Low-energy effective theory in the bulk for transport in a topological phase

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We construct a low-energy effective action for a two-dimensional nonrelativistic topological (i.e., gapped) phase of matter in a continuum, which completely describes all of its bulk electrical, thermal, and stress-related properties in the limit of low frequencies, long distances, and zero temperature, without assuming either Lorentz or Galilean invariance. This is done by generalizing Luttinger's approach to thermoelectric phenomena, via the introduction of a background vielbein (i.e., gravitational) field and spin connection a la Cartan, in addition to the electromagnetic vector potential, in the action for the microscopic degrees of freedom (the matter fields). Crucially, the geometry of spacetime is allowed to have timelike and spacelike torsion. These background fields make all natural invariances—under U(1) gauge transformations, translations in both space and time, and spatial rotations—appear locally, and corresponding conserved currents and the stress tensor can be obtained, which obey natural continuity equations. On integrating out the matter fields, we derive the most general form of a local bulk induced action to first order in derivatives of the background fields, from which thermodynamic and transport properties can be obtained. We show that the gapped bulk cannot contribute to low-temperature thermoelectric transport other than the ordinary Hall conductivity; the other thermoelectric effects (if they occur) are thus purely edge effects. The coupling to "reduced" spacelike torsion is found to be absent in minimally coupled models, and, using a generalized Belinfante stress tensor, the stress response to time-dependent vielbeins (i.e., strains) is the Hall viscosity, which is robust against perturbations and related to the spin current, as in earlier work.

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I. INTRODUCTION

There has been great interest recently in the thermal Hall conductivity of gapped topological phases at temperatures small compared with the bulk energy gap. It has been known for some time, using arguments based on the existence of a gapless edge, that the thermal Hall conductivity κ^H of such systems is given by [1,2]

$$\kappa^H = \frac{\pi T}{6} c, \tag{1.1}$$

where T is the temperature and $c = c_L - c_R$ is the (topological) central charge of the edge theory. It has been hoped that a calculation of κ^H could be carried out which would illustrate the appearance of the central charge from bulk correlation functions. On the other hand, it has been pointed out [2] that the central charge appears as the coefficient of the gravitational Chern-Simons term

$$S_{\rm GCS} = \frac{c}{96\pi} \int d^3x\, \widehat{\epsilon}^{\mu\nu\lambda} \left(\Gamma^\rho_{\mu\sigma} \partial_\nu \Gamma^\sigma_{\nu\rho} + \frac{2}{3} \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\theta} \Gamma^\theta_{\lambda\rho} \right), \tag{1.2}$$

in terms of the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$. This term, however, is of too high an order in derivatives of the metric to describe thermal conductivity directly [3], but is nonetheless connected with the central charge of the edge states. It manifests itself in the bulk rather through the response of the energy-momentum-stress tensor to gradients in curvature.

In this paper we show that the long-wavelength *bulk* thermal transport properties are completely independent of the central charge, and that the only nonvanishing bulk thermoelectric current is the ordinary Hall current (we neglect effects that vanish exponentially in the energy gap over temperature as the temperature goes to zero). We do this by constructing the most general low-energy effective action for the bulk at the correct order in derivatives of external vielbein (i.e., gravitational) and electromagnetic fields. We show that responses to certain gradients of the vielbeins correspond to thermal and thermoelectric response functions. After using general thermodynamic arguments to identify terms in the effective action, we show that the bulk thermal currents in response to gradients of the vielbeins yield purely *magnetization* currents, that is, currents that vanish when integrated along any cross section of the sample. Thus, we show that the bulk contribution to the thermal Hall conductivity is exponentially suppressed due to the gap.

Our formalism treats arbitrary background geometries for a nonrelativistic system that has neither Lorentz nor Galilean invariance. This unified approach allows us also to consider the stress response to background fields, and thus viscosity, on the same footing as the thermoelectric effects and to account for all bulk magnetization effects: number, energy, and momentum magnetizations. When we use the formalism to study Hall viscosity of a topological phase, we find that use of the appropriate Belinfante stress tensor, while not affecting the results for thermoelectric coefficients, has the effect of removing the contribution of spacelike torsion to the Hall viscosity that had been found in a relativistic setting [4,5]. We also point out that for simple nonrelativistic models in which the background fields are minimally coupled, there is no coupling to spacelike torsion in the limit of a trivial spacetime without torsion. Instead, the Hall viscosity and the spin current follow [6,7] purely from the Wen-Zee term [8] in the effective action and are related as in previous work [9,10].

Our analysis of thermoelectric transport will make contact with the formalism developed by Cooper, Halperin, and Ruzin (CHR) [11], so we now recapitulate their main points. They

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consider the number current $\mathbf{J}$ and $\mathbf{J}_{\mathrm{E}}$ in the presence of an applied electric field $\mathbf{E} = -\nabla\phi$ and a fictitious gravitational field ψ coupled to the energy density [12]. By considering linear response to these fields in the bulk via the Kubo formula, they obtain a set of zero frequency and zero wave-vector response functions $L_{ij}^{(n)}$, such that the changes in number and energy current density can be expressed as

$$\begin{aligned}\delta J^i &= -L_{ij}^{(1)}\partial_j\phi - L_{ij}^{(2)}\partial_j\psi,\\ \delta J_{\mathrm{E}}^i &= -L_{ij}^{(3)}\partial_j\phi - L_{ij}^{(4)}\partial_j\psi.\end{aligned} \tag{1.3}$$

(Here $i,\ j = 1,\ 2$ are the space coordinate indices, the summation convention is in effect for these indices, and we also use $\epsilon_{ij} = -\epsilon_{ji}$, with $\epsilon_{12} = 1$.) They note, however, that in the presence of a background magnetic field B (perpendicular to the plane) there exist magnetization number and energy currents, which if the bulk is translation invariant appear only as edge currents. These are given (in our notation for two space dimensions) by

$$\begin{aligned}J_{\mathrm{mag}}^i &= \epsilon_{ij}\partial_j m,\\ J_{\mathrm{E,mag}}^i &= \epsilon_{ij}\partial_j m^{\mathrm{E}},\end{aligned} \tag{1.4}$$

where m is the ordinary magnetization density and m^{E} is a suitably defined "energy magnetization" density. In the presence of the fields ϕ and ψ the magnetizations differ from their unperturbed values m_0 and m_0^{E} by

$$\begin{aligned}m &= (1+\psi)m_0,\\ m^{\mathrm{E}} &= (1+2\psi)m_0^{\mathrm{E}} + \phi m_0.\end{aligned} \tag{1.5}$$

The magnetization currents induced by the external fields must be accounted for in order to obtain the transport current densities $\mathbf{J}_{\mathrm{tr}}$ and $\mathbf{J}_{\mathrm{tr,E}}$; the transport current densities, by definition, give the net current across a section when integrated along it and are defined to occur solely in the bulk. They find

$$J_{\mathrm{tr}}^i = -L_{ij}^{(1)}\partial_j\phi - \left(L_{ij}^{(2)} + m_0\epsilon_{ij}\right)\partial_j\psi, \tag{1.6}$$

$$J_{\mathrm{E,tr}}^i = -\left(L_{ij}^{(3)} + m_0\epsilon_{ij}\right)\partial_j\phi - \left(L_{ij}^{(4)} + 2m_0^{\mathrm{E}}\epsilon_{ij}\right)\partial_j\psi. \tag{1.7}$$

Then CHR used generalized Einstein relations, which say that with chemical potential μ and nonzero temperature T (both of which can be position dependent since the system is not in equilibrium) the transport currents are responses only to the combinations $\nabla\psi + (1/T)\nabla T$ and $\nabla\phi + T\nabla(\mu/T)$. Finally, setting $\psi = 0$ and defining $\xi = \phi + \mu$ and the transport heat current density $\mathbf{J}_{\mathrm{Q,tr}} = \mathbf{J}_E - \xi\mathbf{J}_{\mathrm{tr}}$, CHR showed that

$$J_{\mathrm{tr}}^i = -N_{ij}^{(1)}\partial_j\xi - \frac{1}{T}N_{ij}^{(2)}\partial_j T, \tag{1.8}$$

$$J_{\mathrm{Q,tr}}^i = -N_{ij}^{(3)}\partial_j\xi - \frac{1}{T}N_{ij}^{(4)}\partial_j T, \tag{1.9}$$

with

$$N_{ij}^{(1)} = L_{ij}^{(1)}, \tag{1.10}$$

$$N_{ij}^{(2)} = L_{ij}^{(2)} - \mu L_{ij}^{(1)} + m_0\epsilon_{ij}, \tag{1.11}$$

$$N_{ij}^{(3)} = L_{ij}^{(3)} - \mu L_{ij}^{(1)} + m_0\epsilon_{ij}, \tag{1.12}$$

$$N_{ij}^{(4)} = L^{(4)} - \mu\left(N_{ij}^{(2)} + N_{ij}^{(3)}\right) - \mu^2 L_{ij}^{(1)} + 2m_0^{\mathrm{E}}\epsilon_{ij}. \tag{1.13}$$

Here the coefficient matrices N obey the Onsager relations, for example, that $N_{ij}^{(2)}(B) = N_{ji}^{(3)}(-B)$, (as do the matrices L), whereas the local current responses to $\nabla\mu$ and ∇T do not. We see also that $N_{ij}^{(2)}$ and $N_{ij}^{(4)}$ must vanish faster than T as $T \to 0$, because the corresponding conductivities must vanish in that limit. In what follows, we show how the bulk contributions to these coefficients appear in and can be determined from the low-energy effective action for the bulk of a gapped system. It will follow that for such gapped systems, among these coefficients only $N^{(1)}$ (the Hall conductivity) receives a bulk contribution. The appearance of the central charge in $N^{(4)}$ is due to an edge effect.

We should explain the type of systems to which our formalism applies. We assume that the system is gapped in the bulk, so when we integrate out the matter fields only integrals of local expressions can occur in this "induced" action in the bulk. We assume this action depends only on the background electromagnetic field and on the spacetime geometry, and that it has symmetries under U(1) gauge transformations (because particle number is conserved), coordinate transformations (from translation invariance in both space and time in a flat background, leading to conservation of energy and momentum), and spatial rotations. Thus, we assume that these symmetries are not broken either spontaneously or explicitly. If either occurred, it would be necessary to include further background fields in the induced action that describe the breaking, and for spontaneously broken continuous symmetry in a system with short-range interactions there would be gapless degrees of freedom, so that the induced action is not local. Hence, our approach applies to quantum Hall systems and to insulators (including topological insulators) in a continuum approximation with rotation invariance, but not to fluids or (possibly topological) superconductors. In a model for a superconductor in which particle number is conserved, either (in the case of short-range interactions) it has a gapless Goldstone mode, or with a long-range interaction it can be fully gapped, but then the long range of interaction produces additional problems for us. Without conserved particle number we could simply drop the U(1) gauge field everywhere, but such paired states of fermions in which the pairs have nonzero angular momentum also break rotation symmetry and require a different treatment that will not be given here.

In Sec. II, we explain the geometry to be used and develop our microscopic model for the deformed system, deriving explicit expressions and conservation laws for the currents. After discussing in Sec. III some general facts about different terms in an induced action, we then write in Sec. IV the most general effective (induced) action to linear order in derivatives of the perturbing fields and consistent with the symmetries of the microscopic model. These results allow us to identify number, energy, and momentum magnetizations. Then in Sec. V we turn to linear response. In Sec. V B we calculate the response of the number and heat currents to an electric field and to Luttinger's gravitational field and show explicitly that the bulk contributions to the thermoelectric transport currents vanish. In Sec. V C, we address the stress response. We go over to a generalized Belinfante definition of the energy-momentum-stress tensor, which is described in Appendix A. This eliminates contributions to the Hall viscosity from locally invariant terms in the bulk (the nonrelativistic version of the "torsional Hall viscosity"[5] is one such effect). We observe that in simple models, there is no momentum magnetization. Finally we show that the Wen-Zee term produces the Hall viscosity in agreement with the spin current, in line with previous results.

There seems to be some confusion in the literature about whether the thermal Hall conductivity comes from the bulk. For free fermion systems, one can use linear response theory to derive the thermal conductivity [13–16] in a way that seemingly makes no reference to the edge physics. In Appendix B we review the calculation of the thermoelectric response coefficients for a noninteracting integer quantum Hall system. The key point is that such approaches calculate the response of current densities integrated across sections of a sample. Such integrated currents implicitly contain contributions from edge physics. The calculation thus essentially reduces to the use of the same edge argument to which we already referred [1,2].

II. BACKGROUND FIELDS AND MATTER ACTION

A. Spacetime geometry

Before we begin, let us establish some notational conventions. We work in $d+1$ spacetime dimensions throughout, with coordinates x^μ. We need to distinguish between two different types of indices: ambient spacetime indices, denoted by $\mu,\nu = 0,1,2,\ldots,d$, and similar indices, denoted by $\alpha,\beta = 0,1,2,\ldots,d$ in a flat internal spacetime. When we refer to spacelike directions only, we will use roman letters, $i,j = 1,2,\ldots$ for the ambient indices, and $a,b = 1,2,\ldots$ for the internal indices. We use the summation convention for all four types of indices, adhering from this point on to the conventions of placement of upper and lower indices, and $\partial_\mu = \partial/\partial x^\mu$. In this initial discussion, we keep d general, but later we specialize to $d = 2$.

The geometry of the spacetime that we use does not possess the metric structure of Minkowski spacetime or even of a Galilean analog. The only structure is that at any point we can distinguish between space and time, as if there were a local absolute time coordinate and a local positive-definite spatial metric. These statements do *not* mean that an absolute time coordinate can be defined, even on a small region of spacetime, so neither are there spacelike surfaces of fixed time. (These structures are similar to those used by Wen and Zee [8]; however, they assumed that the spacetime has a global absolute time and that the spatial metric of fixed-time slices was time independent.) In order to precisely define these structures, we prefer to be able to use arbitrary coordinate systems, and to be able to make arbitrary coordinate transformations (diffeomorphisms). The spacetime structures can be introduced using Cartan's vielbein formalism [17] (also called the vierbein or tetrad formalism in the case of $d = 3$ or $3+1$ dimensions). At each point we have a set of one forms with components e^α_μ and a dual set of vector fields e^μ_α that obey the duality relations $e^\mu_\alpha e^\beta_\mu = \delta^\beta_\alpha$ and $e^\mu_\alpha e^\alpha_\nu = \delta^\nu_\mu$. Either set defines a frame at each point in spacetime, that is a preferred basis set of one forms (or the dual set of vectors) indexed by α; these define the structure in a coordinate-independent way. The frames are assumed not to degenerate at any spacetime point; that is, the set of vectors is linearly independent at each x. Actually, the choice of basis (in the internal space) for the spacelike one forms e^a_μ (or for the spacelike vectors e^μ_a) is arbitrary up to a rotation on the internal indices; we incorporate that fact in due course. The vielbeins and their inverses can be used (by contraction) to convert ambient to internal spacetime indices or vice versa.

In particular, we have a one form with components e^0_μ, where the upper index is internal. If there existed a function t of position over regions of spacetime, such that (using the notation of differential forms) $e^0_\mu dx^\mu = dt$, then t would be absolute time, but we do *not* assume this. In order to obtain such an absolute time t, the necessary and sufficient condition is $\partial_\nu e^0_\mu - \partial_\mu e^0_\nu = 0$; in general, we do *not* impose this. We can use the one form to measure amounts of time using a squared line element for the time direction (or a particular degenerate or "partial" metric),

$$\left(e^0_\mu dx^\mu\right)^2. \tag{2.1}$$

Likewise for the analog of time slices, we have the components e^a_μ and e^μ_a, and the internal spacelike components of each are orthogonal to the timelike component of the inverses, for example, $e^\mu_a e^0_\mu = 0$, just as if the vectors e^μ_a were tangent vectors to a fixed-time surface (but no such surfaces exist in general). There is a spatial metric or squared line element,

$$h_{\mu\nu} dx^\mu dx^\nu \equiv e^a_\mu e^a_\nu dx^\mu dx^\nu, \tag{2.2}$$

where $h_{\mu\nu} = e^a_\mu e^a_\nu \equiv e^a_\mu e^b_\nu \eta_{ab}$ and η_{ab} is the standard internal spatial metric, given by the identity matrix or $\eta_{ab} = \delta_{ab}$. (Note the use of notation like $v^a w^a = v^a w^b \eta_{ab}$ with summation convention, as a way of contracting internal spacelike indices, and a similar convention for the case of lower indices.) "Inverse" spatial metrics $h^{\mu\nu}$ and η^{ab} with upper indices can be defined likewise, but notice that the ambient spacetime metrics are degenerate and not truly inverses of each other; instead, $h_{\mu\nu} h^{\nu\lambda} = \delta^\lambda_\mu - e^0_\mu e^\lambda_0$. We assume that both the ambient spatial metrics are positive semidefinite. It may be tempting to combine these timelike and spacelike partial metric tensors into a single spacetime metric, but because of the lack of Lorentz invariance, this is not necessary, nor would it be uniquely defined [18,19] (line elements of time and space have different dimensions; there is no universal scale of speed). Therefore, such a metric will *not* be used, and in general we do not raise or lower any indices (occasionally we do so for internal spacelike indices using η_{ab} or η^{ab}).

One could make different choices of the one forms e^α_μ that differ by a linear transformation of the internal indices α. Because $\alpha = 0$ has been singled out, and because of the choice of internal metric on the space components which we may as well fix, the only possible transformations are SO(d) rotations on the internal spacelike indices a, b only (we neglect improper rotations of negative determinant). These rotations act as internal gauge transformations, as in a Yang-Mills gauge theory. To make expressions containing partial derivatives covariant under such transformations, we need a gauge field or "spin connection" ${\omega_\mu}^\alpha{}_\beta$ (an example of its use will appear in a moment). In the present case, only the internal spacelike components ${\omega_\mu}^a{}_b$ are nonzero. In view of the standard Euclidean metric on internal spacelike indices, we can raise or lower an index a or b, and it makes sense to say that the spin connection is antisymmetric on its internal indices [it is in the Lie algebra of SO(d)]. For $d=2$, the spin connection is effectively a pseudoscalar on the internal indices.

We also need a Christoffel connection with components (or Christoffel symbols) ${\Gamma^\mu}_{\nu\lambda}$ in order to write covariant derivatives in spacetime. As an example of the use of the two connections, the covariant derivative of the one forms is

$$\nabla_\mu e^\alpha_\nu = \partial_\mu e^\alpha_\nu + {\omega_\mu}^\alpha{}_\beta e^\beta_\nu - {\Gamma^\lambda}_{\mu\nu} e^\alpha_\lambda. \tag{2.3}$$

We do *not* impose the symmetry condition ${\Gamma^\mu}_{\nu\lambda} = {\Gamma^\mu}_{\lambda\nu}$, which means that our spacetime generally has torsion; the torsion tensor is $T^\mu_{\nu\lambda} = {\Gamma^\mu}_{\nu\lambda} - {\Gamma^\mu}_{\lambda\nu}$. It frequently appears with an upper internal index α in place of μ. We call the $\alpha = 0$ components of $T^\alpha_{\mu\nu}$ the timelike torsion, and the $\alpha = a$ components the spacelike torsion.

The one forms, spin connection, and Christoffel connection are not necessarily independent. We impose the requirement that

$$\nabla_\mu e^\alpha_\nu = 0, \tag{2.4}$$

so the vielbeins (and their inverses) are covariantly constant. When we come to varying an action, we must specify which variables are viewed as independent, and in the first part of the paper, we choose to view the vielbein and spin connection as the independent variables that describe the spacetime geometry. The covariant-constancy equation can be solved for the Christoffel symbols, and the torsion is

$$T^\alpha_{\mu\nu} = \partial_\mu e^\alpha_\nu - \partial_\nu e^\alpha_\mu + {\omega_\mu}^\alpha{}_\beta e^\beta_\nu - {\omega_\nu}^\alpha{}_\beta e^\beta_\mu. \tag{2.5}$$

The components of the timelike torsion are essentially the curl (or exterior derivative) of the one form e^0_μ; the vanishing of these is precisely the condition above for the existence of an absolute time coordinate. Later in the paper, we also make use of a different point of view, in which the vielbeins e^α_μ and what we call the reduced torsion $\widetilde{T}^a_{\mu\nu}$ (a part of the spacelike torsion that is independent of the vielbeins; the timelike torsion is fully determined by the timelike vielbeins in any case) are viewed as the independent variables; using the covariant-constancy equation, one can express the Christoffel symbols and the spin connection in terms of these. As these expressions are more lengthy, they are given in Appendix A.

Our actions involve integration over spacetime, and we need to use a volume form or measure for the integration. This is simply constructed from the timelike and spacelike metrics above and is written as $d^{d+1}x\,\widehat{\sqrt{g}}$ as usual, where $\widehat{\sqrt{g}}$ (which does not transform as a scalar) is defined (for $d=2$, but other dimensions are similar) by

$$\widehat{\sqrt{g}} = \tfrac{1}{6}\hat{\epsilon}^{\mu\nu\lambda}\epsilon_{\alpha\beta\gamma} e^\alpha_\mu e^\beta_\nu e^\gamma_\lambda, \tag{2.6}$$

which clearly is simply the determinant of the matrix with entries e^α_μ; the non degeneracy condition implies that it is nonzero at all spacetime points, and we assume it is positive. The ambient spacetime ϵ symbol (not tensor) $\hat{\epsilon}^{\mu\nu\lambda}$ is defined in any coordinate system (again for $d=2$) by $\hat{\epsilon}^{012} = 1$, and the internal one (which is an invariant tensor for the internal transformations, essentially spatial rotations, that we use) $\epsilon^{\alpha\beta\gamma}$ likewise. The lower-index ones $\epsilon_{\alpha\beta\gamma}$ and $\check{\epsilon}_{\mu\nu\lambda}$ are defined in the same way. The notation with a hat used here will indicate throughout that the object on which it appears is a tensor density, rather than a tensor, which transforms under coordinate transformation with an additional determinantal factor (as $\widehat{\sqrt{g}}$ does) compared with a tensor with the same ambient spacetime indices; alternatively, a tensor density divided by $\widehat{\sqrt{g}}$ transforms as a tensor. (The lower ambient index ϵ symbol with the check symbol transforms inversely to the upper index one.) We note the useful relation

$$\Gamma^\nu{}_{\mu\nu} = \frac{1}{\widehat{\sqrt{g}}}\partial_\mu \widehat{\sqrt{g}}. \tag{2.7}$$

Without the hat, $\epsilon^{\mu\nu\lambda} = \epsilon^{\alpha\beta\gamma} e^\mu_\alpha e^\nu_\beta e^\lambda_\gamma$ is a tensor, and similarly for $\epsilon_{\mu\nu\lambda}$, which is the volume three form written in components. We also sometimes use the two-index ϵ symbol $\epsilon^{ab} = \epsilon^{0ab}$, and likewise for lower indices, which is natural in view of the singling out of timelike components, and can even be done for the ambient versions as $\epsilon^{\mu\nu} = e^0_\lambda \epsilon^{\lambda\mu\nu}$.

We also define here the Riemann curvature tensor, although it does not appear much in this paper. This can be obtained [17] from the commutator of two covariant derivatives $[\nabla_\mu,\nabla_\nu]$ applied to a vector field with an internal index, say v^α. (By covariant constancy of the vielbein which can be used to convert indices, this determines the Riemann tensor in general.) We have

$$R_{\mu\nu}{}^\alpha{}_\beta = \partial_\mu {\omega_\nu}^\alpha{}_\beta - \partial_\nu {\omega_\mu}^\alpha{}_\beta + {\omega_\mu}^\alpha{}_\gamma {\omega_\nu}^\gamma{}_\beta - {\omega_\nu}^\alpha{}_\gamma {\omega_\mu}^\gamma{}_\beta, \tag{2.8}$$

and so vanishes unless $\alpha = a$, $\beta = b$. In the case $d=2$ of interest in this paper, the nonvanishing components reduce to

$$R_{\mu\nu}{}^a{}_b = \partial_\mu {\omega_\nu}^a{}_b - \partial_\nu {\omega_\mu}^a{}_b, \tag{2.9}$$

which is effectively the curl of the single one form ${\omega_\mu}^1{}_2$, similar to the case in Ref. [8].

Finally, we note that the variation of spacetime tensors under diffeomorphisms is given by the Lie derivative $\mathcal{L}$. Under a diffeomorphism generated by the vector field ξ, we have for scalar functions

$$\mathcal{L}_\xi f = \xi^\nu \partial_\nu f, \tag{2.10}$$

for vectors

$$\mathcal{L}_\xi V^\mu = \xi^\nu \partial_\nu V^\mu - V^\nu \partial_\nu \xi^\mu, \tag{2.11}$$

and for one forms

$$\mathcal{L}_\xi W_\mu = \xi^\nu \partial_\nu W_\mu + W_\nu \partial_\mu \xi^\nu. \tag{2.12}$$

Although we do not need it here, the generalization to higher rank tensors is obtained by demanding that $\mathcal{L}$ satisfies the Liebnitz rule.

In addition to these geometric structures in spacetime, we also use a U(1) gauge potential A_μ, with field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ as usual. For covariant derivatives of fields that carry U(1) charge, as in the following section, ∇_μ denotes the fully covariant derivative that includes the vector potential.

B. Action for matter

We now consider actions for nonrelativistic matter fields as an illustration of the use of the above background fields and to check that the variations with these background fields produce the correct conserved currents. In relation to this, the vielbeins play the role of gauge potentials that enable us, in some sense, to gauge translation invariance, and so variations with respect to them produce the corresponding covariantly conserved currents and densities, just as varying the electromagnetic gauge potential produces the conserved electric current/density (that is, satisfying the continuity equation). As an example, let us consider a minimally coupled second-quantized action for spinless bosons or fermions in flat spacetime,

$$S = \int d^{d+1}x \left[i\varphi^\dagger D_0\varphi - \frac{1}{2m}(D_i\varphi)^\dagger D_i\varphi \right] - \frac{1}{2}\int d^{d+1}x\, d^{d+1}y\, V(x-y)\varphi^\dagger(x)\varphi^\dagger(y)\varphi(x)\varphi(y), \tag{2.13}$$

where φ is a scalar field (either commuting or anticommuting, for bosons or fermions, respectively), $D_\mu = \partial_\mu - iA_\mu$ is the gauge-covariant derivative, and V is an interaction potential in spacetime. For a general spacetime we obtain the covariant action

$$S = \int d^{d+1}x\sqrt{\widehat{g}}\left[\frac{1}{2} i e_0^\mu(\varphi^\dagger \overleftrightarrow{\nabla}_\mu\varphi) - \frac{1}{2m}e_a^\mu e_a^\nu(\nabla_\mu\varphi)^\dagger\nabla_\nu\varphi + \frac{1}{2}\int d^{d+1}y\sqrt{\widehat{g}}\, V(x,y)\varphi^\dagger(x)\varphi(x)\varphi^\dagger(y)\varphi(y)\right]. \tag{2.14}$$

The expression for the interaction term containing $V(x,y)$ requires some care. For the case of contact interactions, where the interaction potential V is given by a differential operator acting on a δ function, there are no issues as the δ function is already a scalar density; we need simply to replace the derivatives acting upon it with covariant derivatives, contracted using the spacelike metric. We note that such contact interactions may be taken to be independent of the spacelike torsion $T^a_{\mu\nu}$.

For finite-range instantaneous interactions, we need to generalize the notion of spacelike distance to curved spacetime with torsion. There are two conceptual difficulties here. First, if the timelike torsion $T^0_{\mu\nu}$ is not identically zero in a region of spacetime, there does not exist an absolute time variable defined in that region, as we mentioned before, and so hypersurfaces of constant absolute time do not exist. However, that condition, which says that e^0_μ is an exact differential, is more restrictive than is necessary for this purpose, and in general, according to a theorem of Frobenius [17,20], hypersurfaces whose tangent vectors e^μ_a are orthogonal to e^0_μ at each point exist if and only if the weaker condition,

$$e^\mu_a e^\nu_b T^0_{\mu\nu} = 0, \tag{2.15}$$

holds throughout a region, that is when the tangent vector fields e^μ_a are integrable. For $d = 2$, this expression can also be written as

$$\epsilon^{\mu\nu\lambda} e^0_\lambda T^0_{\mu\nu} = 0. \tag{2.16}$$

For general d, one can write this simply as $e^0_{[\lambda}T^0_{\mu\nu]} = 0$, where the square brackets surrounding indices mean antisymmetrization. In this form, for all d, the (dual version of the) Frobenius theorem says equivalently that the condition is satisfied if and only if e^0_μ obeys an equation of the form $e^0_\mu dx^\mu = \psi dw$ for some scalar functions $\psi(x)$, $w(x)$; then the spacelike hypersurfaces are surfaces of constant w.

Second, in the presence of torsion we must distinguish between spacelike *geodesics*—paths $r^\mu(\lambda)$ that satisfy both the geodesic equation [17]

$$0 = \frac{d^2 r^\mu}{d\lambda^2} + \Gamma^\mu{}_{\nu\rho}\frac{dr^\nu}{d\lambda}\frac{dr^\rho}{d\lambda} \tag{2.17}$$

and the spacelike constraint

$$0 = e^0_\mu \frac{dr^\mu}{d\lambda} \tag{2.18}$$

—and spacelike paths of minimal distance. The geodesics are those paths which parallel transport their tangent vectors, and in the absence of torsion, these coincide with paths of minimal distance. This can be seen by examining Eq. (A3) for the Christoffel symbols and noting that the Euler-Lagrange equation for minimization of spacelike distance depends only on the contribution of the spacelike metric to the connection. Here we work with spacelike geodesics because they are easier to construct.

Given a point x^μ on our manifold, we denote by $r^\mu_x(v^a,\lambda)$ the parametrized geodesic satisfying the initial condition that its tangent is along a spacelike vector v^a

$$r^\mu_x(v^a,0) = x^\mu, \tag{2.19}$$

$$\left.\frac{dr^\mu_x}{d\lambda}\right|_{\lambda=0} = v^a e^\mu_a, \tag{2.20}$$

Because of the possible reparametrizations of λ, v^a is only defined up to a scalar factor. If our manifold is sufficiently well behaved (i.e., geodesically complete), we may take $\lambda \in (-\infty,\infty)$. Since we do not have a notion of spacelike hypersurfaces, we must make do with the set of all points connected to x^μ by spacelike geodesics. More formally, we consider the open sets U_x defined by

$$U_x = \left\{ r^\mu_x(v^a,\lambda)\right\}. \tag{2.21}$$

Note that the sets U_x are the images of the exponential map acting on the set of spacelike tangent vectors at x, and hence they are proper d-dimensional submanifolds of spacetime [21]. For each $y \in U_x$, we may then define the distance

$$d_x(y) = \left| \int_0^{\lambda_0} d\lambda \sqrt{h_{\mu\nu}\frac{dr^\mu_x}{d\lambda}\frac{dr^\nu_x}{d\lambda}}\right|. \tag{2.22}$$

Note that $d_x(y) = d_y(x)$ by the uniqueness of solutions to the geodesic equation. Using this distance, we may form the covariant interaction term

$$S_{\text{int}} = \frac{1}{2}\int d^{d+1}x\sqrt{\widehat{g}}\int d^{d+1}y\sqrt{\widehat{g}}\{\chi_{U_x}(y)V[d_x(y)] \times \varphi^\dagger(x)\varphi(x)\varphi^\dagger(y)\varphi(y)\}, \tag{2.23}$$

where χ_{U_x} is the characteristic function of the set U_x. This expression is rather cumbersome, and we do not make explicit use of it in the remainder of this work. However, we note that in the absence of torsion, it reduces to a straightforward generalization of the interaction term constructed in Ref. [22].

Returning to our expression Eq. (2.14) for the microscopic action, it is illuminating to assign independent meaning to certain components of e^α_μ or, more precisely, to $\delta e^\alpha_\mu = e^\alpha_\mu - \delta^\alpha_\mu$. By examining the action, we see that δe^0_0 enters exactly as the artificial gravitational potential ψ introduced by Luttinger for calculating thermal response functions [11,12]; when this is the only nonzero component of e, it multiplies the energy density. This is consistent with the standard Newtonian approximation to gravity in the general relativity literature [17]. Notice that the spatial component e^a_i enters similarly as the matrix Λ of Ref. [22]. This is no accident: The matrix Λ presented there is very much just these components of the vielbein, and to first order δe^a_i are the matrices λ^a_i. Because it couples longitudinally to the heat current, δe^0_i can be interpreted as the "gravitomagnetic vector potential" mentioned recently in the literature [23,24].

Next, we outline the general procedure for obtaining equations of motion and the various currents from an action and obtain the conservation laws for the currents from the invariance properties. We use the action above (or the version with $V=0$) as an example with which to check the results. Given an action S involving the background fields and a scalar field φ, and now taking the vielbeins and spin connection as the independent background fields, we can consider the variations

$$\varphi \to \varphi + \delta\varphi, \tag{2.24}$$

$$A_\mu \to A_\mu + \delta A_\mu, \tag{2.25}$$

$$e^\alpha_\mu \to e^\alpha_\mu + \delta e^\alpha_\mu, \tag{2.26}$$

$$\omega_\mu{}^a{}_b \to \omega_\mu{}^a{}_b + \delta\omega_\mu{}^a{}_b, \tag{2.27}$$

to obtain

$$\delta S = \int d^{d+1}x \left[\frac{\delta S}{\delta e^\mu_\alpha}\delta e^\mu_\alpha + \frac{\delta S}{\delta\varphi}\delta\varphi + \frac{\delta S}{\delta A_\mu}\delta A_\mu + \frac{\delta S}{\delta\omega_\mu{}^a{}_b}\delta\omega_\mu{}^a{}_b\right]. \tag{2.28}$$

For the equations of motion of the matter field, the variation of the action with the background fields fixed is set to zero, and so the equations of motion are

$$\frac{\delta S}{\delta\varphi} = 0. \tag{2.29}$$

Now we define several currents. These are the number current (with components $\mu=0$ for density and $\mu=i$ for spatial current),

$$J^\mu = \frac{1}{\sqrt{g}}\frac{\delta S}{\delta A_\mu}, \tag{2.30}$$

and analogously the energy-momentum-stress current (or tensor),

$$\tau^\mu{}_\alpha = -\frac{1}{\sqrt{g}}\frac{\delta S}{\delta e^\alpha_\mu}. \tag{2.31}$$

The latter contains the energy current $J^\mu_E = \tau^\mu_0$ as the $\alpha=0$ components, and the momentum current as the $\alpha=a$ components, of which the $\mu=0$ component is momentum density, and the $\mu=i$ components are the momentum flux or (essentially) the stress. Finally, there is the spin current

$$J_S{}^\mu{}_a{}^b = \frac{1}{\sqrt{g}}\frac{\delta S}{\delta\omega_\mu{}^a{}_b}. \tag{2.32}$$

In $d=2$ dimensions, the spin current is antisymmetric in a, b, and those indices can be dropped.

Next we obtain the conservation laws for these currents from the local symmetries. Considering first an infinitesimal U(1) gauge transformation,

$$\delta\varphi = i\varphi\theta, \tag{2.33}$$

$$\delta A_\mu = \partial_\mu\theta, \tag{2.34}$$

$$\delta e^\alpha_\mu = 0, \tag{2.35}$$

$$\delta\omega_\mu{}^a{}_b = 0, \tag{2.36}$$

with a scalar function $\theta(x)$, we find, after using the equations of motion Eq. (2.29) and the fact that the variation of the action under a symmetry transformation is by definition zero, the number current conservation law

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}J^\mu) = \nabla_\mu J^\mu - T^\nu_{\nu\mu}J^\mu = 0. \tag{2.37}$$

Next we wish to examine local space and time translations. We can do this in one fell swoop by looking at how the action changes under arbitrary infinitesimal diffeomorphisms $x^\mu \to x^\mu + \xi^\mu$, where $\xi^\mu(x)$ is a vector field. This has the effect of modifying all fields by their Lie derivatives, as pointed out above. However, because the Lie derivative is not explicitly covariant, it is useful to modify it to also include a well-chosen U(1) gauge transformation and an internal rotation. That is, to the Lie derivative of charged fields we add an additional gauge transformation by the amount $\xi^\mu A_\mu$, and to the Lie derivative of fields with an internal index we add an additional internal rotation by the amount $\xi^\mu\omega_\mu{}^a{}_b$. We are free to do this since these transformations are themselves symmetries of the action. A short calculation shows that the field variations are then given by the covariant Lie derivatives

$$\delta\varphi = \xi^\mu\nabla_\mu\varphi, \tag{2.38}$$

$$\delta A_\mu = \xi^\nu F_{\nu\mu}, \tag{2.39}$$

$$\delta e^\alpha_\mu = e^\alpha_\nu\nabla_\mu\xi^\nu - T^\alpha_{\mu\nu}\xi^\nu, \tag{2.40}$$

$$\delta\omega_\mu{}^a{}_b = \xi^\nu R_{\mu\nu}{}^a{}_b, \tag{2.41}$$

yielding, after an application of the equations of motion, the energy-momentum conservation law

$$\nabla_\mu\tau^\mu{}_\alpha - T^\lambda_{\lambda\mu}\tau^\mu{}_\alpha = -e^\nu_\alpha\big(J^\mu F_{\mu\nu} + J_S{}^\mu{}_a{}^b R_{\mu\nu}{}^a{}_b + \tau^\mu{}_\beta T^\beta_{\mu\nu}\big). \tag{2.42}$$

The contribution on the right-hand side of the form spin current times Riemann curvature is a known effect that corresponds to a force on spinning bodies due to curvature.

Finally, using an infinitesimal internal rotation,

$$\delta e^a_\mu = \Omega^a{}_b e^b_\mu, \tag{2.43}$$

$$\delta\omega_\mu{}^a{}_b = \Omega^a{}_c\omega_\mu{}^c{}_b - \Omega^c{}_b\omega_\mu{}^a{}_c - \partial_\mu\Omega^a{}_b, \tag{2.44}$$

with $\Omega^a{}_b(x)$ an arbitrary antisymmetric matrix function, we find that the antisymmetric part of the stress tensor satisfies

$$\epsilon^a{}_b\tau^b{}_a = \epsilon^a{}_b\left(\nabla_\mu J_S{}^\mu{}_a{}^b - T^\nu_{\nu\mu}J_S{}^\mu{}_a{}^b\right), \tag{2.45}$$

where

$$\tau^b{}_a = e^b_\mu\tau^\mu{}_a. \tag{2.46}$$

This can also be viewed as the conservation law for the spin current.

With this formalism established, we can now proceed to identify these conserved currents with physical quantities. Here we focus on the flat space $e = Id, \omega = 0$ expressions of the currents, deferring discussion of the more general case (and the associated "contact" terms) until Sec. V. We also set the interaction potential V to zero for brevity. Using the action Eq. (2.14), we find for the number current

$$\begin{aligned} J^\mu &= \frac{1}{\sqrt{g}}\frac{\delta S}{\delta A_\mu}\bigg|_{e=Id,\omega=0} \\ &= \delta^\mu_0\varphi^\dagger\varphi - \frac{i}{2m}[\varphi^\dagger D_i\varphi - (D_i\varphi)^\dagger\varphi]\delta^\mu_i, \end{aligned} \tag{2.47}$$

as expected for a charged field. For the energy-momentum-stress tensor, things are quite a bit more complicated, but we eventually find

$$\begin{aligned} \tau^\mu{}_\alpha &= -\frac{1}{\sqrt{g}}\frac{\delta S}{\delta e^\alpha_\mu}\bigg|_{e=Id,\omega=0} \\ &= \frac{i}{2}\left(\delta^\mu_0\delta^\nu_\alpha - \delta^\mu_\alpha\delta^\nu_0\right)[\varphi^\dagger D_\nu\varphi - (D_\nu\varphi)^\dagger\varphi] \\ &\quad - \frac{1}{2m}\left(\delta^\lambda_\alpha\delta^\nu_a\delta^\mu_a + \delta^\nu_\alpha\delta^\lambda_a\delta^\mu_a - \delta^\mu_\alpha\delta^\nu_a\delta^\lambda_a\right)(D_\lambda\varphi)^\dagger D_\nu\varphi. \end{aligned} \tag{2.48}$$

Unpacking terms, we see (after using the equations of motion to eliminate time derivatives) that the $\alpha = 0$ components of τ give the energy density and spatial energy current consistent with Ref. [11], while the $\alpha = a$ components give *minus* the momentum density and stress tensor consistent with Ref. [22], plus an additional term $\frac{1}{4m}D^2J^0\delta^\mu_\alpha$ due to operator ordering (cf. Ref. [25]). Finally, the spin current is zero because the action does not contain the spin connection.

Readers will have noticed that there is no chemical potential in our action. That is because we work in the canonical ensemble with a fixed particle number N. N is the flux of J^μ across an arbitrary (in principle spacelike) section $\mathcal{A}$; as J^μ obeys a covariant continuity equation, N is invariant under small changes in the section. Precisely, the flux can be written (for $d = 2$; other dimensions are similar)

$$N = \int_{\mathcal{A}}\epsilon_{\mu\nu\lambda}J^\mu dx^\nu dx^\lambda, \tag{2.49}$$

and we note that $\epsilon_{\mu\nu\lambda}J^\mu$ is the set of components of a two form, and the two form (in general, a d form) can be integrated over a d surface without any use of the metric. N is invariant under small changes in the section because conservation implies $\partial_{[\rho}\epsilon_{\nu\lambda]\mu}J^\mu = 0$. Thus, classically, the expression for N has to be imposed as a constraint; quantum mechanically, in an operator formalism, one uses only states that obey this as an initial condition, which is preserved by time evolution; it can be imposed in a functional integral treatment by introducing an integration over an additional variable (actually a gauge potential) to make a functional δ function. In general, the effect of the global constraint is only felt globally, and if we eventually consider response functions in flat spacetime with a translation invariant system, the effect only shows up at zero wave vector $\mathbf{k}$ in responses that couple to the particle number. For quantities of interest we can take the limit as $\mathbf{k} \to 0$ instead of $\mathbf{k} = 0$ when it makes a difference. Thus, in practice, when studying local behavior in a large system, we simply ignore the number constraint. (If desired, it can be incorporated along the lines mentioned.)

III. INDUCED ACTION: GENERALITIES

Our goal is to find the most general induced bulk action,

$$S^{\text{eff}}[A_\mu, e^\alpha_\mu, \omega_\mu{}^a{}_b],$$

one could obtain for a system that is gapped in the bulk (i.e., a topological phase) after integrating out the matter fields. As indicated, this induced action is a functional of the electromagnetic potential A_μ, the vielbeins e^α_μ, and the spin connection $\omega_\mu{}^a{}_b$, which for now we continue to use as the independent background fields. We can expand this functional as the integral of a sum of local terms,

$$\begin{aligned} S^{\text{eff}} = \int d^{d+1}x\,\sqrt{g}\big[&\mathcal{L}^{(0)}(A_\mu, e^\alpha_\mu, \omega_\mu{}^a{}_b) \\ &+ \mathcal{L}^{(1)}(A_\mu, e^\alpha_\mu, \omega_\mu{}^a{}_b) + \cdots\big]. \end{aligned} \tag{3.1}$$

Each such $\mathcal{L}^{(n)}$ is a function of the external fields and their derivatives, and each *integral* must be invariant under coordinate transformations, internal rotations, and electromagnetic gauge transformations up to boundary terms. Very generally, these terms can be divided into two categories. The first category, which we term "locally invariant," consists of those terms in which the $\mathcal{L}^{(n)}$ themselves are invariant under all the aforementioned symmetry transformations. These terms can be written as polynomials in strictly covariant quantities such as the vielbein e^α_μ, the electromagnetic field strength $F_{\mu\nu}$, the torsion $T^\alpha_{\mu\nu}$, the curvature $R_{\mu\nu}{}^\alpha{}_\beta$, and their (covariant) derivatives, with appropriate index contractions. All such terms can be combined into one action,

$$S^{\text{loc}}[e^\alpha_\mu, F_{\mu\nu}, T^0_{\mu\nu}, T^a_{\mu\nu}, R_{\mu\nu}{}^\alpha{}_\beta], \tag{3.2}$$

which is a functional of these covariant tensors and their covariant derivatives. The second category consists of the remaining integrands $\mathcal{L}^{(n)}$ which cannot be made locally invariant by integration by parts, and which for at least one type of transformation are invariant only up to a total derivative. The integrals of such terms are invariant only up to boundary contributions, and, in general, invariance of the total action necessitates the existence of gapless edge degrees of freedom. An example of such a term in $2+1$ dimensions is the familiar U(1) Chern-Simons term

$$\mathcal{L}^{(\mathrm{CS})} = \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda, \tag{3.3}$$

which is not manifestly invariant under a gauge transformation, but changes by a total derivative. A locally covariant term can be multiplied by an arbitrary function of position, and would still be covariant, and this could occur due to changes of parameters with position in the microscopic action. For a Chern-Simons-type term, on the other hand, this cannot be done as it spoils the invariance up to a total derivative. This implies that the coefficient in a Chern-Simons term should usually be robust against changes in the parameters in the underlying "microscopic" action throughout a topological phase that respects the symmetries in question; if it were not, varying microscopic parameters in spacetime would lead to changes in the coefficient and so violate invariance of the induced action. (This field-theoretic argument, often formulated as the nonrenormalization of the coefficients in Chern-Simons-type terms, deserves to be more familiar in condensed-matter physics.) Further Chern-Simons-type terms that can occur in our theories in $2+1$ dimensions are the first and second Wen-Zee terms [8],

$$\mathcal{L}^{(\mathrm{WZ1})} = \epsilon^{\mu\nu\lambda} \omega_\mu \partial_\nu A_\lambda, \tag{3.4}$$

$$\mathcal{L}^{(\mathrm{WZ2})} = \epsilon^{\mu\nu\lambda} \omega_\mu \partial_\nu \omega_\lambda, \tag{3.5}$$

as well as the gravitational Chern-Simons term

$$\mathcal{L}^{(\mathrm{GCS})} = \epsilon^{\mu\nu\lambda}\left(\Gamma^\rho_{\mu\sigma}\partial_\nu\Gamma^\sigma_{\nu\rho} + \tfrac{2}{3}\Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\theta}\Gamma^\theta_{\lambda\rho}\right) \tag{3.6}$$

(here we use θ as an ambient index; as usual with this sort of thing, one very quickly runs out of greek letters). These terms must be treated individually on a case-by-case basis.

Given such an induced action, one can effect the functional derivatives as in the previous section to compute the expectation values of currents in the presence of a given background configuration of the fields A_μ, e^α_μ, and $\omega_\mu{}^a{}_b$. In particular, we can define the number current

$$J^\mu = \frac{1}{\sqrt{g}} \frac{\delta S^{\mathrm{eff}}}{\delta A_\mu}, \tag{3.7}$$

the energy-momentum-stress tensor

$$\tau^\mu{}_\alpha = -\frac{1}{\sqrt{g}} \frac{\delta S^{\mathrm{eff}}}{\delta e^\alpha_\mu}, \tag{3.8}$$

and the spin current

$$J_S{}^\mu{}_a{}^b = \frac{1}{\sqrt{g}} \frac{\delta S^{\mathrm{eff}}}{\delta \omega_\mu{}^a{}_b}, \tag{3.9}$$

where we use the same notation for the currents and their expectation values, as we expect the meaning to be clear from context. The conservation laws obeyed by the currents are the same as in the previous section.

In computing the currents from the induced action, we see that the contributions from the locally invariant terms and from the Chern-Simons-type terms have very different structure. The contributions of the locally invariant terms to the currents have the forms

$$J^\mu_{\mathrm{loc}} = \frac{1}{\sqrt{g}} \partial_\lambda \left(\frac{\delta S^{\mathrm{loc}}}{\delta F_{\mu\lambda}} \right), \tag{3.10}$$

$$\tau^\mu{}_{\alpha,\mathrm{loc}} = -\frac{1}{\sqrt{g}} \frac{\delta S^{\mathrm{loc}}}{\delta e^a_\mu} - \frac{1}{\sqrt{g}} \partial_\lambda \left(\frac{\delta S^{\mathrm{loc}}}{\delta T^\alpha_{\mu\lambda}} \right) - \frac{1}{\sqrt{g}} \omega_\lambda{}^\beta{}_\beta \frac{\delta S^{\mathrm{loc}}}{\delta T^\beta_{\mu\lambda}}, \tag{3.11}$$

$$J_S{}^\mu{}_a{}^b{}_{,\mathrm{loc}} = \frac{1}{\sqrt{g}} e^b_\lambda \frac{\delta S^{\mathrm{loc}}}{\delta T^a_{\mu\lambda}} + \frac{1}{\sqrt{g}} \partial_\lambda \left(\frac{\delta S^{\mathrm{loc}}}{\delta R_{\mu\lambda}{}^a{}_b} \right) + \frac{1}{\sqrt{g}} \omega_\lambda{}^a{}_c \frac{\delta S^{\mathrm{loc}}}{\delta R_{\mu\lambda}{}^c{}_b} - \frac{1}{\sqrt{g}} \omega_\lambda{}^c{}_b \frac{\delta S^{\mathrm{loc}}}{\delta R_{\mu\lambda}{}^a{}_c}. \tag{3.12}$$

Each functional derivative here is taken with the remaining arguments in Eq. (3.2) held fixed. There are two types of terms that appear here. The first type occurs as the first term on the right-hand side of the last two equations, which enter because e^α_μ and $\omega_\mu{}^a{}_b$ can appear in S^{loc} without derivatives. The remaining terms make up the second type and in each case can be combined to produce covariant derivatives of tensor quantities, using, for example, the fact that for any antisymmetric tensor field $A^{\mu\nu}$,

$$\frac{1}{\sqrt{g}} \partial_\nu(\sqrt{g} A^{\mu\nu}) = \nabla_\nu A^{\mu\nu} + T^\lambda_{\nu\lambda} A^{\mu\nu} + \frac{1}{2} T^\mu_{\lambda\nu} A^{\lambda\nu}, \tag{3.13}$$

and similar extensions including the spin connection for tensors with internal indices. The combinations of derivatives and spin connections that appear are, in fact, a covariant form of the curl (the covariant exterior derivative), in view of the antisymmetry of the tensors $F_{\mu\nu}$, $T^\alpha_{\mu\nu}$, and $R_{\mu\nu}{}^a{}_b$ in μ and ν. We refer to such terms as bulk magnetization currents, in analogy with Eqs. (1.4). [This is not entirely appropriate for all the components, because, for example, the field strength $F_{\mu\nu}$ can appear in the familiar combinations $\mathbf{E}^2$ and B^2 (for electric and magnetic fields), with different coefficients, and the former is related to electric polarization, not magnetization. However, we are interested in the spacelike components and zero frequency, and then the term magnetization is appropriate for the terms we obtain, so for simplicity we use it for all the terms.] We identify the covariant form of the bulk (number) magnetization as

$$m_{\mathrm{b}}^{\mu\nu} = \frac{1}{\sqrt{g}} \frac{\delta S^{\mathrm{loc}}}{\delta F_{\mu\nu}}. \tag{3.14}$$

The covariant bulk "energy-momentum magnetization" is

$$m_{\mathrm{b}}^{\mathrm{EM},\mu\nu}{}_\alpha = -\frac{1}{\sqrt{g}} \frac{\delta S^{\mathrm{loc}}}{\delta T^\alpha_{\mu\nu}}, \tag{3.15}$$

the $\alpha = 0$ component of which can be identified with a covariant version of the energy magnetization,

$$m_{\mathrm{b}}^{\mathrm{E},\mu\nu} \equiv m_{\mathrm{b}}^{\mathrm{EM},\mu\nu}{}_0, \tag{3.16}$$

and the $\alpha = a$ components are a "momentum magnetization,"

$$m_{\mathrm{b}}^{\mathrm{M},\mu\nu}{}_a \equiv m_{\mathrm{b}}^{\mathrm{EM},\mu\nu}{}_a. \tag{3.17}$$

We refer to the resulting contribution of the first two to the currents as number or energy magnetization currents, respectively, while for the contribution of the last to the stress tensor we refer to it as the "magnetization stress." We include for completeness the bulk "spin magnetization"

$$m_{\mathrm{b}\;\;\;a}^{\mathrm{S},\mu\lambda\;\;b} = \frac{1}{\widehat{\sqrt{g}}}\frac{\delta S^{\mathrm{loc}}}{\delta R_{\mu\lambda\;\;b}^{\;\;\;\;a}}, \tag{3.18}$$

although we do not need it in this work.

We can point out here that the momentum magnetization appears in both the contributions to the momentum density and stress tensor and the spin current. We will see that this is directly relevant to the issue of so-called torsional Hall viscosity and its relation with the spin density.

When we consider the contribution of such magnetization currents to the total current flowing through a section of the system (the transport current), we must integrate them along a hypersurface. Then we also pick up corresponding δ-function current contributions on the boundary, which arise because of the boundary term when one integrates by parts to obtain the curl form of the bulk magnetization currents (all terms in the action are assumed to vanish outside the boundary). Consequently, as in textbook electrodynamics of media, the magnetization currents give no net contribution to the transport current. On the other hand, when we look at contributions of the Chern-Simons-type terms to the currents, by construction we *cannot* find contributions that can be written as covariant derivatives of covariant tensors. Because of this, the integration of these contributions to the currents across a section of the sample necessarily gives nontrivial contributions to the transport current.

IV. INDUCED ACTION: FIRST ORDER IN DERIVATIVES

For the remainder of this work, we focus on $d = 2$-dimensional systems. We require the induced action to be consistent with spacetime reparametrization invariance, internal spatial rotation symmetry, and electromagnetic gauge invariance, up to boundary terms. To do this, we must establish a consistent filtration scheme on the myriad of terms that one could write. We adopt a derivative counting scheme in which A_μ and e_μ^α are counted as zero derivatives; that is, they are assigned degree 0. The spin connection ω_μ is counted as one derivative in order to ensure that the spacetime covariant derivative and the torsion tensor have well-defined degree 1.

The naive derivative counting scheme above is complicated slightly by the special role played by the background magnetic and gravitomagnetic fields. In a general curved spacetime, the scalar magnetic field felt by the system is

$$B = \tfrac{1}{2}\epsilon^{\mu\nu\lambda}e_\mu^0 F_{\nu\lambda}. \tag{4.1}$$

As noted above, we are also considering perturbations to the "gravitomagnetic potential" e_μ^0, and, noting the similarities to the electromagnetic potential A_μ, we can consider correlation functions in the presence of not only a background magnetic field, but also in the presence of a background "gravitomagnetic field" constructed from the timelike torsion,

$$B_{\mathrm{G}} = \tfrac{1}{2}\epsilon^{\mu\nu\lambda}e_\mu^0 T_{\nu\lambda}^0, \tag{4.2}$$

which we expect to enter thermodynamic quantities similarly to the magnetic field B. Notice that this is the same quantity, Eq. (2.16), that is zero in a region if and only if spacelike hypersurfaces exist there. In applications, B_{G} will be set to zero at the end, but since it plays a similar thermodynamic role to the magnetic field, we treat the two symmetrically for consistency. The equilibrium properties of our system can be arbitrary functions of B and B_{G}, and to capture this, it is necessary for us to retain terms at all orders in B and B_G.

With this in mind, we adopt the following scheme for writing terms in the induced action. Of all possible terms consistent with spacetime reparametrization invariance, internal rotation symmetry, and U(1) gauge symmetry (up to boundary terms), we retain terms to all orders in B and B_G and only terms quadratic and to first order in derivatives in the other combinations of A_μ, e_μ^α, and ω_μ. This leaves us with all of those terms which contribute to linear order in derivatives of $\psi = e_0^0 - 1$ and A_μ to the thermoelectric response functions. Although we could dispense with higher-order terms in the gravitomagnetic field B_G and still fully capture the thermoelectric response properties, we will see that interpreting our results will be made easier by treating it symmetrically with the magnetic field B.

The most general induced action, consistent with the discussion above, is given by

$$S^{\mathrm{eff}} = \int d^3x\,\widehat{\sqrt{g}}\bigg[f(B,B_G) + \gamma(B,B_G)\epsilon^{\mu\nu\lambda}e_\mu^a T_{\nu\lambda}^a + \widetilde{\gamma}(B,B_G)\epsilon^{\mu\nu\lambda}\epsilon_{ab}e_\mu^a T_{\nu\lambda}^b + \frac{\nu}{4\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda\bigg]. \tag{4.3}$$

Here f, γ, and $\widetilde{\gamma}$ are scalar functions of their arguments. We mention here that we could have treated the two scalars constructed from the spacelike torsion, namely,

$$B_T = \tfrac{1}{2}\epsilon^{\mu\nu\lambda}e_\mu^a T_{\nu\lambda}^a, \tag{4.4}$$

$$\widetilde{B}_T = \tfrac{1}{2}\epsilon^{\mu\nu\lambda}\epsilon_{ab}e_\mu^a T_{\nu\lambda}^b, \tag{4.5}$$

in a similar way as B and B_G, keeping terms to all orders and including them in f, instead of only to first order as we did. We have done it this way because it is useful in the later discussion to separate these pieces and also in order to compare with the literature.

The coefficients in the effective action, or their Taylor expansions in their arguments, correspond to response functions, as we will see. While we are assuming that the temperature is zero, we emphasize that if we do allow nonzero temperature, the coefficients will have only exponentially small corrections, due to the gap in the energy spectrum in the bulk.

We would now like to identify the functions appearing in the actions in Eq. (4.3) with certain thermodynamic properties of the system. We start by computing the average currents Eqs. (3.7) and (3.8). We find for the number current

$$J^\mu = \frac{\nu}{4\pi}\epsilon^{\mu\nu\lambda}F_{\nu\lambda} + \frac{1}{\widehat{\sqrt{g}}}\partial_\nu\left[\widehat{\sqrt{g}}\left(\frac{\partial f}{\partial B} + 2B_T\frac{\partial\gamma}{\partial B} + 2\widetilde{B}_T\frac{\partial\widetilde{\gamma}}{\partial B}\right)\epsilon^{\lambda\mu\nu}e_\lambda^0\right], \tag{4.6}$$

from which we identify the bulk magnetization

$$m_{\mathrm{b}}^{\mu\nu} = \left(\frac{\partial f}{\partial B} + 2B_T\frac{\partial\gamma}{\partial B} + 2\widetilde{B}_T\frac{\partial\widetilde{\gamma}}{\partial B}\right)\epsilon^{\lambda\mu\nu}e^0_\lambda. \tag{4.7}$$

For the spin current we find after a little algebra

$$J_{\mathrm{S}}{}^{\mu}{}_{a}{}^{b} = 4\gamma e^\mu_0\epsilon^{ab}. \tag{4.8}$$

We note that to this order in gradients, there is no spin magnetization as the curvature does not enter into the action. Finally, for the energy-momentum-stress tensor we have

$$\begin{aligned}-\tau^{\mu}{}_{\alpha} &= e^\mu_\alpha f + \gamma\epsilon^{\mu\nu\lambda}T^a_{\nu\lambda}\delta^a_\alpha + \widetilde{\gamma}\epsilon^{\mu\nu\lambda}T^b_{\nu\lambda}\epsilon_a b\delta^a_\alpha + \frac{1}{2}\epsilon^{\rho\nu\lambda}\left(\frac{\partial f}{\partial B}F_{\nu\lambda} + \frac{\partial f}{\partial B_G}T^0_{\nu\lambda}\right)\left(\delta^\mu_\rho\delta^0_\alpha - e^\mu_\alpha e^0_\rho\right)\\
&\quad + \epsilon^{\rho\nu\lambda}\left(\frac{\partial\gamma}{\partial B}F_{\nu\lambda} + \frac{\partial\gamma}{\partial B_G}T^0_{\nu\lambda}\right)\left(\delta^\mu_\rho\delta^0_\alpha - e^\mu_\alpha e^0_\rho\right)B_T + \epsilon^{\rho\nu\lambda}\left(\frac{\partial\widetilde{\gamma}}{\partial B}F_{\nu\lambda} + \frac{\partial\widetilde{\gamma}}{\partial B_G}T^0_{\nu\lambda}\right)\left(\delta^\mu_\rho\delta^0_\alpha - e^\mu_\alpha e^0_\rho\right)\widetilde{B}_T\\
&\quad + \frac{1}{\widehat{\sqrt{g}}}\partial_\nu\left[\widehat{\sqrt{g}}\epsilon^{\rho\mu\nu}\left(\frac{\partial f}{\partial B_G} + 2B_T\frac{\partial\gamma}{\partial B_G} + 2\widetilde{B}\frac{\partial\widetilde{\gamma}}{\partial B_G}\right)e^0_\rho\delta^0_\alpha\right] + \frac{1}{\widehat{\sqrt{g}}}\partial_\nu\left(\epsilon^{\rho\mu\nu}\widehat{\sqrt{g}}2\gamma e^a_\rho\delta^a_\alpha\right) + 2\gamma\omega_\nu{}^a{}_c\epsilon^{\rho\mu\nu}e^c_\rho\delta^a_\alpha\\
&\quad + \frac{1}{\widehat{\sqrt{g}}}\partial_\nu\left(\epsilon^{\rho\mu\nu}\widehat{\sqrt{g}}2\widetilde{\gamma}e^a_\rho\epsilon_{ab}\delta^b_\alpha\right) + 2\widetilde{\gamma}\omega_\nu{}^a{}_b\epsilon^{\rho\mu\nu}e^c_\rho\epsilon_{ac}\delta^b_\alpha.\end{aligned} \tag{4.9}$$

The expression for the energy-momentum-stress tensor is covariant, despite its appearance (compare the discussion in the previous section). In it, we identify the energy-momentum magnetization

$$m_{\mathrm{b}}^{\mathrm{EM},\mu\nu}{}_{\alpha} = -\epsilon^{\rho\mu\nu}\left[\left(\frac{\partial f}{\partial B_G} + 2B_T\frac{\partial\gamma}{\partial B_G} + 2\widetilde{B}_T\frac{\partial\widetilde{\gamma}}{\partial B_G}\right)e^0_\rho\delta^0_\alpha + 2\gamma e^a_\rho\delta^a_\alpha + 2\widetilde{\gamma}e^a_\rho\epsilon_{ab}\delta^b_\alpha\right]. \tag{4.10}$$

To get a feeling for the meaning of these functions, we proceed to evaluate the currents in the absence of any perturbations. That is, we set $e = Id$, $\omega = 0$, which implies in particular that $B = B_0 = F_{12}$, $B_G = 0$. We also take B to be uniform in space. In this case, we find for the number current in either ensemble

$$J^\mu(e = Id,\omega = 0) = \frac{\nu B_0}{2\pi}\delta^\mu_0, \tag{4.11}$$

allowing us to identify the unperturbed expectation of the number density

$$\bar{n} \equiv \frac{\nu B_0}{2\pi}. \tag{4.12}$$

Similarly, we have for the spin current

$$J_{\mathrm{S}}{}^{\mu}{}_{a}{}^{b}(e = Id,\omega = 0) = 4\gamma(B_0,0)\delta^\mu_0\epsilon^{ab}, \tag{4.13}$$

from which we can identify the unperturbed spin density,

$$\rho_{\mathrm{S},0} = 4\gamma(B_0,0). \tag{4.14}$$

For the energy-momentum-stress tensor in flat spacetime we have

$$\tau^{\mu}{}_{\alpha} = -\delta^\mu_\alpha\left[f(B_0,0) - \frac{\partial f}{\partial B}(B_0,0)B_0\right] - \delta^\mu_0\delta^0_\alpha\frac{\partial f}{\partial B}(B_0,0)B_0, \tag{4.15}$$

which allows us to identify $-f(B_0,0)$ as the unperturbed energy density (as is clear from the effective action itself, as we are using the canonical ensemble). The unperturbed internal pressure [11,22] is

$$p_{\mathrm{int},0} \equiv f(B_0,0) - \frac{\partial f}{\partial B}(B_0,0)B_0. \tag{4.16}$$

Last, we examine the unperturbed magnetizations for $e = Id$, $\omega = 0$, as these expressions prove useful later. We find for the bulk number magnetization

$$m^{\mu\nu}_{\mathrm{b},0} = \frac{\partial f}{\partial B}(B_0,0)\epsilon^{0\mu\nu}, \tag{4.17}$$

for the bulk energy magnetization

$$m^{\mathrm{E},\mu\nu}_{\mathrm{b},0} = -\epsilon^{0\mu\nu}\frac{\partial f}{\partial B_G}(B_0,0), \tag{4.18}$$

and for the bulk momentum magnetization

$$m^{\mathrm{M},\mu\nu}_{\mathrm{b},0}{}_{a} = 2\epsilon^{b\mu\nu}\widetilde{\gamma}(B_0,0)\epsilon_{ab} - 2\epsilon^{a\mu\nu}\gamma(B_0,0). \tag{4.19}$$

V. LINEAR RESPONSE FROM THE INDUCED ACTION

A. General considerations

With all this formalism established, we now wish to examine the response of the average currents to the external fields to linear order. Before we proceed to expand the expressions Eqs. (4.6)–(4.9) in the external fields, we must connect our currents with those in the statistical physics literature. To do so, we must make contact with the standard view of the perturbing fields δe and ω as externally applied fields [11,12].

While what we have done up to now is valid in any system of coordinates, we must remember that a physical measurement is performed using a fixed choice of coordinates x^μ (the laboratory coordinate system, if you will). We would like to interpret the vielbeins $e^a_\mu(x)$ as externally applied fields in this given coordinate system. If they were held fixed, then this does not cause any issues, but because we wish to vary them and study the response to perturbations in them, it is necessary to be careful about the following point. Given a conserved vector field K^μ (such as the conserved number current J^μ and others), we identify the experimentally relevant current by considering the flux of K^μ through a surface that is *fixed* when the perturbing δe^α_μ. It is of paramount importance to maintain conservation of K^μ; so by considering integrals of the two form,

$$K^\mu \epsilon_{\mu\nu\lambda} dx^\nu dx^\lambda,$$

across an infinitesimal hypersurface, which, as we saw in Sec. II, requires no additional vielbein factors under the integral, we see that it is actually $K^\mu \epsilon_{\mu\nu\lambda}$ that we should utilize. Extracting the coordinate-transformation invariant $\check{\epsilon}_{\mu\nu\lambda}$ symbol, we see that the physically meaningful quantity is the tensor *density*

$$\widehat{K}^\mu = \widehat{\sqrt{g}} K^\mu. \tag{5.1}$$

The practical effect of this is that when we look at the change in $\widehat{K}^\mu$ to linear order in perturbations of the vielbeins, there is an additional term compared with what one obtains using K^μ. (In a microscopic linear response calculation, these show up as "contact terms," that is, contributions to the response that are given by an expectation value of some operator at a single time, rather like the familiar diamagnetic term in conductivity response.)

As an example of how this makes contact with the literature, let us revisit the microscopic number current computed in Sec. II. In the presence of nontrivial e^α_μ, the number current computed from Eq. (2.14) is

$$J^\mu = e^\mu_0 \varphi^\dagger \varphi - \frac{i}{2m} e^\mu_a e^\nu_a [\varphi^\dagger D_\nu \varphi - (D_\nu \varphi)^\dagger \varphi], \tag{5.2}$$

whereas the number current density is given by

$$\widehat{J}^\mu = \widehat{\sqrt{g}} \left\{ e^\mu_0 \varphi^\dagger \varphi - \frac{i}{2m} e^\mu_a e^\nu_a [\varphi^\dagger D_\nu \varphi - (D_\nu \varphi)^\dagger \varphi] \right\}. \tag{5.3}$$

Let us examine this in the case of Luttinger's gravitational perturbation in otherwise flat space, setting

$$\begin{aligned} e^0_\mu &= \delta^0_\mu (1+\psi), \\ e^a_\mu &= \delta^a_\mu. \end{aligned} \tag{5.4}$$

We then find that

$$\widehat{J}^\mu = \delta^\mu_0 \varphi^\dagger \varphi - (1+\psi)\delta^\mu_i \frac{i}{2m} [\varphi^\dagger D_i \varphi - (D_i \varphi)^\dagger \varphi], \tag{5.5}$$

in agreement with the form of the current operator in the presence of the background gravitational field presented in Refs. [11,12].

Similar considerations hold for the energy-momentum-stress energy tensor $\tau^\mu{}_\alpha$. In that case we must also pay attention to the second (lower) index. The physical response corresponds to the tensor density with the second index converted to an ambient spacetime index in the same laboratory coordinate system. For example, in the case of Luttinger's perturbation, this corresponds to the Hamiltonian being the generator of translations along the vector field $\partial/\partial x^{\mu=0}$ rather than along $e^\mu_0 \partial/\partial x^\mu$. Altogether, we must consider the response of the energy-momentum-stress tensor density $\widehat{\tau}^\mu{}_\nu$ to perturbations. As an illustration, if we consider the energy density $\widehat{\tau}^0{}_{\nu=0}$ computed from the microscopic action Eq. (2.14) in the presence of Luttinger's gravitational perturbation Eq. (5.4), we find

$$\widehat{\tau}^0{}_{\nu=0} = (1+\psi)\frac{1}{2m}(D_i\varphi)^\dagger D_i \varphi, \tag{5.6}$$

consistent with the energy density operator used in Refs. [11,12] for calculating thermal transport coefficients.

Following this discussion, the current densities we wish to consider are, from Eqs. (4.6)–(4.9),

$$\widehat{J}^\mu = \frac{\nu}{4\pi} \widehat{\epsilon}^{\mu\nu\lambda} F_{\nu\lambda} + \partial_\nu \widehat{m}^{\mu\nu}_{\mathrm{b}}, \tag{5.7}$$

$$\begin{aligned} \widehat{\tau}^\mu{}_\nu = & -\widehat{\sqrt{g}}\delta^\mu_\nu f - \gamma \widehat{\epsilon}^{\mu\rho\lambda} T^a_{\rho\lambda} e^a_\nu - \widetilde{\gamma}\widehat{\epsilon}^{\mu\rho\lambda} T^b_{\rho\lambda}\epsilon_{ab} e^a_\nu \\ & - \frac{1}{2}\widehat{\epsilon}^{\rho\sigma\lambda}\left(\frac{\partial f}{\partial B}F_{\sigma\lambda} + \frac{\partial f}{\partial B_G}T^0_{\sigma\lambda}\right)\left(e^0_\nu \delta^\mu_\rho - e^0_\rho \delta^\mu_\nu\right) \\ & - B_T \widehat{\epsilon}^{\rho\sigma\lambda}\left(\frac{\partial \gamma}{\partial B}F_{\sigma\lambda} + \frac{\partial \gamma}{\partial B_G}T^0_{\sigma\lambda}\right)\left(e^0_\nu \delta^\mu_\rho - e^0_\rho \delta^\mu_\nu\right) \\ & - \widetilde{B}_T \widehat{\epsilon}^{\rho\sigma\lambda}\left(\frac{\partial \widetilde{\gamma}}{\partial B}F_{\sigma\lambda} + \frac{\partial \widetilde{\gamma}}{\partial B_G}T^0_{\sigma\lambda}\right)\left(e^0_\nu \delta^\mu_\rho - e^0_\rho \delta^\mu_\nu\right) \\ & - e^\alpha_\nu \partial_\sigma \widehat{m}^{\mathrm{EM}\mu\sigma}_{\mathrm{b}}{}_\alpha - \omega_\sigma{}^a{}_c \widehat{m}^{\mathrm{M}\mu\sigma}_{\mathrm{b}}{}_a e^c_\nu. \end{aligned} \tag{5.8}$$

Here we may mention that because of the use of the current densities, the Hall conductivity (the first term in the current density response) comes out as $\nu/(2\pi)$, which is quantized, times the coordinate-independent $\widehat{\epsilon}^{ij}$, showing quantization with no need to extract a factor involving the vielbeins.

B. Thermoelectric response

We first consider number and energy current density response to Luttinger's $\psi = e^0_0 - 1$ and an electric potential $\phi = -A_0$ to obtain the full set of electric, thermal, and cross conductivities. We assume that $F_{12} = B_0$ is independent of space and time coordinates. A convenient fact about this choice of perturbing fields is that $B = B_0$ and $B_G = 0$, independent of ψ. In particular, this ensures that

$$\partial_\mu \frac{\partial f}{\partial B} = \partial_\mu \frac{\partial f}{\partial B_G} = 0 \tag{5.9}$$

in the Luttinger case. Using this fact, we can expand the number and energy current densities, Eqs. (5.7) and (5.8), to first order in ϕ and ψ to find

$$\widehat{J}^i = -\frac{\nu}{2\pi}\widehat{\epsilon}^{ij}\partial_j \phi + \partial_j\left[\widehat{m}^{ij}_{\mathrm{b},0}(1+\psi)\right], \tag{5.10}$$

$$\widehat{J}^i_E \equiv \widehat{\tau}^{\,i}{}_{\nu=0} = \partial_j\left(\widehat{m}^{ij}_{\mathrm{b},0}\phi\right) + \partial_j\left[\widehat{m}^{\mathrm{E},ij}_{\mathrm{b},0}(1+2\psi)\right]. \tag{5.11}$$

Comparing with Eqs. (1.4) and (1.5), we see that, apart from the Hall conductivity term in $\widehat{J}^\mu$, the terms are precisely the magnetization contributions, though in the present case they result from the bulk only. In particular, this allows us to identify the kinetic coefficients $L^{(n)}$ (which obey Onsager reciprocity provided $\widehat{m}^{ij}_{\mathrm{b},0}$ is an odd function of B),

$$L^{(1)}_{ij} = \frac{\nu}{2\pi}\widehat{\epsilon}^{ij}, \tag{5.12}$$

$$L^{(2)}_{ij} = L^{(3)}_{ij} = -\widehat{m}^{ij}_{\mathrm{b},0} = \frac{\partial f}{\partial B}\widehat{\epsilon}^{ij}, \tag{5.13}$$

$$L^{(4)}_{ij} = -2\widehat{m}^{\mathrm{E},ij}_{\mathrm{b},0} = 2\frac{\partial f}{\partial B_G}\widehat{\epsilon}^{ij}. \tag{5.14}$$

This is one of our main results and requires further discussion. The bulk magnetization currents are equilibrium effects and, because of boundary contributions to the current from the same terms in the action, do not contribute to the net current across any section of the system. Neither does $L^{(4)}$ bear any particular relation to the central charge c.

Comparing with the work of CHR, in their case the magnetization includes edge effects, and these are not just the contributions that relate to the bulk magnetization (the latter is temperature independent, up to exponentially small corrections, while there are thermal edge currents of order T^2). Moreover, the transport current densities, which correspond to the net current through a section across the sample, were declared to be due to bulk transport current density with no contribution located on the edge, *by definition*. When this is done, the effect of thermal excitation at the edge that produces the thermal Hall conductivity (related to the central charge) is reassigned as a bulk effect. At the same time the bulk magnetization effects are canceled in the transport current densities by the corresponding part of the edge currents, as we have seen. (As they emphasize, the actual local current density in the bulk, which is what we have studied, is not the same as the transport current density.) In equations, their prescription for the transport coefficients is given above in Eq. (1.13). (At this stage, they are written for the response of the heat, not energy, current density to perturbations that couple to number and heat, not energy.) If we use our results along with the known $O(T^2)$ edge contribution to the energy magnetization, we finally obtain

$$\begin{aligned} N^{(1)}_{ij} &= \frac{\nu}{2\pi}\widehat{\epsilon}^{ij}, \\ N^{(2)}_{ij} &= N^{(3)}_{ij} = 0, \\ N^{(4)}_{ij} &= \frac{\pi c}{6}T^2\widehat{\epsilon}^{ij}. \end{aligned} \tag{5.15}$$

These results are checked explicitly for a noninteracting integer quantum Hall system in Appendix B.

We see then that, excepting the Hall current, the thermoelectric transport currents are due solely to edge effects and flow along the edge, even if gravitational background fields are included. (In the case of the Hall number current density response, in the more general situation in which there is a bulk electric field as well as a chemical potential gradient, there are contributions from both the bulk and the edge, such that the net current through a section is proportional to the change in the electrochemical potential across the sample, which is the statement of the quantized Hall effect. This was well understood in the 1980s, but is the subject of frequent misstatements at present.) The confusion that exists in the literature concerning the thermal Hall conductivity arises from the aforementioned fact that the total magnetization densities, and thus the transport current densities, are typically defined by fiat to include effects from the edge. We have shown here that bulk thermoelectric response is independent of these edge contributions. This result should be contrasted with the recent claims of Ref. [26].

C. Stress response

Finally, we consider the response of the stress tensor to time-varying spatial perturbations $\delta e^a_i(t)$ of the vielbeins, once again with a spatially uniform and time-independent electromagnetic field strength $F_{12} = B_0$.

Expanding Eq. (5.8) to linear order in the perturbing fields, we find

$$\begin{aligned} -\widehat{\tau}^{\,i}_{\ j} &= \delta^i_j\left[p_{\text{int},0} + \left(p_{\text{int},0} + B_0^2\frac{\partial^2 f}{\partial B^2}\right)\text{tr}\big(\delta e^a_k\big)\right] \\ &\quad + B_0\frac{\partial\gamma}{\partial B}\widehat{\epsilon}^{a\nu\lambda}T^a_{\nu\lambda}\delta^i_j + \gamma\widehat{\epsilon}^{i\rho\lambda}T^a_{\rho\lambda}\delta^a_j \\ &\quad + 2\delta^a_j\widehat{\epsilon}^{i\sigma\rho}\big[\partial_\sigma\big(\gamma e^a_\rho\big) + \gamma\omega_\sigma{}^a{}_b e^b_\rho\big]. \end{aligned} \tag{5.16}$$

All terms proportional to $\widetilde{\gamma}$ cancel. Using the structure equation (2.5), we can combine the second, third, and fourth terms to obtain, when $\omega = 0$,

$$\begin{aligned} -\widehat{\tau}^{\,i}_{\ j} &= \delta^i_j\left[p_{\text{int},0} + \left(p_{\text{int},0} + B_0^2\frac{\partial^2 f}{\partial B^2}\right)\text{tr}\big(\delta e^a_k\big)\right] \\ &\quad + 2\gamma\big(\delta^a_j\epsilon^i{}_\ell - \delta^i_\ell\epsilon^a{}_j\big)\partial_0 e^a_\ell \\ &\quad + \left(\gamma - B_0\frac{\partial\gamma}{\partial B}\right)\big(\epsilon^i{}_j\delta^\ell_a - \delta^i_j\epsilon^{\ell a}\big)\partial_0 e^a_\ell. \end{aligned} \tag{5.17}$$

The first term is what is expected for the response to dilations [22] and allows us to identify the inverse internal compressibility,

$$\kappa^{-1}_{\text{int}} = -B^2\frac{\partial^2 f}{\partial B^2}. \tag{5.18}$$

The second arises from the spacelike torsion term with coefficient γ and gives what has been called "torsional Hall viscosity" [5] and is, in fact, equal to one-half the unperturbed spin density as in Refs. [9,10]. The final term breaks the symmetry of the stress tensor and is necessary to ensure that the continuity equation (2.42) is satisfied.

Even though we do not analyze all possible two-derivative terms in this paper, we also examine the contribution of the first Wen-Zee term,

$$S^{\text{WZ1}} = \frac{\nu\mathcal{S}}{4\pi}\int d^3x\sqrt{g}\epsilon^{\mu\nu\lambda}\omega_\mu\partial_\nu A_\lambda, \tag{5.19}$$

to the stress tensor. Here $\mathcal{S}$ is the shift [8]. While of degree 2 in our counting scheme, it should be kept here because we keep the field strength B_0 as if it were of degree 0. We find, trivially, that this term contributes nothing to the stress, as the spin connection and vector potential are independent of the vielbeins. This is contrary to our expectation that the first Wen-Zee term furnishes a Hall viscosity with a coefficient containing $\mathcal{S}/4$ [6,7]. It does, however, contribute to the spin current a term $\frac{\nu\mathcal{S}}{4\pi}\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda$.

To make sense of these results, we now argue that Eq. (5.8) is not the physical stress tensor corresponding to momentum transport, as Eq. (2.45) shows that it is not symmetric even in the absence of torsion. For various reasons [27,28], it is preferable to use a symmetric stress tensor. This is accomplished with the Belinfante "improved" energy-momentum-stress tensor density. We obtain this by changing our view of which variables are independent in the description of the spacetime geometry. Instead of e^α_μ and ω_μ, from which the Christoffel symbols and torsion were derived using covariant constancy of the vielbein, we now change to using e^α_μ (or the corresponding tensors with upper and lower indices interchanged) and the reduced torsion $\widetilde{T}^a_{\mu\nu}$ defined in Appendix A as the independent variables (the reduced torsion has the same number of independent components as the spin connection). We show in Appendix A that the Christoffel symbols, spin connection, and spacelike torsion can be expressed in terms of these (we already know that the timelike torsion can be). Then we make the definition

$$\tau_{\mathrm{B}}{}^{\mu}{}_{\alpha} = -\frac{1}{\sqrt{\widehat{g}}}\left(\frac{\delta S^{\mathrm{eff}}}{\delta e^\alpha_\mu}\right)_{A,\widetilde{T}^a} \tag{5.20}$$

and call this the generalized Belinfante energy-momentum-stress tensor because it resembles the Belinfante improvement procedure (which we do not describe, but it involves adding derivatives of the spin current to the energy-momentum-stress tensor), and it is "generalized" because we include torsion. Further details are in Appendix A; we note here only that the space components are symmetric in the absence of reduced torsion. In general, the change in definition has the consequence that the Belinfante energy-momentum-stress tensor differs from τ^μ_α by a change in the momentum magnetization from the locally invariant terms in S^{eff} and by the appearance of terms coming from the Riemann tensor in the effective action, which, however, we do not have in our first-order S^{eff}. Clearly, we should use this definition throughout, including for the thermoelectric responses in the previous section. However, for those responses, the change in definition makes no difference. Finally, we continue to refer to the spin current with the same definition as before for convenience; however, the formalism leads us to introduce another field which takes the place of the spin current in some expressions, which is

$$\theta^{\mu\nu}{}_{a} = \frac{1}{\sqrt{\widehat{g}}}\left(\frac{\delta S^{\mathrm{eff}}}{\delta \widetilde{T}^a_{\mu\nu}}\right)_{A,e^\alpha}, \tag{5.21}$$

which, in fact, is exactly the part of the momentum magnetization removed in this construction, and so appeared as a term in the spin current (up to a vielbein factor).

Using our first-order effective action, we find for the Belinfante energy-momentum-stress tensor density

$$\begin{aligned}\widehat{\tau}_{\mathrm{B}}{}^{\mu}{}_{\nu} = &-\sqrt{\widehat{g}}\delta^\mu_\nu f - \gamma\widehat{\epsilon}^{\mu\rho\lambda}\widetilde{T}^a_{\rho\lambda}e^a_\nu - 2\widetilde{\gamma}\widehat{\epsilon}^{\mu\rho\lambda}(\partial_\rho e^b_\lambda)\epsilon_{ab}e^a_\nu\\ &-\frac{1}{2}\widehat{\epsilon}^{\rho\sigma\lambda}\left(\frac{\partial f}{\partial B}F_{\sigma\lambda} + \frac{\partial f}{\partial B_G}T^0_{\sigma\lambda}\right)\left(e^0_\nu\delta^\mu_\rho - e^0_\rho\delta^\mu_\nu\right)\\ &- B_T\widehat{\epsilon}^{\rho\sigma\lambda}\left(\frac{\partial \gamma}{\partial B}F_{\sigma\lambda} + \frac{\partial \gamma}{\partial B_G}T^0_{\sigma\lambda}\right)\left(e^0_\nu\delta^\mu_\rho - e^0_\rho\delta^\mu_\nu\right)\\ &- \widetilde{B}_T\widehat{\epsilon}^{\rho\sigma\lambda}\left(\frac{\partial \widetilde{\gamma}}{\partial B}F_{\sigma\lambda} + \frac{\partial \widetilde{\gamma}}{\partial B_G}T^0_{\sigma\lambda}\right)\left(e^0_\nu\delta^\mu_\rho - e^0_\rho\delta^\mu_\nu\right)\\ &- e^b_\nu\partial_\nu\left(\epsilon^{\rho\mu\nu}\sqrt{\widehat{g}}2\widetilde{\gamma}e^a_\rho\epsilon_{ab}\right) - e^0_\nu\partial_\sigma\widehat{m}_{\mathrm{b}}^{\mathrm{E}\mu\sigma},\end{aligned} \tag{5.22}$$

which, as noted above, differs from Eq. (5.8) in that it receives no magnetization stress contribution from the reduced torsion.

Expanding to linear order in the perturbing fields $\delta e^a_i(t)$ and $\widetilde{T}^a_{\mu\nu}$, we find

$$\begin{aligned}-\widehat{\tau}_{\mathrm{B}}{}^{i}{}_{j} = &\ \delta^i_j\left[p_{\mathrm{int},0} + \left(p_{\mathrm{int},0} + B_0^2\frac{\partial^2 f}{\partial B^2}\right)\mathrm{tr}\left(\delta e^a_k\right)\right]\\ &+ B_0\frac{\partial\gamma}{\partial B}\epsilon^{a\nu\lambda}\widetilde{T}^a_{\nu\lambda}\delta^i_j + \gamma\epsilon^{i\rho\lambda}\widetilde{T}^a_{\rho\lambda}\delta^a_j.\end{aligned} \tag{5.23}$$

The first term here is unchanged. From the tensor structure of the second term, we see that it gives a change in pressure in the presence of background reduced spacelike torsion. The last term breaks the symmetry of the stress tensor in the presence of reduced torsion, which is necessary given the symmetrization condition Eq. (A13) derived in Appendix A. Some intuition for these two terms can be gleaned from the fact that they can be reexpressed as

$$B_0\frac{\partial\gamma}{\partial B}\epsilon^{a\nu\lambda}\widetilde{T}^a_{\nu\lambda}\delta^i_j + \gamma\epsilon^{i\rho\lambda}\widetilde{T}^a_{\rho\lambda}\delta^a_j = \frac{1}{2}e^\alpha_j\widetilde{T}^a_{\rho\lambda}\frac{\delta\widehat{\theta}^{\rho\lambda}{}_a}{\delta e^\alpha_i}, \tag{5.24}$$

expanded to linear order in the external fields. We thus see that these two contributions to the Belinfante tensor correspond to the change in momentum magnetization density due to strain. The stress tensor does not receive a contribution of the form of the "torsional Hall viscosity" mentioned above. A similar effect was noted in Ref. [29] for the relativistic case. If we define viscosity as the response to $\partial_0 e^a_i$ at zero reduced spacelike torsion, then we obtain no viscosity terms at all from the first-order action.

In addition, we now make an important point: The minimally coupled microscopic matter action in Sec. II B does not feel reduced torsion at all in the case of noninteracting particles, or of particles with a δ-function interaction, or for general potential interactions at least when $B_G = 0$ (so spacelike hypersurface exist). In all these cases, the microscopic action depends only on the vielbein and vector potential, not on the reduced torsion. Consequently [see Eqs. (A9) and (A10) in Appendix A], for all such cases the coefficient function γ in the first-order effective action is zero, at least for $B_G = 0$,

$$\gamma(B,0) = 0, \tag{5.25}$$

while $\widetilde{\gamma}$ does not have to vanish. Hence, in these cases the unusual contributions to the stress response are simply absent, and both the current $\theta^{\mu\nu}_a$ and the spin current $J_S{}^{\mu}{}_{a}{}^{b}$ resulting from the first-order action are zero.

Finally, our construction of the Belinfante stress tensor allows us to see how the first Wen-Zee term furnishes a Hall viscosity even in the absence of reduced torsion (we already saw that it produces an addition to the spin current). Equation (A8) allows us to express the first Wen-Zee term solely in terms of the electromagnetic field strength, the reduced torsion, and the spacelike vielbeins. Modulo reduced torsion terms that are locally invariant, which we drop, the first Wen-Zee term Eq. (5.19) becomes

$$S^{\mathrm{WZ1}} \sim \frac{\nu\mathcal{S}}{16\pi}\int d^3x\sqrt{\widehat{g}}\epsilon^{\mu\nu\lambda}\epsilon^{ab}F_{\mu\nu}\left(e^\rho_a e^\sigma_b\partial_\sigma h_{\rho\lambda} + e^\rho_a\partial_\lambda e^b_\rho\right). \tag{5.26}$$

Computing the contribution of this term in the action to the Belinfante stress tensor to linear order in the perturbations yields an additional contribution,

$$\Delta\widehat{\tau}_{\mathrm{B}}{}^{i}{}_{j} = \tfrac{1}{4}\overline{n}\mathcal{S}(\epsilon^{i\ell}\delta_{kj} - \epsilon^{kj}\delta_{i\ell})\partial_0 e^{\ell}_{k}. \tag{5.27}$$

As expected, this has the form of a Hall viscosity, with known coefficient [9,10]

$$\eta^{\mathrm{H}} = \tfrac{1}{4}\overline{n}\mathcal{S}. \tag{5.28}$$

Because the Wen-Zee term is not locally invariant, this contribution is not a magnetization stress, and the coefficient $\nu\mathcal{S}$ is robust against perturbations of the model that maintain the gap and preserve all symmetries. This confirms that previous Berry phase [9,10] and linear response [22] calculations of the Hall viscosity yielded a true transport coefficient. The locally invariant contributions which we have ignored in this analysis only add to the expression for $\theta^{\mu\nu}{}_{a}$ appearing in Eq. (5.24). They contribute nothing to the stress tensor when the reduced torsion is zero and cannot arise in the minimally coupled models when B_G is zero. In the latter case, for the action we have considered, the spin current is due solely to the Wen-Zee term as well, and the relation of the Hall viscosity with the spin density [9,10] $\overline{ns} = \overline{n}\mathcal{S}/2$ is found also (we expect this relation to be maintained when other second- or higher-order terms are included as well). We also note that the Wen-Zee term as above does not contribute to the thermoelectric transport.

VI. CONCLUSION

We have found a low-energy induced bulk action for transport in gapped topological phases by allowing the spacetime geometry to include timelike and spacelike torsion, as well as curvature. From this, we derived the bulk thermoelectric transport coefficients and showed that a gapped bulk cannot contribute to thermal conductivity or thermopower, up to exponentially small corrections in temperature. We examined the stress tensor and showed that any torsional Hall viscosity drops out in the appropriate Belinfante improved tensor, leaving the Hall viscosity that is related to the orbital spin density.

A similar approach can be taken for other terms in the action that are higher than first order in derivatives in our counting scheme. These do not contribute directly to transport, but we expect the central charge to appear as a coefficient. We defer the treatment of these terms to a future work.

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APPENDIX A: GENERALIZED BELINFANTE CONSTRUCTION

In this appendix, we generalize the Belinfante construction of a symmetric stress tensor to situations in which spacetime has torsion. Our guiding principle is that we demand that the stress tensor correspond as closely as possible with a variation of the action with respect to the (degenerate) spatial metric

$$h_{\mu\nu} = e^{a}_{\mu}e^{a}_{\nu}, \tag{A1}$$

rather than to a variation of the action with respect to the vielbein with the spin connection held fixed. If the spin connection can be expressed in terms of the vielbein and torsion, and if these are independent (if there are no relations between torsion and vielbeins), then varying the vielbeins with the torsion held fixed will produce such a stress tensor, very much in analogy with the usual case (in particular, when torsion is absent throughout). In our case, the timelike torsion and, as it turns out, also part of the spacelike torsion are determined by the vielbeins alone, independent of the spin connection, so that if we desire (as we do) to have no constraints on the vielbeins, we cannot take all components of torsion as independent, because, for example, they cannot all be set to zero without introducing unwanted constraints on the vielbeins.

First we solve Eq. (2.4) for the Christoffel symbols and the spin connection. To this end, we note first that an immediate consequence of the covariant constancy of the vielbein is the covariant constancy of the degenerate metric $h_{\mu\nu}$, that is,

$$\nabla_\mu h_{\nu\lambda} = 0 = \partial_\mu h_{\nu\lambda} - \Gamma^{\rho}{}_{\mu\nu}h_{\rho\lambda} - \Gamma^{\rho}{}_{\mu\lambda}h_{\nu\rho}. \tag{A2}$$

We can solve this equation for the symmetric part of the Christoffel symbols, while the antisymmetric part is determined solely by the torsion tensor (timelike and spacelike). The result can be expressed most simply as

$$\begin{aligned}\Gamma^{\alpha}{}_{\lambda\nu} \equiv e^{\alpha}_{\mu}\Gamma^{\mu}{}_{\lambda\nu} = \delta^{\alpha}_{0}\partial_\lambda e^{0}_{\nu} + \delta^{\alpha}_{a}\big[\tfrac{1}{2}\eta^{ab}e^{\mu}_{b}(\partial_\nu h_{\lambda\mu}\\ + \partial_\lambda h_{\mu\nu} - \partial_\mu h_{\nu\lambda}) + K^{a}_{\lambda\nu}\big],\end{aligned} \tag{A3}$$

where we have introduced the contorsion tensor

$$K^{a}_{\lambda\nu} = \tfrac{1}{2}\big[T^{a}_{\lambda\nu} + \eta^{ab}\big(e^{\mu}_{b}e^{c}_{\lambda}T^{c}_{\mu\nu} + e^{\mu}_{b}e^{c}_{\nu}T^{c}_{\mu\lambda}\big)\big]. \tag{A4}$$

Next, with explicit expressions for the Christoffel symbols in hand, we wish to solve Eq. (2.4) for the spin connection. Examination of Eq. (2.5) shows that (similar to the case of the timelike torsion) because the spin connection vanishes when either of its internal indices are timelike, there is a part of the spacelike torsion that is independent of the spin connection. This part can be expressed as

$$C^{ab} \equiv e^{\mu}_{0}e^{\nu}_{c}\big(\eta^{ac}T^{b}_{\mu\nu} + \eta^{bc}T^{a}_{\mu\nu}\big) \tag{A5}$$

$$= \big(e^{\mu}_{0}e^{\nu}_{c} - e^{\nu}_{0}e^{\mu}_{c}\big)\big(\eta^{ac}\partial_\mu e^{b}_{\nu} + \eta^{bc}\partial_\mu e^{a}_{\nu}\big). \tag{A6}$$

This allows us to define what we call the *reduced torsion*, which is purely spacelike:

$$\widetilde{T}^{a}_{\mu\nu} \equiv T^{a}_{\mu\nu} - \tfrac{1}{2}\eta_{bc}C^{ab}\big(e^{0}_{\mu}e^{c}_{\nu} - e^{0}_{\nu}e^{c}_{\mu}\big). \tag{A7}$$

The components of the reduced torsion are not all independent; it is defined so that it yields zero when substituted into the definition of C^{ab} [Eq. (A5)]. This is natural: We are seeking a linear relation between the torsion and spin connection, but the latter has $d(d-1)(d+1)/2$ independent components, while spacelike torsion has $d^2(d+1)/2$ independent components. Taking into account the $d(d+1)/2$ constraints from setting Eq. (A5) to zero, we are left with $d(d-1)(d+1)/2$ independent components of reduced spacelike torsion, the same as in the spin connection, as required.

Specializing to $2+1$ dimensions, we can now solve Eq. (2.4) for the spin connection in terms of the reduced torsion and the vielbeins to find

$$\omega_\lambda \equiv \tfrac{1}{2}\epsilon_a{}^b\omega_\lambda{}^a{}_b = \tfrac{1}{2}\epsilon^{ab}e^\mu_a e^\nu_b\big(\partial_\nu h_{\mu\lambda} + \tfrac{1}{2}e^c_\lambda \widetilde{T}^c_{\mu\nu}\big) + \tfrac{1}{2}\epsilon^{ab}e^\mu_a\big(\partial_\lambda e^b_\mu + \widetilde{T}^b_{\mu\lambda}\big). \tag{A8}$$

There are similar expressions in higher dimensions. We thus see that we are free to consider the reduced torsion, instead of the spin connection, as an independent variable along with the vielbeins and the U(1) vector potential A_μ. We also note that the scalars constructed from the torsion that were defined in Sec. IV can be written as

$$B_T = \tfrac{1}{2}\epsilon^{\mu\nu\lambda}e^a_\mu \widetilde{T}^a_{\nu\lambda}, \tag{A9}$$

$$\widetilde{B}_T = \tfrac{1}{2}\eta_{ab}C^{ab}. \tag{A10}$$

The preceding allows us to define the generalized Belinfante energy-momentum-stress tensor $\tau_{\rm B}{}^\mu{}_\alpha$ resulting from an action S as in Eq. (5.20), where the reduced torsion is held fixed in the functional derivative. We claim that τ_B represents the *physical* energy-momentum-stress tensor. To justify this, we must derive the continuity equation that it satisfies. To do so, we also need $\theta^{\mu\nu}_a$ as defined in Eq. (5.21); it is the analog of the spin current (which has the same number of independent components). Examining the variation of a general action under spacetime diffeomorphism as in Sec. II, we find that the generalized Belinfante energy-momentum-stress tensor satisfies the continuity equation (after use of the equations of motion, if any)

$$\begin{aligned}\nabla_\mu\tau_{\rm B}{}^\mu{}_\lambda - T^\rho_{\rho\mu}\tau_{\rm B}{}^\mu{}_\lambda = {}& 2\widetilde{T}^a_{\mu\lambda}\big(\nabla_\nu\theta^{\mu\nu}{}_a - T^\rho_{\nu\rho}\theta^{\mu\nu}{}_a\big) + \theta^{\mu\nu}{}_a T^\rho_{\mu\nu}\widetilde{T}^a_{\rho\lambda}\\ &+ \theta^{\mu\nu}{}_a\big(\nabla_\lambda\widetilde{T}^a_{\mu\nu} + \nabla_\nu\widetilde{T}^a_{\lambda\mu} + \nabla_\mu\widetilde{T}^a_{\nu\lambda}\big)\\ &+ \tau_{\rm B}{}^\mu{}_\nu T^\nu_{\mu\lambda} - J^\mu F_{\mu\lambda}.\end{aligned} \tag{A11}$$

While this expression is quite unwieldy to say the least, it satisfies an important property: It reduces to $\nabla_\mu\tau_{\rm B}{}^\mu{}_\lambda = -J^\mu F_{\mu\lambda}$ when the *full* torsion $T^\lambda_{\mu\nu} = 0$ throughout the spacetime region in question. (To see this, note that the projection of torsion to reduced torsion is linear and that the coefficients involve the vielbeins, which are covariantly constant.) Compared with the continuity equation (2.42), the spin current times curvature tensor term in that equation has disappeared, though the terms $\theta\nabla\widetilde{T}$ are related to it, in view of the second Bianchi identity [17],

$$\partial_{[\mu}T^a_{\nu\lambda]} + \omega_{[\mu}{}^a{}_{|b|}T^b_{\nu\lambda]} = R_{[\mu\nu}{}^a{}_{|b|}e^b_{\lambda]} \tag{A12}$$

(recall that the indices surrounded by vertical bars $|\cdots|$ are *not* antisymmetrized with the others, namely μ, ν, and λ).

Additionally, by considering invariance of a general action under internal spatial rotations, we find that the Belinfante stress tensor satisfies the symmetry condition

$$\epsilon^a{}_b\tau_{\rm B}{}^b{}_a = \epsilon^a{}_b\theta^{\mu\nu}{}_a\widetilde{T}^b_{\mu\nu}, \tag{A13}$$

so that it is symmetric in the absence of reduced torsion, even when the full torsion tensor is nonvanishing. We thus see that our definition of τ_B reduces to the standard Belinfante energy-momentum-stress tensor in the absence of torsion, and is symmetric in the presence of torsion provided the reduced torsion vanishes. Therefore, we claim that it represents the physical stress tensor of a general system.

An important special case that illustrates the significance of reduced torsion and the Belinfante construction is a $2+1$-dimensional manifold with vielbeins that differ from the trivial ones $e^\alpha_\mu = \delta^\alpha_\mu$ only in the space-space components e^a_i and are independent of the space coordinates. If we try to directly solve the Cartan equations (2.5) for the spin connection with the torsion set to zero, we find that they are inconsistent. In fact, for this geometry,

$$C^{ab} = \eta^{cb}e^i_c\partial_0 e^a_i + \eta^{ac}e^i_c\partial_0 e^b_i, \tag{A14}$$

and hence the spacelike torsion does not vanish for any choice of spin connection. The reduced torsion, however, may be set to any arbitrary value, and the Cartan equations can be solved to find a spin connection that is gauge equivalent to Eq. (A8). Note that this geometry is precisely what is usually considered for computations of the viscosity tensor [9,10,22,30], although the nonvanishing of the spacelike torsion has not previously been noted to our knowledge. Our generalized Belinfante construction ensures the existence of a symmetric stress tensor provided one takes the reduced torsion to be zero. This is done implicitly in the condensed-matter literature, since, as noted above, the reduced torsion does not enter into any usual microscopic actions.

There exists an explicit formula for the improvement term needed to convert the canonical stress tensor into this Belinfante form. It can be derived from

$$\tau_{\rm B}{}^\mu{}_\alpha - \tau^\mu{}_\alpha = -\frac{1}{\sqrt{g}}\int d^{d+1}x\sqrt{g}J_{\rm S}{}^\lambda{}_a{}^b\left(\frac{\delta\omega_\lambda{}^a{}_b}{\delta e^\alpha_\mu}\right)_{\widetilde{T}^a}, \tag{A15}$$

although the general expressions are quite cumbersome and unilluminating. We note only that, in the absence of torsion, the improvement term reduces to the known Belinfante improvement term; see, for example, Refs. [27,28].

APPENDIX B: LINEAR RESPONSE CALCULATION OF THERMOELECTRIC COEFFICIENTS FOR NONINTERACTING ELECTRONS

In this appendix, we recapitulate the standard linear response calculation of the response functions for noninteracting electrons in an integer quantum Hall phase.

1. Operator formalism

We consider a model Hamiltonian for a system of electrons in a magnetic field,

$$H_0 = \sum_p \frac{\pi^p_i\pi^p_i}{2m} + V(\mathbf{r}_p), \tag{B1}$$

$$= \sum_p h^p, \tag{B2}$$

where we use $i,j = 1,2$ for spatial indices as above and $p,q = 0,1,\ldots,N$ for particle indices; h^p is the Hamiltonian for the pth particle. The π^p_i are the kinetic momenta, and

$$\left[r_i^p,\pi_j^q\right] = i\delta_{pq}\delta_{ij}, \tag{B3}$$

$$\left[\pi_i^p\pi_j^q\right] = iB\epsilon_{ij}\delta_{pq}, \tag{B4}$$

with B the magnetic field strength. As our goal will be to calculate thermoelectric coefficients, in particular the thermal conductivity tensor κ_{ij}, we need to identify the number current density $J_i(\mathbf{r})$, the heat current density $J_i^{\mathrm{Q}}(\mathbf{r})$, and the perturbations to which they couple.

Following Luttinger and CHR, we introduce a Hamiltonian density $h(\mathbf{r})$, a perturbing electric field $\phi(\mathbf{r})$, and a fictitious gravitational field $\psi(\mathbf{r})$ and identify the perturbed Hamiltonian as

$$H_T = \int d^2r\{[1+\psi(\mathbf{r})]h(\mathbf{r}) + \phi(\mathbf{r})\rho(\mathbf{r})\} \tag{B5}$$

$$= \int d^2r h_T(\mathbf{r}), \tag{B6}$$

where

$$\rho(\mathbf{r}) = \sum_p \delta(\mathbf{r}-\mathbf{r}_i) \tag{B7}$$

is the density operator, and the Hamiltonian density $h(\mathbf{r})$ satisfies

$$\int d^2r h(r) = H_0. \tag{B8}$$

Note that there is an operator ordering ambiguity inherent in any attempt to define the energy density $h(\mathbf{r})$. It is essential later that we adopt the definition

$$h(\mathbf{r}) = \sum_p \left[\frac{1}{2m}\pi_i^p\delta(\mathbf{r}-\mathbf{r}_i)\pi_i^p + V(\mathbf{r_i})\delta(\mathbf{r}-\mathbf{r}_i)\right]. \tag{B9}$$

This differs from the more commonly used expression $h^{\mathrm{CHR}}(\mathbf{r})$ by

$$h(\mathbf{r}) = h^{\mathrm{CHR}}(\mathbf{r}) + \frac{1}{2m}\nabla^2\rho(\mathbf{r}) \tag{B10}$$

and instead corresponds closely with the second-quantized energy density operator of Sec. II used more recently in the literature [15,23]. Note that $h(\mathbf{r})$ yields a positive definite kinetic energy density, while $h^{\mathrm{CHR}}(\mathbf{r})$ does not. This justifies its use, contrary to established convention.

The number and energy currents are determined up to a divergence-free part by the continuity equations

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0, \tag{B11}$$

$$\frac{\partial h_T}{\partial t} + \nabla\cdot\mathbf{J}^{\mathrm{E}} = \left(\frac{\partial\phi}{\partial t} + \frac{\partial\psi}{\partial t}\right)h \tag{B12}$$

[the right-hand side of Eq. (B12) accounts for the fact that the *explicit* time dependence of the perturbing fields breaks energy conservation]. In order to fix the divergence-free pieces of the currents, we demand that the CHR scaling relations

$$\mathbf{J}(\mathbf{r}) = [1+\psi(\mathbf{r})]\mathbf{j}(\mathbf{r}), \tag{B13}$$

$$\mathbf{J}^{\mathrm{E}}(\mathbf{r}) = [1+2\psi(\mathbf{r})]\mathbf{j}^E(\mathbf{r}) + \phi(\mathbf{r})\mathbf{j}(\mathbf{r}), \tag{B14}$$

hold to first order in ϕ and ψ, where $\mathbf{j}(\mathbf{r})$ and $\mathbf{j}^E(\mathbf{r})$ are the unperturbed number and energy currents, respectively. These have exactly the form obtained from the formalism in Sec. II, with $e_\mu^0 = \delta_\mu^0(1+\psi)$. A short calculation for the number current reveals the standard result,

$$j_i(\mathbf{r}) = \frac{1}{2m}\sum_p\left\{\pi_i^p,\delta(\mathbf{r}-\mathbf{r}_i)\right\}, \tag{B15}$$

and for the energy current,

$$j_i^E(\mathbf{r}) = \frac{1}{2m}\sum_p\left\{\pi_i^p,\frac{1}{2m}\pi_j^p\delta(\mathbf{r}-\mathbf{r}_i)\pi_j^p + V(\mathbf{r}_i)\delta(\mathbf{r}-\mathbf{r}_i)\right\}$$
$$-\frac{i}{8m^2}\epsilon_{ij}\partial_j\left[\epsilon_{kl}\pi_k^p\delta(\mathbf{r}-\mathbf{r}_i)\pi_l^p\right]. \tag{B16}$$

We are now interested in the linear response of the total (or integrated) currents to the perturbations $-\nabla\phi$ and $-\nabla\psi$, to lowest order in wave vector $\mathbf{q}$. Denote the integrated currents by

$$\bar{\mathbf{J}} = \frac{1}{V}\int d^2r\mathbf{J}(\mathbf{r}) = \frac{1}{mV}\sum_p\pi^p, \tag{B17}$$

$$\bar{\mathbf{J}}^{\mathrm{E}} = \frac{1}{V}\int d^2r\mathbf{J}^{\mathrm{E}}(\mathbf{r}) = \frac{1}{2mV}\sum_p\{\pi^p,h^p\}, \tag{B18}$$

where V is the volume of the system. Since we are interested in vanishing $\mathbf{q}$, we may take for the perturbations

$$\phi(\mathbf{r}) = -E_i(t)r_i, \tag{B19}$$

$$\psi(\mathbf{r}) = -G_i(t)r_i, \tag{B20}$$

where we have restored the explicit time dependence of the perturbation. Then what we want to compute are the zero frequency response coefficients $R_{ij}^{(n)}$ satisfying

$$\delta\langle\bar{J}_i\rangle = R_{ij}^{(1)}E_j + R_{ij}^{(2)}G_j, \tag{B21}$$

$$\delta\langle\bar{J}_i^E\rangle = R_{ij}^{(3)}E_j + R_{ij}^{(4)}G_j. \tag{B22}$$

Now the response functions $R^{(n)}$ are not simply given by the naive Kubo formulas; the scaling relations Eqs. (B13) and (B14) imply the presence of contact terms. For $R_{ij}^{(1)}$ this is not the case and we have simply that

$$R_{ij}^{(1)} = \sigma_{ij}, \tag{B23}$$

the usual zero-frequency conductivity tensor. For $R_{ij}^{(2)}$ we have in linear response

$$R_{ij}^{(2)} = L_{ij}^{(2)} - \frac{1}{V}\int d^2r\langle r_j j_i(\mathbf{r})\rangle_0, \tag{B24}$$

$$L_{ij}^{(2)} = i\int_0^\infty dt\, e^{i\omega^+t}\left\langle\left[\bar{J}_i(t),D_j^{\mathrm{E}}(0)\right]\right\rangle_0, \tag{B25}$$

where D_i^{E} is the operator which couples to G_i in the Hamiltonian, henceforth referred to as the *energy polarization*:

$$D_i^{\mathrm{E}} = \int d^2r r_i h(\mathbf{r}) \tag{B26}$$

$$= \frac{1}{2}\sum_p\left\{h^p,r_i^p\right\}. \tag{B27}$$

Defining the magnetization density m_0 to be

$$m_0 = \frac{1}{2V}\int d^2r\langle \mathbf{r}\times\mathbf{j}(\mathbf{r})\rangle, \tag{B28}$$

we see, after some elementary manipulations, that the contact term in Eq. (B24) is simply

$$-\frac{1}{V}\int d^2r\langle r_j j_i(\mathbf{r})\rangle_0 = m_0\epsilon_{ij}. \tag{B29}$$

This term must be present due to the scaling relation (B13), although from the form above we see that it is nonvanishing only when time-reversal symmetry is broken. Similarly, we have for $R^{(3)}_{ij}$

$$R^{(3)}_{ij} = L^{(3)}_{ij} - \frac{1}{V}\int d^2r\langle r_j j_i(\mathbf{r})\rangle_0, \tag{B30}$$

$$L^{(3)}_{ij} = i\int_0^\infty dt\, e^{i\omega^+ t}\big\langle\big[\bar{J}^{\mathrm{E}}_i(t),D_j(0)\big]\big\rangle_0, \tag{B31}$$

where D_i is the ordinary polarization, which couples to E_i in the Hamiltonian, i.e.,

$$D_i = \sum_p r^p_i. \tag{B32}$$

Note that the scaling relation (B14) ensures that it is the ordinary magnetization that again appears as the contact term in Eq. (B30).

Up to now, these formulas have all agreed with those of Středa and Smrčka. For $R^{(4)}$, however, we have

$$R^{(4)}_{ij} = L^{(4)}_{ij} - \frac{1}{V}\int d^2r\big\langle r_j j^{\mathrm{E}}_i(\mathbf{r})\big\rangle_0, \tag{B33}$$

$$L^{(4)}_{ij} = i\int_0^\infty dt\, e^{i\omega^+ t}\big\langle\big[\bar{J}^{\mathrm{E}}_i(t),D^{\mathrm{E}}_j(0)\big]\big\rangle_0; \tag{B34}$$

the contact term can again be expressed in terms of a suitably defined energy magnetization,

$$2m^{\mathrm{E}}_0 = \frac{1}{V}\int d^2r\langle \mathbf{r}\times\mathbf{j}^{\mathrm{E}}(\mathbf{r})\rangle_0 \tag{B35}$$

$$= \frac{1}{V}\sum_p \epsilon_{ij}\big\langle h^p\big\{r^p_i,\pi^p_j\big\}\big\rangle_0 - \frac{B\bar{n}}{4m^2}, \tag{B36}$$

as

$$-\frac{1}{V}\int d^2r\big\langle r_j j^{\mathrm{E}}_i(\mathbf{r})\big\rangle_0 = \left(2m^{\mathrm{E}}_0 + \frac{B\bar{n}}{4m^2}\right)\epsilon_{ij}. \tag{B37}$$

It is important to note that in deriving these contact terms and in relating them to the magnetizations, certain integrals and trace identities need to be used which are only valid if the states live in an honest-to-goodness Hilbert space; i.e., if they are normalizable. Thus, the presence of the confining potential $V(\mathbf{r})$ is indispensable at this stage of the calculation. It is only in the final expressions in Sec. B 3 of this appendix, that we are able to take $V\to 0$.

It is also worth mentioning that these Kubo formulas can be put into a different, more suggestive form. Using the identities for the integrated currents

$$\bar{J}_i = \frac{1}{V}\frac{\partial D_i}{\partial t}, \tag{B38}$$

$$\bar{J}^{\mathrm{E}}_i = \frac{1}{V}\frac{\partial D^{\mathrm{E}}_i}{\partial t}, \tag{B39}$$

we can integrate Eqs. (B23), (B24), (B30), and (B33) by parts to find

$$R^{(1)}_{ij} = \frac{\omega^+}{V}\int_0^\infty dt e^{i\omega^+ t}\langle[D_i(t),D_j(0)]\rangle, \tag{B40}$$

$$R^{(2)}_{ij} = \frac{\omega^+}{V}\int_0^\infty dt e^{i\omega^+ t}\big\langle\big[D^{\mathrm{E}}_i(t),D_j(0)\big]\big\rangle, \tag{B41}$$

$$R^{(3)}_{ij} = \frac{\omega^+}{V}\int_0^\infty dt e^{i\omega^+ t}\big\langle\big[D_i(t),D^{\mathrm{E}}_j(0)\big]\big\rangle, \tag{B42}$$

$$R^{(4)}_{ij} = \frac{\omega^+}{V}\int_0^\infty dt e^{i\omega^+ t}\big\langle\big[D^{\mathrm{E}}_i(t),D^{\mathrm{E}}_j(0)\big]\big\rangle, \tag{B43}$$

where the surface terms arising from the partial integration exactly cancel the magnetization contact terms (in the case of the conductivity $R^{(1)}$, both are identically zero). In this form, we know from the projection theorem of Ref. [22] that in the thermodynamic limit as $\omega\to 0$, the $R^{(n)}_{ij}$ will be dominated (if there were no confining potential) by matrix elements of the polarization operators coming from states degenerate with the ground state. In the presence of the confining potential, however, the center-of-mass degeneracy of the Landau levels is broken, and there is no longer an exact degeneracy. On the other hand, in the thermodynamic limit, edge excitations become gapless and, in fact, have a linear dispersion. The sum over matrix elements then, schematically, produces terms like

$$\omega^+\int d\mathbf{k}\rho(\mathbf{k})\frac{F_{0\mathbf{k}}}{\omega^+ - v\mathbf{k}}, \tag{B44}$$

where $\rho(\mathbf{k})$ is the density of states for the edge excitations. The functions $F_{0\mathbf{k}}$ represent the matrix elements of the polarization operators between the ground state and the various edge-excited states. These can be interpreted as moments of the energy density operator on the edge (since in the bulk the states are indistinguishable). This integral, when viewed as a function of ω, has a branch point at the origin, and therefore the limit $\omega\to 0$ must be evaluated carefully: It will be nonvanishing.

This demonstrates clearly the role of edge states in determining the thermoelectric coefficients. One must keep in mind, however, that the conductivity $R^{(1)}$ is fairly special in this regard. In the absence of a confining potential, the polarization-polarization and current-polarization response functions are completely equivalent; the contributions to the conductivity can be viewed alternatively as coming from the center-of-mass degenerate single-particle states in the thermodynamic limit. For the other response functions, the current-polarization form must be added to the magnetization contribution in order to recover the full response function. In the absence of a confining potential, the magnetization term is not well defined: Particles at larger and larger distances contribute more and more to the magnetization. Thus, for these response function, the presence of edge states is essential.

2. Zero-temperature response in the integer quantum Hall regime

We would now like to explicitly calculate the $R^{(n)}$'s for a system with j filled Landau levels at chemical potential $\mu \in [\omega_c(j-1/2),\omega_c(j+1/2)]$ lying in the bulk gap between levels. We are interested primarily in the low-temperature behavior of the response coefficients. However, for single-particle operators we have

$$\langle\mathcal{O}\rangle(\mu,T) = \int_{-\infty}^{\infty} d\eta\, n_{\mathrm{F}}(\eta,T)\mathrm{Tr}[\delta(\eta-h)\mathcal{O}] \tag{B45}$$

$$= -\int_{-\infty}^{\infty} d\eta \frac{dn_{\mathrm{F}}}{d\eta}(\eta,T)\int_{-\infty}^{\eta} d\zeta\,\mathrm{Tr}[\delta(\zeta-h)\mathcal{O}] \tag{B46}$$

$$= -\int_{-\infty}^{\infty} d\eta \frac{dn_{\mathrm{F}}}{d\eta}(\eta,T)\langle\mathcal{O}\rangle(\mu=\eta,T=0), \tag{B47}$$

where the trace is over all single-particle states, h is the single-particle Hamiltonian, and $n_F(\eta,T) = 1/(e^{(\eta-\mu)/T}+1)$ is the Fermi function. Thus, we can determine the behavior of the response coefficients at nonzero temperature once their zero-temperature behavior is known. Hence, in this section we aim to evaluate the $R^{(n)}_{ij}$ at zero temperature. This was done in Ref. [13] using a resolvent formalism; however, here we proceed directly using the Kubo formulas in the time domain. This allows us to illuminate some subtleties in the derivation. For notational simplicity, it should be understood that the limit $\omega \to 0$ is implied in all expressions in this section.

Let us begin by noting that the conductivity σ_{ij} is given by

$$\sigma_{ij}(\mu) = \frac{i}{mV}\int_{-\infty}^{\mu} d\eta \int_0^{\infty} dt e^{i\omega^+ t}\mathrm{Tr}\{\delta(\eta-h)[\pi_i(t),r_j(0)]\} \tag{B48}$$

$$\equiv \int_{-\infty}^{\mu} d\eta A_{ij}(\eta), \tag{B49}$$

where we have used the freedom afforded us by this noninteracting problem to evaluate the averages using only single-particle states. As its name would suggest, this A_{ij} is a generalization of the function introduced by Smrčka and Středa [13].

Next we examine $R^{(3)}_{ij}$ ($R^{(3)} = R^{(2)}$ via Onsager reciprocity [11,13]). We can write the Kubo part of the response function, $L^{(3)}_{ij}$, as

$$L^{(3)}_{ij} = \int_{-\infty}^{\mu} d\eta \left[\eta A_{ij}(\eta) + \frac{1}{2}B_{ij}(\eta)\right], \tag{B50}$$

where we have defined

$$B_{ij}(\eta) = \frac{1}{m^2V}\int_0^{\infty} dt e^{i\omega^+ t}\mathrm{Tr}[\delta(\eta-h)\{\pi_i(t),\pi_j(0)\}]. \tag{B51}$$

After a partial integration, we find that

$$L^{(3)}_{ij} = \mu\sigma_{ij}(\mu) + \int_0^{\mu} d\eta\,(\eta-\mu)\left(A_{ij} - \frac{1}{2}\frac{dB_{ij}}{d\eta}\right). \tag{B52}$$

We can perform a similar analysis for $L^{(4)}_{ij}$ to find

$$L^{(4)}_{ij} = \frac{B\bar{n}}{4m^2}\epsilon_{ij} + \mu^2\sigma_{ij}(\mu) + \int_{-\infty}^{\mu} d\eta(\eta^2-\mu^2)\left[A_{ij}(\eta) - \frac{1}{2}\frac{dB_{ij}}{d\eta}\right]. \tag{B53}$$

Now it can be shown [13] that

$$A_{ij} - \frac{1}{2}\frac{dB_{ij}}{d\eta} = \epsilon_{ij}\frac{dm_0}{d\eta}. \tag{B54}$$

Plugging this into Eqs. (B52) and (B53), we find

$$L^{(3)}_{ij} = \mu\sigma_{ij}(\mu) - m_0\epsilon_{ij}, \tag{B55}$$

$$L^{(4)}_{ij} = \mu^2\sigma_{ij}(\mu) - 2m_0^{\mathrm{E}}\epsilon_{ij}, \tag{B56}$$

where we have used the relation

$$m^{\mathrm{E}} = \int_{-\infty}^{\mu} d\eta\eta m_0(\eta) - \frac{B\bar{n}}{8m^2}, \tag{B57}$$

which follows from Eq. (B36). From our discussion above, we recognize the magnetization terms in Eqs. (B55) and (B56) as precisely the negative of the contact terms in Eqs. (B30) and (B33). Furthermore, we see that the explicit dependence on the chemical potential in the first terms above indicates that these are edge contributions. Upon subtracting the total magnetizations, we have that the L's are given by the *bulk* contributions to the magnetizations, as asserted in Sec. V and consistent with Eqs. (5.12)–(5.14). Thus, putting everything together, we have

$$R^{(1)}_{ij}(\mu) = \sigma_{ij}(\mu), \tag{B58}$$

$$R^{(2)}_{ij}(\mu) = R^{(3)}_{ij}(\mu) = \mu\sigma_{ij}(\mu), \tag{B59}$$

$$R^{(4)}_{ij}(\mu) = \mu^2\sigma_{ij}(\mu), \tag{B60}$$

in agreement with known results.

3. Extension to nonzero temperature

Having derived expressions for the $R^{(n)}$ at zero temperature, we can now use Eq. (B47) to evaluate the transport coefficients for all values of chemical potential μ and temperature T. Let us start with the Hall conductivity $R^{(1)}$. As is well known, at zero temperature we have in the thermodynamic limit (this is the stage at which it is safe to take the limit) and with chemical potential μ in a bulk gap

$$R^{(1)}_{ij}(\mu) = \frac{1}{2\pi}\epsilon_{ij}\sum_n \Theta(\mu-\epsilon_n), \tag{B61}$$

where n indexes the Landau levels and ϵ_n is the Landau level energy. At nonzero temperature, this becomes

$$R^{(1)}_{ij}(\mu,T) = -\int_{-\infty}^{\infty} d\eta \frac{dn_F}{d\eta} R^{(1)}_{ij}(\eta,0)$$
$$= -\epsilon_{ij}\frac{1}{2\pi}\sum_n \int_{-\infty}^{\infty} d\eta \frac{dn_F(\eta)}{d\eta}\Theta(\eta-\epsilon_n) \tag{B62}$$
$$= \frac{1}{2\pi}\sum_n n_F(\epsilon_n)\epsilon_{ij}, \tag{B63}$$

from which we see that corrections to the low-temperature behavior are exponentially suppressed, as expected. In fact, we have for η in a neighborhood of the chemical potential μ that $R^{(1)}(\eta)$ is a slowly varying function when μ is in a gap (actually, it is a constant), and hence for temperatures $T \ll \omega_c$ we can make use of the Sommerfeld expansion

$$-\frac{dn_F}{d\eta} \approx \delta(\mu-\eta) + \frac{\pi^2}{6}T^2\delta''(\mu-\eta) + \cdots, \tag{B64}$$

from whence we see that the corrections to the conductivity at low temperature are nonperturbatively suppressed.

Using similar logic, we find for the thermoelectric transport coefficients [14,31–33]

$$N^{(1)}_{ij}(\mu,T) = \frac{\nu}{2\pi}\epsilon_{ij}, \tag{B65}$$
$$N^{(2)}_{ij}(\mu,T) = 0, \tag{B66}$$
$$N^{(3)}_{ij}(\mu,T) = 0, \tag{B67}$$
$$N^{(4)}(\mu,T) = \frac{\pi\nu T^2}{6}\epsilon_{ij}. \tag{B68}$$

As emphasized throughout, the nonzero contribution to $N^{(4)}$ is purely an edge effect. We have for an integer quantum Hall system at low temperatures

$$\bar{J}_i = -\frac{\nu}{2\pi}\epsilon_{ij}\partial_j\phi, \tag{B69}$$
$$\bar{J}^{\rm Q}_i = -\frac{\pi\nu T}{6}\epsilon_{ij}\partial_j T. \tag{B70}$$

We see directly from this that the thermal Hall conductivity is given by Eq. (1.1), with central charge $c = \nu$ corresponding to ν filled Landau levels. Note also the Wiedemann-Franz relation

$$\kappa_{ij} = \frac{\pi^2 T}{3}\sigma_{ij}. \tag{B71}$$

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