the Dijkgraaf-Witten models since we cannot rule out the possibility that two gauge theories may share the

same invariants but still belong to distinct phases. That being said, a negative answer can be interpreted as circumstantial evidence that the Dijkgraaf-Witten models exhaust all possible Abelian gauge theories.

To address this question, we compare the general constraints derived in the previous section with our explicit computation of the invariants in the Dijkgraaf-Witten models. We first consider the 2D case. In that case, we know that the invariants must obey constraints (53a)–(53d) as well as (55a)–(55c). At the same time, we know that the Dijkgraaf-Witten models can realize any  $(\Theta_i, \Theta_{ij}, \Theta_{ijk})$  of the form given in Eqs. (42a)–(42c). Comparing these two results, one can easily verify that the Dijkgraaf-Witten models can realize all possible values of the invariants that are consistent with the general constraints. We conclude that in the 2D case, the Dijkgraaf-Witten models exhaust all possible values of the invariants.

Next, we consider the 3D Abelian case: that is, 3D gauge theories with gauge group  $G$  and Abelian three-loop statistics. In this case, we know that the invariants must obey the constraints (63). We also know that the Dijkgraaf-Witten models with Abelian statistics [49] can realize any  $(\Theta_{i,l}, \Theta_{ij,l})$  of the form given in Eqs. (49a) and (49b). From these two facts, one can show that the Abelian Dijkgraaf-Witten models realize all possible values of the invariants that are consistent with the general constraints; this derivation is given in Appendix H. One loophole is that the last constraint (63g) is simply a *conjecture*. Therefore, all we can say is that the Dijkgraaf-Witten models exhaust all possible values of the invariants, assuming this conjecture is correct.

Finally, let us consider the general 3D case: that is, 3D gauge theories with gauge group  $G$  and any type of three-loop statistics (Abelian or non-Abelian). In this case, we have only managed to prove very weak constraints on the invariants, as discussed in the previous section. As a result, the Dijkgraaf-Witten models only realize a small subset of the invariants that are consistent with our constraints. Hence, we cannot make any statements as to whether the Dijkgraaf-Witten models exhaust all possible values of the invariants.

### VIII. RELATION BETWEEN THE INVARIANTS AND BRAIDING STATISTICS

In this section, we show that the topological invariants contain the same information as the full set of braiding statistics data, for 2D or 3D gauge theories with gauge group  $G = \prod_{i=1}^K \mathbb{Z}_{N_i}$  and Abelian statistics. We do not know whether a similar result holds for gauge theories with gauge group  $G$  and non-Abelian statistics.

#### A. 2D case

We begin with the 2D case. What we will show is that if two 2D gauge theories with Abelian statistics have the same values for the invariants  $\Theta_i$  and  $\Theta_{ij}$ , then all their braiding statistics must be identical. In this sense, the topological invariants contain all the information about the braiding statistics in these systems.

Before presenting our argument, let us recall our definition for when two gauge theories have the “same” braiding



statistics. As discussed in Sec. II C, we say that two gauge theories have the same braiding statistics if there exists a one-to-one correspondence between the quasiparticle excitations in the two theories that (1) preserves all the algebraic structure associated with braiding statistics, e.g.,  $R$  symbols, fusion rules,  $F$  symbols, etc., and (2) preserves the gauge flux of excitations. In other words, for each excitation in one gauge theory, there should be a corresponding excitation in the other gauge theory that has the same braiding statistics properties and the same gauge flux.

Given the above definition, our task is as follows. Consider two 2D gauge theories with group  $G = \prod_i^K \mathbb{Z}_{N_i}$  and Abelian statistics. Suppose that the gauge theories have the same values for the invariants  $\Theta_i$  and  $\Theta_{ij}$ . We have to show that there exists a one-to-one correspondence between the excitations in the two theories that preserves their exchange statistics, mutual statistics, and gauge flux.

We construct this correspondence as follows. In the first gauge theory, for each  $i = 1, \dots, K$ , we choose one of the  $|G|$  types of vortices that carry unit flux  $\frac{2\pi}{N_i} e_i$ , and denote it by  $\hat{v}_i$ . Similarly, in the second gauge theory, for each  $i = 1, \dots, K$ , we choose one of the  $|G|$  types of vortices carrying unit flux  $\frac{2\pi}{N_i} e_i$  and denote it by  $\hat{w}_i$ . Given that the two gauge theories have identical values of  $\Theta_i$  and  $\Theta_{ij}$ , we know that the exchange statistics and mutual statistics of  $\{\hat{v}_i\}$  and  $\{\hat{w}_i\}$  are related by

$$\theta_{\hat{v}_i} = \theta_{\hat{w}_i} + \frac{2\pi x_i}{N_i}, \quad \theta_{\hat{v}_i \hat{v}_j} = \theta_{\hat{w}_i \hat{w}_j} + \frac{2\pi y_{ij}}{N_{ij}} \quad (73)$$

for some integers  $x_i, y_{ij}$ , with  $y_{ii} = 2x_i$ .

In the next step, we fuse some gauge charge  $q_i = (q_{i1}, \dots, q_{iK})$  onto each vortex  $\hat{w}_i$ , to obtain another unit flux vortex  $\hat{w}'_i$ . We choose the  $q_i$  so that the new vortices  $\hat{w}'_i$  obey

$$\theta_{\hat{v}_i} = \theta_{\hat{w}'_i}, \quad \theta_{\hat{v}_i \hat{v}_j} = \theta_{\hat{w}'_i \hat{w}'_j}. \quad (74)$$

To see that we can always do this, note that

$$\begin{aligned} \theta_{\hat{w}'_i} &= \theta_{\hat{w}_i} + \frac{2\pi q_{ii}}{N_i}, \\ \theta_{\hat{w}'_i \hat{w}'_j} &= \theta_{\hat{w}_i \hat{w}_j} + \frac{2\pi q_{ij}}{N_j} + \frac{2\pi q_{ji}}{N_i} \end{aligned} \quad (75)$$

by the Aharonov-Bohm formula. Hence, we can arrange for Eq. (74) to hold if we choose the gauge charges  $q_i$  so that they satisfy

$$\begin{aligned} q_{ii} &= x_i \pmod{N_i}, \\ \frac{1}{N_{ij}}(N_i q_{ij} + N_j q_{ji}) &= y_{ij} \pmod{N^{ij}}. \end{aligned} \quad (76)$$

We are now ready to construct the desired one-to-one correspondence between the excitations in the two gauge theories. We note that every excitation in the first gauge theory can be written uniquely as a fusion product

$$\alpha = (\hat{v}_1)^{a_1} \times \dots \times (\hat{v}_K)^{a_K} \times q, \quad (77)$$

where  $a_i$  are integers with  $0 \leq a_i \leq N_i - 1$ , and where  $q = (q_1, \dots, q_K)$  is some gauge charge with  $0 \leq q_i \leq N_i - 1$ . Similarly, every excitation in the second gauge theory can be written uniquely as

$$\alpha' = (\hat{w}'_1)^{a_1} \times \dots \times (\hat{w}'_K)^{a_K} \times q. \quad (78)$$

We define a one-to-one correspondence between two sets of excitations by mapping

$$\begin{aligned} \alpha &= (\hat{v}_1)^{a_1} \times \dots \times (\hat{v}_K)^{a_K} \times q \\ \leftrightarrow \alpha' &= (\hat{w}'_1)^{a_1} \times \dots \times (\hat{w}'_K)^{a_K} \times q. \end{aligned} \quad (79)$$

It is clear that this correspondence preserves gauge flux since

$$\phi_\alpha = \left( \frac{2\pi a_1}{N_1}, \dots, \frac{2\pi a_K}{N_K} \right) = \phi_{\alpha'}. \quad (80)$$

To see that this correspondence preserves the exchange statistics and mutual statistics of the excitations, we need to check that

$$\theta_\alpha = \theta_{\alpha'}, \quad \theta_{\alpha\beta} = \theta_{\alpha'\beta'} \quad (81)$$

for any  $\alpha, \beta$  in the first gauge theory and corresponding  $\alpha', \beta'$  in the second gauge theory. These relations follow immediately from (74) together with the linearity relations (54c) and (54d). This completes our argument: we have shown that if two gauge theories have the same values of the invariants  $\Theta_i, \Theta_{ij}$ , then all their braiding statistics is identical.

### B. 3D case

We now repeat the argument in the 3D case. Consider two 3D gauge theories with group  $G = \prod_i^K \mathbb{Z}_{N_i}$  and Abelian three-loop statistics. Suppose that the gauge theories have the same values for the invariants  $\Theta_{i,l}$  and  $\Theta_{ij,l}$ . We will show that the two gauge theories have identical three-loop statistics. More precisely, we will show that for each gauge flux  $\phi$ , there exists a one-to-one correspondence between the looplike excitations in the two theories that are linked with base loops with flux  $\phi$ , such that the corresponding excitations have the same three-loop statistics and the same gauge flux.

The derivation closely follows the 2D case. To begin, we focus on the first gauge theory, and we fix a base loop that carries unit flux  $\frac{2\pi}{N_i} e_l$ . For each  $i = 1, \dots, K$ , we choose one of the  $|G|$  types of vortex loops that carry flux  $\frac{2\pi}{N_i} e_i$  and are linked with the base loop and we denote it by  $\hat{v}_i$ . We repeat this process for the second gauge theory. That is, we fix a base loop with flux  $\frac{2\pi}{N_i} e_l$  and we choose vortex loops  $\{\hat{w}_i\}$  that are linked with the base loop and carry flux  $\frac{2\pi}{N_i} e_i$ . Using the fact that the two gauge theories have identical values of  $\Theta_{i,l}$  and  $\Theta_{ij,l}$ , we know that the three-loop statistics of the  $\{\hat{v}_i\}$  and  $\{\hat{w}_i\}$  loops are related by

$$\theta_{\hat{v}_i, e_l} = \theta_{\hat{w}_i, e_l} + \frac{2\pi x_{i,l}}{N_i}, \quad \theta_{\hat{v}_i \hat{v}_j, e_l} = \theta_{\hat{w}_i \hat{w}_j, e_l} + \frac{2\pi y_{ij,l}}{N_{ij}}$$

for some integers  $x_{i,l}, y_{ij,l}$ , with  $y_{ii,l} = 2x_{i,l}$ . Note that the integers  $x_{i,l}$  and  $y_{ij,l}$  may depend on  $l$ , that is, they may take different values for different base loops.

We next fuse some gauge charge  $q_i = (q_{i1}, \dots, q_{iK})$  onto each vortex loop  $\hat{w}_i$ , to obtain another vortex loop  $\hat{w}'_i$ . We choose the  $q_i$  so that the new vortex loops  $\hat{w}'_i$  obey

$$\theta_{\hat{v}_i, e_l} = \theta_{\hat{w}'_i, e_l}, \quad \theta_{\hat{v}_i \hat{v}_j, e_l} = \theta_{\hat{w}'_i \hat{w}'_j, e_l}. \quad (82)$$

The fact that we can always find such a  $q_i$  follows from the same reasoning as in the 2D case.

We now construct a one-to-one correspondence between the loop excitations in the two gauge theories, focusing on the

excitations that are linked with a base loop with flux  $\frac{2\pi}{N_i} e_i$ . The construction is identical to the 2D case. First, we note that every loop excitation  $\alpha$  in the first gauge theory can be written uniquely as a fusion product of the  $\hat{v}_i$  vortex loops together with some gauge charge, as in Eq. (77). Similarly, every loop excitation in the second gauge theory can be written uniquely as a product of the  $\hat{w}'_i$  vortex loops as in Eq. (78). We can therefore define a one-to-one correspondence using the mapping in Eq. (79). For the same reasons as in the 2D case, it is clear that this correspondence preserves the statistics of the loop excitations, as well as their gauge flux.

To finish the derivation, we need to generalize the above one-to-one correspondence to the case where the base loop carries arbitrary flux  $\phi$ . This generalization is easy to prove since we can construct base loops with arbitrary flux by fusing together base loops with unit flux using the “ $\circ$ ” fusion process (see Fig. 7). Furthermore, using the linearity relations (64e) and (64f) we can see that the three-loop statistics associated with these more general base loops is completely determined by the three-loop statistics for the unit flux base loops. Putting these pieces together, we can construct a similar one-to-one correspondence between loop excitations that are linked with base loops with arbitrary flux. This completes our argument and proves that the two gauge theories have identical three-loop statistics.

## IX. CONCLUSION

In this paper, we have studied the braiding statistics of 2D and 3D gauge theories with group  $G = \prod_{i=1}^K \mathbb{Z}_{N_i}$ , and we have defined topological invariants that summarize some of the most important aspects of this braiding structure. In the 2D case, these invariants consist of three tensors  $\{\Theta_i, \Theta_{ij}, \Theta_{ijk}\}$ , while in the 3D case, they consist of three tensors  $\{\Theta_{i,l}, \Theta_{ij,l}, \Theta_{ijk,l}\}$ . These tensors are defined in terms of certain composite braiding processes involving vortices and vortex loops.

Using these invariants, we have obtained two results. First, we have shown that the invariants distinguish all 2D and 3D Dijkgraaf-Witten models (= gauged group cohomology models) with group  $G$ . Second, we have shown that the Dijkgraaf-Witten models with group  $G$  exhaust all possible values of the invariants in the 2D case and we have derived similar, but weaker, results in the 3D case.

So far, our discussion has focused on gauge theories or, more precisely, gauged SPT models. We now return to the questions raised in the Introduction, and discuss the implications of our findings for *ungauged* SPT models. We begin with our result that the invariants take different values for each of the 2D and 3D Dijkgraaf-Witten models (= gauged group cohomology models) with group  $G$ . This result has an immediate implication: the 2D and 3D group cohomology models with group  $G$  all belong to distinct phases.

Next, we consider our finding that the 2D Dijkgraaf-Witten models exhaust all possible values for the invariants. While we cannot draw rigorous conclusions from this result, it is at least consistent with the possibility that the 2D group cohomology models realize all possible SPT phases with symmetry group  $G$ . Our 3D results on this topic are also consistent with this possibility, though they are somewhat weaker.

In short, our results support the group cohomology classification conjecture of Chen, Gu, Liu, and Wen [10] for the case of finite, Abelian symmetry group  $G$ . In addition, our results allow us to go further: they provide a simple diagnostic for determining whether a specific microscopic Hamiltonian belongs to the same phase as a specific group cohomology model. The diagnostic involves gauging the Hamiltonian and then computing the topological invariants of the associated gauge theory. If the invariants for the microscopic Hamiltonian are different from that of the group cohomology model, then we may conclude that the Hamiltonian belongs to a distinct phase. If the invariants are the same, then it may belong to the same phase, though we cannot be certain because we do not know if the invariants are complete in the sense that they distinguish all possible SPT phases.

We see this work as a first step towards answering the three questions raised in the Introduction. We have managed to make some progress on these questions for the special case of Abelian symmetry groups, but many issues remain unresolved. First, although we have partially answered questions 1 and 2 from the Introduction, we have said nothing at all about question 3, which asks whether the braiding statistics data can distinguish all possible SPT phases. Another important direction for future work is to understand the braiding statistics in gauge theories with *non-Abelian* gauge group and see if similar topological invariants could be defined in that context. Finally, it would be interesting to study gauge theories associated with fermionic SPT models. Such systems may have an even richer braiding statistics structure than the bosonic case analyzed here.

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## APPENDIX A: GAUGING PRESCRIPTION

In this Appendix, we give a general prescription for how to gauge a lattice boson model with an Abelian symmetry group  $G = \prod_{i=1}^K \mathbb{Z}_{N_i}$ . The procedure we describe follows the usual minimal coupling scheme for lattice gauge theories [36]. The only nonstandard element is that we will set the gauge coupling constant to zero in order to maximize our control over the models, as in Refs. [15,50]. More precisely, what we mean by this is that the Hamiltonians for the gauged models commute with the flux operators that measure the gauge flux through each plaquette in the lattice. This property has a nice consequence: the gauge theories we construct are guaranteed to be gapped and deconfined as long as the original boson models are gapped and do not break the symmetry spontaneously.

For concreteness, we will focus on a particular kind of boson model with  $\prod_{i=1}^K \mathbb{Z}_{N_i}$  symmetry. Specifically, we will focus on lattice boson models built out of  $K$  species of bosons, where the particle number of the  $i$ th species is conserved modulo  $N_i$  and where the different species of bosons live on the sites  $p$  of some 2D or 3D lattice. We denote the boson creation operator for the  $i$ th species on lattice site  $p$  by  $b_{p,i}^\dagger$ . We assume that the boson model has local interactions, so that the Hamiltonian of

the bosons can be written as

$$H = \sum_A H_A(\{b_{p,i}\}), \quad (\text{A1})$$

where the sum is taken over localized regions  $A$ , and where  $H_A$  is some operator composed out of boson creation and annihilation operators acting on region  $A$ .

We now discuss our procedure for gauging such a Hamiltonian. The first step is to introduce a Hilbert space  $\mathcal{H}_{pq}$  of dimension  $|G| = \prod_{i=1}^K N_i$  for each link  $pq$  of the lattice. This Hilbert space is spanned by basis states  $\{|m\rangle\}$ , where  $m = (m_1, \dots, m_K)$  with  $0 \leq m_i \leq N_i - 1$ . Along with the Hilbert space, we introduce lattice gauge field operators  $\mu_{pq,i}$  for each species  $i = 1, \dots, K$ . These operators are defined by

$$\mu_{pq,i}|m\rangle = e^{\pm i \frac{2\pi}{N_i} m_i} |m\rangle \quad (\text{A2})$$

with a  $+$  or  $-$  sign depending on whether  $pq$  is parallel or antiparallel to some fixed orientation that we assign to every link of the lattice. Likewise, we define a set of unitary shift operators  $S_{pq,a}$  for each group element  $a \in G$ :

$$S_{pq,a}|m\rangle = |m \pm a\rangle \quad (\text{A3})$$

with a  $+$  or  $-$  sign depending on whether  $pq$  is parallel or antiparallel to the prescribed orientation. Here, we have used  $a = (a_1, \dots, a_K)$ ,  $0 \leq a_i \leq N_i - 1$ , to denote the group elements of  $G = \prod_{i=1}^K \mathbb{Z}_{N_i}$ .

In the second step, we replace each operator  $H_A(\{b_{p,i}\})$  by a gauged operator  $\tilde{H}_A(\{b_{p,i}, \mu_{pq,i}\})$  following the minimal coupling procedure:

$$H_A(\{b_{p,i}\}) \rightarrow \tilde{H}_A(\{b_{p,i}, \mu_{pq,i}\}). \quad (\text{A4})$$

For example, a nearest-neighbor hopping term undergoes the following substitution under minimal coupling:

$$b_{p,i} b_{q,i}^\dagger \rightarrow b_{p,i} b_{q,i}^\dagger \mu_{pq,i}. \quad (\text{A5})$$

For more complicated terms involving multiple sites, the substitution contains a product of  $\mu$  operators acting on a path that connects the sites. One subtlety is that there is an ambiguity for how to choose the path. This ambiguity is eliminated by the third step in the gauging procedure. In this step, we multiply  $\tilde{H}_A$  by a projection operator  $P_A$  which projects onto the states that have vanishing gauge flux through each plaquette that belong to  $A$ . That is, the projector  $P_A$  can be written as a product

$$P_A = \prod_{\langle pqr \rangle \in A} P_{\langle pqr \rangle}, \quad (\text{A6})$$

where  $P_{\langle pqr \rangle}$  projects onto states with vanishing flux through a particular plaquette  $\langle pqr \rangle$ , which we have assumed to be triangular for concreteness. The projector  $P_{\langle pqr \rangle}$  can be explicitly written as

$$P_{\langle pqr \rangle} = \frac{1}{|G|} \prod_{i=1}^K \left( \sum_{k=0}^{N_i-1} (\mu_{pq,i} \mu_{qr,i} \mu_{rp,i})^k \right). \quad (\text{A7})$$

After multiplying  $\tilde{H}_A$  by  $P_A$ , one can show that all paths enclosed by  $A$  lead to the same term so that the minimal coupling procedure is unambiguous. It is not hard to see that  $P_{\langle pqr \rangle}$  is a Hermitian operator.

In the last step of the gauging procedure, we add a term of the form  $-\Delta P_{\langle pqr \rangle}$  to the Hamiltonian for each plaquette, so that it costs finite energy to create vortex excitations. After applying these steps, the final gauged Hamiltonian is

$$\tilde{H} = \sum_A \tilde{H}_A(\{b_{p,i}, \mu_{pq,i}\}) P_A - \Delta \sum_{\langle pqr \rangle} P_{\langle pqr \rangle}, \quad (\text{A8})$$

where we assume  $\Delta$  is large and positive. This Hamiltonian is defined on a Hilbert space consisting of gauge-invariant states, i.e., states  $|\Psi\rangle$  satisfying

$$T_{p,a}|\Psi\rangle = |\Psi\rangle \quad (\text{A9})$$

for every  $p, a$ , where  $T_{p,a}$  is the gauge transformation associated with group element  $a \in G$  and site  $p$ :

$$T_{p,a} = e^{i \sum_i \frac{2\pi a_i}{N_i} b_{p,i}^\dagger} \prod_{q \in \text{neigh.}(p)} S_{qp,a}. \quad (\text{A10})$$

This constraint is the analog of the usual Gauss's law of electromagnetism  $\nabla \cdot E = \rho$ .

The gauging prescription we have just described has a special property: the Hamiltonian  $\tilde{H}$  commutes with the flux through each plaquette of the lattice. That is,  $[\tilde{H}, \mu_{pq,i} \mu_{qr,i} \mu_{rp,i}] = 0$  for every plaquette  $\langle pqr \rangle$  and every  $i = 1, \dots, K$ . This property has an important consequence: the gauge theory  $\tilde{H}$  is guaranteed to be gapped and deconfined as long as  $H$  is gapped and does not break the symmetry spontaneously. To see this, note that  $[\tilde{H}, P_{\langle pqr \rangle}] = 0$ , so the eigenstates of  $\tilde{H}$  are also eigenstates of  $P_{\langle pqr \rangle}$ . As long as  $\Delta$  is large, then all the low-energy eigenstates  $|\Psi\rangle$  will have vanishing flux, that is,  $P_{\langle pqr \rangle}|\Psi\rangle = |\Psi\rangle$ . At the same time, it is easy to see that, within the zero flux sector, the energy spectrum of  $\tilde{H}$  is identical to the energy spectrum of the original boson model  $H$ . We conclude that  $\tilde{H}$  and  $H$  have identical low-energy spectra. Hence,  $\tilde{H}$  is guaranteed to be gapped if  $H$  is gapped. Similar reasoning shows that  $\tilde{H}$  is deconfined as long as  $H$  does not break the symmetry spontaneously.

## APPENDIX B: GROUP COHOMOLOGY

In this Appendix, we review the basic ingredients of the cohomology of finite groups [10,37,39]. We focus on the cohomology group  $H^n[G, U(1)]$ .

Let  $G$  be a finite group. The basic objects that group cohomology studies are *n-cochains*. An *n-cochain* is a  $U(1)$  valued function  $c(g_1, \dots, g_n)$ :

$$c : \underbrace{G \times G \times \dots \times G}_{n \text{ times}} \rightarrow U(1).$$

The collection of *n-cochains* form an Abelian group  $\mathcal{C}^n$ , where the group operation is defined by

$$(c_1 \cdot c_2)(g_1, \dots, g_n) = c_1(g_1, \dots, g_n) \cdot c_2(g_1, \dots, g_n).$$

The *coboundary operator*  $\delta$  is a map  $\delta : \mathcal{C}^n \rightarrow \mathcal{C}^{n+1}$ , defined by

$$\begin{aligned} \delta c(g_1, \dots, g_{n+1}) &= c(g_2, \dots, g_{n+1}) c(g_1, \dots, g_n)^{(-1)^{n+1}} \\ &\times \prod_{i=1}^n [c(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1})]^{(-1)^i}. \end{aligned} \quad (\text{B1})$$

It is easy to check that the coboundary operator satisfies  $\delta(c_1 \cdot c_2) = \delta c_1 \cdot \delta c_2$ . More importantly, one can check that  $\delta$  is nilpotent:  $\delta^2 = 1$ .

With the help of the coboundary operator, we can now define *n-cocycles* and *n-coboundaries*. An *n-cocycle* is an *n-cochain*  $\omega$  that satisfies  $\delta\omega = 1$ . For example, 3-cocycles satisfy

$$\frac{\omega(g_2, g_3, g_4)\omega(g_1, g_2, g_3, g_4)\omega(g_1, g_2, g_3)}{\omega(g_1, g_2, g_3, g_4)\omega(g_1, g_2, g_3, g_4)} = 1, \quad (\text{B2})$$

and 4-cocycles satisfy

$$\frac{\omega(g_2, g_3, g_4, g_5)\omega(g_1, g_2, g_3, g_4, g_5)\omega(g_1, g_2, g_3, g_4, g_5)}{\omega(g_1, g_2, g_3, g_4, g_5)\omega(g_1, g_2, g_3, g_4, g_5)\omega(g_1, g_2, g_3, g_4)} = 1. \quad (\text{B3})$$

Likewise, an *n-coboundary* is an *n-cochain*  $\nu$  that can be written as  $\nu = \delta c$  where  $c \in C^{n-1}$ . The nilpotence of  $\delta$  implies that a coboundary must also be a cocycle. This allows us to define an equivalence relation for the cocycles: two *n-cocycles*  $\omega_1$  and  $\omega_2$  are said to be *cohomologically equivalent* if and only if  $\omega_1 = \omega_2 \cdot \delta c$ , for some  $c \in C^{n-1}$ . The equivalence classes of the *n-cocycles* form an Abelian group, called the *nth cohomology group*, which is denoted by  $H^n[G, U(1)]$ .

In this paper, we will only need the cohomology group  $H^n[G, U(1)]$  for  $G = \prod_{i=1}^K \mathbb{Z}_{N_i}$  and for  $n = 3$  and  $4$ . These cohomology groups can be computed explicitly using the Kunnet formula [10,39]

$$H^3[G, U(1)] = \prod_i \mathbb{Z}_{N_i} \prod_{i < j} \mathbb{Z}_{N_{ij}} \prod_{i < j < k} \mathbb{Z}_{N_{ijk}}, \quad (\text{B4})$$

$$H^4[G, U(1)] = \prod_{i < j} (\mathbb{Z}_{N_{ij}})^2 \prod_{i < j < k} (\mathbb{Z}_{N_{ijk}})^2 \prod_{i < j < k < l} \mathbb{Z}_{N_{ijkl}}. \quad (\text{B5})$$

## APPENDIX C: GROUP COHOMOLOGY MODELS AND DIJKGRAAF-WITTEN MODELS

In this Appendix, we briefly review the group cohomology models of Ref. [10], as well as the Dijkgraaf-Witten models of Ref. [35]. In addition, we show that coupling the group cohomology models to a lattice gauge field gives exactly the Dijkgraaf-Witten models. For convenience, we describe these models as well as the gauging procedure using a path-integral formulation in Euclidean space-time. This is different from the gauging procedure in Appendix A which is described in a Hamiltonian formulation. We expect the similar results can be derived in a Hamiltonian formulation (e.g., see Refs. [51,52] for a Hamiltonian description of Dijkgraaf-Witten models).

### 1. Group cohomology models

The basic data needed to construct a  $d + 1$ -dimensional group cohomology model with (finite) group  $G$  is (1) a  $d + 1$ -cocycle  $\omega$  together with (2) a triangulation of  $d + 1$ -dimensional Euclidean space-time. To build the model, we label the vertices of the triangulation by an ordered sequence  $i, j, \dots$ , the links by  $[ij], [jk], \dots$ , and the triangular plaquettes by  $[ijk], \dots$ , etc. We will refer to the vertices as “0-simplices,” the links as “1-simplices,” the triangular plaquettes

as “2-simplices,” and so on. The basic degrees of freedom in the model are group elements  $g_i \in G$  that live on the vertices  $i$  of the triangulation. For every space-time configuration  $\{g_i\}$ , we assign a local weight  $[\omega(g_i^{-1}g_j, \dots, g_k^{-1}g_l)]^{\sigma_{ij\dots kl}}$  to each  $d + 1$ -simplex  $[ij\dots kl]$  ( $i < j < \dots < k < l$ ) where  $\sigma_{ij\dots kl} = \pm 1$  is the chirality of the simplex  $[ij\dots kl]$ . The action corresponding to  $\{g_i\}$  is given by the product of the local weights

$$e^{-S_1(\{g_i\})} = \prod_{[ij\dots kl]} [\omega(g_i^{-1}g_j, \dots, g_k^{-1}g_l)]^{\sigma_{ij\dots kl}}. \quad (\text{C1})$$

Summing over all the configurations  $\{g_i\}$ , we obtain the partition function

$$Z_1 = \frac{1}{|G|^{N_v}} \sum_{\{g_i\}} e^{-S_1(\{g_i\})}, \quad (\text{C2})$$

where  $\frac{1}{|G|^{N_v}}$  is a normalization factor,  $|G|$  is the size of the group, and  $N_v$  is the number of vertices. One can easily check that the action (C1) is invariant under the global symmetry

$$g_i \rightarrow gg_i \quad (\text{C3})$$

for all  $g \in G$ . According to the arguments in Ref. [10], the ground states of these models are gapped and short-range entangled for any  $G$  and  $\omega$ . Moreover, two cocycles that differ by a coboundary define the same group cohomology model. Thus, the group cohomology models are labeled by equivalence classes of cocycles, i.e., by elements of the cohomology group  $H^{d+1}[G, U(1)]$ .

### 2. Dijkgraaf-Witten models

The basic data needed to construct a  $d + 1$ -dimensional Dijkgraaf-Witten model with (finite) group  $G$  is the same as that for a group cohomology model: (1) a  $d + 1$ -cocycle  $\omega$  together with (2) a triangulation of  $d + 1$ -dimensional Euclidean space-time. Unlike the group cohomology SPT models, the basic degrees of freedom in a Dijkgraaf-Witten model are group elements  $h_{ij} \in G$  that live on the *links*  $[ij]$  of the triangulation. For every space-time configuration  $\{h_{ij}\}$ , the corresponding action  $e^{-S_2(\{h_{ij}\})}$  is defined as follows. First, one needs to determine if the configuration is flat, i.e.,  $h_{ij}h_{jk}h_{ki} = 1$  for every 2-simplex  $[ijk]$ . If it is flat, we assign a local weight  $[\omega(h_{ij}, \dots, h_{kl})]^{\sigma_{ij\dots kl}}$  to each  $d + 1$ -simplex  $[ij\dots kl]$  ( $i < j < \dots < k < l$ ) where  $\sigma_{ij\dots kl} = \pm 1$  is the chirality of the simplex  $[ij\dots kl]$ . The action is then given by

$$e^{-S_2(\{h_{ij}\})} = \prod_{[ij\dots kl]} [\omega(h_{ij}, \dots, h_{kl})]^{\sigma_{ij\dots kl}}. \quad (\text{C4})$$

If the gauge configuration is not flat, then  $e^{-S_2(\{h_{ij}\})} = 0$ . Summing over all the configurations  $\{h_{ij}\}$ , we obtain the partition function

$$Z_2 = \frac{1}{|G|^{N_v}} \sum'_{\{h_{ij}\}} \prod_{[ij\dots kl]} [\omega(h_{ij}, \dots, h_{kl})]^{\sigma_{ij\dots kl}}, \quad (\text{C5})$$

where  $\frac{1}{|G|^{N_v}}$  is again a normalization factor, and  $\sum'$  is a summation over flat gauge configurations. The partition function  $Z_2$  describes the Dijkgraaf-Witten models. One can check that

if two cocycles  $\omega, \omega'$  differ by a coboundary, then they define the same Dijkgraaf-Witten models. Thus, the Dijkgraaf-Witten models are labeled by elements of  $H^{d+1}[G, U(1)]$ , just like the group cohomology models.

### 3. Connection between the two classes of models

We now show that the Dijkgraaf-Witten model with group  $G$  and cocycle  $\omega$  is equivalent to the group cohomology model of the same group  $G$  and cocycle  $\omega$  after the global symmetry of the latter is gauged.

To gauge the symmetry in the group cohomology model, we introduce lattice gauge fields  $h_{ij} \in G$  that live on the links  $[ij]$  of the triangulation. We then couple the matter degrees of freedom  $\{g_i\}$  and gauge degrees of freedom  $\{h_{ij}\}$  by replacing each  $g_i^{-1}g_j$  in the action (C1) by  $g_i^{-1}h_{ij}g_j$ , following the minimal coupling procedure. After this step, the action becomes

$$e^{-\tilde{S}_1(\{g_i\}, \{h_{ij}\})} = \prod_{[ij\dots kl]} [\omega(g_i^{-1}h_{ij}g_j, \dots, g_k^{-1}h_{kl}g_l)]^{\sigma_{ij\dots kl}}. \quad (\text{C6})$$

The next step is to choose a value for the gauge coupling constant. Here, as in Appendix A, we choose the gauge coupling constant to be 0. This means that we set  $e^{-\tilde{S}_1(\{g_i\}, \{h_{ij}\})} = 0$  if the gauge configuration is not flat. With this choice of coupling constant, the gauged partition function acquires the form

$$\tilde{Z}_1 = \frac{1}{|G|^{2N_v}} \sum_{\{g_i\}} \sum'_{\{h_{ij}\}} \prod_{[ij\dots kl]} [\omega(g_i^{-1}h_{ij}g_j, \dots, g_k^{-1}h_{kl}g_l)]^{\sigma_{ij\dots kl}}, \quad (\text{C7})$$

where the summation  $\sum'$  is taken only over flat gauge configurations, and where we have included a normalization factor  $\frac{1}{|G|^{2N_v}}$ . By construction, the partition function  $\tilde{Z}_1$  has a local gauge symmetry

$$g_i \rightarrow \alpha_i g_i, \quad h_{ij} \rightarrow \alpha_i h_{ij} \alpha_j^{-1} \quad (\text{C8})$$

for all  $\alpha_i \in G$  and all  $i, j$ .

To see the connection between the gauged group cohomology models  $\tilde{Z}_1$  and the Dijkgraaf-Witten models  $Z_2$ , we fix the gauge in  $\tilde{Z}_1$ . The gauge that we choose is

$$g_i = 1 \quad \text{for all } i. \quad (\text{C9})$$

In this gauge, the partition function  $\tilde{Z}_1$  becomes

$$\tilde{Z}_1 = \frac{1}{|G|^{N_v}} \sum'_{\{h_{ij}\}} \prod_{[ij\dots kl]} [\omega(h_{ij}, \dots, h_{kl})]^{\sigma_{ij\dots kl}}, \quad (\text{C10})$$

where the numerical factor  $|G|^{N_v}$  comes from performing the sum over  $\{g_i\}$ . We can see that  $\tilde{Z}_1$  is identical to  $Z_2$ , proving that the gauged group cohomology model with group  $G$  and cocycle  $\omega$  is equivalent to the Dijkgraaf-Witten model with the same  $G$  and  $\omega$ .

## APPENDIX D: PROPERTIES OF FUSION RULES IN ABELIAN DISCRETE GAUGE THEORIES

In this Appendix, we derive some properties of the fusion rules of excitations in 2D gauge theories with group  $G = \prod_i \mathbb{Z}_{N_i}$ .

The first property is that any excitation  $\gamma$  that appears in the fusion product  $\alpha \times \beta = \sum_\gamma N_{\alpha\beta}^\gamma \gamma$  must obey  $\phi_\gamma = \phi_\alpha + \phi_\beta$ . This property is clear from the following thought experiment. Imagine braiding an arbitrary charge  $q$  around  $\alpha$  and  $\beta$ . One can braid  $q$  around  $\alpha$  and  $\beta$  sequentially, which gives a phase  $q \cdot (\phi_\alpha + \phi_\beta)$ . Or, one can first fuse  $\alpha$  and  $\beta$  into some  $\gamma$ , then braid  $q$  around  $\gamma$ , leading to a phase  $q \cdot \phi_\gamma$ . Clearly, the two processes should give the same phase, so  $q \cdot (\phi_\alpha + \phi_\beta) = q \cdot \phi_\gamma$ . Since  $q$  is arbitrary, we have  $\phi_\alpha + \phi_\beta = \phi_\gamma$ .

Another property of the fusion rules is that when an excitation  $\alpha$  is fused with a charge  $q$ , there is exactly one fusion outcome:

$$q \times \alpha = \alpha', \quad (\text{D1})$$

where  $\alpha'$  might be the same as  $\alpha$ . To prove this property, we imagine four excitations  $\alpha, \bar{\alpha}$  and  $q, \bar{q}$ , where  $\bar{\alpha}$  denotes the antiparticle of  $\alpha$  and we suppose that the overall fusion channel for the four excitations is the vacuum. We now count the degeneracy of this four-excitation space in two different ways. First, we note that  $q, \bar{q}$  are Abelian particles, so they must fuse to the vacuum, which in turn forces  $\alpha, \bar{\alpha}$  to fuse to the vacuum. So, the four-excitation space is nondegenerate. On the other hand, we can also fuse the particles in a different order: we first fuse  $q$  with  $\alpha$  and fuse  $\bar{q}$  with  $\bar{\alpha}$ , then fuse the resulting particles. If we fuse the particles in this way, we can see that the degeneracy is given by the number of different fusion outcomes in  $q \times \alpha$ . We conclude that there is a unique fusion outcome in  $q \times \alpha$ . Therefore, (D1) holds.

The third property is that an excitation  $\alpha$  and its antiparticle  $\bar{\alpha}$  can only fuse to charges, i.e.,

$$\alpha \times \bar{\alpha} = \emptyset + q_1 + q_2 + \dots, \quad (\text{D2})$$

where  $\emptyset$  denotes the vacuum. Moreover, the coefficient of each  $q_i$  appearing in the fusion rule is 1. The first statement is easy to prove: all particles on the right side must be charges since  $\phi_\alpha + \phi_{\bar{\alpha}} = 0$ . To prove that the coefficient associated with each  $q_i$  is 1, we use the fact that  $N_{\alpha\bar{\alpha}}^\gamma = N_{\alpha\bar{\alpha}}^\beta$  from the general algebraic theory of anyons [34]. From this fact, we derive  $N_{\alpha\bar{\alpha}}^{q_i} = N_{\alpha\bar{\alpha}}^{q_j} = 0, 1$ , where the last equality follows from property (D1). As a corollary, we see that the charges  $q_i$  appearing in Eq. (D2) are exactly those that follow the fusion rule  $\alpha \times q_i = \alpha$ .

With the above properties, we now prove two claims. The first claim states that for any two excitations  $\alpha, \alpha'$  with  $\phi_{\alpha'} = \phi_\alpha$ , there exists at least one charge  $q$  with  $\alpha' = \alpha \times q$ . The second claim is more complicated. To explain it, consider two excitations  $\alpha, \beta$  in a fusion channel  $\gamma$ , and two other excitations  $\alpha', \beta'$  in a fusion channel  $\gamma'$ . The claim states if  $\phi_{\alpha'} = \phi_\alpha$  and  $\phi_{\beta'} = \phi_\beta$  then there exist charges  $q_1$  and  $q_2$  such that  $\alpha' = \alpha \times q_1$ ,  $\beta' = \beta \times q_2$ , and  $\gamma' = \gamma \times q_1 \times q_2$ .

To prove the first claim, consider an arbitrary excitation  $\alpha'$  with  $\phi_{\alpha'} = \phi_\alpha$ . Imagine fusing together  $\alpha, \bar{\alpha}, \alpha'$ . We can do the fusion in two different orders

$$(\alpha \times \bar{\alpha}) \times \alpha' = (\emptyset + q_1 + \dots) \times \alpha' \quad (\text{D3})$$

and

$$\alpha \times (\bar{\alpha} \times \alpha') = \alpha \times (q'_1 + q'_2 + \dots), \quad (\text{D4})$$

where the “...” means some charges. Since the order of fusion cannot affect the final result, we conclude that  $\alpha' = \alpha \times q'_i$ , where  $q'_i$  is one of the charges in Eq. (D4). Hence, the claim holds.

To prove the second claim, let  $\alpha, \beta, \alpha', \beta'$  be any excitations with  $\phi_{\alpha'} = \phi_\alpha$  and  $\phi_{\beta'} = \phi_\beta$ . Let  $\gamma$  be one of the fusion channels of  $\alpha, \beta$  and let  $\gamma'$  be one of the fusion channels of  $\alpha', \beta'$ . Our task is to show that, given any state in  $\mathbb{V}'_{\alpha\beta}$ , we can construct at least one state in  $\mathbb{V}'_{\alpha'\beta'}$  by fusing charges onto  $\alpha$  and  $\beta$ . To show this, we note that  $\gamma' = \gamma \times q$  for some charge  $q$ , by the first claim, proven above. Let us consider the fusion product  $(\alpha' \times \beta') \times (\bar{\alpha} \times \bar{\beta}) \times \bar{q}$ . The vacuum fusion channel must appear at least once in this fusion product since

$$(\alpha' \times \beta') \times (\bar{\alpha} \times \bar{\beta}) \times \bar{q} = (\gamma' + \dots) \times (\bar{\gamma} + \dots) \times \bar{q} \quad (\text{D5})$$

and  $\gamma' \times \bar{\gamma} \times \bar{q}$  contains the vacuum. We can now reorder the fusion product as

$$(\alpha' \times \bar{\alpha}) \times (\beta' \times \bar{\beta}) \times \bar{q}. \quad (\text{D6})$$

This reordering should not change the result so the vacuum must also appear in this product. We conclude  $q$  can be written as a product  $q = q_1 \times q_2$  where  $q_1$  appears in the fusion product of  $\alpha' \times \bar{\alpha}$  and  $q_2$  belongs to the fusion product of  $\beta' \times \bar{\beta}$ . Clearly,  $q_1$  and  $q_2$  satisfy  $q_1 \times \alpha = \alpha'$  and  $q_2 \times \beta = \beta'$ . Also, we know that  $q_1 \times q_2 \times \gamma = q \times \gamma = \gamma'$ . This proves the claim.

#### APPENDIX E: A PROPERTY OF $R_{\alpha\alpha}^\beta$

In this Appendix, we show that when an excitation is braided around another identical excitation in a 2D gauge theory with group  $G = \prod_{i=1}^K \mathbb{Z}_{N_i}$ , the resulting unitary transformation is a pure phase, i.e., proportional to the identity matrix. Note that this result only holds for a full braiding: the unitary transformation associated with an exchange of two identical excitations need not be a pure phase.

Consider an arbitrary excitation  $\alpha$  in a 2D gauge theory with group  $G = \prod_{i=1}^K \mathbb{Z}_{N_i}$ . In the first step, we show that the unitary transformation associated with braiding  $\alpha$  around its antiparticle  $\bar{\alpha}$  is a pure phase. To see this, note that according to Appendix D, the only fusion outcomes for  $\alpha$  and  $\bar{\alpha}$  are charges:  $\alpha \times \bar{\alpha} = \emptyset + q + \dots$ . For any  $q$  that appears in this fusion rule, we can use the formula (9) to derive

$$R_{\bar{\alpha}\alpha}^q R_{\alpha\bar{\alpha}}^q = e^{i2\pi(s_q - s_\alpha - s_{\bar{\alpha}})} = e^{-i4\pi s_\alpha}, \quad (\text{E1})$$

where we have used the facts  $\dim(\mathbb{V}'_{\alpha\bar{\alpha}}) = 1$ ,  $s_q = 0$ , and  $s_\alpha = s_{\bar{\alpha}}$ . Examining the above identity, we can see that the statistical phase associated with braiding  $\alpha$  around  $\bar{\alpha}$  is independent of the fusion channel  $q$ . Hence, braiding  $\alpha$  around  $\bar{\alpha}$  gives a pure phase.

Next, we show that all the charges  $q$  that appear in the fusion product  $\alpha \times \bar{\alpha}$  have vanishing braiding statistics with  $\alpha$ , i.e.,  $\theta_{\alpha q} = 0$ . To see this, imagine we have two excitations  $\alpha$  and  $\bar{\alpha}$  in the vacuum fusion channel. If we now fuse  $q$  to  $\bar{\alpha}$ , the excitation  $\bar{\alpha}$  will remain unchanged (i.e.,  $q \times \bar{\alpha} = \bar{\alpha}$ )

but after this fusion process, the two excitations  $\alpha$  and  $\bar{\alpha}$  will be in the fusion channel  $q$ . Let us imagine braiding  $\alpha$  around  $\bar{\alpha}$  before and after fusing  $q$  into  $\bar{\alpha}$ . Clearly, the two processes will differ by a phase factor  $e^{i\theta_{\alpha q}}$ . Hence,

$$R_{\bar{\alpha}\alpha}^q R_{\alpha\bar{\alpha}}^q = R_{\bar{\alpha}\alpha}^\beta R_{\alpha\bar{\alpha}}^\beta e^{i\theta_{\alpha q}}. \quad (\text{E2})$$

Then, since  $R_{\bar{\alpha}\alpha}^q R_{\alpha\bar{\alpha}}^q$  is independent of  $q$ , we derive  $\theta_{\alpha q} = 0$ .

To complete the argument, we consider a process in which  $\alpha$  is braided around a pair of  $\alpha$  and  $\bar{\alpha}$  excitations. Independent of the fusion channel of the  $\alpha$  and  $\bar{\alpha}$  excitations, the unitary transformation associated with this process must be the identity since  $\theta_{\alpha q} = 0$ . At the same time, this braiding process can be divided into two pieces: first  $\alpha$  is braided around  $\bar{\alpha}$ , and then around another  $\alpha$ . Since the first piece is a pure phase  $e^{-i4\pi s_\alpha}$ , the second must also be a pure phase. This proves the claim. In addition, we have derived the formula

$$R_{\alpha\alpha}^\beta R_{\alpha\alpha}^\beta = e^{i4\pi s_\alpha}, \quad (\text{E3})$$

where  $\beta$  is any excitation in the fusion product  $\alpha \times \alpha$ .

#### APPENDIX F: PROVING $\Theta_{ijk}$ IS WELL DEFINED

In this Appendix, we prove that  $\Theta_{ijk}$  is a well-defined quantity. More specifically, we show that (i) the unitary transformation associated with the braiding process defining  $\Theta_{ijk}$  is always an Abelian phase even if the vortices are non-Abelian; we also show that (ii) the Abelian phase is a function of  $i, j, k$  only and does not depend on the choice of vortices  $\alpha, \beta, \gamma$  as long as they carry fluxes  $\frac{2\pi}{N_i} e_i, \frac{2\pi}{N_j} e_j, \frac{2\pi}{N_k} e_k$ , respectively.

To prove points (i) and (ii), we make use of a diagrammatic technique to compute the unitary matrix associated with the braiding process in the definition of  $\Theta_{ijk}$  (for more details about this diagrammatic technique, see Ref. [34].) The technique uses space-time trajectories, where the arrow of time is drawn upward. We will not use the technique to carry out an actual calculation but only to show that the unitary transformation associated with  $\Theta_{ijk}$  is a pure phase. We will make use of two diagrammatic relations in our proof. The first relation is

$$\begin{array}{c} \alpha \\ \downarrow \\ \bar{\alpha} \\ \downarrow \\ \alpha \end{array} = u_\alpha \begin{array}{c} \alpha \\ \downarrow \\ \alpha \end{array} \quad (\text{F1})$$

This relation allows us to turn downward a trajectory of  $\alpha$  by introducing its antiparticle  $\bar{\alpha}$ , with a compensation of a complex factor  $u_\alpha$ . The factor  $u_\alpha$  is related to the quantum dimension  $d_\alpha$  by  $d_\alpha = |u_\alpha|^{-1}$ . The second relation is

$$\begin{array}{c} \alpha \\ \downarrow \\ \alpha \end{array} \begin{array}{c} \beta \\ \downarrow \\ \beta \end{array} = \sum_{\gamma, n} \begin{array}{c} \alpha \quad \beta \\ \diagdown \quad \diagup \\ \gamma \quad n \\ \diagup \quad \diagdown \\ \alpha \quad \beta \end{array} \quad (\text{F2})$$

which means that the propagation of two particles  $\alpha, \beta$  can be decomposed into a sum over their possible fusion channels,

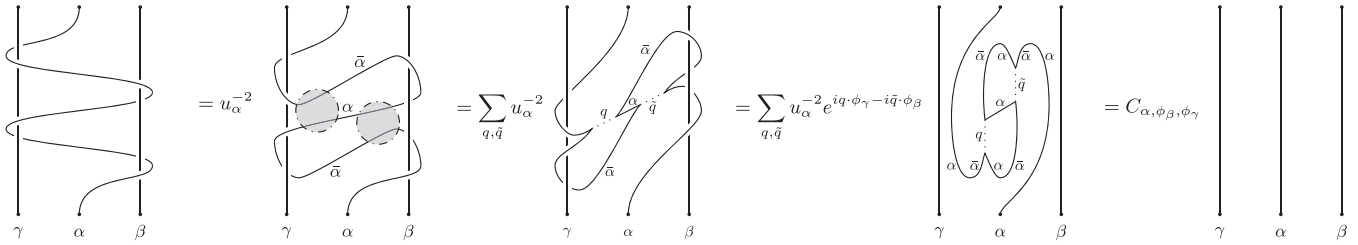
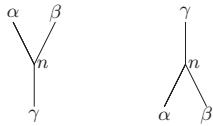


FIG. 8. Diagrammatic proof that the unitary matrix associated with the braiding process in the definition of  $\Theta_{ijk}$  is an Abelian phase.

where  $\gamma$  ranges over the fusion channels in  $\alpha \times \beta = \sum_{\gamma} N_{\alpha\beta}^{\gamma} \gamma$  and  $n$  ranges over  $1, \dots, N_{\alpha\beta}^{\gamma}$ . The vertices



mean the  $n$ th way of splitting and fusing  $\alpha, \beta$  into  $\gamma$ , respectively.

Now, let  $\alpha, \beta, \gamma$  be vortices carrying unit flux  $\frac{2\pi}{N_i} e_i, \frac{2\pi}{N_j} e_j, \frac{2\pi}{N_k} e_k$ , respectively. Consider the space-time trajectories of  $\alpha, \beta, \gamma$  associated with the braiding process in the definition of  $\Theta_{ijk}$  (Fig. 8). Making use of the relation (F1), we establish the first equation in Fig. 8. The second equation is established by using the relation (F2) in the two shaded regions in the second diagram in Fig. 8. The charges  $q, \tilde{q}$  are those appearing in the fusion rule of  $\alpha \times \bar{\alpha}$ . We have used the fact that  $N_{\alpha\bar{\alpha}}^q = 1$  for any  $q$ , which is proven in Appendix D. To establish the third equation, we notice that  $q, \tilde{q}$  have Abelian statistics with the vortices  $\beta, \gamma$ . Winding  $q$  around  $\gamma$  in the counterclockwise direction gives rise to a phase  $e^{iq \cdot \phi_{\gamma}}$ , and winding  $\tilde{q}$  around  $\beta$  in the clockwise direction gives rise to a phase  $e^{-i\tilde{q} \cdot \phi_{\beta}}$ . In the fourth diagram in Fig. 8, we see that  $\beta$  and  $\gamma$  are decoupled from  $\alpha$ , while  $\alpha$  still has some “self-interaction.” However complicated this “self-interaction” is, it only depends on  $\alpha, q, \tilde{q}$ , but not on  $\beta, \gamma$ . Therefore, we can denote everything as a complex number  $C_{\alpha, \phi_{\beta}, \phi_{\gamma}}$ , after the summation over  $q, \tilde{q}$  has been performed. We see that  $C_{\alpha, \phi_{\beta}, \phi_{\gamma}}$  only depends on the flux of  $\beta, \gamma$ , but not on the choice of  $\beta, \gamma$ , nor on their fusion channel. Since the overall transformation must be unitary,  $C_{\alpha, \phi_{\beta}, \phi_{\gamma}}$  is a pure phase.

To complete the argument, we have to show that the factor  $C_{\alpha, \phi_{\beta}, \phi_{\gamma}} = e^{i\Theta_{ijk}}$  only depends on the flux of  $\alpha$  but not on the choice of  $\alpha$ . This can be proven by fusing charges to  $\alpha$ . Let the outcome of the fusion be a vortex  $\alpha'$ . According to the Aharonov-Bohm law, one can easily see that

$$\begin{aligned} C_{\alpha', \phi_{\beta}, \phi_{\gamma}} &= C_{\alpha, \phi_{\beta}, \phi_{\gamma}} e^{iq \cdot \phi_{\beta} + i\tilde{q} \cdot \phi_{\gamma} - i\tilde{q} \cdot \phi_{\beta} - iq \cdot \phi_{\gamma}} \\ &= C_{\alpha, \phi_{\beta}, \phi_{\gamma}}. \end{aligned} \quad (\text{F3})$$

With this, we have shown that  $\Theta_{ijk}$  only depends on the flux of  $\alpha, \beta, \gamma$ , i.e., it only depends on  $i, j, k$ . Hence,  $\Theta_{ijk}$  is well defined.

## APPENDIX G: CORRESPONDENCE BETWEEN LABELS $(a, \rho)$ AND PHYSICAL NOTIONS OF GAUGE FLUX AND GAUGE CHARGE

In this Appendix, we show how to translate between the mathematical labels  $\alpha = (a, \rho)$ , used to denote excitations in 2D Dijkgraaf-Witten models, and the physical notions of gauge flux and gauge charge. As discussed in the main text, the basic outline of correspondence is simple: the first component  $a$  describes the amount of flux carried by the excitation  $\alpha$ , while the second component  $\rho$  is related to the amount of charge attached to  $\alpha$ . We now explain how this works in more detail.

For each group element  $a = (a_1, \dots, a_K)$ , the corresponding gauge flux is given by  $\phi = (\phi_1, \dots, \phi_K)$  where  $\phi_i = \frac{2\pi}{N_i} a_i$ . Likewise, for each representation  $\rho$  we should define a corresponding gauge charge  $q = (q_1, \dots, q_K)$ . This correspondence is easy to define for the case where  $\alpha$  is a pure charge excitation:  $\alpha = (0, \rho)$ . Indeed in this case, Eq. (22) implies that  $\rho$  is a *linear* representation of  $G$ , provided that we choose a “gauge” [53] such that  $\omega(a, b, c) = 1$  if any of  $a, b, c$  is 0. It follows that  $\rho$  can be written in the form  $\rho(h) = \exp(\sum_k \frac{2\pi i}{N_k} q_k h_k)$  for some integer vector  $(q_1, \dots, q_K)$ . This defines the desired correspondence  $\rho \leftrightarrow q = (q_1, \dots, q_K)$ .

How does the correspondence work for vortex excitations  $\alpha = (a, \rho)$  where  $a \neq 0$ ? In this case,  $\rho$  is a *projective* representation, so there is no natural way to translate  $\rho$  into an integer vector  $(q_1, \dots, q_K)$ . This is related to the general point made in Sec. III A1: we do not know a physically meaningful way to define the absolute charge carried by a vortex excitation. On the other hand, if we *compare* two vortex excitations with the same flux,  $\alpha = (a, \rho)$  and  $\alpha' = (a, \rho')$ , and we find that  $\rho, \rho'$  are related by  $\rho'(h) = \rho(h) \exp(\sum_k \frac{2\pi i}{N_k} q_k h_k)$  for some  $(q_1, \dots, q_k)$ , then we can say that  $\alpha'$  can be obtained from  $\alpha$  by attaching charge  $q = (q_1, \dots, q_K)$ .

## APPENDIX H: SHOWING THAT ALL SOLUTIONS TO THE CONSTRAINTS (63) CAN BE WRITTEN IN THE FORMS (49a) AND (49b)

In this section, we show that if  $\Theta_{i,l}$  and  $\Theta_{j,l}$  obey the constraints (63a)–(63g), then they can be written in the forms (49a) and (49b), i.e.,

$$\begin{aligned} \Theta_{i,l} &= \frac{2\pi}{N_{il}} (M_{ili} - M_{lji}), \\ \Theta_{ij,l} &= \frac{2\pi N^{ij}}{N_{il} N_j} (M_{ilj} - M_{lij}) + \frac{2\pi N^{ij}}{N_{jl} N_i} (M_{jli} - M_{lji}) \quad (\text{H1}) \end{aligned}$$

for some integer tensor  $M_{ijk}$ . The first step is to rewrite the second equation as

$$\Theta_{ij,l} = \frac{2\pi N^{ijl}}{N^{jl}N_{ijl}}(M_{ilj} - M_{lij}) + \frac{2\pi N^{ijl}}{N^{il}N_{ijl}}(M_{jli} - M_{lji}). \quad (\text{H2})$$

This new form of the second equation can be derived from the relations

$$\frac{N^{ij}}{N_{il}N_j} = \frac{N^{ijl}}{N^{jl}N_{ijl}}, \quad \frac{N^{ij}}{N_{jl}N_i} = \frac{N^{ijl}}{N^{il}N_{ijl}}, \quad (\text{H3})$$

which in turn follow from the identities

$$N^{ij} = \frac{N_i N_j}{N_{ij}}, \quad N^{ijl} = \frac{N^{ij} N^{il} N^{jl} N_{ijl}}{N_i N_j N_l}. \quad (\text{H4})$$

We now construct the integer tensor  $M_{ijk}$ . First, we set  $M_{iii} = 0$  for all  $i$ , and we set

$$M_{lii} = M_{iil} = 0, \quad M_{ili} = \frac{N_{il}}{2\pi} \Theta_{i,l} \quad (\text{H5})$$

for all  $i \neq l$ . Next, for each  $i < j < l$ , we define

$$\begin{aligned} M_{ilj} &= -b \frac{N_{ijl}}{2\pi} \Theta_{ijl}, \\ M_{jil} &= \frac{N_{ijl}}{2\pi} (a \Theta_{jli} + b \Theta_{lij}), \\ M_{lji} &= -a \frac{N_{ijl}}{2\pi} \Theta_{ijl}, \\ M_{lij} &= 0, \quad M_{jli} = 0, \quad M_{ijl} = 0, \end{aligned} \quad (\text{H6})$$

where  $a$  and  $b$  are integers such that

$$a \frac{N^{ijl}}{N^{il}} - b \frac{N^{ijl}}{N^{jl}} = 1. \quad (\text{H7})$$

(The existence of  $a, b$  will be established in the following.) If one substitutes the tensor  $M_{ijk}$  into (H1) and (H2), it is straightforward to check that the resulting expressions exactly reproduce the invariants  $\Theta_{i,l}$  and  $\Theta_{ij,l}$  as long as these invariants obey the constraints (63).

At this point, we have successfully constructed a tensor  $M_{ijk}$  that satisfies equations (H1) and (H2). However, there are two gaps in our derivation that need to be addressed. First, we have to show that the components of  $M_{ijk}$  are all integers. Second, we have to show that we can always find integers  $a, b$  satisfying (H7). The fact that the components of  $M_{ijk}$  are all integers is easy to prove, as it follows immediately from the two constraints (63d) and (63c). As for the second statement, this will follow if we can show that  $N^{ijl}/N^{il}$  and  $N^{ijl}/N^{jl}$  are relatively prime. The latter property can be derived from a simple observation: we note that the only prime factors appearing in  $N^{ijl}/N^{il}$  are those that divide into  $j$  more times than either  $i$  or  $l$ . Similarly, the only prime factors appearing in  $N^{ijl}/N^{jl}$  are those that divide into  $i$  more times than  $j$  or  $l$ . We conclude that these two numbers do not share any prime factors so they are relatively prime.

## APPENDIX I: COUNTING THE NUMBER OF VALUES OF $\Theta_{i,l}, \Theta_{ij,l}, \Theta_{ijk,l}$

In this section, we consider the formulas (49a)–(49c), reprinted below for convenience:

$$\Theta_{i,l} = \frac{2\pi}{N_{il}}(M_{ili} - M_{lii}), \quad (\text{I1a})$$

$$\Theta_{ij,l} = \frac{2\pi N^{ijl}}{N^{jl}N_{ijl}}(M_{ilj} - M_{lij}) + \frac{2\pi N^{ijl}}{N_{il}N_{ijl}}(M_{jli} - M_{lji}), \quad (\text{I1b})$$

$$\Theta_{ijk,l} = \frac{2\pi}{N_{ijkl}} \sum_{\hat{p}} \text{sgn}(\hat{p}) L_{\hat{p}(i)\hat{p}(j)\hat{p}(k)\hat{p}(l)}. \quad (\text{I1c})$$

What we will show is that invariants  $\Theta_{i,l}, \Theta_{ij,l}, \Theta_{ijk,l}$  take on at least

$$\prod_{i < j} (N_{ij})^2 \prod_{i < j < l} (N_{ijl})^2 \prod_{i < j < k < l} N_{ijkl} \quad (\text{I2})$$

different values when  $M_{ijk}$  ranges over all integer tensors, and  $L_{ijkl}$  ranges over all integer tensors obeying  $L_{ijkl} = 0$  if  $i, j, k$  are not all distinct. [The reader may notice that the second equation (I1b) differs slightly from Eq. (49b) in the main text. The equivalence between these two equations follows from simple identities and is explained in Appendix H.]

To perform our counting, we consider separately each of the components of  $\Theta_{i,l}, \Theta_{ij,l}$ . First, we consider the invariant  $\Theta_{i,l}$  for  $i \neq l$ . From Eq. (I1a) we can see that  $\Theta_{i,l}$  can take on  $N_{il}$  different values as we vary  $M_{ili}, M_{lii}$ . We can also see that the different  $\Theta_{i,l}$  invariants with  $i \neq l$  are independent of one another. Hence, all together the  $\Theta_{i,l}$  invariants can take on at least

$$\mathcal{N}_1 = \prod_{i \neq l} N_{il} = \prod_{i < l} (N_{il})^2 \quad (\text{I3})$$

different values. Next, we fix  $i < j < l$  and consider the three invariants  $\Theta_{ij,l}, \Theta_{jl,i}, \Theta_{li,j}$ . We will show that these invariants can take on at least  $(N_{ijl})^2$  different values. To see this, we set

$$\begin{aligned} M_{ilj} &= -bx, \quad M_{lji} = -ax, \quad M_{jil} = y, \\ M_{lij} &= 0, \quad M_{jli} = 0, \quad M_{ijl} = 0, \end{aligned} \quad (\text{I4})$$

where  $x, y$  are arbitrary integers and  $a$  and  $b$  are chosen so that

$$a \frac{N^{ijl}}{N^{il}} - b \frac{N^{ijl}}{N^{jl}} = 1.$$

(One can always find such  $a, b$  since  $\frac{N^{ijl}}{N^{il}}$  and  $\frac{N^{ijl}}{N^{jl}}$  are relatively prime, as explained at the end of Appendix H.) Substituting these expressions for  $M$  into (I1b), it is straightforward to derive the following two formulas:

$$\begin{aligned} \Theta_{ij,l} &= 2\pi x / N_{ijl}, \\ a \Theta_{jli} + b \Theta_{lij} &= 2\pi y / N_{ijl}. \end{aligned} \quad (\text{I5})$$

From these formulas it is clear that  $\Theta_{ij,l}, \Theta_{jl,i}, \Theta_{li,j}$  take on different values for each  $x, y$  with  $0 \leq x, y \leq N_{ijl} - 1$ . Hence,  $\Theta_{ij,l}, \Theta_{jl,i}, \Theta_{li,j}$  can take on at least  $N_{ijl}^2$  different values, as claimed above. It is also clear that the values for different  $i, j, l$



are independent of one another. Hence, all together the  $\Theta_{ij,l}$  invariants can take on at least

$$\mathcal{N}_2 = \prod_{i < j < l} (N_{ijl})^2 \quad (16)$$

different values. Finally, we consider the invariants  $\Theta_{ijk,l}$  for  $i < j < k < l$ . From Eq. (11c), it is clear that  $\Theta_{ijk,l}$  can take on  $N_{ijkl}$  different values as we vary  $L_{ijkl}$ . Hence, all together,

the number of distinct values of  $\{\Theta_{ijk,l}\}$  is at least

$$\mathcal{N}_3 = \prod_{i < j < k < l} N_{ijkl}. \quad (17)$$

Combining all the cases, we conclude that the total number of values that the invariants can take on is at least  $\mathcal{N}_1 \mathcal{N}_2 \mathcal{N}_3$ , which is what we wanted to show. (In fact, with more careful accounting, one can show that this inequality is actually an equality, but we will not need this sharper result here.)

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- [1] M. Z. Hasan and C. L. Kane, *Rev. Mod. Phys.* **82**, 3045 (2010).  
 [2] X.-L. Qi and S.-C. Zhang, *Rev. Mod. Phys.* **83**, 1057 (2011).  
 [3] Z.-C. Gu and X.-G. Wen, *Phys. Rev. B* **80**, 155131 (2009).  
 [4] F. Pollmann, A. M. Turner, E. Berg, and M. Oshikawa, *Phys. Rev. B* **81**, 064439 (2010).  
 [5] L. Fidkowski and A. Kitaev, *Phys. Rev. B* **83**, 075103 (2011).  
 [6] X. Chen, Z.-C. Gu, and X.-G. Wen, *Phys. Rev. B* **83**, 035107 (2011).  
 [7] X. Chen, Z.-C. Gu, and X.-G. Wen, *Phys. Rev. B* **84**, 235128 (2011).  
 [8] N. Schuch, D. Perez-Garcia, and I. Cirac, *Phys. Rev. B* **84**, 165139 (2011).  
 [9] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, *Science* **338**, 1604 (2012).  
 [10] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, *Phys. Rev. B* **87**, 155114 (2013).  
 [11] F. Verstraete, J. I. Cirac, J. I. Latorre, E. Rico, and M. M. Wolf, *Phys. Rev. Lett.* **94**, 140601 (2005).  
 [12] G. Vidal, *Phys. Rev. Lett.* **99**, 220405 (2007).  
 [13] X. Chen, Z.-C. Gu, and X.-G. Wen, *Phys. Rev. B* **82**, 155138 (2010).  
 [14] X. Chen, Z.-X. Liu, and X.-G. Wen, *Phys. Rev. B* **84**, 235141 (2011).  
 [15] M. Levin and Z.-C. Gu, *Phys. Rev. B* **86**, 115109 (2012).  
 [16] Y.-M. Lu and A. Vishwanath, *Phys. Rev. B* **86**, 125119 (2012).  
 [17] A. Vishwanath and T. Senthil, *Phys. Rev. X* **3**, 011016 (2013).  
 [18] A. Kapustin, [arXiv:1403.1467](https://arxiv.org/abs/1403.1467).  
 [19] X.-G. Wen, [arXiv:1410.8477](https://arxiv.org/abs/1410.8477).  
 [20] O. M. Sule, X. Chen, and S. Ryu, *Phys. Rev. B* **88**, 075125 (2013).  
 [21] X.-G. Wen, *Phys. Rev. B* **89**, 035147 (2014).  
 [22] L.-Y. Hung and X.-G. Wen, *Phys. Rev. B* **89**, 075121 (2014).  
 [23] M. Cheng and Z.-C. Gu, *Phys. Rev. Lett.* **112**, 141602 (2014).  
 [24] C. Wang and M. Levin, *Phys. Rev. Lett.* **113**, 080403 (2014).  
 [25] S. Jiang, A. Mesaros, and Y. Ran, *Phys. Rev. X* **4**, 031048 (2014).  
 [26] J. C. Wang and X.-G. Wen, *Phys. Rev. B* **91**, 035134 (2015).  
 [27] C. Xu, *Phys. Rev. B* **87**, 144421 (2013).  
 [28] Z. Bi, A. Rasmussen, K. Slagle, and C. Xu, *Phys. Rev. B* **91**, 134404 (2015).  
 [29] P. Ye and Z.-C. Gu, [arXiv:1410.2594](https://arxiv.org/abs/1410.2594).  
 [30] J. C. Wang, Z.-C. Gu, and X.-G. Wen, *Phys. Rev. Lett.* **114**, 031601 (2015).  
 [31] M. Barkeshli, P. Bonderson, M. Cheng, and Z. Wang, [arXiv:1410.4540](https://arxiv.org/abs/1410.4540).  
 [32] M. P. Zaletel, *Phys. Rev. B* **90**, 235113 (2014).  
 [33] We explain the logic behind this claim in Sec. II C.  
 [34] A. Kitaev, *Ann. Phys. (NY)* **321**, 2 (2006).  
 [35] R. Dijkgraaf and E. Witten, *Commun. Math. Phys.* **129**, 393 (1990).  
 [36] J. B. Kogut, *Rev. Mod. Phys.* **51**, 659 (1979).  
 [37] K. S. Brown, *Cohomology of Groups* (Springer, New York, 1982).  
 [38] J. Preskill, Lecture notes: topological quantum computation, <http://www.theory.caltech.edu/~preskill/ph219/topological.pdf>.  
 [39] M. de Wild Propitius, Ph.D. thesis, available at [arXiv:hep-th/9511195](https://arxiv.org/abs/hep-th/9511195).  
 [40] Aside from simple loops such as circles or linked circles, loops can be knotted or even branched. In this paper, we only consider simple loops without knotting or branching.  
 [41] C. Aneziris, A. P. Balachandran, L. Kauffman, and A. M. Srivastava, *Int. J. Mod. Phys. A* **6**, 2519 (1991).  
 [42] M. G. Alford, K.-M. Lee, J. March-Russell, and J. Preskill, *Nucl. Phys. B* **384**, 251 (1992).  
 [43] J. C. Baez, D. K. Wise, and A. S. Crans, *Adv. Theor. Math. Phys.* **11**, 707 (2007).  
 [44] C.-M. Jian and X.-L. Qi, *Phys. Rev. X* **4**, 041043 (2014).  
 [45] Z. Bi, Y.-Z. You, and C. Xu, *Phys. Rev. B* **90**, 081110 (2014).  
 [46] Note that this correspondence between loop and particle braiding only holds if the loops are linked to a base loop  $\gamma$ ; the braid group for unlinked loops is different from the 2D case, as discussed in Ref. [43].  
 [47] X. Chen, Y.-M. Lu, and A. Vishwanath, *Nat. Commun.* **5**, 3507 (2014).  
 [48] R. Dijkgraaf, V. Pasquier, and P. Roche, *Nucl. Phys. B* **18**, 60 (1991).  
 [49] It can be shown that the Dijkgraaf-Witten model corresponding to the cocycle  $\omega$  (48) is Abelian if  $L_{ijkl} = 0$ .  
 [50] Z.-C. Gu and M. Levin, *Phys. Rev. B* **89**, 201113 (2014).  
 [51] Y. Hu, Y. Wan, and Y.-S. Wu, *Phys. Rev. B* **87**, 125114 (2013).  
 [52] Y. Wan, J. Wang, and H. He, [arXiv:1409.3216](https://arxiv.org/abs/1409.3216).  
 [53] It is not hard to see that for any cohomology class in  $H^3[G, U(1)]$ , we can always find a representative cocycle  $\omega$  with the property that  $\omega(a, b, c) = 1$  if any of  $a, b, c$  is 0.