Gapped quantum liquids and topological order, stochastic local transformations and emergence of unitarity

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In this work, we present some new understanding of topological order, including three main aspects. (1) It was believed that classifying topological orders corresponds to classifying gapped quantum states. We show that such a statement is not precise. We introduce the concept of *gapped quantum liquid* as a special kind of gapped quantum states that can "dissolve" any product states on additional sites. Topologically ordered states actually correspond to gapped quantum liquids with stable ground-state degeneracy. Symmetry-breaking states for on-site symmetry are also gapped quantum liquids, but with unstable ground-state degeneracy. (2) We point out that the universality classes of generalized local unitary (gLU) transformations (without any symmetry) contain both topologically ordered states and symmetry-breaking states. This allows us to use a gLU invariant—topological entanglement entropy—to probe the symmetry-breaking properties hidden in the exact ground state of a finite system, which does not break any symmetry. This method can probe symmetry- breaking orders even without knowing the symmetry and the associated order parameters. (3) The universality classes of topological orders and symmetry-breaking orders can be distinguished by *stochastic local (SL) transformations* (i.e., *local invertible transformations*): small SL transformations can convert the symmetry-breaking classes to the trivial class of product states with finite probability of success, while the topological-order classes are stable against any small SL transformations, demonstrating a phenomenon of emergence of unitarity. This allows us to give a definition of long-range entanglement based on SL transformations, under which only topologically ordered states are long-range entangled.

I. INTRODUCTION

Topological order was first introduced as a new kind of order beyond Landau symmetry breaking theory [1–3]. At the beginning, it was defined by (a) the topology-dependent ground-state degeneracy [1,2] and (b) the non-Abelian geometric phases of the degenerate ground states [3,4], where both of them are *robust against any local perturbations* that can break any symmetry [2]. This is just like that superfluid order is given by zero-viscosity and quantized vorticity, which are robust against any local perturbations that preserve the U(1) symmetry. Chiral spin liquids [5,6], integral/fractional quantum Hall states [7–9], $\mathbb{Z}_2$ spin liquids [10–12], non-Abelian fractional quantum Hall states [13–16], etc. are examples of topologically ordered phases.

Microscopically, superfluid order is originated from boson or fermion-pair condensation. So it is natural for us to ask: what is the microscopic origin of topological order? What is the microscopic origin of robustness against *any* local perturbations? Recently, it was found that, microscopically, topological order is related to long-range entanglement [17,18]. In fact, we can regard topological order as patterns of long-range entanglement [19] defined through local unitary (LU) transformations [20–22].

In this paper, we will discuss in more detail the relation between topological order and many-body quantum entanglement. We first point out that the topologically ordered states are not arbitrary gapped states, but belong to a special kind of gapped quantum states, called *gapped quantum liquids*. We

will give a definition of gapped quantum liquids. Haah's cubic model may be an example of gapped quantum states that are not a gapped quantum liquid [23].

The notion of gapped quantum liquids can also be applied to solve the problem of how to take the thermodynamic limit for systems without translation symmetry. In general, in the presence of strong randomness, the thermodynamic limit is not well defined (without impurity average). We show that for gapped quantum liquids, the thermodynamic limit is well defined even without impurity average. Consequently, the notions of quantum phases and quantum phase transitions are well defined for gapped quantum liquids.

We then show that the equivalence classes of gLU transformations, not only describe topologically ordered states, but also include the ground states of symmetry-breaking phases, where the exact symmetric ground states have entanglement of the Greenberger-Horne-Zeilinger [24] (GHZ) form. This allows us to use a gLU invariant—topological entanglement entropy—to probe the symmetry-breaking properties hidden in the exact ground state of a finite system, which is invariant under the symmetry transformation. Note that, to use the topological entanglement entropy to probe the symmetry breaking states, *we do not need to know the symmetry or the symmetry-breaking order parameters*. Usually, one needs to identify the symmetry-breaking order parameters and compute their long-range correlation functions to probe the symmetry-breaking properties hidden in the symmetric exact ground-state wave function.

We further show that many-body states with GHZ-form entanglement are convertible to product states with a finite probability under stochastic local (SL) transformations, which are local invertible transformations that are not necessarily unitary. In contrast, topologically ordered states are not convertible to product states via small SL transformations. This allows us to give a definition of long-range entanglement based on SL convertibility to product states, under which only topologically ordered states have long-range entanglement. Moreover, we show that the topological entanglement entropy for topological orders is stable under small SL transformations but unstable for symmetry-breaking orders.

For topologically ordered states, the robustness of the ground-state degeneracy and the robustness of the unitary non-Abelian geometric phases against any (small) SL transformations (i.e., local nonunitary transformations) reveal the phenomenon of *emergence of unitarity*: even when the bare quantum evolution at lattice scale is nonunitary, the induced adiabatic evolution in the ground-state subspace is still unitary. In this sense, topological order can be defined as states with emergent unitarity from nonunitary quantum evolutions. The phenomenon of emergence of unitarity may have deep impact in the foundation of quantum theory, and in the elementary particle theory, since the emergence/unification of gauge interaction and Fermi statistics is closely related to topological order and long-range entanglement [25]. (The emergence of unitarity is also discussed in the $N = 4$ supersymmetric Yang-Mills scattering amplitudes in the planar limit [26].)

II. GAPPED QUANTUM LIQUIDS AND TOPOLOGICAL ORDER

A. Gapped quantum system and gapped quantum phase

Topologically ordered states are gapped quantum states. To clarify the concept of gapped quantum states, we first define a gapped quantum system. Since a gapped system may have gapless excitations on the boundary (such as quantum Hall systems), so to define gapped Hamiltonians, we need to put the Hamiltonian on a space with no boundary. Also, a system with certain sizes may contain nontrivial excitations (such as a spin liquid state of spin-1/2 spins on a lattice with an ODD number of sites), so we have to specify that the system has a certain sequence of sizes when we take the thermodynamic limit.

Definition 1. Gapped quantum system. Consider a local Hamiltonian of a qubit system on graphs with no boundary, with finite spatial dimension D. If there is a sequence of sizes of the system N_k, $N_k \to \infty$, as $k \to \infty$, such that the size-N_k system has the following "gap property," then the system, defined by the Hamiltonian sequence $\{H_{N_k}\}$, is said to be gapped. Here N_k can be viewed as the number of qubits in the system.

Definition 2. Gap property. There is a fixed Δ (i.e., independent of N_k) such that (1) the size-N_k Hamiltonian has no eigenvalue in an energy window of size Δ; (2) the number of eigenstates below the energy window does not depend on N_k; and (3) the energy splitting of those eigenstates below the energy window approaches zero as $N_k \to \infty$.
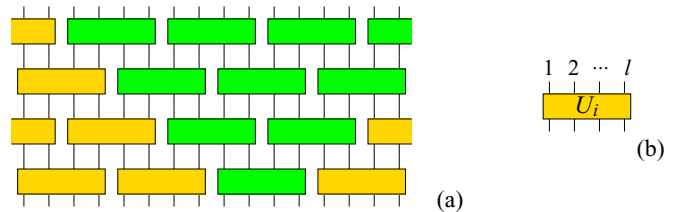

FIG. 1. (Color online) (a) A graphic representation of a quantum circuit, which is formed by (b) unitary operations on blocks of finite size l. The green shading represents a causal structure.

Note that the notion of "gapped quantum system" cannot be even defined for a single Hamiltonian. It is a property of a sequence of Hamiltonians, $\{H_{N_k}\}$, in the large size limit $N_k \to \infty$. In this paper, the term "a gapped quantum system" refers to a sequence of Hamiltonians, $\{H_{N_k}\}$, that satisfies the above properties. Now we can give a precise definition for the ground-state degeneracy and ground-state subspace.

Definition 3. Ground-state degeneracy and ground-state subspace. The number of eigenstates below the energy window becomes the ground-state degeneracy of the gapped system. (This is how the ground-state degeneracy of a topologically ordered state is defined [1–3,27].) The states below the energy window span the *ground-state subspace*, which is denoted as $\mathcal{V}_{N_k}$.

Now, we would like to define gapped quantum phase. First, we introduce the local unitary (LU) transformation.

Definition 4. Local unitary (LU) transformation [19]. An LU transformation can be given by a quantum circuit as shown in Fig. 1. An LU transformation is given by finite layers (i.e., the number of layers is a constant that is independent of the system size) of piecewise local unitary transformations:

$$U_{\text{circ}}^M = U_{\text{pwl}}^{(1)} U_{\text{pwl}}^{(2)} \cdots U_{\text{pwl}}^{(M)},$$

where each layer has a form

$$U_{\text{pwl}} = \prod_i U^i.$$

Here, $\{U^i\}$ is a set of unitary operators that act on nonoverlapping regions. The size of each region is less than a finite number l.

Two gapped systems connected by an LU transformation can deform into each other smoothly without closing the energy gap, and thus belong to the same phase. This leads us to define a gapped quantum phase.

Definition 5. Gapped quantum phase. Two gapped quantum systems $\{H_{N_k}\}$ and $\{H'_{N_k}\}$ are equivalent if the *ground-state subspaces* of H_{N_k} and H'_{N_k} are connected by LU transformations for all N_k. The equivalence classes of the above equivalence relation are the gapped quantum phases (see Fig. 2).

It is highly desired to identify topological orders as gapped quantum phases, since both concepts do not involve symmetry. In the following, we will show that gapped quantum phases, sometimes, are not well behaved in the thermodynamic limit. As a result, it is not proper to define topological orders as gapped quantum phases. To fix this problem, we will introduce the concept of gapped quantum liquid phase.
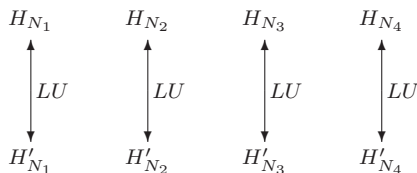
FIG. 2. The two rows of Hamiltonians describe two gapped quantum systems. The two rows connected by LU transformations represent the equivalence relation between the two gapped quantum systems, whose equivalence classes are gapped quantum phases. We may view H_N as a projector that defines its ground-state subspace. The ground-state subspaces of two equivalent systems are connected by the LU transformations.

B. Gapped quantum liquid system and gapped quantum liquid phase

Why gapped quantum systems may not be well behaved in the thermodynamic limit? This is because Hamiltonians with different sizes are not related (see Fig. 2) in our definition of gapped quantum systems. As a result, we are allowed to choose totally different H_{N_k} and $H_{N_{k+1}}$ as long as the Hamiltonians have the same ground-state degeneracy. For example, one can be topologically ordered and the other can be symmetry breaking. To fix this problem, we choose a subclass of gapped quantum systems that is well-behaved in the thermodynamic limit. Those gapped quantum systems are "shapeless" and can "dissolve" any product states on additional sites to increase its size. Such gapped quantum systems are called *gapped quantum liquid systems*.

Definition 6. Gapped quantum liquid system. A gapped quantum liquid system is a gapped quantum system, described by the sequence $\{H_{N_k}\}$, with two additional properties: (1) $0 < c_1 < (N_{k+1} - N_k)/N_k < c_2$, where c_1 and c_2 are constants that do not depend on the system size, and (2) the ground-state subspaces of H_{N_k} and $H_{N_{k+1}}$ are connected by a generalized local unitary (gLU) transformation (see Fig. 3).

Figure 4 explains how we transform H_{N_k} to $H_{N_{k+1}}$ via a gLU transformation. For the system H_{N_k}, we first need to add $N_{k+1} - N_k$ qubits. We would like to do this addition "locally." That is, the distribution of the added qubits may not be uniform in space but maintains a finite density (number of qubits per unit volume). We then define how to write Hamiltonians after adding particles to the system.

Definition 7. Local addition (LA) transformation. For adding $N_{k+1} - N_k$ qubits to the system H_{N_k} locally, we
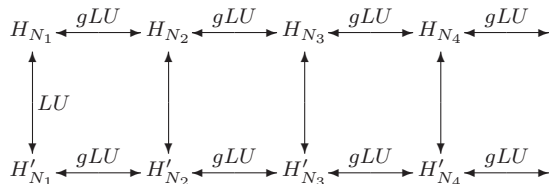

FIG. 3. The two rows define two gapped quantum liquid systems via gLU transformations. The two rows connected by LU transformations represent the equivalence relation between two gapped quantum liquid systems, whose equivalence classes are gapped quantum liquid phases.
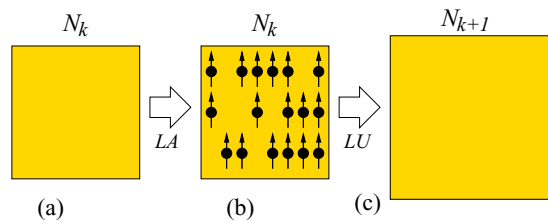

FIG. 4. (Color online) Two systems (a) and (c), with size N_k and N_{k+1}, are described by H_{N_k} and $H_{N_{k+1}}$ respectively. (a) $\rightarrow$ (b) is an LA transformation where we add $N_{k+1} - N_k$ qubits to the system H_{N_k} to obtain the Hamiltonian $H_{N_k} + \sum_i Z_i$ for the combined system (b). Under the LA transformation, the ground states of H_{N_k} is tensored with a product state to obtain the ground states of $H_{N_k} + \sum_i Z_i$. In (b) $\rightarrow$ (c), we transform the ground-state subspace of $H_{N_k} + \sum_i Z_i$ to the ground-state subspace of $H_{N_{k+1}}$ via an LU transformation.

consider the Hamiltonian $H_{N_k} + \sum_{i=1}^{N_{k+1} - N_k} Z_i$ for the combined system [see Fig. 4(b)], where Z_i is the Pauli Z operator acting on the ith qubit. This defines an LA transformation from H_{N_k} to $H_{N_k} + \sum_{i=1}^{N_{k+1} - N_k} Z_i$.

Definition 8. gLU transformation. If for any LA transformation from H_{N_k} to $H_{N_k} + \sum_{i=1}^{N_{k+1} - N_k} Z_i$, the ground-state subspace of $H_{N_k} + \sum_{i=1}^{N_{k+1} - N_k} Z_i$ can be transformed into the ground-state subspace of $H_{N_{k+1}}$ via an LU transformation, then we say H_{N_k} and $H_{N_{k+1}}$ are connected by a gLU transformation.

According to our definition, the sequence of following Hamiltonians

$$H_{N_k}^{\text{trivial-liquid}} = -\sum_{i=1}^{N_k} Z_i, \tag{1}$$

gives rise to a gapped quantum liquid system. The topologically ordered toric code Hamiltonian $H_{N_k}^{\text{toric}}$ is also a gapped quantum liquid, as illustrated in Fig. 5. This reveals one important feature of a gapped quantum liquid—the corresponding lattice in general do not have a "shape" (i.e., the system can be defined on an arbitrary lattice with a meaningful thermodynamic limit).
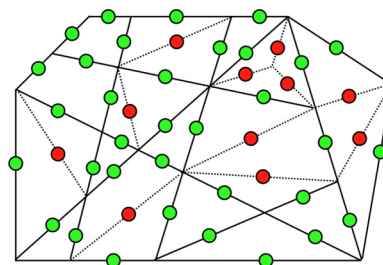

FIG. 5. (Color online) Toric code as a gapped quantum liquid: toric code of N_k qubits on an arbitrary 2D lattice, where the green dots represent qubits sitting on the link of the lattice (given by solid lines). By adding $N_{k+1} - N_k$ qubits (red dots), the gLU transformation $H_{N_k} \rightarrow H_{N_{k+1}}$ "dissolves" the red qubits in the new lattice (with both the solid lines and dashed lines).

To have an example of a gapped system that is not a quantum liquid, consider another sequence of Hamiltonians:

$$H_{N_k}^{\text{nonliquid}} = -\sum_{i=1}^{N_k-1} Z_i. \qquad (2)$$

It describes a gapped quantum system with twofold degenerate ground states (coming from the N_k^{th} qubit that carries no energy). However, such a gapped quantum system is not a gapped quantum liquid system. Because the labeling of the N_{k+1} qubit is essentially arbitrary, for some LA transformations, the map from $H_{N_k} + \sum_{i=1}^{N_{k+1}-N_k} Z_i$ to $H_{N_{k+1}}$ cannot be local.

Through the above example, we see that a gapped quantum system may not have a well defined thermodynamic limit (because the low-energy property—the degenerate ground states, is given by an isolated qubit, which is not a thermodynamic property.) Similarly, the gapped quantum phase (as defined in Definition 5) is not a good concept, since it is not a thermodynamic property sometimes. In contrast, the gapped quantum liquid system and gapped quantum liquid phase (defined below in Definition 9) are a good concepts, since they are always thermodynamic properties.

We also believe that the cubic code of the Haah model is another example of a gapped quantum system that is not a gapped quantum liquid system [23]. There exists a sequence of linear sizes of the cube: $L_k \to \infty$, where the ground-state degeneracy is two, provided that $L_k = 2^k - 1$ (or $L_k = 2^{2k+1} - 1$) for any integer k, and correspondingly $N_k = L_k^3$. However, we do not think that $H_{N_k}^{\text{Haah}}$ and $H_{N_{k+1}}^{\text{Haah}}$ are connected by an gLU transformation. Here, $H_{N_k}^{\text{Haah}}$ is the Hamiltonian of the cubit code of size N_k. We can now define a gapped quantum liquid phase.

Definition 9. Gapped quantum liquid phase. Two gapped quantum liquid systems $\{H_{N_k}\}$ and $\{H'_{N_k}\}$ are equivalent if the ground-state subspaces of H_{N_k} and H'_{N_k} are connected by LU transformations for all N_k. The equivalence classes of the above equivalence relation are the gapped quantum liquid phases (see Fig. 3).

C. Topological order

Using the notion of gapped quantum liquid phase, we can have a definition of topological order. First, we introduce a stable gapped quantum system.

Definition 10. Stable gapped quantum system. If the ground-state degeneracy of a gapped quantum system is stable against any local perturbation (in the large N_k limit), then the gapped quantum system is stable.

An intimately related fact to this definition is that the ground-state subspace of a stable gapped quantum system (in the large N_k limit) is a quantum error-correcting code with macroscopic distance [27]. This is to say, for any orthonormal basis $\{|\Phi_i\rangle\}$ of the ground-state subspace, for any local operator M, we have

$$\langle \Phi_i | M | \Phi_j \rangle = C_M \delta_{ij}, \qquad (3)$$

where C_M is a constant which only depends on M [28–31].

Note that a gapped quantum liquid system may not be a stable gapped quantum system. A symmetry-breaking system is an example of a gapped quantum liquid system that is not a stable gapped quantum system (the ground-state degeneracy can be lifted by symmetry-breaking perturbations). Also a stable gapped quantum system may not be a gapped quantum liquid system—a non-Abelian quantum Hall state [13,14] with traps [32] that traps non-Abelian quasiparticles is an example of this. Since the ground state with traps contains non-Abelian quasiparticles, the resulting degeneracy is robust against any local perturbations. So the system is a stable gapped quantum system. However, for such a system, H_{N_k} and $H_{N_{k+1}}$ are not connected via gLU transformations, hence it is not a gapped quantum liquid system. Now we can define topological order (or different phases of topologically ordered states).

Definition 11. Topological order. The topological orders are stable gapped quantum liquid phases.

We remark that we in fact define different topological orders as different equivalent classes. One of these equivalent classes represents the trivial (topological) order. In Definition 11, we put trivial and nontopological order together to have a simple definition. This is similar to symmetry transformations, which usually include both trivial and nontrivial transformations, so that we can say symmetry transformations form a group. Similarly, if we include the trivial one, then we can say that topological orders form a monoid under the stacking operation [33].

There are also unstable gapped quantum liquid systems. They can be defined via the definition of a first-order phase transition for gapped quantum liquid systems.

Definition 12. First-order phase transition for gapped quantum liquid systems. A deformation of a gapped quantum liquid system experiences a first-order phase transition if the Hamiltonian remains gapped along the deformation path and if the ground-state degeneracy at a point on the deformation path is different from its neighbours. That point is the transition point of the first-order phase transition.

The first-order phase-transition point is also an unstable gapped quantum liquid system. Physically and generically, an unstable gapped quantum liquid system is a system with accidental degenerate ground states.

From the above discussions, we see that topological orders are the universality classes of stable gapped quantum liquid systems that are separated by gapless quantum systems or unstable gapped quantum systems. Moving from one universality class to another universality class by passing through a gapless system corresponds to a continuous phase transition. Moving from one universality class to another universality class by passing through an unstable gapped system corresponds to a first-order phase transition.

D. Gapped quantum liquid

We would like to emphasize that the topological order is a notion of universality classes of local Hamiltonians (or more precisely, gapped quantum systems). In the following, we will introduce the universality classes of many-body wave functions. We can also use the universality classes of many-body wave functions to understand topological orders.

Definition 13. Gapped quantum state. A gapped quantum system is defined by a sequence of Hamiltonians $\{H_{N_k}\}$. Let $\mathcal{V}_{N_k}$ be the ground-state subspace of H_{N_k}. The sequence of
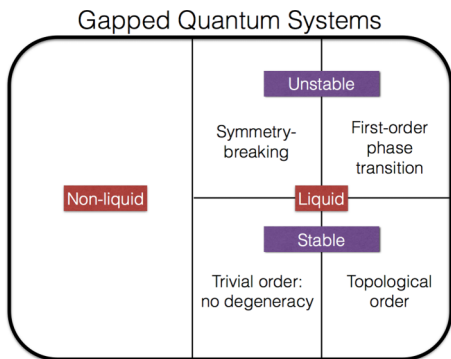
FIG. 6. (Color online) Summary of gapped quantum systems: gapped quantum systems include gapped quantum liquid systems, and systems that are not liquid (nonliquid). For gapped quantum liquids, there are stable systems (including the trivial systems given by, e.g., the Hamiltonian $H_{N_k}^{\text{nonliquid}}$ and the topologically ordered systems) and unstable systems (including symmetry-breaking systems and first-order phase transitions).

ground-state subspaces $\{\mathcal{V}_{N_k}\}$ is referred as a gapped quantum state.

Note that a gapped quantum state is not described by a single wave function, but by a sequence of ground-state subspaces $\{\mathcal{V}_{N_k}\}$. Similarly, we define a gapped quantum liquid and the associated phases.

Definition 14. Gapped quantum liquid. The sequence of ground-state subspaces $\{\mathcal{V}_{N_k}\}$ of a gapped quantum liquid system defined by $\{H_{N_k}\}$ is referred to as a gapped quantum liquid.

Definition 15. Gapped quantum liquid phase and topologically ordered phase. Two gapped quantum liquids, defined by two sequences of ground-state subspaces $\{\mathcal{V}_{N_k}\}$ and $\{\mathcal{V}'_{N'_k}\}$ (on space with no boundary), are equivalent if they can be connected via gLU transformations, i.e., we can map $\mathcal{V}_{N_k}$ into $\mathcal{V}'_{N'_k}$ and map $\mathcal{V}'_{N'_k}$ into $\mathcal{V}_{N_k}$ via gLU transformations (assuming $N_k \sim N'_k$). The equivalence classes of gapped quantum liquids are gapped quantum liquid phases. The equivalence classes of stable gapped quantum liquids are topologically ordered phases.

In the next section, we will show that gapped liquid phases contain both symmetry-breaking phases and topologically ordered phases. We summarize the different kinds of gapped quantum systems in Fig. 6.

III. LOCAL UNITARY TRANSFORMATIONS AND SYMMETRY-BREAKING ORDER

To study the universality classes of many-body wave functions, a natural idea is from the LU transformations [19]. In this section, we will analyze the classes of wave functions under LU transformations, or more generally, gLU transformation.

As discussed above, the gLU transformations define an equivalence relation among many-body ground-state subspaces. The equivalence classes defined by such an equivalence relation will be called the gLU classes. The gLU classes of gapped quantum liquids correspond to gapped quantum liquid phases.

We now ask the following question. Since the definition of the gLU classes does not require symmetry, then do the gLU classes of a gapped quantum liquid have a one-to-one correspondence with topological orders (as defined in Definition 11)?

We will show that the answer is no, i.e., there are *unstable* gapped quantum liquids. Only the gLU classes for *stable* gapped quantum liquids have a one-to-one correspondence with topological orders.

A. Symmetry-breaking orders

An example of unstable gapped quantum liquids is given by symmetry breaking states. Those unstable gapped quantum liquids are in a different gLU class from the trivial phase, and thus are nontrivial gapped quantum liquid phases.

Let us consider an example of the unstable gapped quantum liquids, the 1D transverse Ising model with the Hamiltonian (with periodic boundary condition)

$$H_{N_k}^{\text{tIsing}}(B) = -\sum_{i=1}^{N_k} Z_i Z_{i+1} + B \sum_{i=1}^{N_k} X_i, \qquad (4)$$

where Z_i and X_i are the Pauli Z/X operators acting on the ith qubit. The Hamiltonian $H_{N_k}^{\text{tIsing}}(B)$ has a $\mathbb{Z}_2$ symmetry, which is given by $\prod_{i=1}^{N_k} X_i$. The gapped ground states are nondegenerate for $B > 1$. For $0 \leqslant B < 1$, the gapped ground states are twofold degenerate. The degeneracy is unstable against perturbation that breaks the $\mathbb{Z}_2$ symmetry.

The phase for $B > 1$ is a trivial gapped liquid phase. The phase for $0 < B < 1$ is a nontrivial gapped liquid phase. This due to a very simple reason: the two phases have different group state degeneracy, and the ground-state degeneracy is a gLU invariant. Gapped quantum liquids with different ground-state degeneracy always belong to different gapped liquid phases.

Now, let us make a more nontrivial comparison. Here, we view $H_{N_k}^{\text{tIsing}}(B)$ (with $0 < B < 1$) as a gapped quantum system (rather than a gapped quantum liquid system). We compare it with another gapped quantum system $H_{N_k}^{\text{nonliquid}}$ [see (2)] discussed before. Both gapped systems have twofold degenerate ground states. Do the two systems belong to the same gapped quantum phase (as defined in Definition 5)?

Consider $H_{N_k}^{\text{tIsing}}(B)$ for any $0 < B < 1$ and any size $N_k < \infty$. The (symmetric) exact ground state $|\Psi_+(B)\rangle$ is an adiabatic continuation of the GHZ state:

$$|\text{GHZ}_+\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N_k} + |1\rangle^{\otimes N_k}), \qquad (5)$$

i.e., $|\Psi_+(B)\rangle$ is in the same gLU class of $|\text{GHZ}_+\rangle$. There is another state $|\Psi_-(B)\rangle$ below the energy window Δ, which is an adiabatic continuation of the state

$$|\text{GHZ}_-\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N_k} - |1\rangle^{\otimes N_k}). \qquad (6)$$

The energy splitting of $|\Psi_+(B)\rangle$ and $|\Psi_-(B)\rangle$ approaches zero as $N_k \to \infty$.

However, we know that the GHZ state $|\text{GHZ}_+\rangle$ (hence $|\Psi_+(B)\rangle$) and the product state $|0\rangle^{\otimes N_k}$ belong to two different gLU classes. Both states are regarded to have the same

trivial topological order. So gLU transformations assign GHZ states, or symmetry-breaking many-body wave functions, to nontrivial classes. Therefore by studying the gLU classes of gapped quantum liquids, we can study both the topologically ordered states and the symmetry-breaking states.

To be more precise, the ground-state subspace of $H_{N_k}^{\text{tIsing}}(B)(0 < B < 1)$ contain nontrivial GHZ states. On the other hand, the ground-state subspace of $H_{N_k}^{\text{nonliquid}}$ contain only product states. There is no GHZ states. This makes the two systems $H_{N_k}^{\text{tIsing}}(B)$ and $H_{N_k}^{\text{nonliquid}}$ to belong to two different gapped quantum phases, even though the two systems have the same ground-state degeneracy. We can now define a gapped symmetry-breaking quantum system.

Definition 16. Gapped symmetry-breaking quantum system. A gapped symmetry-breaking system is a gapped quantum liquid system with certain symmetry and degenerate ground states, where the symmetric ground states have the GHZ form of entanglement.

We remark that the ground-state subspace of a gapped symmetry-breaking quantum system is a "classical" error-correcting code with macroscopic distance, correcting errors that do not break the symmetry. This is to say, for any orthonormal basis $\{|\Phi_i\rangle\}$ of the ground-state subspace, for any local operator M_s that does not break symmetry, we have

$$\langle \Phi_i | M_s | \Phi_j \rangle = C_{M_s} \delta_{ij}, \tag{7}$$

where M_s is a constant that only depends on M_s.

Here by "classical" we mean the following. For the ground-state subspace, there exists a basis $\{|\Phi_i\rangle\}$ that is connected by symmetry. In this basis, the ground-state subspace is a classical error-correcting code of macroscopic distance, in the sense that for any local operator M, we have

$$\langle \Phi_i | M | \Phi_j \rangle = 0, \quad i \neq j. \tag{8}$$

Notice that Eq. (8) does not contain the coherence condition for $i = j$, which is the requirement to make the subspace a "quantum" code.

The transverse Ising mode is an example of such a special case with $\mathbb{Z}_2$ symmetry. The basis that is connected by the $\mathbb{Z}_2$ symmetry is $|\Psi_\pm(B)\rangle$. And it is obvious that $\langle \Psi_+(B)|M|\Psi_-(B)\rangle = 0, i \neq j$.

We have now shown that gapped liquid phases also contain symmetry-breaking phases. We summarize the LU classes for ground states of local Hamiltonians in Fig. 7.
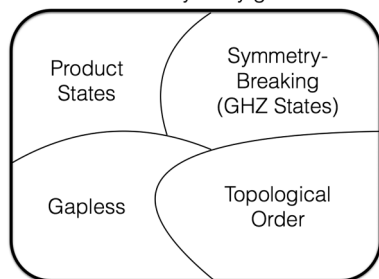

FIG. 7. LU classes for ground states (many-body wave functions) of local Hamiltonians.
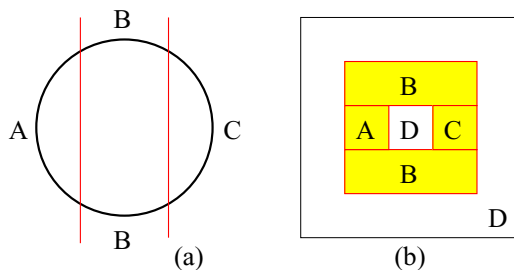

FIG. 8. (Color online) (a) Cutting a ring into three parts A, B, C. (b) Cutting a 2D system into four parts A, B, C, D.

B. Topological entanglement entropy

Topological entanglement entropy is an invariant of gLU transformations [17]. This allows us to use topological entanglement entropy to detect if a gapped quantum liquid belongs to a nontrivial gLU class or not, hence we can study both topological orders and symmetry-breaking orders.

Here we define a new type of topological entanglement entropy on a graph with no boundary (for simplicity we consider the topology of a D-dimensional sphere $\mathcal{S}^D$), by dividing the entire system into *three* nonoverlapping parts A, B, C.

Definition 17. The tritopological entanglement entropy on $\mathcal{S}^D$ is given by

$$S_{\text{topo}}^{\text{tri}} = S(AB) + S(BC) - S(B) - S(ABC), \tag{9}$$

where the parts A, B, C are illustrated in Fig. 8(a) for the case of $d = 1$. And $S(*)$ is the von Neumann entropy of the reduced density matrix of the part $*$.

We note that, in the original definition of topological entropy (denoted as $S_{\text{topo}}^{\text{qua}}$), the entire system is cut into *four* unconnected nonloverlapping pieces A, B, C, D [17]. $S_{\text{topo}}^{\text{qua}}$ is defined on the part of the system A, B, C with nontrivial topology [see Fig. 8(b)]:

$$S_{\text{topo}}^{\text{qua}} = S(AB) + S(BC) - S(B) - S(ABC). \tag{10}$$

We remark that the topological entanglement entropy given by $S_{\text{topo}}^{\text{qua}}$ has the reversed sign compared to the original definition [17,18]. Also, both $S_{\text{topo}}^{\text{tri}}$ and $S_{\text{topo}}^{\text{qua}}$ are guaranteed to be non-negative due to the strong subadditivity of quantum entropy [34].

Notice that for product states $S_{\text{topo}}^{\text{tri}} = S_{\text{topo}}^{\text{qua}} = 0$. However, we have $S_{\text{topo}}^{\text{tri}}(|\text{GHZ}_+\rangle) = 1$. And one expects the same $S_{\text{topo}}^{\text{tri}} = 1$ for $|\Psi_+(B)\rangle$ when $N_k \to \infty$ and $0 < B < 1$, which is not an ideal GHZ state. That is, $S_{\text{topo}}^{\text{tri}}$ can detect the states with GHZ form of entanglement. Since the product state $|0\rangle^{\otimes N_k}$ has zero tri-topological entropy, it belongs to a different gLU class.

The long-range correlation of the wave function may be better seen by the matrix product state (MPS) renormalization group, where the isometric form is given by [35]

$$\sum_\alpha |\alpha, \ldots, \alpha\rangle \otimes |\omega_{D_\alpha}\rangle^{\otimes N_k}, \tag{11}$$
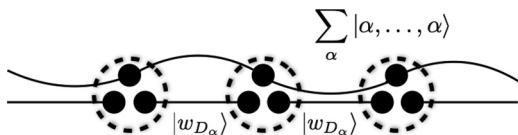
FIG. 9. Isometric form for MPS with GHZ form of entanglement [35].

as illustrated in Fig. 9. Here the bond state is

$$|\omega_{D_\alpha}\rangle = \frac{1}{\sqrt{D_\alpha}} \sum_{j=1}^{D_\alpha} |jj\rangle, \quad (12)$$

and the bond dimension D_α can couple to the value of α.

The term $\sum_\alpha |\alpha, \dots, \alpha\rangle$ captures the GHZ form of entanglement that contributes to $S_{\text{topo}}^{\text{tri}}$, and the term $|\omega_{D_\alpha}\rangle^{\otimes N_k}$ is the short-range entangled part that has $S_{\text{topo}}^{\text{tri}} = 0$.

Notice that, if the state of the system ABC is a pure state, we have

$$\begin{aligned} S(AB) &= S(C), \\ S(BC) &= S(A), \\ S(B) &= S(AC), \\ S(ABC) &= 0. \end{aligned} \quad (13)$$

In this case the right-hand side of Eq. (9) becomes the mutual information between the parts A and C, i.e.,

$$I(A:C) = S(A) + S(C) - S(AC). \quad (14)$$

This gives an alternative explanation that a nonzero mutual information of two disconnected parts of a pure state indicates long-range correlation, as is observed in Ref. [36].

However, in general, a nonzero $I(A:C)$ does not indicate a GHZ form of entanglement. For instance, the state

$$\rho = \frac{1}{2}(|\bar{0}\rangle\langle\bar{0}| + |\bar{1}\rangle\langle\bar{1}|), \quad (15)$$

where $|\bar{0}\rangle = |0\rangle^{\otimes N_k}$ and $|\bar{1}\rangle = |1\rangle^{\otimes N_k}$, contains only classical correlation but no entanglement. For ρ, we have $I(A:C) = 1$ but $S_{\text{topo}}^{\text{tri}} = 0$.

We remark that, $S_{\text{topo}}^{\text{tri}}$ is evaluated for a quantum state of the region ABC. For finite systems, we will focus on the value of $S_{\text{topo}}^{\text{tri}}$ on the exact ground state, which is nondegenerate and does not break any symmetry. We refer to Refs. [28,29,37,38] for some other approaches proposed to detect orders of the systems based on density matrices.

As an example to demonstrate the use of $S_{\text{topo}}^{\text{tri}}$ to determine the quantum phase transitions, we calculate $S_{\text{topo}}^{\text{tri}}$ for the ground state of the transverse Ising model. We rescale the Ising Hamiltonian $H(B)$ as

$$H(\alpha) = -\cos\alpha \sum_i Z_i Z_{i+1} + \sin\alpha \sum_i X_i, \quad (16)$$

where $\alpha \in [0, \pi/2]$.

We choose the area A, C and each connected component of the area B to have 1,2,3,4,5 qubits, so we compute $S_{\text{topo}}^{\text{tri}}$ for total $n = 4,8,12,16,20$ qubits. The results are shown in Fig. 10. The five curves intersect at $2\alpha/\pi = 1/2$, i.e., $\alpha = \pi/4$, which corresponds to the well-known phase transition at $B = 1$.
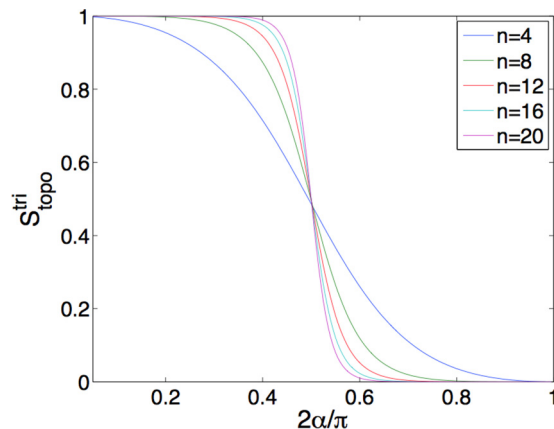

FIG. 10. (Color online) $S_{\text{topo}}^{\text{tri}}$ for the transverse Ising model. The horizontal axis is the angle $2\alpha/\pi$ for the Hamiltonian $H(\alpha)$ as given in Eq. (16). A similar result is presented in Ref. [39], from a different viewpoint.

We emphasize that for both the symmetric ground states of the symmetry breaking phase and the trivial phase, the tri-topological entanglement entropy $S_{\text{topo}}^{\text{tri}}$ is quantized. In this sense, it is similar to the topological entanglement entropy $S_{\text{topo}}^{\text{qua}}$ of topologically ordered ground states. However, the symmetry-breaking classes are quite different from topological classes: $S_{\text{topo}}^{\text{tri}} \neq 0$ and $S_{\text{topo}}^{\text{qua}} = 0$ for symmetry-breaking classes, while $S_{\text{topo}}^{\text{tri}} = 0$ and $S_{\text{topo}}^{\text{qua}} \neq 0$ for topological classes (with nontrivial topological excitations [33]). We see that, for symmetry-breaking classes, the original definition $S_{\text{topo}}^{\text{qua}}$ fails to detect different gLU classes. This is because for a symmetry-breaking class, the information of the nontrivial entanglement is only contained in the wave function for the entire system. Reduced density matrices of any part of the system do not contain that information.

We remark that, at the transition point $\alpha = \pi/4$ where the system is gapless, the five curves in Fig. 10 intersect at $S_{\text{topo}}^{\text{tri}} \sim 0.5$. However, this value of $S_{\text{topo}}^{\text{tri}}$ is not a constant, which depends on the shape of the areas A, B, C. For instance,
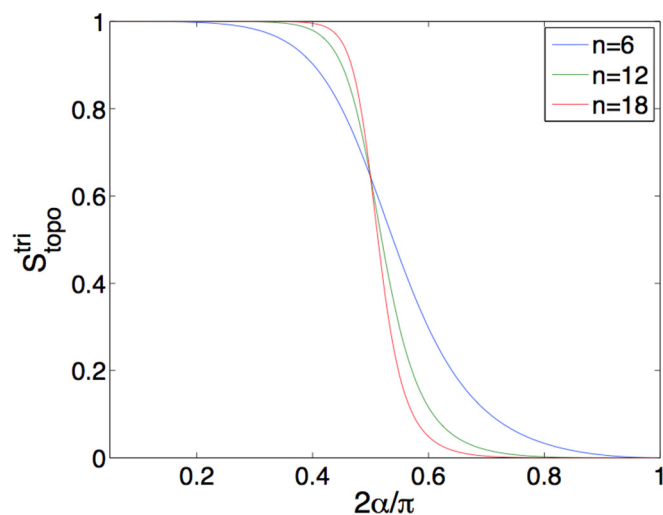

FIG. 11. (Color online) $S_{\text{topo}}^{\text{tri}}$ for the transverse Ising model. The ratio $r = 2 : 1$, and the system sizes are $n = 6, 12, 18$ qubits.

if we choose the ratio

$$r = \frac{\# \text{ in each of the area A,C}}{\# \text{ in each connected component of } B} \quad (17)$$

$2:1$, where $\#$ means the number of qubits and to have $1,2,3$ qubits for each connected component of the area B, then we can compute $S_{\text{topo}}^{\text{tri}}$ for total $n = 6,12,18$ qubits, as shown in Fig. 11. This ratio dependence is typical in critical systems [36,40], and our results are consistent with the conformal field theory predictions [36,41].

IV. STOCHASTIC LOCAL TRANSFORMATIONS AND LONG-RANGE ENTANGLEMENT

We have seen that the nontrivial equivalence classes of many-body wave functions under the gLU transformations contain both topologically ordered phases and symmetry-breaking phases (described by the symmetric many-body wave functions with GHZ form of entanglement). In this section, we will introduce generalized stochastic local (gSL) transformations, which are local invertible transformations that are not necessarily unitary. The term "stochastic" means that these transformations can be realized by generalized local measurements with a finite probability of success [42].

We show that the many-body wave functions for symmetry-breaking phases (i.e., the states of GHZ form of entanglement) are convertible to the product states under the gSL transformations with a finite probability, but in contrast the topological ordered states are not. This allows us to give a new definition of long-range entanglement under which only topologically ordered states are long-range entangled. We further show that the topological orders are stable against small stochastic local transformations, while the symmetry-breaking orders are not.

A. Stochastic local transformations

The idea for using gSL transformations is simple. The topologically *stable* degenerate ground states for a topologically ordered system are not only stable under real-time evolutions (which are described by gLU transformations), they are also stable and are the fixed points under imaginary-time evolutions. The imaginary-time evolutions of the ground states are given by the gSL transformations (or local nonunitary transformations), therefore the topological orders are robust under (small) gSL transformations.

On the other hand, the states of GHZ form an entanglement that is not robust under small gSL transformations, and can be converted into product states with a finite probability. Thus there is no emergence of unitarity for symmetry-breaking states.

To define gSL transformations, we start from reviewing the most general form of quantum operations (also known as quantum channels), which are complete-positive trace-preserving maps [43]. A quantum operation $\mathcal{E}$ acting on any density matrix ρ has the form

$$\mathcal{E}(\rho) = \sum_{k=1}^{r} A_k \rho A_k^{\dagger}, \quad (18)$$

with

$$\sum_{k=1}^{r} A_k^{\dagger} A_k = I, \quad (19)$$

where I is the identity operator.

The operators A_k are called Kraus operators of ρ and satisfies

$$A_k^{\dagger} A_k \leqslant I. \quad (20)$$

This means that the operation $A_k \rho A_k^{\dagger}$ can be realized with probability $\text{Tr}(A_k \rho A_k^{\dagger})$ for a normalized state $\text{Tr}\rho = 1$. In the following, we will drop the label k for the measurement outcome.

We will now definite gSL transformations along a similar line as the definition of gLU transformations. Let us first define a layer of SL transformation that has a form

$$W_{\text{pwl}} = \prod_i W^i,$$

where $\{W^i\}$ is a set of invertible operators that act on nonoverlapping regions, and each W^i satisfies

$$W^{i\dagger} W^i \leqslant I. \quad (21)$$

The size of each region is less than a finite number l. The invertible operator W_{pwl} defined in this way is called a layer of piecewise stochastic local transformation with a range l.

A stochastic local (SL) transformation is then given by a finite layers of piecewise local invertible transformation:

$$W_{\text{circ}}^M = W_{\text{pwl}}^{(1)} W_{\text{pwl}}^{(2)} \cdots W_{\text{pwl}}^{(M)}.$$

We note that such a transformation does not change the degree of freedom of the state.

Similarly to the gLU transformations, we can also have a transformation that can change the degree of freedom of the state, by a tensor product of the state with another product state $|\Psi\rangle \rightarrow (\otimes_i |\psi_i\rangle) \otimes |\Psi\rangle$, where $|\psi_i\rangle$ is the wave function for the ith qubit. A finite combination of the above two types of transformations is then a generalized stochastic local (gSL) transformation.

We remark that, despite the simplicity of the idea, similar to the gLU transformations, gSL transformations are more subtle to deal with. First of all, notice that gSL transformations do not preserve the norm of quantum states [i.e., not trace-preserving, as given by Eq. (21)]. Furthermore, as we are dealing with the thermodynamic limit ($N_k \rightarrow \infty$), we are applying gSL transformations on a system of infinite-dimensional Hilbert space. In this case, even if each W^i is invertible, $W_{\text{pwl}} = \prod_i W^i$ may be noninvertible due to the thermodynamic limit. We will discuss these issues in more detail in the next section.

B. Short-range entanglement and symmetry-breaking orders

It is known in fact that the SL convertibility in infinite-dimensional systems is subtle, and to avoid technical difficulties dealing with the infinite dimensional Hilbert space, we would instead start from borrowing the idea in Ref. [44] to use ϵ convertibility instead of talking about the exact convertibility of states under gSL. For simplicity, we will omit the notation "ϵ" and still name it "gSL convertibility."

Definition 18. Convertibility by gSL transformation. We say that $|\Psi\rangle$ is convertible to $|\Phi\rangle$ by a gSL transformation, if for any $\epsilon > 0$, there exists an integer N, a probability $0 < p < 1$, and gSL transformations W_{N_k}, such that for any $N_k > N$, W_{N_k} satisfy the condition

$$\left\| \frac{W_{N_k}(|\Psi\rangle\langle\Psi|)W_{N_k}^\dagger}{\mathrm{Tr}\big(W_{N_k}(|\Psi\rangle\langle\Psi|)W_{N_k}^\dagger\big)} - \frac{|\Phi\rangle\langle\Phi|}{\mathrm{Tr}(|\Phi\rangle\langle\Phi|)} \right\|_{\mathrm{tr}} < \epsilon, \qquad (22)$$

where $\|\cdot\|_{\mathrm{tr}}$ is the trace norm and

$$\frac{\mathrm{Tr}\big(W_{N_k}|\Psi\rangle\langle\Psi|W_{N_k}^\dagger\big)}{\mathrm{Tr}(|\Psi\rangle\langle\Psi|)} > p. \qquad (23)$$

The idea underlying Definition 18 is that $|\Psi\rangle$ can be transformed to any neighborhood of $|\Phi\rangle$, though not $|\Phi\rangle$ itself, and these neighborhood states become indistinguishable from $|\Phi\rangle$ in the thermodynamic limit.

Using the idea of gSL transformations, we can have a new definition for short-range and long-range entanglement ("new" in a sense that the previous definition was given by gLU transformations).

Definition 19. Short/long-range entanglement. A state is short-range entangled (SRE) if it is convertible to a product state by a gSL transformation. Otherwise, the state is long-range entangled (LRE).

Under this new definition, the states that can be transformed to product states by gLU transformations are SRE. However, the SRE states under gSL transformations will also include some of the states that cannot be transformed to product states by gLU transformations.

As an example, the state

$$|\mathrm{GHZ}_+(a)\rangle = a|0\rangle^{\otimes N_k} + b|1\rangle^{\otimes N_k} \qquad (24)$$

with $|a|^2 + |b|^2 = 1$ cannot be transformed to product states under gLU transformations. However, if one allows gSL transformations, then all the $|\mathrm{GHZ}_+(a)\rangle$ are convertible to $|\mathrm{GHZ}_+(1)\rangle$, i.e., the product state $|0\rangle^{\otimes N_k}$. To see this, one only needs to apply the gSL transformation

$$W_{N_k} = \prod_{i=1}^{N_k} O_i, \qquad (25)$$

where O_i is the invertible operator

$$\begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix} \qquad (26)$$

acting on the i the qubit, and $0 < \gamma < 1$. And we have

$$\begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}^\dagger \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix} \leqslant \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I. \qquad (27)$$

That is,

$$W_{N_k}|\mathrm{GHZ}_+(a)\rangle = a|0\rangle^{\otimes N_k} + b\gamma^{N_k}|1\rangle^{\otimes N_k}$$
$$= |\mathcal{V}_{N_k}\rangle. \qquad (28)$$

Obviously, the right-hand side of Eq. (28) can be arbitrarily close to the product state $|0\rangle^{\otimes N_k}$ as long as N_k is large enough. Furthermore, $\mathrm{Tr}(|\mathcal{V}_{N_k}\rangle\langle\mathcal{V}_{N_k}|) > |a|^2$ for any N_k. Therefore according to Definition 18, $|\mathrm{GHZ}_+(a)\rangle$ is convertible to the product state $|0\rangle^{\otimes N_k}$ by the gSL transformation W_{N_k}.

As another example, we can see how to convert a ground state of any 1D gapped quantum liquid to a product state by gSL transformations. Hence there are no long-range entangled states (i.e., no topological order) in 1D systems. We again use the MPS isometric form [35]

$$\sum_\alpha |\alpha, \dots, \alpha\rangle \otimes |\omega_{D_\alpha}\rangle^{\otimes N_k}. \qquad (29)$$

This state is the convertible to a product state by gSL transformations via two steps: the first step is a gLU transformation to convert the $|\omega_{D_\alpha}\rangle^{\otimes N_k}$ part to a product state and end up with a GHZ state. The next step is to apply the gSL transformation W_{N_k} as given in Eq. (28), which transforms the GHZ state to a product state with a finite probability.

If $|\Psi\rangle$ is convertible to $|\Phi\rangle$ by a gSL transformation, we write

$$|\Psi\rangle \xrightarrow{\mathrm{gSL}} |\Phi\rangle. \qquad (30)$$

Notice that $|\Psi\rangle \xrightarrow{\mathrm{gSL}} |\Phi\rangle$ does not mean $|\Phi\rangle \xrightarrow{\mathrm{gSL}} |\Psi\rangle$. For example, while we have

$$|\mathcal{V}_{N_k}\rangle \xrightarrow{\mathrm{gSL}} |0\rangle^{\otimes N_k}, \qquad (31)$$

$|0\rangle^{\otimes N_k}$ is not gSL convertible to $|\mathcal{V}_{N_k}\rangle$, where $|\mathcal{V}_{N_k}\rangle$ is given in Eq. (28).

That is, the gSL convertibility is not an equivalence relation. It instead defines a partial order (in terms of set theory) on all the quantum states. That is, if $|\Psi\rangle \xrightarrow{\mathrm{gSL}} |\Phi\rangle$ and $|\Phi\rangle \xrightarrow{\mathrm{gSL}} |\Omega\rangle$, then $|\Psi\rangle \xrightarrow{\mathrm{gSL}} |\Omega\rangle$. And there exists $|\Psi\rangle$ and $|\Phi\rangle$ that is not comparable under gSL, i.e., neither $|\Psi\rangle$ is gSL convertible to $|\Phi\rangle$, nor $|\Phi\rangle$ is gSL convertible to $|\Psi\rangle$. Based on this partial order we can further define equivalent classes.

Definition 20. gSL equivalent states. We say that two states $|\Psi\rangle$ and $|\Phi\rangle$ are equivalent under gSL transformations if they are convertible to each other by gSL transformations. That is, $|\Psi\rangle \xrightarrow{\mathrm{gSL}} |\Phi\rangle$ and $|\Phi\rangle \xrightarrow{\mathrm{gSL}} |\Psi\rangle$.

Under this definitions, all the states $|\mathrm{GHZ}_+(a)\rangle$ are in the same gSL class unless $a = 0,1$. The product states with $a = 0,1$ are not in the same gSL class, but any $|\mathrm{GHZ}_+(a)\rangle$ is convertible to the product states by gSL transformations. The converse is not true, that a product state is not convertible to $|\mathrm{GHZ}_+(a)\rangle$ with $a = 0,1$ by gSL transformations.

That is to say, the states with GHZ-form of entanglement are indeed "more entangled" than product states, but they are "close enough" to produce states under gSL transformations. Furthermore, the topological entanglement entropy $S_{\mathrm{topo}}^{\mathrm{tri}}$ for these types of states is unstable under small gSL transformations. In this sense, we can still treat the GHZ-form of entanglement as product states, i.e., states with no long-range entanglement.

C. Long-range entanglement and topological order

We can now define topologically ordered states based on gSL transformations (notice that Definition 11 defines topological order through properties of the Hamiltonian).

Definition 21. Topologically ordered states. Topologically ordered states are LRE gapped quantum liquids. In other words, a ground state $|\Psi\rangle$ of a gapped Hamiltonian has a

nontrivial topological order if it is not convertible to a product state by any gSL transformation.

Not all LRE states can be transformed into each other via gSL transformations. Thus LRE states can belong to different phases, i.e., the LRE states that are not connected by gSL transformations belong to different phases. When we restrict ourselves to LRE gapped quantum liquids, those different phases are nothing but the topologically ordered phases [1–4].

Definition 22. Topologically ordered phases. Topologically ordered phases are equivalence classes of LRE gapped quantum liquids under the gSL transformations.

We believe the following observation is true, which provides a support to the above picture and definition of topologically ordered phases.

Observation 1. The topological entanglement entropy $S_{\text{topo}}^{\text{qua}}$ for topological order is stable under small gSL transformations. Furthermore, $S_{\text{topo}}^{\text{qua}}$ is an invariant for any gSL equivalence class of topological orders.

The first sentence of Observation 1 is in fact similar to stating that topological orders are stable under imaginary time evolution. We can also see this as a direct consequence of the quantum error correction condition given by Eq. (3). Consider any small λ and a local Hamiltonian H, for any local operator M and small λ, the equation

$$\langle \Psi_i | e^{\lambda H} M e^{\lambda H} | \Psi_j \rangle = C_M \delta_{ij} \tag{32}$$

remains to be valid (notice that the constant C_M may change, but the independence of C_M on the subscripts i, j would remain unchanged). To see this, one only needs to write $e^{\lambda H}$ as $1 + \lambda H$.

Similarly, for symmetry-breaking orders, we have the following.

Corollary 1. The tri-topological entanglement entropy $S_{\text{topo}}^{\text{tri}}$ for symmetry-breaking orders is stable under small gSL transformations that do not break symmetry. But unstable under small gSL transformations that break the symmetry. Furthermore, $S_{\text{topo}}^{\text{tri}}$ is not an invariant for any gSL equivalence class of symmetry-breaking orders.

As an example, in the transverse Ising model, the gSL transformation that transforms $|\text{GHZ}_+(a)\rangle$ of different a breaks the $\mathbb{Z}_2$ symmetry. However, $|\text{GHZ}_+(a)\rangle$ of different a are in the same gSL equivalent class, yet with different topological entanglement entropy.

The second sentence of Observation 1 is more subtle, as the topological entanglement entropy $S_{\text{topo}}^{\text{qua}}$ for topological order is not an invariant of gSL transformations [as a finite probability p as given in Eq. (23) may not exist]. This is because that unlike gLU transformations, gSL transformations can be taken arbitrarily close to a noninvertible transformation. For instance, take the gSL transformation W_{N_k} as given in Eq. (25). If we allow γ to be arbitrarily close to zero, then for any wave function $|\mathcal{V}_{N_k}\rangle$, when applying $W_{N_k}|\mathcal{V}_{N_k}\rangle$, it is "as if" we are just projecting everything to $|0\rangle^{N_k}$, which should not protect any topological order in $|\mathcal{V}_{N_k}\rangle$.

On the other hand, the option to choose γ arbitrarily small does not mean any quantum state is gSL convertible to a product state. The key point here is the existence of a finite probably p that is independent of system size N_k, as given in Definition 18. For states with GHZ form of entanglement, we know that we can always find such a finite probability p.

However, for topological ordered states, there does not exist such a finite probability p. In fact, we have $p \to 0$ when $N_k \to \infty$, and furthermore the speed of p approaching 0 may be exponentially fast in terms of the growth of N_k. Therefore $S_{\text{topo}}^{\text{tri}}$ shall remain invariant within any gSL equivalent class.

The above idea is further supported by the results known for geometrical entanglement of topological ordered states [45]. More precisely, let us divide the system to m nonoverlapping local parts, as illustrated in Fig. 1 for one layer. Label each part by i and write the Hilbert space of the system by $\mathcal{H} = \bigotimes_{i=1}^{M} \mathcal{H}_i$. Now for any normalized wave function $|\Psi\rangle \in \mathcal{H}$, the goal is to determine how far $|\Psi\rangle$ is from a normalized product state

$$|\Phi\rangle = \otimes_{i=1}^{M} |\phi_i\rangle \tag{33}$$

with $|\phi_i\rangle \in \mathcal{H}_i$.

The geometric entanglement $E_G(|\Psi\rangle)$ is then revealed by the maximal overlap

$$\Lambda_{\max}(|\Psi\rangle) = \max_{|\Phi\rangle} |\langle \Phi | \Psi \rangle|, \tag{34}$$

and is given by

$$E_G(|\Psi\rangle) = -\log \Lambda_{\max}^2(|\Psi\rangle). \tag{35}$$

Notice that for $\Lambda_{\max}(|\Psi\rangle)$, the maximum is also taken for all the partition of the system into local parts.

It is shown in Ref. [45] for a topologically ordered state $|\Psi\rangle$, $E_G(|\Psi\rangle)$ is proportional to the number of qubits in the system. This means that the probability to project $|\Psi\rangle$ to any product state is exponentially small in terms of the system size N_k. Therefore one shall not expect $|\Psi\rangle$ to be convertible to any product state with a finite probability p.

In contract, the geometric entanglement for states with GHZ-form of entanglement is a constant independent of the system size N_k. And it remains to be the case for the entire symmetric-breaking phase (see, e.g., Ref. [46]), which indicates that these GHZ-form states are convertible to product states with some finite probability p.

D. Emergence of unitarity

The example of toric code discussed in Sec. IV C indicates that the gSL and gLU shall give the same equivalent classes for topological orders, if we restrict ourselves in the case of LRE states. We believe that this holds in general and summarize it as the following observation.

Observation 2. For the LRE gapped quantum liquids, topologically ordered wave functions are equivalence classes of gLU transformations.

This statement is highly nontrivial since, in the above, the concept of LRE and topologically ordered wave functions are defined via nonunitary gSL transformations. Observation 2 reflect one aspect of emergence of unitarity in topologically ordered states.

The locality structure of the total Hilbert space is described by the tensor product decomposition: $\mathcal{H} = \otimes_{i=1}^{m} \mathcal{H}_i$, where $\mathcal{H}_i$ is the local Hilbert space on ith site. The inner product on $\mathcal{H}$ is compatible with the locality structure if it is induced from the inner product on $\mathcal{H}_i$. A small deformation of the inner product on $\mathcal{H}_i$ can be induced by a small gSL transformation.

Consider an orthonormal basis $\{|\Psi\rangle_i\}$ of a topologically ordered degenerate ground-state subspace, where $\langle\Psi_i|\Psi_j\rangle = \delta_{ij}$. Since a small gSL transformation does not change the orthonormal property $\langle\Psi_i|\Psi_j\rangle = \delta_{ij}$, a small deformation of the inner product also does not change this orthonormal property. This is another way of stating that small gSL transformations can be realized as gLU transformations for topologically ordered degenerate ground states, which represent another aspect of emergence of unitarity.

We can then summarize the above argument as the following observation.

Observation 3. For an orthonormal basis $\{|\Psi_i\rangle\}$ of a topologically ordered degenerate ground-state subspace, the orthonormal property $\langle\Psi_i|\Psi_j\rangle = \delta_{ij}$ for $i \neq j$ is invariant under a small deformation of the inner product, as long as the inner product is compatible with the locality structure of the total Hilbert space, that is, for a given orthonormal basis $\{|\Psi_i\rangle\}$, $\langle\Psi_i|\Psi_j\rangle$ does not change, up to an overall factor, under a small deformation of the inner product.

We can also view the emergence of unitarity from the viewpoint of imaginary-time evolution. In particular, if one imaginary-time evolution leads to degenerate ground states for a topologically ordered phase, a slightly different imaginary-time evolution will lead to another set of degenerate ground states for the same topologically ordered phase. The two sets of degenerate ground states are related by a unitary transformation. In this sense, topological order realizes the emergence of unitarity at low energies.

V. SUMMARY AND DISCUSSION

In this work, we have introduced the concept of gapped quantum liquids, which is a special kind of gapped quantum states. There exist gapped quantum states that are not gapped quantum liquids, such as 3D gapped states formed by stacking 2D $\nu = 1/3$ fractional quantum Hall states. The cubic code may provide such an example. We show that topologically ordered states, whose Hamiltonians have stable ground-state degeneracy against any local perturbations, belong to gapped quantum liquids. On-site-symmetry-breaking states are also gapped quantum liquids, whose Hamiltonians have unstable ground-state degeneracy. This result implies that it is incorrect to regard every gapped state without symmetry as a topologically ordered state. There are more exotic gapped states than

topologically ordered states. This result also allows us to give a more precise definition of topological order.

We have shown that gLU classes for stable gapped quantum liquids have a one-to-one correspondence to topological orders. For unstable gapped quantum liquids, gLU transformations assign symmetry-breaking orders to nontrivial classes. We have introduced a new topological entanglement entropy $S_{\text{topo}}^{\text{tri}}$ that can detect symmetry-breaking orders. As topological entanglement entropies $S_{\text{topo}}^{\text{tri}}$ and $S_{\text{topo}}^{\text{qua}}$ are invariants under gLU transformations, we can use them to study both topological orders and symmetry-breaking orders.

We introduce the idea of gSL transformations and define gSL convertibility of quantum states. This convertibility is a partial order (in terms of set theory) on quantum states and it connects symmetry-breaking ground states to the product states. In this sense we re-define the concept of short/long range entanglement and have shown that only topologically ordered states are long-range entangled, in a sense that they are not convertible to product states by gSL transformations.

We show that the topological entanglement entropies $S_{\text{topo}}^{\text{tri}}$ and $S_{\text{topo}}^{\text{qua}}$ for topological order are stable under gSL transformations, and are invariants within any gSL equivalent class, although it is not an invariant of gSL transformations in general (which may be arbitrarily close to a projection onto a product state). On the other hand, $S_{\text{topo}}^{\text{tri}}$ is not stable for symmetry-breaking orders. We further show that for the LRE gapped quantum liquids (i.e., topological orders), gSL equivalent classes coincide with the gLU equivalent classes. This is consistent with the observation that gLU classes for stable gapped quantum liquids have a one-to-one correspondence to topological orders, which realizes the emergence of unitarity at low energies. This result reveals to true essence of topological order and long-range entanglement: the emergence of unitarity at low energies.

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